Journal of Scientific Computing, Vol. 32, No. 2, August 2007 (© 2007)
DOI: 10.1007/s10915-007-9136-x

A Nonoverlapping Domain Decomposition Method for Legendre Spectral Collocation Problems

Bernard Bialecki[1] **and Andreas Karageorghis**[2,3]

Received October 19, 2006; accepted (in revised form) March 17, 2007; Published online May 21, 2007

We consider the Dirichlet boundary value problem for Poisson's equation in an L-shaped region or a rectangle with a cross-point. In both cases, we approximate the Dirichlet problem using Legendre spectral collocation, that is, polynomial collocation at the Legendre–Gauss nodes. The L-shaped region is partitioned into three nonoverlapping rectangular subregions with two interfaces and the rectangle with the cross-point is partitioned into four rectangular subregions with four interfaces. In each rectangular subregion, the approximate solution is a polynomial tensor product that satisfies Poisson's equation at the collocation points. The approximate solution is continuous on the entire domain and its normal derivatives are continuous at the collocation points on the interfaces, but continuity of the normal derivatives across the interfaces is not guaranteed. At the cross point, we require continuity of the normal derivative in the vertical direction. The solution of the collocation problem is first reduced to finding the approximate solution on the interfaces. The discrete Steklov–Poincaré operator corresponding to the interfaces is self-adjoint and positive definite with respect to the discrete inner product associated with the collocation points on the interfaces. The approximate solution on the interfaces is computed using the preconditioned conjugate gradient method. A preconditioner is obtained from the discrete Steklov–Poincaré operators corresponding to pairs of the adjacent rectangular subregions. Once the solution of the discrete Steklov–Poincaré equation is obtained, the collocation solution in each rectangular subregion is computed using a matrix decomposition method. The

[1] Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401, USA.
[2] Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus.
[3] To whom correspondence should be addressed. E-mail: andreask@ucy.ac.cy

total cost of the algorithm is $O(N^3)$, where the number of unknowns is proportional to N^2.

KEY WORDS: Dirichlet problem; Poisson's equation; nonoverlapping domain decomposition; Legendre spectral collocation; preconditioned conjugate gradient method.
AMS(MOS) SUBJECT CLASSIFICATIONS: 65N35; 65N55.

1. INTRODUCTION

Nonoverlapping domain decomposition methods for solving boundary value problems with finite difference, finite element, and spectral element disretizations are surveyed in [5, 15, 17, 20]. According to [1], there are two nonoverlapping domain decomposition approaches for spectral discretizations: one based on the variational formulation of the continuous problem and the other one based on the strong formulation of the continuous problem. The spectral element method [11] and the spectral mortar element method [1], which fall into the first category, require constructions of H^1 and mortar subspaces, respectively, for the whole domain of the problem. On the other hand, the spectral collocation method [13], which belongs to the second category, relies on the construction of independent subspaces on each constituent subdomain. This paper is concerned with a nonoverlapping domain decomposition spectral collocation method. In contrast to the spectral element and spectral mortar element methods (see [2] and references therein), the literature on nonoverlapping domain decomposition spectral collocation methods is limited. For Helmholtz's equation on a rectangle partitioned into two subrectangles, a nonoverlapping domain decomposition method is analyzed in [8] for computing the spectral collocation solution with the collocation points being the nodes of either the Legendre–Gauss–Lobatto or the Chebyshev–Gauss–Lobatto quadrature. In [14], following the approach of [7], a modification of the method of [8] and its analysis are given for a rectangle partitioned into several subrectangles. In comparison to [8], where the jump in the normal derivative is zero at the interface collocation points, the approach of [14] requires that such jumps be equal to "a suitable linear combination of the residual of the equation." It is claimed in [14] that this modified method can also be formulated for a rectangular polygon, that is, a region which is a union of rectangles with sides parallel to the x- and y-coordinate axes. The iterative Dirichlet–Neumann domain decomposition methods of [8, 14] require the dynamical selection of relaxation parameters. In [9], spectral collocation at the Legendre–Gauss–Lobatto nodes is combined with the so-called projection decomposition method to solve Helmholtz's equation in a rectangular polygon. In this method, the continuous Steklov–Poincaré equation corresponding to the interfaces is first solved using the Galerkin method with piecewise-polynomials. The use of special basis functions for the Galerkin problem leads to a symmetric, positive definite, and well-conditioned linear system which is solved by the conjugate gradient method. Using the decomposition of [9] and finite element preconditioning techniques, several preconditioners are discussed in [12] for the preconditioned conjugate gradient (PCG) solution of the Schur complement system arising from the spectral element method [11] applied to Helmholtz's equation in a rectangular polygon partitioned into many rectangular subregions. It should be noted that the schemes in [11, 9, 12], derived from the variational formulation of the continuous problem, lead to the standard collocation equations at the interior Legendre–Gauss–Lobatto nodes in each rectangular subregion. On the other hand, the development of and the solution procedure for our nonvariational spectral collocation scheme are based on a general idea described in Sec. 1.1 of [15] for the continuous problem. The same idea was used successfully in [3] for orthogonal spline collocation and an L-shaped region.

Although our approach is applicable to Helmholtz's equation with a variable coefficient and a rectangular polygon partitioned into many rectangular subregions, for the sake of simplicity, we consider Poisson's equation in an L-shaped region or a rectangle with a single cross point. We use a nonoverlapping domain decomposition technique to first define and then to compute the spectral collocation solution with the collocation points being the nodes of the Legendre–Gauss, rather than, the Legendre–Gauss–Lobatto quadrature. As in [8], we require that the jump in the normal derivative be zero at the interface collocation points. At a cross-point, we require the continuity of the normal derivative in the vertical direction. (Our treatment of a cross-point seems to contradict the statement at the bottom of page 86 in [11], where it is speculated that in the patching method approach, the sense in which the normal derivative is to be interpreted at internal corners is much less obvious than in the spectral element method.) In contrast to the approach based on the use of the Legendre–Gauss–Lobatto nodes, our approach leads, in a natural way, to a self-adjoint and positive definite interface problem which is solved by the PCG method. It should be noted that our previous numerical tests indicate (see, e.g., Example 1 in [4]) that, in the continuous maximum norm, the Legendre–Gauss nodes are less accurate than the Legendre–Gauss–Lobatto nodes by only a factor of 2. Hence, while not being significantly less accurate than the Legendre–Gauss–Lobatto nodes, the Legendre–Gauss nodes allow for the more efficient solution of the resulting discrete problem.

In this paper, the rectangular subregions Ω_i and the interfaces Γ_i are defined by (see Fig. 2)

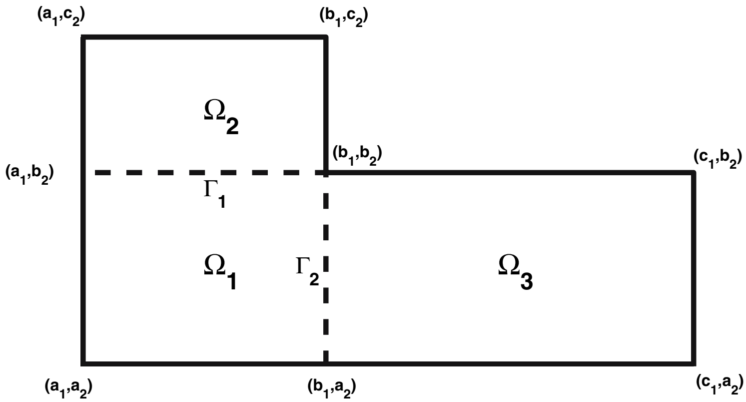

Fig. 1. Domain decomposition of L-shaped domain.

$$\begin{aligned}\Omega_1 &= (a_1, b_1) \times (a_2, b_2), \quad \Omega_2 = (a_1, b_1) \times (b_2, c_2), \quad \Omega_3 = (b_1, c_1) \times (a_2, b_2),\\ &\quad \Omega_4 = (b_1, c_1) \times (b_2, c_2),\\ \Gamma_1 &= (a_1, b_1) \times \{b_2\}, \quad \Gamma_2 = \{b_1\} \times (a_2, b_2), \quad \Gamma_3 = \{b_1\} \times (b_2, c_2),\\ &\quad \Gamma_4 = (b_1, c_1) \times \{b_2\}.\end{aligned}$$

We consider the model Dirichlet boundary value problem for Poisson's equation

$$\Delta u = f \quad \text{in} \quad \Omega, \qquad u = g \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where

$$\Omega = \bigcup_{i=1}^{3} \Omega_i \cup \bigcup_{i=1}^{2} \Gamma_i \tag{1.2}$$

is the L-shaped region (see Fig. 1) or

$$\Omega = \bigcup_{i=1}^{4} \Omega_i \cup \bigcup_{i=1}^{4} \Gamma_i \cup \{(b_1, b_2)\} \tag{1.3}$$

is the rectangle with the cross-point (b_1, b_2) (see Fig. 2). We approximate (1.1) using domain decomposition with a spectral collocation discretization. In each rectangular subregion Ω_i, the collocation solution is a tensor product polynomial that satisfies Poisson's equation at the collocation points. The collocation solution is continuous in $\overline{\Omega}$ and its normal derivatives are continuous at the collocation points on the interfaces Γ_i.

However, the continuity of the normal derivatives across the interfaces is not guaranteed. At a cross-point, we require continuity of the normal derivative in the vertical direction. We prove existence and uniqueness of the collocation solution. The solution of the collocation problem is first reduced to finding the collocation solution on the interfaces. The discrete Steklov–Poincaré operator corresponding to the interfaces is self-adjoint and positive definite with respect to the discrete inner product associated with the collocation points on the interfaces. The right-hand side in the discrete Steklov–Poincaré operator equation is obtained by solving a collocation problem with Dirichlet boundary conditions in each rectangular subregion. With the use of the matrix decomposition method of [4] this is accomplished at a cost of $O(N^3)$, where the number of unknowns in the collocation solution is $O(N^2)$. The collocation solution on the interfaces is computed using the PCG method with a preconditioner obtained from the discrete Steklov–Poincaré operators corresponding to pairs of the adjacent rectangular subregions. The cost of each PCG iteration is $O(N^2)$. (In comparison, the cost of each PCG iteration in [9, 12] appears to be proportional to N^4 and N^3, respectively.) Once the solution of the discrete Steklov–Poincaré equation is available, the collocation solution in each rectangular subregion is computed at a cost $O(N^3)$ using the matrix decomposition method of [4]. The total cost of the algorithm is $O(N^3) + O(mN^2)$, where m is the number of PCG iterations required to solve the interface problem. For the L-shaped region, our preconditioner is spectrally equivalent to the interface operator with spectral constants independent of N. Hence in this case, the number m should be proportional to $\ln(1/\epsilon)$, where ϵ is the factor by which the initial error is to be reduced. With $\epsilon = O(N^{-k})$, which corresponds to the convergence rate of our spectral collocation method, $m = O(k \ln N)$. Clearly, even with $m = O(N)$, the total cost of our algorithm is $O(N^3)$.

An outline of this paper is as follows. In Sec. 2, we introduce certain spectral collocation concepts, state and prove some necessary results. The spectral collocation problem for the L-shaped region is defined and analyzed in Sec. 3. In Sec. 4, we formulate an algorithm for solving the collocation problem in the L-shaped region. The solution of the interface problem is discussed in Sec. 5. The cost of solving the collocation problem in the L-shaped region is given in Sec. 6. In Sec. 7, we consider spectral collocation for the rectangle with a cross point. Extensions of our method are discussed in Sec. 8. Numerical results are presented in Sec. 9 and conclusions are given in Sec. 10.

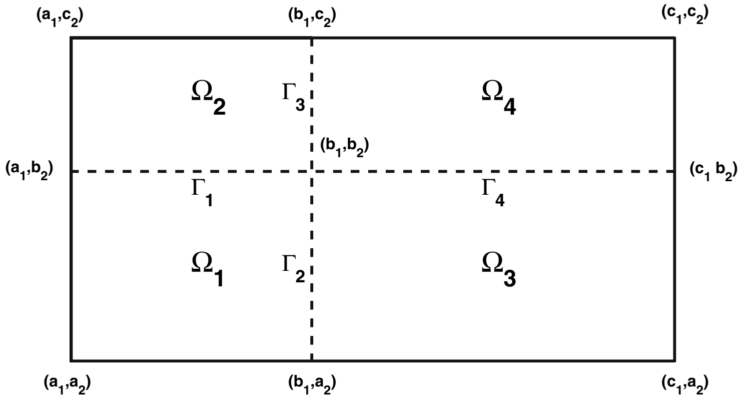

Fig. 2. Domain decomposition of rectangle with cross-point.

