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A Rectangular Finite Volume Element Method for a Semilinear Elliptic Equation

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Abstract In this paper we extend the idea of interpolated coefficients for semilinear problems to the finite volume element method based on rectangular partition. At first we introduce bilinear finite volume element method with interpolated coefficients for a boundary value problem of semilinear elliptic equation. Next we derive convergence estimate in H^1-norm and superconvergence of derivative. Finally an example is given to illustrate the effectiveness of the proposed method.

Keywords Semilinear elliptic equation · Rectangular mesh · Finite volume element with interpolated coefficients

1 Introduction

Finite volume element method uses a volume integral formulation of the differential equation with a finite partitioning set of volume to discretize the equation, then restricts the

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admissible functions to a linear finite element space to discretize the solution [3, 12–15]. The method has been widely used in computational fluid mechanics because it keeps the mass conservation. As far as the method is concerned, it is identical to the special case of the generalized difference method or GDM proposed by Li-Chen-Wu [9].

The finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal [21]. Later Larsson-Thomee-Zhang [8] studied the semidiscrete linear triangular finite element with interpolated coefficients and Chen-Larsson-Zhang [6] derived almost optimal order convergence on piecewise uniform triangular meshes by use of superconvergence techniques. Xiong-Chen studied superconvergence of triangular quadratic finite element and superconvergence of rectangular finite element for semilinear elliptic problem, respectively, and illustrated the effectiveness of the proposed method in some examples [16–18]. Recently Xiong-Chen first put the interpolation idea into the finite volume element method and studied the finite volume element with interpolated coefficients of the two-point boundary problem [19].

On the quadrilateral grid, Li-Li and Zhu-Li studied the finite volume element method for elliptic problems [10, 20], and in the case of rectangular grid and under stronger assumptions on the smoothness of the solutions, Zhu-Li obtained a superconvergence result in a discrete norm [20]. Wang studied a mixed finite volume element method based on rectangular mesh for biharmonic equations [15]. Recently Man-Shi discussed the finite volume element method of P_1-nonconforming quadrilateral element for elliptic problems and obtain optimal error estimates for general quadrilateral partition [11]. In this paper, we shall put the excellent interpolating coefficients idea into the finite volume element methods on rectangular mesh for a semilinear elliptic equation.

We denote Sobolev space and its norm by $W^{m,p}(\Omega)$ and $\|\cdot\|_{m,p}$, respectively [1]. For $p=2$ denote $H^m(\Omega)=W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ to be the subspace of $H^1(\Omega)$ consisting of functions with vanishing on $\partial\Omega$. $\|\cdot\|_m=\|\cdot\|_{m,2}$ and $\|\cdot\|=\|\cdot\|_{0,2}$. Further we shall denote with p' the adjoint of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1,\ p,p'\geq 1$. We shall assume that the exact solution u is sufficiently smooth for our purpose. The constants C, C_1, C_2, etc. are generic in the paper.

The rest of the paper is organized as follow. First we shall introduce the finite volume element method with interpolated coefficients in Sect. 2 and give preliminaries and some lemmas in Sect. 3. Next we derive optimal order H^1-norm estimate in Sect. 4 and superconvergence of the finite volume element derivative in Sect. 5. Finally the theoretical results are tested by a numerical example in Sect. 6.

2 Finite Volume Element Method with Interpolated Coefficients

Consider the semilinear elliptic boundary value problem

$$-\Delta u+f(u)=g,\quad \text{in } \Omega,\qquad u=0,\quad \text{on } \partial\Omega. \tag{2.1}$$

It is assumed that $f'(s)>0$ for $s\in(-\infty,+\infty)$ and $f''(s)$ is continuous with respect to s. For convenience, assume $\Omega=(0,1)\times(0,1)$.

Let $V\subset\Omega$ be any control volume with piecewise smooth boundary ∂V. Integrate (2.1) over control volume V, then by the Green's formula, the conservative integral of (2.1) reads, finding u, such that

$$-\int_{\partial V}\frac{\partial u}{\partial n}\mathrm{d}s+\int_V f(u)\mathrm{d}x\mathrm{d}y=\int_V g\mathrm{d}x\mathrm{d}y,\quad V\subset\Omega. \tag{2.2}$$

The FVE method of (2.2) consists of replacing by finite-dimensional space of piecewise smooth function and using a finite set of volumes. In this paper, we shall consider rectangular partition of Ω and piecewise bilinear interpolation with interpolated coefficients, for u.

Give a quasi-uniform rectangular partition $\mathcal{J}_h$ for Ω and the nodes are (x_i, y_j), $i = 0, 1, \dots, N$, $j = 0, 1, 2, \dots, M$. Let $\overline{\Omega}_h = \{(x_i, y_j), 0 \le i \le N, 0 \le j \le M\}$ and denote all interior nodes by Ω_h. Further let $h_i = x_i - x_{i-1}, i = 1, 2, \dots, N$, $k_j = y_j - y_{j-1}, j = 1, 2, \dots, M$, $x_{i-1/2} = x_i - h_i/2$, $y_{j-1/2} = y_j - k_j/2$ and $\tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, then $V_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ is a control volume or dual element of node (x_i, y_j). For boundary nodes, their control volumes should be modified correspondingly. All the control volumes constitute the dual partition $\mathcal{J}_h^*$.