2. PRELIMINARIES

Let $P_N(a,b)$ denote the set of polynomials of degree $\leq N$ on $[a,b]$, and let

$$P_N^0(a,b) = \{p \in P_N(a,b) : p(a) = p(b) = 0\}.$$

Let $\mathcal{G}_{a,b} = \{\xi_i^{a,b}\}_{i=1}^{N-1}$ and $\{w_i^{a,b}\}_{i=1}^{N-1}$ be, respectively, the nodes and weights of the $N-1$-point Legendre–Gauss quadrature on (a,b). Note that

$$\xi_i = \frac{2\xi_i^{a,b} - a - b}{b-a}, \quad w_i = \frac{2}{b-a} w_i^{a,b}, \quad i = 1, \ldots, N-1,$$

where $\{\xi_i\}_{i=1}^{N-1}$ and $\{w_i\}_{i=1}^{N-1}$ are, respectively, the nodes and weights of the $N-1$-point Legendre–Gauss quadrature on $(-1,1)$. Let

$$D = \operatorname{diag}(w_1, \ldots, w_{N-1}). \tag{2.1}$$

Let $\{L_k(t)\}_{k=0}^{\infty}$, $t \in [-1,1]$, be the set of Legendre polynomials, that is,

$$L_0(t) = 1, \quad L_1(t) = t, \quad L_k(t) = \frac{2k-1}{k} t L_{k-1}(t) - \frac{k-1}{k} L_{k-2}(t), \quad k = 2, \ldots.$$

Let $\{\phi_k(t)\}_{k=0}^{N}$, $t \in [-1,1]$, be the basis for $P_N(-1,1)$ defined by (cf. (3.12a), (3.12c) in [19], (2.7) in [16], and (4.3), (4.5) in [4])

$$\phi_0(t) = \frac{1}{2} L_0(t) - \frac{1}{2} L_1(t), \quad \phi_1(t) = \frac{1}{2} L_0(t) + \frac{1}{2} L_1(t), \tag{2.2}$$

$$\phi_k(t) = c_k[L_{k-2}(t) - L_k(t)], \quad k = 2, \ldots, N, \tag{2.3}$$

where

$$c_k = [4k-2]^{-1/2}, k = 2, \ldots, N-1, \quad c_N = [(4N-2)(2-1/N)]^{-1/2}.$$

Note that $\{\phi_k(t)\}_{k=2}^N$ is a basis for $P_N^0(-1,1)$. Let A and B be two dense $(N-1) \times (N-1)$ matrices defined by (see (4.13) in [4])

$$A = (-\phi_k''(\xi_i))_{i=1,k=2}^{N-1,N}, \quad B = (\phi_k(\xi_i))_{i=1,k=2}^{N-1,N}, \tag{2.4}$$

where i and k are the row and column indices, respectively. With D of (2.1), we introduce

$$A' = B^T DA, \quad B' = B^T DB. \tag{2.5}$$

It follows from (4.30) in [4] that

$$A' = I, \tag{2.6}$$

where I denotes the identity matrix. It also follows from (4.9), (4.17), and (4.21) in [4] that the symmetric, positive definite, pentadiagonal matrix B' splits into two tridiagonal matrices whose entries are given by (4.16), (4.18), (4.19), (4.20), and (4.22) in [4]. Since B' is symmetric and positive definite, there exist (see Theorem 8.1.1 in [10]) a real $(N-1) \times (N-1)$ matrix $Z = (z_{k,n})_{k,n=2}^N$ and a real matrix

$$\Lambda = \text{diag}(\lambda_2, \ldots, \lambda_N), \quad \lambda_n > 0, \quad n = 2, \ldots, N \tag{2.7}$$

such that

$$\Lambda = Z^T B' Z, \quad Z^T Z = I. \tag{2.8}$$

For any V, W defined on $\mathcal{G}_{a,b}$, we introduce

$$\langle V, W \rangle_{a,b} = \sum_{i=1}^{N-1} w_i^{a,b} (VW)(\xi_i^{a,b}). \tag{2.9}$$

Lemma 3.1 in [6] implies that

$$-\langle p'', q \rangle_{a,b} = \int_a^b (p'q')(s)\, ds - p'q|_a^b + C_N^{a,b} p^{(N)} q^{(N)}, \quad p, q \in P_N(a,b), \tag{2.10}$$

where $C_N^{a,b}$ denotes a generic positive constant that depends on N and a, b. The property (2.10) plays a key role in proving that the discrete Steklov–Poincaré operator and its preconditioner are self-adjoint and positive definite with respect to the discrete inner product associated with the collocation points on the interfaces.

The lemmata and the remark in the remaining part of this section are important for the efficient solution with the preconditioner and the efficient multiplication by the discrete Steklov–Poincaré operator, both of which rely on the use of separation of variables.

Lemma 2.1. There exist linearly independent functions $\psi_n \in P_N^0(a,b)$, $n = 2, \dots, N$, and the corresponding positive numbers γ_n, $n = 2, \dots, N$, such that

$$-\gamma_n \psi_n''(\xi) = \psi_n(\xi), \quad \xi \in \mathcal{G}_{a,b},$$
$$-\langle \psi_n'', \psi_m \rangle_{a,b} = \tfrac{2}{b-a}\delta_{n,m}, \quad n, m = 2, \dots, N,$$

where $\delta_{n,m}$ is the Kronecker delta.

Proof. Let $\{\phi_k^{a,b}\}_{k=2}^N$ be the basis for $P_N^0(a,b)$ defined by

$$\phi_k^{a,b}(s) = \phi_k\left(\frac{2s-a-b}{b-a}\right), \quad k = 2, \dots, N, \tag{2.11}$$

where $\{\phi_k\}_{k=2}^N$ are given by (2.3). Then $\psi_n \in P_N^0(a,b)$ can be written as

$$\psi_n = \sum_{k=2}^{N} \psi_{k,n}\phi_k^{a,b}. \tag{2.12}$$

Thus, the two equations in Lemma 2.1 are equivalent to

$$\gamma_n \frac{4}{(b-a)^2} A\vec{\psi}_n = B\vec{\psi}_n, \quad \frac{2}{b-a}(DA\vec{\psi}_n, B\vec{\psi}_m)_{R^{N-1}} = \frac{2}{b-a}\delta_{n,m},$$

where $\vec{\psi}_n = [\psi_{2,n}, \dots, \psi_{N,n}]^T$ and D, A, B are defined in (2.1), (2.4). It follows from (2.5) and (2.6) that the last two equations are equivalent to

$$\gamma_n \frac{4}{(b-a)^2}\vec{\psi}_n = B'\vec{\psi}_n, \quad (\vec{\psi}_n, \vec{\psi}_m)_{R^{N-1}} = \delta_{n,m}.$$

Hence (2.7) and (2.8) show that the required equations are satisfied with

$$\gamma_n = \lambda_n \frac{(b-a)^2}{4}, \quad n = 2, \dots, N, \quad \psi_{k,n} = z_{k,n}, \quad k, n = 2, \dots, N. \tag{2.13}$$

The linear independence of the $\{\psi_n\}_{n=2}^N$ follows from that of the $\{\phi_k^{a,b}\}_{k=2}^N$ and the invertibility of Z. $\square$

Lemma 2.2. For any $\gamma_n > 0$, $n = 2, \ldots, N$, there exists a unique $v_n \in P_N(a,b)$ such that

$$\gamma_n v_n''(\xi) = v_n(\xi), \quad \xi \in \mathcal{G}_{a,b}, \quad v_n(a) = 0, \quad v_n(b) = 1.$$

Proof. Let $\{\phi_k^{a,b}\}_{k=0}^N$ be the basis for $P_N(a,b)$ defined by (2.11) and by

$$\phi_k^{a,b}(s) = \phi_k\left(\frac{2s-a-b}{b-a}\right), \quad k = 0, 1, \tag{2.14}$$

where ϕ_0, ϕ_1 are given in (2.2). Since $\phi_0^{a,b}(a) = 1$, $\phi_k^{a,b}(a) = 0$, $k \neq 0$, and $\phi_1^{a,b}(b) = 1$, $\phi_k^{a,b}(b) = 0$, $k \neq 1$, the boundary conditions in Lemma 2.2 imply that

$$v_n = \sum_{k=2}^N v_{k,n}\phi_k^{a,b} + \phi_1^{a,b}. \tag{2.15}$$

Since $\phi_1^{a,b}$ is a linear function, the problem reduces to finding $v_{k,n}$, $k = 2, \ldots, N$, such that

$$\sum_{k=2}^N \left\{\gamma_n[-\phi_k^{a,b}]''(\xi) + \phi_k^{a,b}(\xi)\right\} v_{k,n} = -\phi_1^{a,b}(\xi), \quad \xi \in \mathcal{G}_{a,b}.$$

Introducing $\vec{v}_n = [v_{2,n}, \ldots, v_{N,n}]^T$, $\vec{c}_1 = [\phi_1^{a,b}(\xi_1^{a,b}), \ldots, \phi_1^{a,b}(\xi_N^{a,b})]^T$, and using (2.4), we obtain

$$\left(\gamma_n \frac{4}{(b-a)^2} A + B\right) \vec{v}_n = -\vec{c}_1.$$

It follows from (2.5) and (2.6) that the last linear system becomes

$$\left(\gamma_n \frac{4}{(b-a)^2} I + B'\right) \vec{v}_n = -B^T D \vec{c}_1. \tag{2.16}$$

The linear system (2.16) has a unique solution $\vec{v}_n$ since $\gamma_n > 0$ and since B' is symmetric and positive definite. □

Lemma 2.3. For any $\gamma_n > 0$, $n = 2, \ldots, N$, there exists a unique $w_n \in P_N(a,b)$ such that

$$\gamma_n w_n''(\xi) = w_n(\xi), \quad \xi \in \mathcal{G}_{a,b}, \quad w_n(a) = 1, \quad w_n(b) = 0.$$

Proof. Let v_n be as in Lemma 2.2, and let $w_n(t) = v_n(a+b-t)$, $t \in [a,b]$. Then $w_n \in P_N(a,b)$ and $w_n(a)=1$, $w_n(b)=0$. Symmetry of $\mathcal{G}_{a,b}$ about $(a+b)/2$ implies that $a+b-\xi \in \mathcal{G}_{a,b}$ if $\xi \in \mathcal{G}_{a,b}$. Hence

$$\gamma_n w_n''(\xi) = \gamma_n v_n''(a+b-\xi) = v_n(a+b-\xi) = w_n(\xi), \quad \xi \in \mathcal{G}_{a,b}.$$

□

Lemma 2.4. Let the linearly independent functions $\psi_n \in P_N^0(a,b)$, $n = 2, \ldots, N$, be as in Lemma 2.1. Assume $v \in P_N^0(a,b)$ and hence $v = \sum_{n=2}^{N} \alpha_n \psi_n$. If $\vec{\alpha} = [\alpha_2, \ldots, \alpha_N]^T$, then

$$[v(\xi_1^{a,b}), \ldots, v(\xi_{N-1}^{a,b})]^T = BZ\vec{\alpha}, \quad \vec{\alpha} = Z^T[B']^{-1}B^T D[v(\xi_1^{a,b}), \ldots, v(\xi_{N-1}^{a,b})]^T,$$

where the matrices D, B, B', and Z are those in (2.1), (2.4), (2.5), and (2.8).

Proof. Using (2.12) and (2.13) we have

$$v(\xi) = \sum_{n=2}^{N} \alpha_n \sum_{k=2}^{N} z_{k,n} \phi_k^{a,b}(\xi) = \sum_{k=2}^{N} \phi_k^{a,b}(\xi) \sum_{n=2}^{N} z_{k,n}\alpha_n, \quad \xi \in \mathcal{G}_{a,b},$$

which implies the first required equation. The second equation is obtained from the first one using $(BZ)^{-1} = Z^T B^{-1}$ and $B' = B^T DB$. □

Noting that D is a diagonal matrix, B' splits into two tridiagonal matrices, and B, Z are dense matrices, we have the following remark.

Remark 2.1. The cost of multiplying a vector by BZ is $O(N^2)$ and the cost of multiplying a vector by $Z^T[B']^{-1}B^T D$ is $O(N^2)$.

3. SPECTRAL COLLOCATION PROBLEM FOR L-SHAPED REGION

With Ω defined in (1.2), we introduce

$$Q_1 = P_N(a_1,b_1) \otimes P_N(a_2,b_2), \quad Q_2 = P_N(a_1,b_1) \otimes P_N(b_2,c_2),$$
$$Q_3 = P_N(b_1,c_1) \otimes P_N(a_2,b_2),$$

$$X_i = \{v \in Q_i : v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}, \quad i = 1,2,3.$$

Let

$$\mathcal{G}_1 = \mathcal{G}_{a_1,b_1} \times \mathcal{G}_{a_2,b_2}, \quad \mathcal{G}_2 = \mathcal{G}_{a_1,b_1} \times \mathcal{G}_{b_2,c_2}, \quad \mathcal{G}_3 = \mathcal{G}_{b_1,c_1} \times \mathcal{G}_{a_2,b_2}$$

be the sets of the collocation points in Ω_1, Ω_2, Ω_3, respectively. For $i = 1, 2, 3$, let $\tilde{\mathcal{G}}_i$ be the set of collocation points on $\partial\Omega \cap \partial\Omega_i$; for example,

$$\tilde{\mathcal{G}}_1 = \{(a_1, \xi) : \xi \in \mathcal{G}_{a_2,b_2} \cup \{a_2, b_2\}\} \cup \{(\xi, a_2) : \xi \in \mathcal{G}_{a_1,b_1} \cup \{b_1\}\} \cup \{(b_1, b_2)\}.$$

For $i = 1, 2$, let $\mathcal{G}'_i$ be the set of collocation points on Γ_i; for example, $\mathcal{G}'_1 = \{(\xi, b_2) : \xi \in \mathcal{G}_{a_1,b_1}\}$.

The Legendre spectral collocation problem for (1.1) and (1.2) involves finding $U_i \in Q_i$, $i = 1, 2, 3$, such that

$$\Delta U_i(\xi) = f(\xi), \quad \xi \in \mathcal{G}_i, \quad i = 1, 2, 3, \tag{3.1}$$

$$U_i(\xi) = g(\xi), \quad \xi \in \tilde{\mathcal{G}}_i, \quad i = 1, 2, 3 \tag{3.2}$$

and such that

$$\frac{\partial^j U_1}{\partial y^j}(\xi) = \frac{\partial^j U_2}{\partial y^j}(\xi), \quad \xi \in \mathcal{G}'_1, \quad \frac{\partial^j U_1}{\partial x^j}(\xi) = \frac{\partial^j U_3}{\partial x^j}(\xi), \quad \xi \in \mathcal{G}'_2, \quad j = 0, 1. \tag{3.3}$$

While (3.2) and (3.3) with $j = 0$ imply

$$U_1|_{\overline{\Gamma}_1} = U_2|_{\overline{\Gamma}_1}, \quad U_1|_{\overline{\Gamma}_2} = U_3|_{\overline{\Gamma}_2} \tag{3.4}$$

in general

$$\frac{\partial U_1}{\partial y}(b_1, b_2) \neq \frac{\partial U_2}{\partial y}(b_1, b_2), \quad \frac{\partial U_1}{\partial x}(b_1, b_2) \neq \frac{\partial U_3}{\partial x}(b_1, b_2).$$

In the remainder of this section, we prove existence and uniqueness of the spectral collocation solution. For $v_i \in X_i$, $i = 1, 2, 3$, we introduce

$$\|(v_1, v_2, v_3)\|^2 = \sum_{i=1}^{3} \|v_i\|_i^2, \tag{3.5}$$

where

$$\|v_1\|_1^2 = \sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\| \frac{\partial v_1}{\partial x}(\cdot, \xi_j^{a_2,b_2}) \right\|_{L^2(a_1,b_1)}^2 + \sum_{i=1}^{N-1} w_i^{a_1,b_1} \left\| \frac{\partial v_1}{\partial y}(\xi_i^{a_1,b_1}, \cdot) \right\|_{L^2(a_2,b_2)}^2,$$

$$\|v_2\|_2^2 = \sum_{j=1}^{N-1} w_j^{b_2,c_2} \left\| \frac{\partial v_2}{\partial x}(\cdot, \xi_j^{b_2,c_2}) \right\|_{L^2(a_1,b_1)}^2 + \sum_{i=1}^{N-1} w_i^{a_1,b_1} \left\| \frac{\partial v_2}{\partial y}(\xi_i^{a_1,b_1}, \cdot) \right\|_{L^2(b_2,c_2)}^2,$$

$$\|v_3\|_3^2 = \sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\| \frac{\partial v_3}{\partial x}(\cdot, \xi_j^{a_2,b_2}) \right\|_{L^2(b_1,c_1)}^2 + \sum_{i=1}^{N-1} w_i^{b_1,c_1} \left\| \frac{\partial v_3}{\partial y}(\xi_i^{b_1,c_1}, \cdot) \right\|_{L^2(a_2,b_2)}^2.$$

It is easy to verify that $\|\cdot\|$ defined by (3.5) is a norm on $X_1 \times X_2 \times X_3$.

Lemma 3.1. For any $U_i \in X_i$, $i=1,2,3$, satisfying (3.3), we have

$$\|(U_1, U_2, U_3)\|^2 \leq \sum_{i=1}^{3} \langle -\Delta U_i, U_i \rangle_i,$$

where

$$\langle V, W \rangle_1 = \sum_{i=1}^{N-1} w_i^{a_1,b_1} \langle V(\xi_i^{a_1,b_1}, \cdot), W(\xi_i^{a_1,b_1}, \cdot) \rangle_{a_2,b_2} = \sum_{j=1}^{N-1} w_j^{a_2,b_2} \langle V(\cdot, \xi_j^{a_2,b_2}), W(\cdot, \xi_j^{a_2,b_2}) \rangle_{a_1,b_1} \tag{3.6}$$

$\langle \cdot, \cdot \rangle_i$, $i=2,3$, are defined in a similar way, and where $\langle \cdot, \cdot \rangle_{a,b}$ is given in (2.9).

Proof. Using the definitions of $\langle \cdot, \cdot \rangle_i$, $i=1,2,3$, we have

$$\sum_{i=1}^{3} \langle -\Delta U_i, U_i \rangle_i = I_x + I_y, \tag{3.7}$$

where

$$I_x = -\sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\{ \left\langle \frac{\partial^2 U_1}{\partial x^2}(\cdot, \xi_j^{a_2,b_2}), U_1(\cdot, \xi_j^{a_2,b_2}) \right\rangle_{a_1,b_1} - \left\langle \frac{\partial^2 U_3}{\partial x^2}(\cdot, \xi_j^{a_2,b_2}), U_3(\cdot, \xi_j^{a_2,b_2}) \right\rangle_{b_1,c_1} \right\} - \sum_{j=1}^{N-1} w_j^{b_2,c_2} \left\langle \frac{\partial^2 U_2}{\partial x^2}(\cdot, \xi_j^{b_2,c_2}), U_2(\cdot, \xi_j^{b_2,c_2}) \right\rangle_{a_1,b_1},$$

$$I_y = -\sum_{i=1}^{N-1} w_i^{a_1,b_1} \left\{ \left\langle \frac{\partial^2 U_1}{\partial y^2}(\xi_i^{a_1,b_1}, \cdot), U_1(\xi_i^{a_1,b_1}, \cdot) \right\rangle_{a_2,b_2} - \left\langle \frac{\partial^2 U_2}{\partial y^2}(\xi_i^{a_1,b_1}, \cdot), U_2(\xi_i^{a_1,b_1}, \cdot) \right\rangle_{b_2,c_2} \right\} - \sum_{i=1}^{N-1} w_i^{b_1,c_1} \left\langle \frac{\partial^2 U_3}{\partial y^2}(\xi_i^{b_1,c_1}, \cdot), U_3(\xi_i^{b_1,c_1}, \cdot) \right\rangle_{a_2,b_2}.$$

It follows from (2.10) and (3.3) that

$$\begin{aligned} I_x = {} & \sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\{ \left\| \frac{\partial U_1}{\partial x}(\cdot, \xi_j^{a_2,b_2}) \right\|^2_{L^2(a_1,b_1)} + C_N^{a_1,b_1} \left[U_1^{(N,0)}(\xi_j^{a_2,b_2}) \right]^2 \right\} \\ & + \sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\{ \left\| \frac{\partial U_3}{\partial x}(\cdot, \xi_j^{a_2,b_2}) \right\|^2_{L^2(b_1,c_1)} + C_N^{b_1,c_1} \left[U_3^{(N,0)}(\xi_j^{a_2,b_2}) \right]^2 \right\} \\ & + \sum_{j=1}^{N-1} w_j^{b_2,c_2} \left\{ \left\| \frac{\partial U_2}{\partial x} \|(\cdot, \xi_j^{b_2,c_2}) \right\|^2_{L^2(a_1,b_1)} + C_N^{a_1,b_1} \left[U_2^{(N,0)}(\xi_j^{b_2,c_2}) \right]^2 \right\}, \end{aligned} \tag{3.8}$$

where

$$U_1^{(N,0)}(\xi) = \frac{\partial^N U_1}{\partial x^N}(\cdot, \xi), \quad U_3^{(N,0)}(\xi) = \frac{\partial^N U_3}{\partial x^N}(\cdot, \xi), \quad U_2^{(N,0)}(\xi) = \frac{\partial^N U_2}{\partial x^N}(\cdot, \xi).$$

The required inequality is a consequence of (3.8) and a similar formula for I_y. □

Theorem 3.1. The Legendre spectral collocation problem (3.1)–(3.3) has a unique solution.

Proof. Consider the collocation problem (3.1)–(3.3) with $f = 0$ and $g = 0$. Then $U_i \in X_i$, $i = 1, 2, 3$, and since $\| \cdot \|$ is a norm on $X_1 \times X_2 \times X_3$, Lemma 3.1 gives $U_i = 0$, $i = 1, 2, 3$. This proves the existence and uniqueness of the collocation solution for any f and g since the number of degrees of freedom in the collocation problem is equal to the number of constraints. □

4. ALGORITHM FOR SOLVING COLLOCATION PROBLEM IN *L*-SHAPED REGION

Assume that $U_i \in Q_i$, $i = 1, 2, 3$, satisfy (3.1)–(3.3). As in the case of the continuous problem, the idea behind our algorithm for obtaining the U_i is based on representing each U_i as the sum of two approximate solutions, one satisfying Poisson's equation on $\mathcal{G}_i$ and the other satisfying Laplace's equation on $\mathcal{G}_i$. Let $U_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, be defined by (cf. (3.4))

$$\begin{aligned} U_{\Gamma_1}(\xi) &= U_1(\xi, b_2) = U_2(\xi, b_2), \quad \xi \in \mathcal{G}_{a_1,b_1}, \\ U_{\Gamma_2}(\xi) &= U_1(b_1, \xi) = U_3(b_1, \xi), \quad \xi \in \mathcal{G}_{a_2,b_2}. \end{aligned} \tag{4.1}$$

For $i = 1, 2, 3$, let $\hat{U}_i \in Q_i$ be such that (cf. (3.1)–(3.3))

$$\Delta \hat{U}_i(\xi) = f(\xi), \quad \xi \in \mathcal{G}_i, \quad i = 1, 2, 3, \tag{4.2}$$

$$\hat{U}_i(\xi) = g(\xi), \quad \xi \in \tilde{\mathcal{G}}_i, \quad i = 1, 2, 3, \tag{4.3}$$

$$\hat{U}_1(\xi) = \hat{U}_2(\xi) = 0, \quad \xi \in \mathcal{G}_1', \quad \hat{U}_1(\xi) = \hat{U}_3(\xi) = 0, \quad \xi \in \mathcal{G}_3'. \tag{4.4}$$

Let

$$\tilde{U}_i = U_i - \hat{U}_i, \quad i = 1, 2, 3. \tag{4.5}$$

Then it follows from (4.5), (3.1)–(3.2), and (4.1)–(4.4) that $\tilde{U}_i \in X_i$, $i = 1, 2, 3$, and that

$$\Delta \tilde{U}_1(\xi) = 0, \quad \xi \in \mathcal{G}_1, \quad \tilde{U}_1|_{\overline{\Gamma}_1} = U_{\Gamma_1}, \quad \tilde{U}_1|_{\overline{\Gamma}_2} = U_{\Gamma_2}, \tag{4.6}$$

$$\Delta \tilde{U}_2(\xi) = 0, \quad \xi \in \mathcal{G}_2, \quad \tilde{U}_2|_{\overline{\Gamma}_1} = U_{\Gamma_1}, \tag{4.7}$$

$$\Delta \tilde{U}_3(\xi) = 0, \quad \xi \in \mathcal{G}_3, \quad \tilde{U}_3|_{\overline{\Gamma}_2} = U_{\Gamma_2}. \tag{4.8}$$

Moreover, (3.3) with $j = 1$ and (4.5) give

$$\begin{aligned} \frac{\partial \tilde{U}_1}{\partial y}(\xi) - \frac{\partial \tilde{U}_2}{\partial y}(\xi) &= \frac{\partial \hat{U}_2}{\partial y}(\xi) - \frac{\partial \hat{U}_1}{\partial y}(\xi), \quad \xi \in \mathcal{G}_1', \\ \frac{\partial \tilde{U}_1}{\partial x}(\xi) - \frac{\partial \tilde{U}_3}{\partial x}(\xi) &= \frac{\partial \hat{U}_3}{\partial x}(\xi) - \frac{\partial \hat{U}_1}{\partial x}(\xi), \quad \xi \in \mathcal{G}_2'. \end{aligned} \tag{4.9}$$

We obtain the following algorithm for solving the problem (3.1)–(3.3).

Algorithm.

Step 1. With $\hat{U}_i \in Q_i$, $i=1,2,3$, defined by (4.2)–(4.4), compute the right sides of (4.9).
Step 2. Compute $U_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i=1,2$, such that $\tilde{U}_i \in X_i$, $i=1,2,3$, satisfy (4.6)–(4.9).
Step 3. Compute $U_i \in Q_i$, $i=1,2,3$, satisfying (3.1)–(3.2), (4.1).

5. INTERFACE PROBLEM FOR L-SHAPED REGION

In this section, we discuss in more detail step 2 of the Algorithm in Sec. 4.