Let $S_h \subset H^1(\Omega)$ and $S_{0h} \subset H_0^1(\Omega)$ be both the piecewise bilinear finite element subspace over the partition $\mathcal{J}_h$, and S_h^* be the piecewise constant space over the dual partition $\mathcal{J}_h^*$. Suppose $P(x_i, y_j)$ is an arbitrary node in $\overline{\Omega}_h$. Denote ϕ_P by basic function of S_h at the node P. Denote $V_P = V_{ij}$ by the corresponding dual element of the node P and χ_P by characteristic function over V_P. Define standard Lagrangian interpolation operator $\mathrm{I}_h : C(\Omega) \to S_h$ by

$$\mathrm{I}_h \varphi = \sum_{P \in \overline{\Omega}_h} \varphi(P) \phi_P, \quad \forall \varphi \in C(\Omega), \tag{2.3}$$

and interpolation operator $\mathrm{I}_h^* : C(\Omega) \to S_h^*$ by

$$\mathrm{I}_h^* \varphi = \sum_{P \in \overline{\Omega}_h} \varphi(P) \chi_P, \quad \forall \varphi \in C(\Omega). \tag{2.4}$$

The standard finite volume element scheme of (2.2) can read, finding $\bar{u}_h \in S_{0h}$, such that

$$\begin{aligned} &-\int_{\partial V_{ij}} \frac{\partial \bar{u}_h}{\partial n} \mathrm{d}s + \int_{V_{ij}} f(\bar{u}_h) \mathrm{d}x\mathrm{d}y = \int_{V_{ij}} g \mathrm{d}x\mathrm{d}y, \\ &i = 1, 2, \dots, N-1, j = 1, 2, \dots, M-1. \end{aligned} \tag{2.5}$$

For the sake of simplicity, we now define bilinear finite volume element scheme with interpolated coefficients, finding $u_h \in S_{0h}$, such that

$$\begin{aligned} &-\int_{\partial V_{ij}} \frac{\partial u_h}{\partial n} \mathrm{d}s + \int_{V_{ij}} \mathrm{I}_h f(u_h) \mathrm{d}x\mathrm{d}y = \int_{V_{ij}} g \mathrm{d}x\mathrm{d}y, \\ &i = 1, 2, \dots, N-1, j = 1, 2, \dots, M-1. \end{aligned} \tag{2.6}$$

Equation (2.6) can be further written as difference equations. Denote by $u_{ij} = u_h(x_i, y_j)$, $f_{ij} = f(u_{ij}) = f(u(x_i, y_j))$. For a uniform partition with $M = N$ and $h_i = k_j = h$, (2.6) can be written as

$$\begin{aligned} &\frac{1}{4}\Big[12u_{ij} - (u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}) \\ &\quad - 2(u_{i,j-1} + u_{i+1,j} + u_{i,j+1} + u_{i-1,j})\Big] \\ &\quad + \frac{h^2}{64}\Big[36 f_{ij} + (f_{i-1,j-1} + f_{i+1,j-1} + f_{i+1,j+1} + f_{i-1,j+1}) \end{aligned}$$

$$
\begin{aligned}
&+ 6(f_{i,j-1} + f_{i+1,j} + f_{i,j+1} + f_{i-1,j})\Big] \\
&= \int_{V_{ij}} g\mathrm{d}x\mathrm{d}y, \quad i, j = 1, 2, \ldots, N-1,
\end{aligned} \tag{2.7}
$$

which is a nonlinear system with respect to u_{ij}. For nonregular inner nodes (x_i, y_j), by boundary condition the above equation should be modified correspondingly. For instance,

$$
\begin{aligned}
&\frac{1}{4}[12u_{11} - u_{22} - 2(u_{21} + u_{12})] + \frac{h^2}{64}[36f_{11} + f_{22} + 6(f_{21} + f_{12})] \\
&= \int_{V_{11}} g\mathrm{d}x\mathrm{d}y - \frac{15h^2}{64} f(0).
\end{aligned}
$$

Obviously, the nonlinear system of equations (2.7) is simpler than that of standard finite volume element method. One can be solved by the Newton iteration method in which its tangent matrix can be calculated simply.

3 Preliminaries and Lemmas

In preceding section, we have given the finite volume element scheme with interpolated coefficients. We give preliminary work and some lemmas in this section. Letting

$$
a(u, \mathrm{I}_h^*\varphi_h) = -\sum_{P\in\Omega_h} \varphi_h(P) \int_{\partial V_P \setminus \partial\Omega_h} \frac{\partial u}{\partial n} \mathrm{d}s, \quad \varphi_h \in S_{0h},
$$

$$
(u, \mathrm{I}_h^*\varphi_h) = \sum_{P\in\Omega_h} \varphi_h(P) \int_{V_P} u\mathrm{d}x\mathrm{d}y, \quad \varphi_h \in S_{0h}
$$

and taking $V = V_P$, (2.2) can be rewritten as

$$
a(u, \mathrm{I}_h^*\varphi_h) + (f(u), \mathrm{I}_h^*\varphi_h) = (g, \mathrm{I}_h^*\varphi_h), \quad \forall\varphi_h \in S_{0h}. \tag{3.1}
$$

Analogously, (2.6) is equivalent to finding $u_h \in S_{0h}$, such that

$$
a(u_h, \mathrm{I}_h^*\varphi_h) + (\mathrm{I}_h f(u_h), \mathrm{I}_h^*\varphi_h) = (g, \mathrm{I}_h^*\varphi_h), \quad \forall\varphi_h \in S_{0h}. \tag{3.2}
$$

As $\mathcal{J}_h$ is the quasi-uniformly rectangular partition, there exists positive constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, such that

$$
\gamma_1 \max_i h_i \le \min_i h_i, \qquad \gamma_2 \max_j k_j \le \min_j k_j, \qquad \gamma_3 k_j \le h_i \le \gamma_4 k_j.
$$

Let $h = \max(\max h_i, \max k_j)$. Depicted as in Fig. 1, we convert the integral on the edge of dual partition to the related element τ, then

$$
a(u, \mathrm{I}_h^*\varphi_h) = -\sum_{\tau\in\mathcal{J}_h} \sum_{l=1}^{4} (\varphi_h(P_l) - \varphi_h(P_{l+1})) \int_{\overline{M_l Q}} \frac{\partial u}{\partial n} \mathrm{d}s, \quad \forall\varphi_h \in S_{0h}, \tag{3.3}
$$

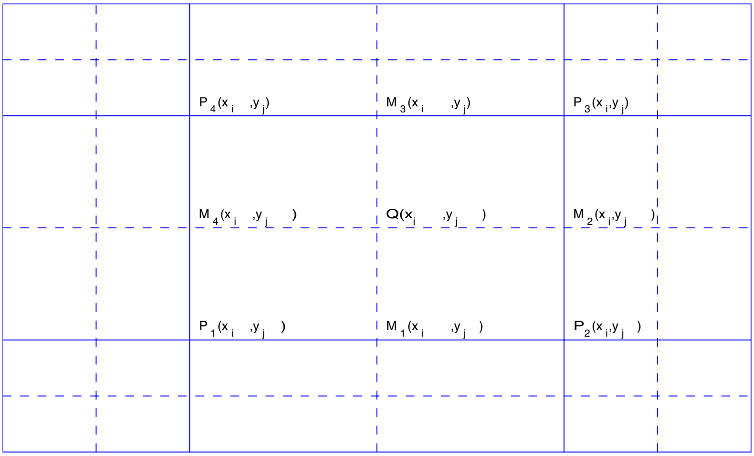

Fig. 1 Illustration for an element τ and its modes

where $P_5 = P_1$. Similarly we can obtain

$$(u, \mathrm{I}_h^* \varphi_h) = \sum_{\tau \in \mathcal{J}_h} \int_\tau u \mathrm{I}_h^* \varphi_h \mathrm{d}x\mathrm{d}y = \sum_{\tau \in \mathcal{J}_h} \sum_{l=1}^{4} \varphi_h(P_l) \int_{\tau \cap V_{P_l}} u \mathrm{d}x\mathrm{d}y. \tag{3.4}$$

Denote $\|\cdot\|_s$ and $|\cdot|_s$ be continuous norm and continuous semi-norm of order s in Sobolev space $H^s(\Omega)$, respectively. Define discrete H^1 semi-norm and discrete L^2 norm, respectively, by

$$|\varphi_h|_{1,h} = \left\{ \sum_{\tau \in \mathcal{J}_h} |\varphi_h|_{1,h,\tau}^2 \right\}^{1/2}, \quad \|\varphi_h\|_{0,h} = \left\{ \sum_{\tau \in \mathcal{J}_h} \|\varphi_h\|_{0,h,\tau}^2 \right\}^{1/2}, \quad \varphi_h \in S_{0h}, \tag{3.5}$$

where $\tau = \overline{P_1 P_2 P_3 P_4} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, shown as in Fig. 1 and

$$|\varphi_h|_{1,h,\tau}^2 = \frac{k_j}{2h_i} \sum_{l=1,3} (\varphi_h(P_{l+1}) - \varphi_h(P_l))^2 + \frac{h_i}{2k_j} \sum_{l=2,4} (\varphi_h(P_{l+1}) - \varphi_h(P_l))^2,$$

$$\|\varphi_h\|_{0,h,\tau}^2 = \frac{h_i k_j}{4} \sum_{l=1}^{4} \varphi_h(P_l)^2.$$

From [15], we have the following lemmas.

Lemma 3.1 *For* $\forall \varphi_h \in S_{0h}$, $|\varphi_h|_{1,h}$ *is equivalent to* $|\varphi_h|_1$ *and* $\|\varphi_h\|_{0,h}$ *is equivalent to* $\|\varphi_h\|_0$, *that is, the following inequalities hold*

$$\frac{\sqrt{3}}{3} |\varphi_h|_{1,h} \le |\varphi_h|_1 \le |\varphi_h|_{1,h}, \qquad \frac{1}{3} \|\varphi_h\|_{0,h} \le \|\varphi_h\|_0 \le \|\varphi_h\|_{0,h}. \tag{3.6}$$

Proof See [15]. □

Lemma 3.2

$$a(\varphi_h, \mathrm{I}_h^*\varphi_h) \ge \frac{1}{2}|\varphi_h|_1^2, \quad \forall \varphi_h \in S_{0h}, \tag{3.7}$$

$$|a(u-\mathrm{I}_h u, \mathrm{I}_h^*\varphi_h)| \le Ch\|u\|_2|\varphi_h|_1, \quad \forall u \in H_0^1(\Omega),\ \varphi_h \in S_{0h}. \tag{3.8}$$

Proof See [15]. □

From [4, 5, 9], we have two lemmas.

Lemma 3.3 *The semi-norm* $|\cdot|_1$ *and the norm* $\|\cdot\|_1$ *are equivalent in the space* $H_0^1(\Omega)$, *that is, there exists positive constants* C *such that*

$$|\varphi_h|_1 \le \|\varphi_h\|_1 \le C|\varphi_h|_1. \tag{3.9}$$

Proof See [9]. □

Lemma 3.4 *The interpolation operator* I_h^* *has the following properties*

$$\int_\tau \mathrm{I}_h^* v_h \mathrm{d}x\mathrm{d}y = \int_\tau v_h \mathrm{d}x\mathrm{d}y \quad \forall v_h \in S_{0h},\ \textit{for any}\ \tau \in \overline{\Omega}_h, \tag{3.10}$$

$$\int_{P_lP_{l+1}} \mathrm{I}_h^* v_h \mathrm{d}s = \int_{P_lP_{l+1}} v_h \mathrm{d}s \quad \forall v_h \in S_{0h},\ \textit{for any}\ \tau \in \overline{\Omega}_h, \tag{3.11}$$

$$\|\varphi_h - \mathrm{I}_h^*\varphi_h\|_{0,p,\tau} \le Ch|\varphi_h|_{1,p,\tau}, \quad \forall \varphi_h \in S_{0h},\ 1 \le p \le \infty. \tag{3.12}$$