5.1. Discrete Steklov–Poincaré Operator

Let $K: P_N^0(a_1, b_1) \times P_N^0(a_2, b_2) \to P_N^0(a_1, b_1) \times P_N^0(a_2, b_2)$ be defined for $V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i=1,2$, by

$$K(V_{\Gamma_1}, V_{\Gamma_2}) = (W_{\Gamma_1}, W_{\Gamma_2}), \tag{5.1}$$

where $W_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i=1,2$, are uniquely determined by

$$\begin{aligned} W_{\Gamma_1}(\xi) &= \frac{\partial V_1}{\partial y}(\xi, b_2) - \frac{\partial V_2}{\partial y}(\xi, b_2), \xi \in \mathcal{G}_{a_1,b_1}, \\ W_{\Gamma_2}(\xi) &= \frac{\partial V_1}{\partial x}(b_1, \xi) - \frac{\partial V_3}{\partial x}(b_1, \xi), \xi \in \mathcal{G}_{a_2,b_2} \end{aligned} \tag{5.2}$$

with $V_i \in X_i$, $i=1,2,3$, satisfying

$$\Delta V_1(\xi) = 0, \quad \xi \in \mathcal{G}_1, \quad V_1|_{\overline{\Gamma}_1} = V_{\Gamma_1}, \quad V_1|_{\overline{\Gamma}_2} = V_{\Gamma_2}, \tag{5.3}$$

$$\Delta V_2(\xi) = 0, \quad \xi \in \mathcal{G}_2, \quad V_2|_{\overline{\Gamma}_1} = V_{\Gamma_1}, \tag{5.4}$$

$$\Delta V_3(\xi) = 0, \quad \xi \in \mathcal{G}_3, \quad V_3|_{\overline{\Gamma}_2} = V_{\Gamma_2}. \tag{5.5}$$

Then step 2 of the Algorithm is equivalent to finding $U_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i=1,2$, such that

$$K(U_{\Gamma_1}, U_{\Gamma_2}) = (F_{\Gamma_1}, F_{\Gamma_2}), \tag{5.6}$$

where, with $\hat{U}_i \in Q_i$, $i = 1, 2, 3$, satisfying (4.2)–(4.4), $F_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, are given by

$$F_{\Gamma_1}(\xi) = \frac{\partial \hat{U}_2}{\partial y}(\xi, b_2) - \frac{\partial \hat{U}_1}{\partial y}(\xi, b_2), \xi \in \mathcal{G}_{a_1,b_1},$$
$$F_{\Gamma_2}(\xi) = \frac{\partial \hat{U}_3}{\partial x}(b_1, \xi) - \frac{\partial \hat{U}_1}{\partial x}(b_1, \xi), \xi \in \mathcal{G}_{a_2,b_2}. \tag{5.7}$$

The inner product in $P_N^0(a_1, b_1) \times P_N^0(a_2, b_2)$ is defined by

$$\langle (V_{\Gamma_1}, V_{\Gamma_2}), (W_{\Gamma_1}, W_{\Gamma_2}) \rangle = \sum_{i=1}^{2} \langle V_{\Gamma_i}, W_{\Gamma_i} \rangle_{a_i,b_i}, \tag{5.8}$$

where $\langle \cdot, \cdot \rangle_{a,b}$ is given in (2.9).

Theorem 5.1. The operator $K : P_N^0(a_1, b_1) \times P_N^0(a_2, b_2) \to P_N^0(a_1, b_1) \times P_N^0(a_2, b_2)$ defined by (5.1), (5.2) is self-adjoint and positive definite with respect to the inner product (5.8).

Proof. To show that K is self-adjoint, we have to verify that

$$\langle K(V_{\Gamma_1}, V_{\Gamma_2}), (W_{\Gamma_1}, W_{\Gamma_2}) \rangle = \langle (V_{\Gamma_1}, V_{\Gamma_2}), K(W_{\Gamma_1}, W_{\Gamma_2}) \rangle,$$
$$V_{\Gamma_i}, W_{\Gamma_i} \in P_N^0(0, 1), \quad i = 1, 2.$$

It follows from (5.8), (5.1), and (5.2) that

$$\langle K(V_{\Gamma_1}, V_{\Gamma_2}), (W_{\Gamma_1}, W_{\Gamma_2}) \rangle = \left\langle \frac{\partial V_1}{\partial y}(\cdot, b_2) - \frac{\partial V_2}{\partial y}(\cdot, b_2), W_{\Gamma_1} \right\rangle_{a_1,b_1} + \left\langle \frac{\partial V_1}{\partial x}(b_1, \cdot) - \frac{\partial V_3}{\partial x}(b_1, \cdot), W_{\Gamma_2} \right\rangle_{a_2,b_2}, \tag{5.9}$$

where $V_i \in X_i$, $i = 1, 2, 3$, satisfy (5.3)–(5.5). Let $W_i \in X_i$, $i = 1, 2, 3$, satisfy (5.3)–(5.5) with W_{Γ_i} in place of V_{Γ_i}, that is,

$$\Delta W_1(\xi) = 0, \quad \xi \in \mathcal{G}_1, \quad W_1|_{\overline{\Gamma}_1} = W_{\Gamma_1}, \quad W_1|_{\overline{\Gamma}_2} = W_{\Gamma_2}, \tag{5.10}$$

$$\Delta W_2(\xi) = 0, \quad \xi \in \mathcal{G}_2, \quad W_2|_{\overline{\Gamma}_1} = W_{\Gamma_1}, \tag{5.11}$$

$$\Delta W_3(\xi) = 0, \quad \xi \in \mathcal{G}_3, \quad W_3|_{\overline{\Gamma}_2} = W_{\Gamma_2}. \tag{5.12}$$

Using (5.3), (3.6), (2.10), and (5.10), we obtain

$$
\begin{aligned}
0 = \langle -\Delta V_1, W_1\rangle_1 &= -\sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\langle \frac{\partial^2 V_1}{\partial x^2}(\cdot, \xi_j^{a_2,b_2}), W_1(\cdot, \xi_j^{a_2,b_2}) \right\rangle_{a_1,b_1} \\
&\quad - \sum_{i=1}^{N-1} w_i^{a_1,b_1} \left\langle \frac{\partial^2 V_1}{\partial y^2}(\xi_i^{a_1,b_1}, \cdot), W_1(\xi_i^{a_1,b_1}, \cdot) \right\rangle_{a_2,b_2} \\
&= I_1(V_1, W_1) - \left\langle \frac{\partial V_1}{\partial x}(b_1, \cdot), W_{\Gamma_2} \right\rangle_{a_2,b_2} - \left\langle \frac{\partial V_1}{\partial y}(\cdot, b_2), W_{\Gamma_1} \right\rangle_{a_1,b_1}, \quad (5.13)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(V_1, W_1) &= \int_{a_1}^{b_1} \left\langle \frac{\partial V_1}{\partial x}(x, \cdot), \frac{\partial W_1}{\partial x}(x, \cdot) \right\rangle_{a_2,b_2} dx + C_N^{a_1,b_1} \left\langle V_1^{(N,0)}, W_1^{(N,0)} \right\rangle_{a_2,b_2} \\
&\quad + \int_{a_2}^{b_2} \left\langle \frac{\partial V_1}{\partial y}(\cdot, y), \frac{\partial W_1}{\partial y}(\cdot, y) \right\rangle_{a_1,b_1} dy + C_N^{a_2,b_2} \left\langle V_1^{(0,N)}, W_1^{(0,N)} \right\rangle_{a_1,b_1},
\end{aligned}
\quad (5.14)
$$

$$
\begin{aligned}
V_1^{(N,0)}(\xi) &= \frac{\partial^N V_1}{\partial x^N}(x, \xi), \; W_1^{(N,0)}(\xi) = \frac{\partial^N W_1}{\partial x^N}(x, \xi), \quad x \in [a_1, b_1], \quad \xi \in \mathcal{G}_{a_2,b_2}, \\
V_1^{(0,N)}(\xi) &= \frac{\partial^N V_1}{\partial y^N}(\xi, y), \; W_1^{(0,N)}(\xi) = \frac{\partial^N W_1}{\partial y^N}(\xi, y), \quad \xi \in \mathcal{G}_{a_1,b_1}, \quad y \in [a_2, b_2].
\end{aligned}
$$

In a similar way, using (5.4), (5.5), (2.10), (5.11), and (5.12), we obtain

$$
0 = \langle -\Delta V_2, W_2\rangle_2 = I_2(V_2, W_2) + \left\langle \frac{\partial V_2}{\partial y}(\cdot, b_2), W_{\Gamma_1} \right\rangle_{a_1,b_1}, \quad (5.15)
$$

$$
0 = \langle -\Delta V_3, W_3\rangle_3 = I_3(V_3, W_3) + \left\langle \frac{\partial V_3}{\partial x}(b_1, \cdot), W_{\Gamma_2} \right\rangle_{a_2,b_2}, \quad (5.16)
$$

where

$$
\begin{aligned}
I_2(V_2, W_2) &= \int_{a_1}^{b_1} \left\langle \frac{\partial V_2}{\partial x}(x, \cdot), \frac{\partial W_2}{\partial x}(x, \cdot) \right\rangle_{b_2,c_2} dx + C_N^{a_1,b_1} \left\langle V_2^{(N,0)}, W_2^{(N,0)} \right\rangle_{b_2,c_2} \\
&\quad + \int_{b_2}^{c_2} \left\langle \frac{\partial V_2}{\partial y}(\cdot, y), \frac{\partial W_2}{\partial y}(\cdot, y) \right\rangle_{a_1,b_1} dy + C_N^{b_2,c_2} \left\langle V_2^{(0,N)}, W_2^{(0,N)} \right\rangle_{a_1,b_1},
\end{aligned}
\quad (5.17)
$$

$$I_3(V_3, W_3) = \int_{b_1}^{c_1} \left\langle \frac{\partial V_3}{\partial x}(x,\cdot), \frac{\partial W_3}{\partial x}(x,\cdot) \right\rangle_{a_2,b_2} dx + C_N^{b_1,c_1} \left\langle V_3^{(N,0)}, W_3^{(N,0)} \right\rangle_{a_2,b_2}$$
$$+ \int_{a_2}^{b_2} \left\langle \frac{\partial V_3}{\partial y}(\cdot, y), \frac{\partial W_3}{\partial y}(\cdot, y) \right\rangle_{b_1,c_1} dy + C_N^{a_2,b_2} \left\langle V_3^{(0,N)}, W_3^{(0,N)} \right\rangle_{b_1,c_1}. \tag{5.18}$$

Equations (5.9), (5.13), (5.15), and (5.16) yield

$$\langle K(V_{\Gamma_1}, V_{\Gamma_2}), (W_{\Gamma_1}, W_{\Gamma_2}) \rangle = \sum_{i=1}^{3} I_i(V_i, W_i) \tag{5.19}$$

and hence (5.14), (5.17), and (5.18) imply that K is self-adjoint.

Equations (5.19), (5.14), (5.17), and (5.18) imply that $\langle K(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, V_{\Gamma_2}) \rangle \geq 0$. To show that K is positive definite, we assume $\langle K(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, V_{\Gamma_2}) \rangle = 0$. Then using (5.14), we obtain

$$\frac{\partial V_1}{\partial x}(x, \xi) = 0, \quad x \in [a_1, b_1], \quad \xi \in \mathcal{G}_{a_2,b_2},$$
$$\frac{\partial V_1}{\partial y}(\xi, y) = 0, \xi \in \mathcal{G}_{a_1,b_1}, \quad y \in [a_2, b_2].$$

Since $V_1 \in X_1$, we have

$$V_1(a_1, \xi) = 0, \quad \xi \in \mathcal{G}_{a_2,b_2}, \quad V_1(\xi, a_2) = 0, \quad \xi \in \mathcal{G}_{a_1,b_1}.$$

Hence,

$$V_1(x, \xi) = 0, \quad x \in [a_1, b_1], \quad \xi \in \mathcal{G}_{a_2,b_2}, \quad V_1(\xi, y) = 0, \quad \xi \in \mathcal{G}_{a_1,b_1}, \quad y \in [a_2, b_2].$$

Taking $x = b_1$ and $y = b_2$, and using (5.3) and that $V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, we obtain $V_{\Gamma_1} = V_{\Gamma_2} = 0$. □

5.2. Preconditioner

Let $P: P_N^0(a_1, b_1) \times P_N^0(a_2, b_2) \to P_N^0(a_1, b_1) \times P_N^0(a_2, b_2)$ be defined for $V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, by

$$P(V_{\Gamma_1}, V_{\Gamma_2}) = (W_{\Gamma_1}, W_{\Gamma_2}), \tag{5.20}$$

where the $W_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, are uniquely determined by (cf. (5.2))

$$W_{\Gamma_1}(\xi) = \frac{\partial V_1^h}{\partial y}(\xi, b_2) - \frac{\partial V_2}{\partial y}(\xi, b_2), \xi \in \mathcal{G}_{a_1,b_1},$$
$$W_{\Gamma_2}(\xi) = \frac{\partial V_1^v}{\partial x}(b_1, \xi) - \frac{\partial V_3}{\partial x}(b_1, \xi), \xi \in \mathcal{G}_{a_2,b_2} \tag{5.21}$$

and where the $V_i \in X_i$, $i=2,3$, satisfy (5.4), (5.5), and $V_1^h, V_1^v \in X_1$ satisfy (cf. (5.3))

$$\Delta V_1^h(\xi)=0, \quad \xi \in \mathcal{G}_1, \quad V_1^h|_{\overline{\Gamma}_1}=V_{\Gamma_1}, \quad V_1^h|_{\overline{\Gamma}_2}=0, \tag{5.22}$$

$$\Delta V_1^v(\xi)=0, \quad \xi \in \mathcal{G}_1, \quad V_1^v|_{\overline{\Gamma}_1}=0, \quad V_1^v|_{\overline{\Gamma}_2}=V_{\Gamma_2}. \tag{5.23}$$

Clearly, our definition of P involves two pairs of adjacent problems: (5.22), (5.4) and (5.23), (5.5). In terms of matrices, the matrix representation of P consists of the diagonal blocks in the matrix representation of K.