Proof For $v_h \in S_h$ in $\tau \in \mathcal{J}_h$, write v_h as

$$v_h = v_h(P_1)\frac{(x_i-x)(y_j-y)}{h_ik_j} + v_h(P_2)\frac{(x-x_{i-1})(y_j-y)}{h_ik_j} + v_h(P_3)\frac{(x-x_{i-1})(y-y_{j-1})}{h_ik_j} + v_h(P_4)\frac{(x_i-x)(y-y_{j-1})}{h_ik_j} \triangleq \sum_{l=1}^4 v_h(P_l)\varphi_l,$$

then we have

$$\int_\tau \mathrm{I}_h^* v_h \mathrm{d}x\mathrm{d}y = \sum_{l=1}^4 v_h(P_l)\int_{\tau\cap V_{P_l}} \mathrm{d}x\mathrm{d}y = \frac{1}{4}h_ik_j\sum_{l=1}^4 v_h(P_l),$$

$$\int_\tau v_h \mathrm{d}x\mathrm{d}y = \sum_{l=1}^4 \int_\tau v_h(P_l)\varphi_l \mathrm{d}x\mathrm{d}y = \frac{1}{4}h_ik_j\sum_{l=1}^4 v_h(P_l).$$

The desired result (3.10) is derived from two above formulations. From [5] we also obtain (3.11) and (3.12). □

In addition in [5], for the interpolation operator I_h, the following lemma was derived.

Lemma 3.5 *Assume* a, w *are sufficiently smooth functions. Let* $\mathrm{I}_h w \in S_{0h}$ *be the Lagrangian interpolation of* w, *then*

$$|(a\,(w-\mathrm{I}_h w), \varphi_h)| \le Ch^2\|w\|_{3,p}\|\varphi_h\|_{1,p'}, \quad \forall \varphi_h \in S_{0h}, \tag{3.13}$$

for $\frac{1}{p}+\frac{1}{p'}=1$, $1<p\le\infty$.

Proof See [5]. □

Lemma 3.6 *Assume* $w\in H_0^1(\Omega)\cap H^2(\Omega)$, *then there exists a positive constant* C, *independent of the mesh size* h, *such that*

$$|(w-\mathrm{I}_h w,\mathrm{I}_h^*\varphi_h)|\le Ch^2\|w\|_2\|\varphi_h\|_1,\quad \forall\varphi_h\in S_{0h}. \tag{3.14}$$

Proof In view of Lemma 3.4 and the Schwartz inequality, we find

$$\begin{aligned}
&|(w-\mathrm{I}_h w,\varphi_h)-(w-\mathrm{I}_h w,\mathrm{I}_h^*\varphi_h)|\\
&\quad\le\sum_{\tau\in\mathcal{J}_h}\int_\tau|w-\mathrm{I}_h w||\varphi_h-\mathrm{I}_h^*\varphi_h|\mathrm{d}x\mathrm{d}y\\
&\quad\le\sum_{\tau\in\mathcal{J}_h}C_\tau h^3\|w\|_{2,\tau}\|\varphi_h\|_1\le Ch^3\|w\|_2\|\varphi_h\|_1.
\end{aligned}\tag{3.15}$$

Combining of this, and (3.13) with $a\equiv 1$ and $p=2$, gives the desired (3.14). □

4 Error Estimate of the Finite Volume Element

We have given the definition of the finite volume element scheme with interpolated coefficients. Now we analyze the error of the scheme. To start our analysis, we introduce an auxiliary bilinear form

$$A(u;w,\mathrm{I}_h^*\varphi_h)=a(w,\mathrm{I}_h^*\varphi_h)+(f'(u)w,\mathrm{I}_h^*\varphi_h),$$

where u is the exact solution in (2.2). For the auxiliary bilinear form $A(u;\cdot,\cdot)$, we have following positive definite properties.

Lemma 4.1 *For* $u\in H_0^1(\Omega)$, $A(u;w_h,\mathrm{I}_h^*w_h)$ *is positive definite for sufficiently small* h, *i.e., there exists a positive constant* α, *such that*

$$A(u;w_h,\mathrm{I}_h^*w_h)\ge\alpha(u,f)\|w_h\|_1^2. \tag{4.1}$$

Proof Rewrite $A(u;w_h,\mathrm{I}_h^*w_h)$ as

$$\begin{aligned}
A(u;w_h,\mathrm{I}_h^*w_h)&=a(w_h,\mathrm{I}_h^*w_h)+(f'(u)w_h,w_h)-[(f'(u)w_h,w_h)\\
&\quad-(f'(u)w_h,\mathrm{I}_h^*w_h)].
\end{aligned}\tag{4.2}$$

Application of Lemmas 3.2 and 3.3 yields

$$a(w_h,\mathrm{I}_h^*w_h)\ge C_1\|w_h\|_1^2. \tag{4.3}$$

Note that $f'(s)>0$ and let $C_2=\inf_{P\in\Omega}f'(u(P))$, then we have

$$(f'(u)w_h,w_h)\ge C_2\|w_h\|_0^2\ge 0. \tag{4.4}$$

In terms of (3.10) in Lemma 3.4, we obtain

$$\begin{aligned}
&\left|(f'(u)w_h, w_h) - (f'(u)w_h, \mathrm{I}_h^* w_h)\right| \\
&= \left|\sum_{\tau\in\mathcal{J}_h}\int_\tau f'(u)w_h(w_h - \mathrm{I}_h^* w_h)\mathrm{d}x\mathrm{d}y\right| \\
&= \left|\sum_{\tau\in\mathcal{J}_h}\int_\tau (f'(u)w_h - C_\tau)(w_h - \mathrm{I}_h^* w_h)\mathrm{d}x\mathrm{d}y\right| \le \sum_{\tau\in\mathcal{J}_h} Ch|f'(u)w_h|_{1,\tau} h|w_h|_{1,\tau} \\
&\le \max_\Omega(|f''(u)\nabla u|, |f'(u)|)\sum_{\tau\in\mathcal{J}_h} Ch^2\|w_h\|_{1,\tau}^2 \le C_3 h^2\|w_h\|_1^2, \qquad (4.5)
\end{aligned}$$

where C_τ is the value of $f'(u)w_h$ at the center point in $\tau\in\mathcal{J}_h$. Together (4.3), (4.4) with (4.5) yields

$$A(u; w_h, \mathrm{I}_h^* w_h) \ge C_1\|w_h\|_1^2 - C_3 h^2\|w_h\|_1^2 = (C_1 - C_3 h^2)\|w_h\|_1^2,$$

which implies the desired result (4.1) for sufficiently small h. □

Now we state the main result of this section.

Theorem 4.1 *Assume $u \in H_0^1(\Omega)\cap H^2(\Omega)$ is the solution of (2.1) and $\mathcal{J}_h$ is quasi-uniformly rectangular partition of domain Ω, then the approximate solution $u_h \in S_{0h}$ of finite volume element method (2.7) with interpolated coefficients converges to the exact solution u has the following estimate*

$$\|u - u_h\|_1 \le C(u, f)h, \qquad (4.6)$$

for sufficiently small h.

Proof Subtracting (3.2) from (3.1), we obtain the following error equation

$$a(u - u_h, \mathrm{I}_h^*\varphi_h) + (f(u) - \mathrm{I}_h f(u_h), \mathrm{I}_h^*\varphi_h) = 0. \qquad (4.7)$$

We analyze the following function

$$f(u) - \mathrm{I}_h f(u_h) = (f(u) - \mathrm{I}_h f(u)) + \mathrm{I}_h(f(u) - f(u_h)) = r_1 + r_2. \qquad (4.8)$$

By expansion in $\tau\in\mathcal{J}_h$, we have

$$\begin{aligned}
r_2 = \mathrm{I}_h(f(u) - f(u_h)) &= \sum_{P\in\tau}(f(u(P)) - f(u_h(P)))\phi_P \\
&= f'(u)(\mathrm{I}_h u - u_h) + \delta_1\max_\tau|\mathrm{I}_h u - u_h| + \delta_2\max_\tau|\mathrm{I}_h u - u_h|^2 \\
&= f'(u)(\mathrm{I}_h u - u_h) + r_3, \qquad (4.9)
\end{aligned}$$

where

$$r_3 = \delta_1\max_\tau|\mathrm{I}_h u - u_h| + \delta_2\max_\tau|\mathrm{I}_h u - u_h|^2,$$

$$\delta_1 = C \max_{P',P''\in\tau} |f'(u(P')) - f'(u(P''))| = \mathcal{O}(h),$$

$$\delta_2 = \frac{1}{2} f''(\xi) = \mathcal{O}(1).$$

Substituting (4.9) into (4.7), we find

$$A(u; u_h - \mathrm{I}_h u_h, \mathrm{I}_h^* \varphi_h) = a(u - \mathrm{I}_h u, \mathrm{I}_h^* \varphi_h) + (r_1 + r_3, \mathrm{I}_h^* \varphi_h),$$

where $r_1 = f(u) - \mathrm{I}_h f(u)$. Letting $\theta = u_h - \mathrm{I}_h u \in S_{0h}$ and taking $\varphi_h = \theta$, application of Lemma 4.1, and Lemma 3.6 yields

$$\alpha\|\theta\|_1^2 \le Ch\|\theta\|_1 + C(h\|\theta\|_{0,\infty} + \|\theta\|_{0,\infty}^2)\|\theta\|_{0,1}.$$

Recalling for Bramble [2] that

$$\|\theta\|_{0,\infty} \le C|\ln h|^{1/2}\|\nabla\theta\| \le C|\ln h|^{1/2}\|\theta\|_1$$

holds for $\theta \in S_{0h}$ and by use of the well known Sobolev inequality

$$\|v\|_{0,p} \le C\|v\|_1, \quad 1 \le p < \infty,$$

we get

$$\alpha\|\theta\|_1^2 \le Ch\|\theta\|_1 + C(h|\ln h|^{1/2}\|\theta\|_1 + |\ln h|\|\theta\|_1^2)\|\theta\|_1.$$

Omitting the common factor $\|\theta\|_1$, this implies

$$\alpha\|\theta\|_1 \le Ch + Ch|\ln h|^{1/2}\|\theta\|_1 + C|\ln h|\|\theta\|_1^2. \tag{4.10}$$

For $h \le h'$, omitting the second term of the right-side implies

$$\|\theta\|_1 \le C_1 h + C_2|\ln h|\|\theta\|_1^2. \tag{4.11}$$

Now adopting a continuity argument by imitating the method by Frehse-Rannacher [7], we show

$$\|\theta\|_1 \le \|\mathrm{I}_h u - u_h\|_1 \le 2C_1 h. \tag{4.12}$$

For $s \in [0, 1]$ considering the auxiliary semilinear elliptic problems (P^s): Find u^s such that

$$-\Delta u^s + sf(u^s) = sg, \quad \text{in } \Omega, \qquad u^s = 0, \quad \text{on } \partial\Omega. \tag{4.13}$$

Obviously, for $s = 1$ this is our original problem (2.1) and for $s = 0$ we have $u^0 = 0$. We shall assume the following condition on $\Omega = (0, 1) \times (0, 1)$. For any $s \in [0, 1]$, there is a solution u^s of problem (P^s) and there is a constant Γ such that set

$$N_\Gamma = \left\{\omega|\omega \in H^2(\Omega) \cap H_0^1(\Omega), \max_{\Omega}|u - \omega| < \Gamma\right\}$$

is some neighborhood of exact solution u in (2.1).