Theorem 5.2. The operator $P: P_N^0(a_1,b_1)\times P_N^0(a_2,b_2)\to P_N^0(a_1,b_1)\times P_N^0(a_2,b_2)$ defined by (5.20), (5.21) is self-adjoint and positive definite with respect to the inner product (5.8).

Proof. Following the proof of Theorem 5.1, using (5.8), (5.20), and (5.21), we have

$$\langle P(V_{\Gamma_1},V_{\Gamma_2}),(W_{\Gamma_1},W_{\Gamma_2})\rangle = \left\langle \frac{\partial V_1^h}{\partial y}(\cdot,b_2)-\frac{\partial V_2}{\partial y}(\cdot,b_2),\, W_{\Gamma_1}\right\rangle_{a_1,b_1} + \left\langle \frac{\partial V_1^v}{\partial x}(b_1,\cdot)-\frac{\partial V_3}{\partial x}(b_1,\cdot),\, W_{\Gamma_2}\right\rangle_{a_2,b_2}, \tag{5.24}$$

where the $V_i\in X_i$, $i=2,3$, satisfy (5.4), (5.5), and V_1^h, $V_1^v\in X_1$ satisfy (5.22), (5.23). Let $W_i\in X_i$, $i=2,3$, satisfy (5.11), (5.12), and let W_1^h, $W_1^v\in X_1$ be such that

$$\Delta W_1^h(\xi)=0, \quad \xi \in \mathcal{G}_1, \quad W_1^h|_{\overline{\Gamma}_1}=W_{\Gamma_1}, \quad W_1^h|_{\overline{\Gamma}_2}=0, \tag{5.25}$$

$$\Delta W_1^v(\xi)=0, \quad \xi \in \mathcal{G}_1, \quad W_1^v|_{\overline{\Gamma}_1}=0, \quad W_1^v|_{\overline{\Gamma}_2}=W_{\Gamma_2}. \tag{5.26}$$

Using (5.22), (5.23), (3.6), (2.10), (5.25), and (5.26), we obtain (cf. (5.13))

$$0=\langle -\Delta V_1^h, W_1^h\rangle_1 = -\sum_{j=1}^{N-1} w_j^{a_2,b_2}\left\langle \frac{\partial^2 V_1^h}{\partial x^2}(\cdot,\xi_j^{a_2,b_2}),\, W_1^h(\cdot,\xi_j^{a_2,b_2})\right\rangle_{a_1,b_1} - \sum_{i=1}^{N-1} w_i^{a_1,b_1}\left\langle \frac{\partial^2 V_1^h}{\partial y^2}(\xi_i^{a_1,b_1},\cdot),\, W_1^h(\xi_i^{a_1,b_1},\cdot)\right\rangle_{a_2,b_2}$$

$$= \int_{a_1}^{b_1} \left\langle \frac{\partial V_1^h}{\partial x}(x,\cdot), \frac{\partial W_1^h}{\partial x}(x,\cdot) \right\rangle_{a_2,b_2} dx + C_N^{a_1,b_1} \left\langle V_1^{h(N,0)}, W_1^{h(N,0)} \right\rangle_{a_2,b_2}$$
$$+ \int_{a_2}^{b_2} \left\langle \frac{\partial V_1^h}{\partial y}(\cdot,y), \frac{\partial W_1^h}{\partial y}(\cdot,y) \right\rangle_{a_1,b_1} dy - \left\langle \frac{\partial V_1^h}{\partial y}(\cdot,b_2), W_{\Gamma_1} \right\rangle_{a_1,b_1}$$
$$+ C_N^{a_2,b_2} \left\langle V_1^{h(0,N)}, W_1^{h(0,N)} \right\rangle_{a_1,b_1} \tag{5.27}$$

and

$$0 = \langle -\Delta V_1^v, W_1^v \rangle_1 = -\sum_{j=1}^{N-1} w_j^{a_2,b_2} \left\langle \frac{\partial^2 V_1^v}{\partial x^2}(\cdot, \xi_j^{a_2,b_2}), W_1^v(\cdot, \xi_j^{a_2,b_2}) \right\rangle_{a_1,b_1}$$
$$- \sum_{i=1}^{N-1} w_i^{a_1,b_1} \left\langle \frac{\partial^2 V_1^v}{\partial y^2}(\xi_i^{a_1,b_1},\cdot), W_1^v(\xi_i^{a_1,b_1},\cdot) \right\rangle_{a_2,b_2}$$
$$= \int_{a_1}^{b_1} \left\langle \frac{\partial V_1^v}{\partial x}(x,\cdot), \frac{\partial W_1^v}{\partial x}(x,\cdot) \right\rangle_{a_2,b_2} dx - \left\langle \frac{\partial V_1^v}{\partial x}(b_1,\cdot), W_{\Gamma_2} \right\rangle_{a_2,b_2}$$
$$+ C_N^{a_1,b_1} \left\langle V_1^{v(N,0)}, W_1^{v(N,0)} \right\rangle_{a_2,b_2} + \int_{a_2}^{b_2} \left\langle \frac{\partial V_1^v}{\partial y}(\cdot,y), \frac{\partial W_1^v}{\partial y}(\cdot,y) \right\rangle_{a_1,b_1} dy$$
$$+ C_N^{a_2,b_2} \left\langle V_1^{v(0,N)}, W_1^{v(0,N)} \right\rangle_{a_1,b_1}. \tag{5.28}$$

Since V_i, W_i, $i=2,3$, are the same as in the proof of Theorem 5.1, (5.15) and (5.16) are satisfied. Hence (5.24), (5.27), (5.28), (5.15), and (5.16) give (cf. (5.19))

$$\langle P(V_{\Gamma_1}, V_{\Gamma_2}), (W_{\Gamma_1}, W_{\Gamma_2}) \rangle = I_1(V_1^h, W_1^h) + I_1(V_1^v, W_1^v) + \sum_{i=2}^{3} I_i(V_i, W_i), \tag{5.29}$$

where the I_i, $i=1,2,3$, are defined in (5.14), (5.17), (5.18). Equations (5.29) and (5.14), (5.17), (5.18) imply that P is self-adjoint and positive definite. □

We show the spectral equivalence of the operators K and P in a special case.

Theorem 5.3. Assume that $b_i - a_i = c_i - b_i$, $i=1,2$. Then K and P are spectrally equivalent with respect to the inner product (5.8). Specifically, for $V_{\Gamma_i} \in P_N^0(a_i,b_i)$, $i=1,2$,

$$\frac{1}{2}\langle P(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, V_{\Gamma_2}) \rangle \le \langle K(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, V_{\Gamma_2}) \rangle$$
$$\le 2\langle P(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, V_{\Gamma_2}) \rangle. \tag{5.30}$$

Proof. We take $V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$. Then (5.19) gives

$$\langle K(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, V_{\Gamma_2})\rangle = \sum_{i=1}^{3} I_i(V_i, V_i), \tag{5.31}$$

where $V_i \in X_i$, $i = 1, 2, 3$, satisfy (5.3)–(5.5) and I_i, $i = 1, 2, 3$, are defined in (5.14), (5.17), (5.18). It follows from (5.29) that

$$\langle P(V_{\Gamma_1}, V_{\Gamma_2}), (V_{\Gamma_1}, W_{\Gamma_2})\rangle = I_1(V_1^h, V_1^h) + I_1(V_1^v, V_1^v) + \sum_{i=2}^{3} I_i(V_i, V_i), \tag{5.32}$$

where V_i, $i = 1, 2$, satisfy (5.4), (5.5), and V_1^h, V_1^v satisfy (5.22), (5.23). Using (5.4), (5.22), and (5.23), we have $V_1 = V_1^h + V_2^v$. Hence (5.14) and the inequality $(\alpha + \beta)^2 \le 2(\alpha^2 + \beta^2)$, $\alpha, \beta \in R$, give

$$I_1(V_1, V_1) \le 2[I_1(V_1^h, V_1^h) + I_1(V_1^v, V_1^v)].$$

Therefore (5.31), (5.32), and the last inequality imply the second inequality in (5.30).

It follows from (5.4), (5.22), and symmetry of the collocation points in the y-direction about b_2 that $V_1^h(x, y) = V_2(x, b_2 - y)$. Hence (5.14), (5.17), and symmetry of the collocation points give

$$I_1(V_1^h, V_1^h) = I_2(V_2, V_2).$$

In a similar way, we obtain

$$I_1(V_1^v, V_1^v) = I_3(V_3, V_3).$$

Hence (5.32), the last two equations, and (5.31) yield the first inequality in (5.30). $\square$

Our numerical tests indicate that the assumption $b_i - a_i = c_i - b_i$, $i = 1, 2$, in Theorem 5.3 is technical rather than essential. If the assumption is not satisfied, the operators K and P remain spectrally equivalent with the spectral constants depending on $b_i - a_i$, $c_i - b_i$, $i = 1, 2$.

5.3. Solving with P and Multiplying by K

The definition (5.20)–(5.21) of the operator P implies that, given $W_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, the solution of

$$P(V_{\Gamma_1}, V_{\Gamma_2}) = (W_{\Gamma_1}, W_{\Gamma_2})$$

for $V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i=1,2$, consists of solving two independent problems. The first problem involves finding $V_{\Gamma_1} \in P_N^0(a_1, b_1)$ such that

$$\frac{\partial V_1^h}{\partial y}(\xi, b_2) - \frac{\partial V_2}{\partial y}(\xi, b_2) = W_{\Gamma_1}(\xi), \quad \xi \in \mathcal{G}_{a_1,b_1}, \tag{5.33}$$

where $V_1^h \in X_1$ satisfies (5.22) and $V_2 \in X_2$ satisfies (5.4). The second problem involves finding $V_{\Gamma_2} \in P_N^0(a_2, b_2)$ such that

$$\frac{\partial V_1^v}{\partial x}(b_1, \xi) - \frac{\partial V_3}{\partial x}(b_1, \xi) = W_{\Gamma_2}(\xi), \quad \xi \in \mathcal{G}_{a_2,b_2},$$

where $V_1^v \in X_1$ satisfies (5.23) and $V_3 \in X_3$ satisfies (5.5). We show how to solve (5.33); the second problem can be solved in a similar way. To this end let ψ_n, γ_n, $n=2,\ldots,N$, be as in Lemma 2.1 for $a=a_1$, $b=b_1$. Let v_n, $n=2,\ldots,N$, be as in Lemma 2.2 for $a=a_2$, $b=b_2$, and let w_n, $n=2,\ldots,N$, be as in Lemma 2.3 for $a=b_2$, $b=c_2$. Then for arbitrary $\{\alpha_n\}_{n=2}^N$, V_1^h defined by

$$V_1^h(x,y) = \sum_{n=2}^N \alpha_n \psi_n(x) v_n(y), \quad x \in [a_1, b_1], \quad y \in [a_2, b_2]$$

is in X_1 and it satisfies $\Delta V_1^h(\xi) = 0$, $\xi \in \mathcal{G}_1$, and $V_1^h|_{\overline{\Gamma}_2} = 0$. Similarly, V_2 given by

$$V_2(x,y) = \sum_{n=2}^N \alpha_n \psi_n(x) w_n(y), \quad x \in [a_1, b_1], \quad y \in [b_2, c_2]$$

is in X_2 and it satisfies $\Delta V_2(\xi) = 0$, $\xi \in \mathcal{G}_2$. Moreover,

$$V_{\Gamma_1}(x) = V_1^h(x, b_2) = V_2(x, b_2) = \sum_{n=2}^N \alpha_n \psi_n(x), \quad x \in [a_1, b_1]. \tag{5.34}$$

Since $W_{\Gamma_1} \in P_N^0(a_1, b_1)$, we have

$$W_{\Gamma_1}(x) = \sum_{n=2}^N \beta_n \psi_n(x), \quad x \in [a_1, b_1]. \tag{5.35}$$

Hence (5.33) becomes

$$\sum_{n=2}^N \alpha_n \psi_n(\xi)[v_n'(b_2) - w_n'(b_2)] = \sum_{n=2}^N \beta_n \psi_n(\xi), \quad \xi \in \mathcal{G}_{a_1,b_1},$$

which yields

$$\alpha_n = \frac{\beta_n}{v_n'(b_2) - w_n'(b_2)}, \quad n = 2, \ldots, N.$$

It follows from (2.15), (2.16) with $a = a_2$, $b = b_2$, that all $v_n'(b_2)$, $n = 2, \ldots, N$, can be precomputed with the cost $O(N^2)$. Similarly, it follows from the proof of Lemma 2.3, that all $w_n'(b_2)$, $n = 2, \ldots, N$, can be precomputed with the cost $O(N^2)$ using (2.15), (2.16) with $a = b_2$, $b = c_2$, and then taking $w_n'(b_2) = -v_n'(c_2)$.