We approximate problem (P^s) by the discrete problems (P_h^s): Find $u_h^s \in S_{0h}$ such that

$$a(u_h^s, \mathrm{I}_h^* v_h) + s(\mathrm{I}_h f(u_h^s), \mathrm{I}_h^* v_h) = s(g, \mathrm{I}_h^* v_h), \quad \forall v_h \in S_{0h}. \tag{4.14}$$

We intend to show that (P_h^s) is solvable. For each h, we define the set $E_h \subset [0, 1]$ by

$$E_h = \{s \in [0,1] | (\mathrm{P}_h^s) \text{ has a solution } u_h^s \in N_\Gamma \text{ and there holds } \|\mathrm{I}_h u^s - u_h^s\|_1 \le 2C_1 h\},$$

where C_1 is the constant appearing in (4.11).

(i) E_h is not empty. In fact, for $s = 0$, $u^s = 0$ and $u_h^s = 0$ are the solutions of continuous and the discrete problem, respectively.

(ii) E_h is open in [0, 1]. In fact, if $s \in E_h$ then (P_h^s) is solvable and using the monotonicity condition, we obtain the solvability of (P_h^s) for all t in a neighborhood of s via the implicit function theorem. By the implicit function theorem u_h^t depends continuously on t. Thus properly shorten the neighborhood such that the strict inequality $\|\mathrm{I}_h u^s - u_h^s\|_1 < 2C_1 h$ and $u_h^s \in N_\Gamma$ is still valid and we have $t \in E_h$ for these t.

(iii) E_h is closed. Let $s(j) \in E_h$ and $s(j) \to s$, $j \to \infty$. Since $u_h^{s(j)} \in N_\Gamma$ there is a cluster point u_h^s which is the unique solution of (P_h^s) and satisfies $\|\mathrm{I}_h u^s - u_h^s\|_1 \le 2C_1 h$. Recalling for (4.11) we conclude

$$\|\mathrm{I}_h u^s - u_h^s\|_1 \le C_1 h + 4C_2 C_1^2 |\ln h| h^2 \le C_1(1 + 4C_1 C_2 |\ln h| h) h,$$

then for $h \le h'' = h''(C_1, C_2)$, we have $4C_1 C_2 |\ln h| h < 1$ and $\|\mathrm{I}_h u^s - u_h^s\|_1 < 2C_1 h$, i.e. the strict inequality.

From (i)–(iii), we know that for $h \le \min(h', h'')$ the set E_h is not empty, closed and open with respect to [0, 1] and thus must coincide with [0, 1]. Noting for $s = 1$, (P_h^1) is solvable. We prove that inequality (4.12) and $u_h \in N_\Gamma$ hold for appropriately small h.

Finally, the desired result (4.6) follows from (4.12) and the interpolation property

$$\|u - \mathrm{I}_h u\|_1 \le Ch\|u\|_2.$$

□

5 Superconvergence of Derivative

In this section we show superconvergence of derivative of the finite volume element by (2.6). First we give superclose property of derivative for $\mathrm{I}_h u$.

Lemma 5.1 *For sufficiently smooth function u, let $\mathrm{I}_h u \in S_h$ be the Lagrange interpolation of u, then we have superclose property*

$$|\nabla(u - \mathrm{I}_h u)| = \mathcal{O}(h^2), \tag{5.1}$$

for the center in all $\tau \in \mathcal{J}_h$.

Proof See [5]. □

For the finite volume element with interpolated coefficients on the rectangular mesh for the semilinear elliptic problem (2.6), we have the following a superconvergence result of derivative.

Theorem 5.1 *Assume that $u \in W_\infty^3(\Omega)$. Assume that $f'(s) > 0$ for $s \in (-\infty, +\infty)$ and $f''(s)$ is continuous with respect to s, and $g \in W_0^1(\Omega)$. Let the rectangular partition $\mathcal{J}_h$ of* *the domain Ω be quasi uniform. Then the finite volume element with interpolated coefficients for the semilinear elliptic problem* (2.6) *has superconvergence of derivative, i.e.*,

$$|\nabla(u-u_h)| = \mathcal{O}(h^2|\ln h|)$$

hold for centers $(x_{i-1/2}, y_{j-1/2})$ *in every element* τ_{ij} $(i=1,2,\dots,N,\ j=1,2,\dots,M)$.

Proof Choosing $v=v_h\in S_{0h}$, we get

$$a(u,v_h)+(f(u),v_h)=(g,v_h). \tag{5.2}$$

Subtracting (3.2) from (5.2), we have

$$\begin{aligned}
&a(u-u_h,v_h)+[a(u,v_h)-a(u_h,\mathrm{I}_h^*v_h)]+[a(u_h-u,v_h)-a(u_h-u,\mathrm{I}_h^*v_h)]\\
&\quad+(f(u)-\mathrm{I}_h f(u),v_h)+[(\mathrm{I}_h f(u),v_h)-(\mathrm{I}_h f(u),\mathrm{I}_h^*v_h)]\\
&\quad+(\mathrm{I}_h(f(u)-f(u_h)),\mathrm{I}_h^*v_h)\\
&=(g,v_h)-(g,\mathrm{I}_h^*v_h).
\end{aligned}\tag{5.3}$$

Let $R=u-\mathrm{I}_h u$, $\theta=u_h-\mathrm{I}_h u$. It follows from (5.3), (4.8) and (4.9) that

$$\begin{aligned}
A(u;\theta,v_h)&=a(\theta,v_h)+(f'(u)\theta,v_h)\\
&=a(R,v_h)+[a(u,v_h)-a(u,\mathrm{I}_h^*v_h)]+[a(u_h-u,v_h)-a(u_h-u,\mathrm{I}_h^*v)]\\
&\quad+(r_1,v_h)+[(\mathrm{I}_h f(u),v_h)-(\mathrm{I}_h f(u),\mathrm{I}_h^*v_h)]+(r_3,\mathrm{I}_h^*v_h)\\
&\quad+[(g,\mathrm{I}_h^*v_h)-(g,v_h)].
\end{aligned}\tag{5.4}$$

Recalling [5], we get

$$|a(R,v_h)|=\left|\int_\Omega \nabla R\nabla v_h \mathrm{d}x\mathrm{d}y\right|\le Ch^2\|u\|_{3,\infty}\|v_h\|_{1,1}. \tag{5.5}$$

By Lemma 3.5 with $p=+\infty$,

$$|(r_1,v_h)|=|(f(u)-\mathrm{I}_h f(u),v_h)|\le Ch^2\|f(u)\|_{2,\infty}\|v_h\|_{1,1}. \tag{5.6}$$

In terms of Lemma 3.4 and the trace theorem, we have

$$\begin{aligned}
&|a(u,v_h)-a(u,I_h^*v_h)|\\
&=\left|\sum_{V\in\mathcal{J}_h^*}\left(\int_V \nabla u\nabla v_h\mathrm{d}x\mathrm{d}y-v_h(P)\int_{\partial V}\frac{\partial u}{\partial n}\mathrm{d}s\right)\right|\\
&=\left|\sum_{V\in\mathcal{J}_h^*}\int_{\partial V}\frac{\partial u}{\partial n}(v_h-v_h(P))\mathrm{d}s\right|=\left|\sum_{\tau\in\mathcal{J}_h}\sum_{l=1}^{4}\int_{P_lP_{l+1}}\left(\frac{\partial u}{\partial n}-w_\tau\right)(v_h-\mathrm{I}_h^*v_h)\mathrm{d}s\right|\\
&\le\sum_{\tau\in\mathcal{J}_h}C_\tau h^2\|u\|_{2,\infty,P_lP_{l+1}}\|v_h\|_{1,1,P_lP_{l+1}}\le Ch^2\|u\|_{2,\infty}\|v_h\|_{1,1}
\end{aligned}\tag{5.7}$$

where w_τ is the value of $\frac{\partial u}{\partial n}$ at midpoint in the $P_l P_{l+1}$. Analogously,

$$\begin{aligned}
&|a(u_h - u, v_h) - a(u_h - u, I_h^* v_h)| \\
&\quad = \left| \sum_{\tau\in\mathcal{J}_h} \sum_{l=1}^{4} \int_{P_l P_{l+1}} \frac{\partial (u_h - u)}{\partial n} (v_h - \mathrm{I}_h^* v_h) \mathrm{d}s \right| \\
&\quad \le Ch \|u_h - u\|_{1,\infty} \|v_h\|_{0,1} \le Ch \|u_h - u\|_{1,\infty} \|v_h\|_{1,1}.
\end{aligned} \tag{5.8}$$

In terms of Lemma 3.4 we derive another two estimates

$$|(\mathrm{I}_h f(u), v_h) - (\mathrm{I}_h f(u), \mathrm{I}_h^* v_h)| \le Ch^2 \|f(u)\|_{1,\infty} \|v_h\|_1, \tag{5.9}$$

$$|(g, v_h) - (g, \mathrm{I}_h^* v_h)| \le Ch^2 \|g\|_{1,\infty} \|v_h\|_{1,1}. \tag{5.10}$$

Substituting from (5.5)–(5.10) into (5.4) implies

$$A(u; \theta, v_h) \le Ch^2 \|v_h\|_{1,1} + Ch \|u_h - u\|_{0,\infty} \|v_h\|_{1,1} + C\|r_3\|_{0,\infty} \|v_h\|_{1,1}.$$

Noting

$$\|u_h - u\|_{0,\infty} \le \|R\|_{0,\infty} + \|\theta\|_{0,\infty} \le C(u)h^2 + \|\theta\|_{0,\infty}$$

and

$$\|r_2\|_{0,\infty} \le Ch\|\theta\|_{0,\infty} + C\|\theta\|_{0,\infty}^2,$$

this yields

$$A(u; \theta, v_h) \le Ch^2 \|v_h\|_{1,1} + Ch\|\theta\|_{1,+\infty} \|v_h\|_{1,1} + C\|\theta\|_{1,+\infty}^2 \|v_h\|_{1,1}. \tag{5.11}$$

Choosing $v_h = G_h$ as discrete derivative type Green function with respect to bilinear $A(u_h; \cdot, \cdot)$ and noting $\|G_h\|_{1,1} \le |\ln h|$ (see [5]), we can obtain

$$\|\theta\|_{1,\infty} \le Ch^2 |\ln h| + Ch\|\theta\|_{1,\infty} |\ln h| + C\|\theta\|_{1,\infty}^2 |\ln h|.$$

For sufficiently small h, omitting the second term of the right-side implies

$$\|\theta\|_{1,\infty} \le Ch^2 |\ln h| + C\|\theta\|_{1,\infty}^2 |\ln h|. \tag{5.12}$$

Analogously adopting the continuity argument in the proof of Theorem 4.1 in Sect. 4, we have

$$|\nabla\theta|_{0,\infty} \le \|\theta\|_{1,\infty} \le Ch^2 |\ln h|. \tag{5.13}$$

Combining this with (5.1) of Lemma 5.1 yields

$$|\nabla(u - u_h)| \le |\nabla(u - \mathrm{I}_h u)| + |\nabla\theta| = |\nabla(u - \mathrm{I}_h u)| + \mathcal{O}(h^2 |\ln h|),$$

i.e.,

$$|\nabla(u - u_h)| = \mathcal{O}(h^2 |\ln h|),$$

for the center $(x_{i-1/2}, y_{j-1/2})$ of every $\tau_{i,j} \in \mathcal{J}_h$. This completes the proof of the theorem. □