Remark 5.1. If $b_2 - a_2 = c_2 - b_2$, then $w_n'(b_2) = -v_n(b_2)$ and hence

$$\alpha_n = \frac{\beta_n}{2v_n'(b_2)}, \quad n = 2, \ldots, N.$$

We introduce

$$\vec{\alpha} = [\alpha_2, \ldots, \alpha_N]^T, \quad \vec{\beta} = [\beta_2, \ldots, \beta_N]^T.$$

Then (5.35), (5.34), and Lemma 2.4 give

$$\begin{aligned} \vec{\beta} &= Z^T [B']^{-1} B^T D [W_{\Gamma_1}(\xi_1^{a_1,b_1}), \ldots, W_{\Gamma_1}(\xi_{N-1}^{a_1,b_1})]^T, \\ [V_{\Gamma_1}(\xi_1^{a_1,b_1}), &\ldots, V_{\Gamma_1}(\xi_{N-1}^{a_1,b_1})]^T = BZ\vec{\alpha}. \end{aligned} \tag{5.36}$$

Thus, given $W_{\Gamma_1}(\xi)$, $\xi \in \mathcal{G}_{a_1,b_1}$, (5.36) and Remark 2.1 imply that $V_{\Gamma_1}(\xi)$, $\xi \in \mathcal{G}_{a_1,b_1}$, can be computed with the cost $O(N^2)$ provided that the matrix Z is known. It follows from the discussion at the end of Sec. 4 in [4] that Z can be precomputed at the cost of $O(N^2)$.

The definition (5.1)–(5.2) of the operator K implies that, given $V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, the computation of $W_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, such that $(W_{\Gamma_1}, W_{\Gamma_2}) = K(V_{\Gamma_1}, V_{\Gamma_2})$, involves solving the collocation problems (5.3)–(5.5). Let $V_1, V_1^h, V_1^v \in X_1$ be respectively solutions of (5.3), (5.22), (5.23). Then $V_1 = V_1^h + V_1^v$. Hence, it follows from (5.2) that

$$\begin{aligned} W_{\Gamma_1}(\xi) &= \frac{\partial V_1^h}{\partial y}(\xi, b_2) + \frac{\partial V_1^v}{\partial y}(\xi, b_2) - \frac{\partial V_2}{\partial y}(\xi, b_2), \quad \xi \in \mathcal{G}_{a_1,b_1}, \\ W_{\Gamma_2}(\xi) &= \frac{\partial V_1^h}{\partial x}(b_1, \xi) + \frac{\partial V_1^v}{\partial x}(b_1, \xi) - \frac{\partial V_3}{\partial x}(b_1, \xi), \quad \xi \in \mathcal{G}_{a_2,b_2}. \end{aligned}$$

We show how to compute $\frac{\partial V_1^h}{\partial y}(\xi, b_2)$, $\xi \in \mathcal{G}_{a_1,b_1}$, and $\frac{\partial V_1^h}{\partial x}(b_1, \xi)$, $\xi \in \mathcal{G}_{a_2,b_2}$. (All remaining partial derivatives can be computed in a similar way.) Let

ψ_n, γ_n, $n=2,\ldots,N$, be as in Lemma 2.1 for $a=a_1$, $b=b_1$, and let v_n, $n=2,\ldots,N$, be as in Lemma 2.2 for $a=a_2$, $b=b_2$. Since $V_{\Gamma_1}\in P_N^0(a_1,b_1)$, we have

$$V_{\Gamma_1}(x)=\sum_{n=2}^{N}\alpha_n\psi_n(x),\quad x\in[a_1,b_1]. \tag{5.37}$$

Moreover, V_1^h defined by

$$V_1^h(x,y)=\sum_{n=2}^{N}\alpha_n\psi_n(x)v_n(y),\quad x\in[a_1,b_1],\quad y\in[a_2,b_2] \tag{5.38}$$

is a solution of (5.22). Hence

$$\frac{\partial V_1^h}{\partial y}(x,b_2)=\sum_{n=2}^{N}\alpha_n v_n'(b_2)\psi_n(x),\quad x\in[a_1,b_1], \tag{5.39}$$

where all $v_n'(b_2)$, $n=2,\ldots,N$, have been precomputed. We introduce

$$\vec{\alpha}=[\alpha_2,\ldots,\alpha_N]^T,\quad \vec{\beta}=[\alpha_2 v_2'(b_2),\ldots,\alpha_N v_N'(b_2)]^T.$$

Then (5.37), (5.39), and Lemma 2.4 yield

$$\vec{\alpha}=Z^T[B']^{-1}B^TD[V_{\Gamma_1}(\xi_1^{a_1,b_1}),\ldots,V_{\Gamma_1}(\xi_{N-1}^{a_1,b_1})]^T,$$

$$\left[\frac{\partial V_1^h}{\partial y}(\xi_1^{a_1,b_1},b_2),\ldots,\frac{\partial V_1^h}{\partial y}(\xi_{N-1}^{a_1,b_1},b_2)\right]^T=BZ\vec{\beta}.$$

Thus, given $V_{\Gamma_1}(\xi)$, $\xi\in\mathcal{G}_{a_1,b_1}$, Remark 2.1 implies that $\frac{\partial V_1^h}{\partial y}(\xi,b_2)$, $\xi\in\mathcal{G}_{a_1,b_1}$, can be computed with cost $O(N^2)$. Equation (5.38) gives

$$\frac{\partial V_1^h}{\partial x}(b_1,\xi)=\sum_{n=2}^{N}\alpha_n\psi_n'(b_1)v_n(\xi),\quad \xi\in\mathcal{G}_{a_2,b_2}.$$

All $\psi_n'(b_1)$, $n=2,\ldots,N$, can be precomputed with cost $O(N^2)$ using (2.12) and (2.13) with $a=a_1$, $b=b_1$. Also, all $v_n(\xi)$, $\xi\in\mathcal{G}_{a_2,b_2}$, $n=2,\ldots,N$, can be precomputed with cost $O(N^3)$ using (2.15), (2.16) with $a=a_2$, $b=b_2$. Hence $\frac{\partial V_1^h}{\partial x}(b_1,\xi)$, $\xi\in\mathcal{G}_{a_2,b_2}$, can be obtained with a cost $O(N^2)$ by computing the product of the matrix $C=(c_{i,n})_{i=1,n=2}^{N-1,N}$, $c_{i,n}=v_n(\xi_i^{a_2,b_2})$, and the vector $[\alpha_2\psi_2'(b_1),\ldots,\alpha_N\psi_N'(b_1)]^T$.

6. COST OF SOLVING THE COLLOCATION PROBLEM FOR L-SHAPED REGION

First, with cost $O(N^2)$, we precompute $\{\xi_i\}_{i=1}^N$, $\phi_k(\xi_i)$, $k=0,\ldots,N$, $i=1,\ldots,N-1$, of (2.2) and (2.3), the matrices Λ and Z of (2.7) and (2.8), and $\psi_n'(b_1)$, $n=2,\ldots,N$, of (2.12). Also, with cost $O(N^3)$, we precompute $v_n(\xi)$, $\xi\in\mathcal{G}_{a_2,b_2}$, $n=2,\ldots,N$.

We perform step 1 of the Algorithm in Sec. 4 using the matrix decomposition algorithm of [4]. For example, it follows from [4] that the coefficients $\{\hat{u}_{k,l}^{(1)}\}_{k,l=0}^N$ in

$$\hat{U}_1(x,y)=\sum_{k,l=0}^N \hat{u}_{k,l}^{(1)}\phi_k^{a_1,b_1}(x)\phi_l^{a_2,b_2}(y)$$

are computed with a cost of $O(N^3)$. Thus the cost of step 1 is $O(N^3)$. In step 2 of the Algorithm in Sec. 4, we solve (5.6), (5.7) using PCG with the P of (5.20), (5.21) as a preconditioner and (5.8) as an inner product. It follows from the discussion in Sec. 5.3 that the cost of a PCG step is $O(N^2)$. Hence if the number of PCG steps equal to m, the cost of step 2 is $O(mN^2)$. Finally, step 3 of the Algorithm in Sec. 4 is performed using the matrix decomposition algorithm of [4]. For example, it follows from [4] that the coefficients $\{u_{k,l}^{(1)}\}_{k,l=0}^N$ in

$$U_1(x,y)=\sum_{k,l=0}^N u_{k,l}^{(1)}\phi_k^{a_1,b_1}(x)\phi_l^{a_2,b_2}(y)$$

are computed with a cost of $O(N^3)$. Thus the cost of step 3 is $O(N^3)$. It follows that the total cost of the Algorithm in Sec. 4 for solving the collocation problem (3.1), (3.3) is $O(N^3)+O(mN^2)$.

7. SPECTRAL COLLOCATION FOR RECTANGLE WITH CROSS-POINT

With Ω as in (1.3), let Q_i, X_i, $\mathcal{G}_i$, $\tilde{\mathcal{G}}_i$, $i=1,2,3$, and $\mathcal{G}_i'$, $i=1,2$, be defined as in Sec. 3. Note, for example, that now

$$\tilde{\mathcal{G}}_1=\{(a_1,\xi):\xi\in\mathcal{G}_{a_2,b_2}\cup\{a_2,b_2\}\}\cup\{(\xi,a_2):\xi\in\mathcal{G}_{a_1,b_1}\cup\{b_1\}\}.$$

Let

$$\begin{aligned} Q_4 &= P_N(b_1,c_1)\otimes P_N(b_2,c_2), \quad X_4=\{v\in Q_4: v=0 \text{ on } \partial\Omega\cap\partial\Omega_4\},\\ \mathcal{G}_4 &= \mathcal{G}_{b_1,c_1}\times\mathcal{G}_{b_2,c_2}.\end{aligned}$$

Let $\tilde{\mathcal{G}}_4$ be the set of collocation points on $\partial\Omega \cap \partial\Omega_4$, and let for $i = 3, 4$, $\mathcal{G}'_i$ be the set of collocation points on Γ_i.

The Legendre spectral collocation problem for (1.1) and (1.3) consists of finding $U_i \in Q_i$, $i = 1, 2, 3, 4$, satisfying (3.1)–(3.3),

$$\Delta U_4(\xi) = f(\xi), \quad \xi \in \mathcal{G}_4, \tag{7.1}$$

$$U_4(\xi) = g(\xi), \quad \xi \in \tilde{\mathcal{G}}_4, \tag{7.2}$$

$$\frac{\partial^j U_2}{\partial x^j}(\xi) = \frac{\partial^j U_4}{\partial x^j}(\xi), \quad \xi \in \mathcal{G}'_3, \quad \frac{\partial^j U_3}{\partial y^j}(\xi) = \frac{\partial^j U_4}{\partial y^j}(\xi), \quad \xi \in \mathcal{G}'_4, \quad j = 0, 1,$$

$$U_1(b_1, b_2) = U_2(b_1, b_2) = U_3(b_1, b_2) = U_4(b_1, b_2)$$

and

$$\frac{\partial U_1}{\partial y}(b_1, b_2) = \frac{\partial U_2}{\partial y}(b_1, b_2). \tag{7.3}$$

Our cross-point equation (7.3) appears to be much simpler than the corresponding equation in the spectral element method (see the discussion at the bottom of page 86 in [11] or (28) in [12].) It can be shown that the spectral collocation problem defined in this way has a unique solution.

In addition to (3.4), we have

$$U_2|_{\overline{\Gamma}_3} = U_4|_{\overline{\Gamma}_3}, \quad U_3|_{\overline{\Gamma}_4} = U_4|_{\overline{\Gamma}_4}.$$

Let Y be the space defined by

$$Y = \{(V_1, V_2, V_3, V_4) : V_1 \in P_N(a_1, b_1), V_2 \in P_N(a_2, b_2), V_3 \in P_N(b_2, c_2), \\ V_4 \in P_N(b_1, c_1), V_1(a_1) = V_2(a_2) = V_3(c_2) = V_4(c_1) = 0, \\ V_1(b_1) = V_2(b_2) = V_3(b_2) = V_4(b_1), V'_2(b_2) = V'_3(b_2)\}.$$

It follows from Lemma 2.3 in [6] that any (V_1, V_2, V_3, V_4) in Y is uniquely determined by the values of V_1, V_2, V_3, and V_4 on $\mathcal{G}_{a_1,b_1}$, $\mathcal{G}_{a_2,b_2}$, $\mathcal{G}_{b_2,c_2}$, and $\mathcal{G}_{b_1,c_1}$, respectively.