Assume that M, N are both even integers. Consider a coarser rectangular partition $\mathcal{J}_{2h}$ for Ω, each rectangle element $K_{ij} \in \mathcal{J}_{2h}$ contains four small rectangles: $\tau_{2i-1,2j-1}$, $\tau_{2i-1,2j}$, $\tau_{2i,2j-1}$ and $\tau_{2i,2j}$ of $\mathcal{J}_h$. We can construct a biquadratic interpolation $\mathcal{Q}u_h$ of the finite volume element approximation u_h by

$$\mathcal{Q}u_h = \sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{m,n}(x, y)u_{2i-3+m,2j-3+n}, \quad \forall (x, y) \in K_{ij} \in \mathcal{J}_{2h}, \tag{5.14}$$

where $\lambda_{m,n}(x, y)$ are the Lagrangian shape functions at node (x_{2i-3+m}, y_{2j-3+n}), $m, n = 1, 2, 3$ of $K_{ij} \in \mathcal{J}_{2h}$, $i = 1, 2, \ldots, M/2$, $j = 1, 2, \ldots, N/2$.

Theorem 5.2 *Under the assumptions of Theorem 5.1, the biquadratic interpolation $\mathcal{Q}u_h$ has uniform superconvergence*

$$\|\nabla(u - \mathcal{Q}u_h)\|_{0,\infty} = \mathcal{O}(h^2|\ln h|). \tag{5.15}$$

Proof Let $e = u - u_h$, then we have

$$\mathcal{Q}e = \sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{m,n}(x, y)e_{2i-3+m,2j-3+n} \tag{5.16}$$

$$= \sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{m,n}(x, y)\theta_{2i-3+m,2j-3+n}. \tag{5.17}$$

Noticing that $\sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{m,n}(x, y) = 1$, we have

$$\nabla \mathcal{Q}e = \sum_{m=1}^{3}\sum_{n=1}^{3} \nabla\lambda_{m,n}(x, y)\theta_{2i-3+m,2j-3+n}$$

$$= \sum_{m=1}^{3}\sum_{n=1}^{3} \nabla\lambda_{m,n}(x, y)(\theta_{2i-3+m,2j-3+n} - \theta_{2i-1,2j-1}).$$

Recalling (5.13) and using the mean value theorem yields

$$\theta_{2i-3+m,2j-3+n} - \theta_{2i-1,2j-1}$$
$$= \nabla\theta(\xi_m, \eta_n) \cdot (x_{2i-3+m} - x_{2i-1}, y_{2j-3+n} - x_{2j-1}) = \mathcal{O}(h^3|\ln h|).$$

Together with $\nabla\lambda_{m,n}(x, y) = \mathcal{O}(h^{-1})$ for all $(x, y) \in K_{ij}$ and the interpolation property

$$|\nabla(u - \mathcal{Q}u_h)| = \mathcal{O}(h^2),$$

we have

$$\nabla(u - \mathcal{Q}u_h) = \nabla(u - \mathcal{Q}u) + \nabla(\mathcal{Q}(u - u_h))$$
$$= \mathcal{O}(h^2) + \mathcal{O}(h^{-1}) \cdot \mathcal{O}(h^3|\ln h|) = \mathcal{O}(h^2|\ln h|).$$

This completes the proof of the theorem. □

Table 1 The mean absolute error and relative error at all nodes

	E_u	R_u		E_u	R_u
$N=4$	5.9228×10^{-4}	5.4901×10^{-2}	$N=8$	6.2638×10^{-5}	1.3907×10^{-2}
$N=16$	7.2758×10^{-6}	3.5284×10^{-3}	$n=32$	8.7911×10^{-7}	8.9107×10^{-4}

Table 2 The mean error of derivative at dual partition nodes

	E_{Du}	Rate		E_{Du}	Rate
$N=4$	5.9491×10^{-3}	–	$N=8$	1.4476×10^{-3}	2.0390
$N=16$	3.6494×10^{-4}	1.9879	$N=32$	9.2876×10^{-5}	1.9743

6 Numerical Example

In this section we present a numerical experiment to verify the theoretical investigations. We consider the following semilinear elliptic problem

$$-\Delta u+u^3=g,\quad \text{in } \Omega=(0,1)\times(0,1),\qquad u=0,\quad \text{on } \partial\Omega, \tag{6.1}$$

where the function g is chosen as

$$\begin{aligned}g(x,y)&=2(x(1-x)+y(1-y))\cos(x(1-y))\\&\quad+y(1-x)(x^2+(1-y)^2)\sin(x(1-y))+y^3(1-x)^3\sin^3(x(1-y)),\end{aligned}$$

such that the known solution is

$$u(x,y)=y(1-x)\sin(x(1-y)).$$

The domain $\overline{\Omega}=[0,1]\times[0,1]$ is equally divided into $N\times N$ elements $\tau_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j],\ i,j=1,2,\dots,N$, here $x_i=\frac{i}{N},\ y_j=\frac{j}{N},\ i,j=0,1,2,\dots,N$.

Compute it by the bilinear finite volume element method with interpolated coefficients. The results are listed in Table 1.

In Table 1, E_u, R_u denote average absolute error and relative error of u_h at partition nodes respectively. From Table 1, one can see that the bilinear finite volume element with interpolated coefficients can converge.

Furthermore, we compute the derivatives of u_h at the midpoints of every element, i.e., at the dual partition nodes by the bilinear finite volume element method with interpolated coefficients. The results are shown in Table 2.

In Table 2, E_{Du} denotes mean absolute error about derivative of u_h at dual partition nodes. It is shown from Table 2 that derivative of the bilinear finite volume element with interpolated coefficients converges of $\mathcal{O}(h^2|\ln h|)$ which is superconvergent and conforms to our preceding theoretical analysis.

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