Assume that $U_i \in X_i$, $i = 1, 2, 3, 4$, are solutions to the collocation problem with a cross-point. Let $(U_{\Gamma_1}, U_{\Gamma_2}, U_{\Gamma_3}, U_{\Gamma_4}) \in Y$ be defined by (4.1) and

$$\begin{aligned} U_{\Gamma_3}(\xi) &= U_2(b_1, \xi) = U_4(b_1, \xi), \quad \xi \in \mathcal{G}_{b_2,c_2}, \\ U_{\Gamma_4}(\xi) &= U_3(\xi, b_2) = U_4(\xi, b_2), \quad \xi \in \mathcal{G}_{b_1,c_1}. \end{aligned} \tag{7.4}$$

Let $\hat{U}_i \in Q_i$, $i=1,2,3,4$, be defined by (4.2)–(4.4),

$$\Delta\hat{U}_4(\xi) = f(\xi), \quad \xi \in \mathcal{G}_4, \tag{7.5}$$

$$\hat{U}_4(\xi) = g(\xi), \quad \xi \in \tilde{\mathcal{G}}_4, \tag{7.6}$$

$$\hat{U}_2(\xi) = \hat{U}_4(\xi) = 0, \quad \xi \in \mathcal{G}_3', \quad \hat{U}_3(\xi) = \hat{U}_4(\xi) = 0, \quad \xi \in \mathcal{G}_4', \tag{7.7}$$

$$\hat{U}_1(b_1, b_2) = \hat{U}_2(b_1, b_2) = \hat{U}_3(b_1, b_2) = \hat{U}_4(b_1, b_2) = 0. \tag{7.8}$$

Let $\tilde{U}_i \in X_i$, $i=1,2,3$, be defined by (4.5) and let $\tilde{U}_4 \in X_4$ be such that

$$\tilde{U}_4 = U_4 - \hat{U}_4.$$

Then, in addition to (4.6), we have

$$\Delta\tilde{U}_2(\xi) = 0, \quad \xi \in \mathcal{G}_2, \quad \tilde{U}_2|_{\overline{\Gamma}_1} = U_{\Gamma_1}, \quad \tilde{U}_2|_{\overline{\Gamma}_3} = U_{\Gamma_3}, \tag{7.9}$$

$$\Delta\tilde{U}_3(\xi) = 0, \quad \xi \in \mathcal{G}_3, \quad \tilde{U}_3|_{\overline{\Gamma}_2} = U_{\Gamma_2}, \quad \tilde{U}_3|_{\overline{\Gamma}_4} = U_{\Gamma_4}, \tag{7.10}$$

$$\Delta\tilde{U}_4(\xi) = 0, \quad \xi \in \mathcal{G}_4, \quad \tilde{U}_4|_{\overline{\Gamma}_3} = U_{\Gamma_3}, \quad \tilde{U}_4|_{\overline{\Gamma}_4} = U_{\Gamma_4}. \tag{7.11}$$

Also, in addition to (4.9), we have

$$\begin{aligned} \frac{\partial\tilde{U}_2}{\partial x}(\xi) - \frac{\partial\tilde{U}_4}{\partial x}(\xi) &= \frac{\partial\hat{U}_4}{\partial x}(\xi) - \frac{\partial\hat{U}_2}{\partial x}(\xi), \quad \xi \in \mathcal{G}_3', \\ \frac{\partial\tilde{U}_3}{\partial y}(\xi) - \frac{\partial\tilde{U}_4}{\partial y}(\xi) &= \frac{\partial\hat{U}_4}{\partial y}(\xi) - \frac{\partial\hat{U}_3}{\partial y}(\xi), \quad \xi \in \mathcal{G}_4'. \end{aligned} \tag{7.12}$$

We thus obtain the following algorithm for solving the collocation problem with a cross-point.

Algorithm.

Step 1. With $\hat{U}_i \in Q_i$, $i=1,2,3,4$, defined by (4.2)–(4.4), (7.5)–(7.8), compute the right sides of (4.9), (7.12).

Step 2. Compute $(U_{\Gamma_1}, U_{\Gamma_2}, U_{\Gamma_3}, U_{\Gamma_4}) \in Y$ such that $\tilde{U}_i \in X_i$, $i = 1,2,3,4$, satisfy (4.6), (7.9)–(7.11), (4.9), (7.12).

Step 3. Compute $U_i \in Q_i$, $i=1,2,3,4$, satisfying (3.1), (3.2), (7.1), (7.2), (4.1), (7.4).

Let $K: Y \to Y$ be defined for $(V_{\Gamma_1}, V_{\Gamma_2}, V_{\Gamma_3}, V_{\Gamma_4})$ in Y by

$$K(V_{\Gamma_1}, V_{\Gamma_2}, V_{\Gamma_3}, V_{\Gamma_4}) = (W_{\Gamma_1}, W_{\Gamma_2}, W_{\Gamma_3}, W_{\Gamma_4}),$$

where $(W_{\Gamma_1}, W_{\Gamma_2}, W_{\Gamma_3}, W_{\Gamma_4})$ in Y is uniquely defined by (5.2) and

$$\begin{aligned} W_{\Gamma_3}(\xi) &= \frac{\partial V_2}{\partial x}(b_1, \xi) - \frac{\partial V_4}{\partial x}(b_1, \xi), \quad \xi \in \mathcal{G}_{b_2,c_2}, \\ W_{\Gamma_4}(\xi) &= \frac{\partial V_3}{\partial y}(\xi, b_2) - \frac{\partial V_4}{\partial y}(\xi, b_2), \quad \xi \in \mathcal{G}_{b_1,c_1} \end{aligned}$$

with V_i in X_i, $i = 1, 2, 3, 4$, satisfying (5.3) and

$$\begin{aligned} \Delta V_2(\xi) &= 0, \quad \xi \in \mathcal{G}_2, \quad V_2|_{\overline{\Gamma}_1} = V_{\Gamma_1}, \quad V_2|_{\overline{\Gamma}_3} = V_{\Gamma_3}, \\ \Delta V_3(\xi) &= 0, \quad \xi \in \mathcal{G}_3, \quad V_3|_{\overline{\Gamma}_2} = V_{\Gamma_2}, \quad V_3|_{\overline{\Gamma}_4} = V_{\Gamma_4}, \\ \Delta V_4(\xi) &= 0, \quad \xi \in \mathcal{G}_4, \quad V_4|_{\overline{\Gamma}_3} = V_{\Gamma_3}, \quad V_4|_{\overline{\Gamma}_4} = V_{\Gamma_4}. \end{aligned}$$

Then step 2 of the Algorithm in this section is equivalent to finding $(U_{\Gamma_1}, U_{\Gamma_2}, U_{\Gamma_3}, U_{\Gamma_4})$ in Y such that

$$K(U_{\Gamma_1}, U_{\Gamma_2}, U_{\Gamma_3}, U_{\Gamma_4}) = (F_{\Gamma_1}, F_{\Gamma_2}, F_{\Gamma_3}, F_{\Gamma_4}), \tag{7.13}$$

where, with $\hat{U}_i$ in Q_i, $i = 1, 2, 3, 4$, satisfying (4.2)–(4.4), (7.5)–(7.8), $(F_{\Gamma_1}, F_{\Gamma_2}, F_{\Gamma_3}, F_{\Gamma_4})$ in Y is given by (5.7), and

$$\begin{aligned} F_{\Gamma_3}(\xi) &= \frac{\partial \hat{U}_4}{\partial x}(b_1, \xi) - \frac{\partial \hat{U}_2}{\partial x}(b_1, \xi), \quad \xi \in \mathcal{G}_{b_2,c_2}, \\ F_{\Gamma_4}(\xi) &= \frac{\partial \hat{U}_4}{\partial y}(\xi, b_2) - \frac{\partial \hat{U}_3}{\partial y}(\xi, b_2), \quad \xi \in \mathcal{G}_{b_1,c_1}. \end{aligned}$$

In addition to K, we introduce $\tilde{K}: Y \to Y$ defined for $(V_{\Gamma_1}, V_{\Gamma_2}, V_{\Gamma_3}, V_{\Gamma_4})$ in Y by

$$\tilde{K}(V_{\Gamma_1}, V_{\Gamma_2}, V_{\Gamma_3}, V_{\Gamma_4}) = (W_{\Gamma_1}, W_{\Gamma_2}, W_{\Gamma_3}, W_{\Gamma_4}), \tag{7.14}$$

where $(W_{\Gamma_1}, W_{\Gamma_2}, W_{\Gamma_3}, W_{\Gamma_4})$ in Y is uniquely defined by

$$\begin{aligned} W_{\Gamma_1}(\xi) &= \frac{\partial \tilde{V}_1}{\partial y}(\xi, b_2) - \frac{\partial \tilde{V}_2}{\partial y}(\xi, b_2), \xi \in \mathcal{G}_{a_1,b_1}, \\ W_{\Gamma_2}(\xi) &= \frac{\partial \tilde{V}_1}{\partial x}(b_1, \xi) - \frac{\partial \tilde{V}_3}{\partial x}(b_1, \xi), \xi \in \mathcal{G}_{a_2,b_2}, \end{aligned} \tag{7.15}$$

$$W_{\Gamma_3}(\xi)=\frac{\partial \tilde V_2}{\partial x}(b_1,\xi)-\frac{\partial \tilde V_4}{\partial x}(b_1,\xi),\ \xi\in\mathcal{G}_{b_2,c_2},$$
$$W_{\Gamma_4}(\xi)=\frac{\partial \tilde V_3}{\partial y}(\xi,b_2)-\frac{\partial \tilde V_4}{\partial y}(\xi,b_2),\ \xi\in\mathcal{G}_{b_1,c_1} \tag{7.16}$$

with $\tilde V_i$ in X_i, $i=1,2,3,4$, satisfying

$$\begin{aligned}
&\Delta\tilde V_1(\xi)=0, \quad \xi\in\mathcal{G}_1, \quad \tilde V_1|_{\overline{\Gamma}_1}=\tilde V_{\Gamma_1}, \quad \tilde V_1|_{\overline{\Gamma}_2}=\tilde V_{\Gamma_2},\\
&\Delta\tilde V_2(\xi)=0, \quad \xi\in\mathcal{G}_2, \quad \tilde V_2|_{\overline{\Gamma}_1}=\tilde V_{\Gamma_1}, \quad \tilde V_2|_{\overline{\Gamma}_3}=\tilde V_{\Gamma_3},\\
&\Delta\tilde V_3(\xi)=0, \quad \xi\in\mathcal{G}_3, \quad \tilde V_3|_{\overline{\Gamma}_2}=\tilde V_{\Gamma_2}, \quad \tilde V_3|_{\overline{\Gamma}_4}=\tilde V_{\Gamma_4},\\
&\Delta\tilde V_4(\xi)=0, \quad \xi\in\mathcal{G}_4, \quad \tilde V_4|_{\overline{\Gamma}_3}=\tilde V_{\Gamma_3}, \quad \tilde V_4|_{\overline{\Gamma}_4}=\tilde V_{\Gamma_4},
\end{aligned}$$

where $\tilde V_{\Gamma_i}\in P_N^0(a_i,b_i)$, $i=1,2$, $\tilde V_{\Gamma_3}\in P_N^0(b_2,c_2)$, $\tilde V_{\Gamma_4}\in P_N^0(b_1,c_1)$, are defined by

$$\tilde V_{\Gamma_i}(\xi)=V_{\Gamma_i}(\xi), \quad \xi\in\mathcal{G}_{a_i,b_i}, \quad i=1,2, \tag{7.17}$$

$$\tilde V_{\Gamma_3}(\xi)=V_{\Gamma_3}(\xi), \quad \xi\in\mathcal{G}_{b_2,c_2}, \quad \tilde V_{\Gamma_4}(\xi)=V_{\Gamma_4}(\xi), \quad \xi\in\mathcal{G}_{b_1,c_1}. \tag{7.18}$$

It can be shown that for every (V_1, V_2, V_3, V_4) in Y, we have

$$K(V_1,V_2,V_3,V_4)=\tilde K(V_1,V_2,V_3,V_4),$$

that is, the application of K to (V_1, V_2, V_3, V_4) in Y does not depend on the cross-point value $V_1(b_1)=V_2(b_2)=V_3(b_2)=V_4(b_1)$. Hence it follows that the interface problem (7.13) is equivalent to

$$\tilde K(U_{\Gamma_1},U_{\Gamma_2},U_{\Gamma_3},U_{\Gamma_4})=(F_{\Gamma_1},F_{\Gamma_2},F_{\Gamma_3},F_{\Gamma_4}). \tag{7.19}$$

We define the inner product in Y by

$$\begin{aligned}
&\langle (V_{\Gamma_1},V_{\Gamma_2},V_{\Gamma_3},V_{\Gamma_4}),(W_{\Gamma_1},W_{\Gamma_2},W_{\Gamma_3},W_{\Gamma_4})\rangle\\
&\quad=\sum_{i=1}^{2}\langle V_{\Gamma_i},W_{\Gamma_i}\rangle_{a_i,b_i}+\langle V_{\Gamma_3},W_{\Gamma_3}\rangle_{b_2,c_2}+\langle V_{\Gamma_4},W_{\Gamma_4}\rangle_{b_1,c_1},
\end{aligned} \tag{7.20}$$

where $\langle\cdot,\cdot\rangle_{a,b}$ is given in (2.9). It can be shown that the operator $\tilde K\colon Y\to Y$ defined by (7.14)–(7.16) is self-adjoint and positive definite with respect to the inner product (7.20).

The preconditioner $\tilde P\colon Y\to Y$ for $\tilde K$ is defined for $(V_{\Gamma_1},V_{\Gamma_2},V_{\Gamma_3},V_{\Gamma_4})$ in Y by

$$\tilde P(V_{\Gamma_1},V_{\Gamma_2},V_{\Gamma_3},V_{\Gamma_4})=(W_{\Gamma_1},W_{\Gamma_2},W_{\Gamma_3},W_{\Gamma_4}), \tag{7.21}$$

where $(W_{\Gamma_1}, W_{\Gamma_2}, W_{\Gamma_3}, W_{\Gamma_4})$ in Y is uniquely defined by

$$W_{\Gamma_1}(\xi) = \frac{\partial \tilde V_1^h}{\partial y}(\xi, b_2) - \frac{\partial \tilde V_2^h}{\partial y}(\xi, b_2), \quad \xi \in \mathcal{G}_{a_1,b_1},$$
$$W_{\Gamma_2}(\xi) = \frac{\partial \tilde V_1^v}{\partial x}(b_1, \xi) - \frac{\partial \tilde V_3^v}{\partial x}(b_1, \xi), \quad \xi \in \mathcal{G}_{a_2,b_2}, \tag{7.22}$$

$$W_{\Gamma_3}(\xi) = \frac{\partial \tilde V_2^v}{\partial x}(b_1, \xi) - \frac{\partial \tilde V_4^v}{\partial x}(b_1, \xi), \quad \xi \in \mathcal{G}_{b_2,c_2},$$
$$W_{\Gamma_4}(\xi) = \frac{\partial \tilde V_3^h}{\partial y}(\xi, b_2) - \frac{\partial \tilde V_4^h}{\partial y}(\xi, b_2), \quad \xi \in \mathcal{G}_{b_1,c_1} \tag{7.23}$$

with $\tilde V_i^v, \tilde V_i^h$ in X_i, $i = 1, 2, 3, 4$, satisfying

$$\begin{aligned}
&\Delta \tilde V_1^h(\xi) = 0, \quad \xi \in \mathcal{G}_1, \quad \tilde V_1^h|_{\overline{\Gamma}_1} = \tilde V_{\Gamma_1}, \quad \tilde V_1^h|_{\overline{\Gamma}_2} = 0,\\
&\Delta \tilde V_1^v(\xi) = 0, \quad \xi \in \mathcal{G}_1, \quad \tilde V_1^v|_{\overline{\Gamma}_1} = 0, \quad \tilde V_1^v|_{\overline{\Gamma}_2} = \tilde V_{\Gamma_2},\\
&\Delta \tilde V_2^h(\xi) = 0, \quad \xi \in \mathcal{G}_2, \quad \tilde V_2^h|_{\overline{\Gamma}_1} = \tilde V_{\Gamma_1}, \quad \tilde V_2^h|_{\overline{\Gamma}_3} = 0,\\
&\Delta \tilde V_2^v(\xi) = 0, \quad \xi \in \mathcal{G}_2, \quad \tilde V_2^v|_{\overline{\Gamma}_1} = 0, \quad \tilde V_2^v|_{\overline{\Gamma}_3} = \tilde V_{\Gamma_3},\\
&\Delta \tilde V_3^h(\xi) = 0, \quad \xi \in \mathcal{G}_3, \quad \tilde V_3^h|_{\overline{\Gamma}_2} = 0, \quad \tilde V_3^h|_{\overline{\Gamma}_4} = \tilde V_{\Gamma_4},\\
&\Delta \tilde V_3^v(\xi) = 0, \quad \xi \in \mathcal{G}_3, \quad \tilde V_3^v|_{\overline{\Gamma}_2} = \tilde V_{\Gamma_2}, \quad \tilde V_3^v|_{\overline{\Gamma}_4} = 0,\\
&\Delta \tilde V_4^h(\xi) = 0, \quad \xi \in \mathcal{G}_4, \quad \tilde V_4^h|_{\overline{\Gamma}_3} = 0, \quad \tilde V_4^h|_{\overline{\Gamma}_4} = \tilde V_{\Gamma_4},\\
&\Delta \tilde V_4^v(\xi) = 0, \quad \xi \in \mathcal{G}_4, \quad \tilde V_4^v|_{\overline{\Gamma}_3} = \tilde V_{\Gamma_3}, \quad \tilde V_4^v|_{\overline{\Gamma}_4} = 0,
\end{aligned}$$

where $\tilde V_{\Gamma_i} \in P_N^0(a_i, b_i)$, $i = 1, 2$, $\tilde V_{\Gamma_3} \in P_N^0(b_2, c_2)$, $\tilde V_{\Gamma_4} \in P_N^0(b_1, c_1)$ are defined in (7.17), (7.18). Again, our definition of $\tilde P$ involves pairs of adjacent problems and, in terms of matrices, the matrix representation of $\tilde P$ consists of the diagonal blocks in the matrix representation of $\tilde K$. It can be shown that the operator $\tilde P: Y \to Y$ defined by (7.21)–(7.23) is self-adjoint and positive definite with respect to the inner product (7.20).

The implementation of the Algorithm of this section is similar to the implementation of the Algorithm in Sec. 4. In particular, in step 2, we solve (7.19) using PCG, with $\tilde P$ as a preconditioner and (7.20) as an inner product, to obtain $U_{\Gamma_i}(\xi)$, $\xi \in \mathcal{G}_{a_i,b_i}$, $i = 1, 2$, $U_{\Gamma_3}(\xi)$, $\xi \in \mathcal{G}_{b_2,c_2}$, $U_{\Gamma_4}(\xi)$, $\xi \in \mathcal{G}_{b_1,c_1}$. In order to carry out step 3, we need to evaluate

$$U_1(b_1, b_2) = U_2(b_1, b_2) = U_3(b_1, b_2) = U_4(b_1, b_2).$$

This evaluation is equivalent to computing $V_2(b_2)$ given

$$V_2(a_2), \quad V_2(\xi_i^{a_2,b_2}), \quad i = 1, \ldots, N-1, \quad V_3(\xi_i^{b_2,c_2}), \quad i = 1, \ldots, N-1, \quad V_3(c_2),$$

where $V_2 \in P_N(a_2, b_2)$ and $V_3 \in P_N(b_2, c_2)$ are such that $V_2^{(j)}(b_2) = V_3^{(j)}(b_2)$, $j = 0, 1$. Let $\{\phi_k^{a_2,b_2}\}_{k=0}^N$ be the basis for $P_N(a_2, b_2)$ defined by (2.11) and (2.14). Similarly, let $\{\phi_k^{b_2,c_2}\}_{k=0}^N$ be the basis for $P_N(b_2, c_2)$. Then

$$V_2 = \sum_{k=0}^N \alpha_k \phi_k^{a_2,b_2}, \quad V_3 = \sum_{k=0}^N \beta_k \phi_k^{b_2,c_2}$$

satisfy

$$\sum_{k=0}^N \alpha_k \phi_k^{a_2,b_2}(a_2) = V_2(a_2), \tag{7.24}$$

$$\sum_{k=0}^N \alpha_k \phi_k^{a_2,b_2}(\xi_i^{a_2,b_2}) = V_2(\xi_i^{a_2,b_2}), \quad i = 1, \ldots N-1, \tag{7.25}$$

$$\sum_{k=0}^N \alpha_k \phi_k^{a_2,b_2}(b_2) - \sum_{k=0}^N \beta_k \phi_k^{b_2,c_2}(b_2) = 0, \tag{7.26}$$

$$\sum_{k=0}^N \alpha_k [\phi_k^{a_2,b_2}]'(b_2) - \sum_{k=0}^N \beta_k [\phi_k^{b_2,c_2}]'(b_2) = 0, \tag{7.27}$$

$$\sum_{k=0}^N \beta_k \phi_k^{b_2,c_2}(\xi_i^{b_2,c_2}) = V_3(\xi_i^{b_2,c_2}), \quad i = 1, \ldots N-1, \tag{7.28}$$

$$\sum_{k=0}^N \beta_k \phi_k^{b_2,c_2}(c_2) = V_3(c_2). \tag{7.29}$$

Equations (7.25) can be written as the $(N-1) \times (N+1)$ system

$$\left[\, \vec{b}_0 \middle| \vec{b}_1 \middle| B \,\right] \vec{\alpha} = \vec{v}_2,$$

where $\vec{\alpha} = [\alpha_0, \ldots, \alpha_N]^T$ and $\vec{v}_2 = [V_2(\xi_1^{a_2,b_2}), \ldots, V_2(\xi_{N-1}^{a_2,b_2})]^T$. Premultiplication by $B^T D$ yields

$$\left[\, \vec{b}_0' \middle| \vec{b}_1' \middle| B' \,\right] \vec{\alpha} = \vec{v}_2',$$

where $\vec{b}'_j = B^T D\vec{b}_j$, $j = 0, 1$, $B' = B^T DB$, $\vec{v}'_2 = B^T D\vec{v}_2$. Equivalently, we have

$$[\alpha_2, \ldots, \alpha_N]^T = [B']^{-1}\vec{v}'_2 - \alpha_0[B']^{-1}\vec{b}'_0 - \alpha_1[B']^{-1}\vec{b}'_1. \tag{7.30}$$

Using (7.28), we obtain a similar expression for $[\beta_2, \ldots, \beta_N]^T$. Substitution of these expressions into (7.24), (7.26), (7.27), and (7.29) yields a system of four equations in α_0, α_1, β_0, and β_1. Having solved this system, we obtain $[\alpha_2, \ldots, \alpha_N]^T$ from (7.30) and hence $V_2(b_2)$ is given by

$$V_2(b_2) = \sum_{k=0}^{N} \alpha_k \phi_k^{a_2, b_2}(b_2).$$

It should be noted that the costs of multiplying by B^T and solving with B' are $O(N^2)$ and $O(N)$, respectively. Hence the cost of computing $V_2(b_2)$ is $O(N^2)$.

8. EXTENSIONS

The algorithms of Secs. 4 and 7 generalize to Robin boundary conditions with constant coefficients. For example, Eq. (2.9) still guarantees that the operators K and P are self-adjoint and positive definite. The collocation solution in each rectangular subregion is obtained using the matrix decomposition method of [4] which allows for Robin boundary conditions.

Assume that a rectangular polygon Ω is partitioned into l rectangular subregions Ω_i. Then the definitions of the Legendre spectral collocation problem, the operators $\tilde{K}$ and $\tilde{P}$ are similar to those in Sec. 7. For example, the collocation problem consists of finding $U_i \in Q_i$, $i = 1, \ldots, l$, satisfying:

$$\Delta U_i(\xi) = f(\xi), \quad \xi \in \mathcal{G}_i, \quad U_i(\xi) = g(\xi), \quad \xi \in \tilde{\mathcal{G}}_i, \quad i = 1, \ldots, l$$

continuity conditions (also involving the normal derivatives) at the collocation points on each interface; continuity conditions (involving also the partial derivative in the y-direction) at each cross point. For each interface, $\tilde{P}$ is defined in terms of a jump, at the collocation points on the interface, of the normal derivative of spectral harmonic extensions corresponding to two rectangular subregions adjacent to the interface. It is very likely that, as in the case of the finite element Galerkin method, the preconditioner $\tilde{P}$ may be spectrally equivalent to the operator $\tilde{K}$ with the spectral constants depending on the polynomial degree and the number of the rectangular subregions. A coarse grid modification of the preconditioner $\tilde{P}$ may be necessary to reduce the dependence of the spectral constants on the number of the rectangular subregions.

9. NUMERICAL RESULTS

The algorithms of Secs. 4 and 7 were used to solve problem (1.1) for the L-shaped region Ω given by (1.2) and the rectangle Ω with a cross-point given by (1.3). All computations were carried out in double precision on an IBM RS6000 (375 MHz) workstation. The initial guess in the PCG part of each algorithm was taken to be 0. In one test, the PCG method for solving (5.6) and (7.13) was terminated using the stopping criterion

$$\sqrt{\langle r^{(k)}, r^{(k)}\rangle} \le 10^{-13}\sqrt{\langle r^{(0)}, r^{(0)}\rangle},$$

where $r^{(k)}$ is the residual in the kth PCG iteration and $\langle\cdot,\cdot\rangle$ is defined in (5.8) for the L-shaped region and in (7.20) for the rectangle with a cross-point. In the second test, the number of PCG iterations was set to $3\log_2 N$.

In $\Omega_1 = (a_1, b_1)\times(a_2, b_2)$, we computed the error $e_{1,N}$, approximating the maximum norm, using the formula

$$e_{1,N} = \max_{0\le k,l\le 100} |u(x_k, y_l) - U_1(x_k, y_l)|,$$

where $x_k = a_1 + k(b_1 - a_1)/100$, $y_l = a_2 + l(b_2 - a_2)/100$. In a similar way, we computed the error $e_{i,N}$ in Ω_i for $i = 2, 3, 4$. For the L-shaped region Ω and the rectangle Ω with a cross-point, the maximum errors e_N were taken to be

$$e_N = \max\{e_{1,N}, e_{2,N}, e_{3,N}\}$$

and

$$e_N = \max\{e_{1,N}, e_{2,N}, e_{3,N}, e_{4,N}\},$$

respectively.

We considered the following problems for the L-shaped region Ω:

Problem 1. The exact solution $u(x, y) = \cos(3x + 4y)$ and Ω defined by $a_1 = -0.5$, $b_1 = 0$, $c_1 = 1$, $a_2 = 0$, $b_2 = 1$, $c_2 = 2.5$.

Problem 2. The exact solution $u(x, y) = \cosh(3x + 4y)$ and Ω defined by $a_1 = -0.5$, $b_1 = 0$, $c_1 = 1$, $a_2 = -1$, $b_2 = 0.5$, $c_2 = 1$.

Table I. Errors and Numbers of PCG Iterations for Problem 1 in L-shaped Region

N	Error e_N	Iterations with stopping criterion	Error e_N	$3N\log_2 N$ iterations
6	1.10-02	8	1.10-02	8
8	3.55-04	9	3.55-04	9
10	7.44-06	10	7.44-06	10
12	1.09-07	10	1.09-07	11
14	1.19-09	11	1.19-09	12
16	9.93-12	11	9.93-12	12
18	7.16-14	11	7.16-14	13

Table II. Errors and Numbers of PCG Iterations for Problem 2 in L-shaped Region

N	Error e_N	Iterations with stopping criterion	Error e_N	$3N\log_2 N$ iterations
6	1.56-01	8	1.56-01	8
8	4.61-03	9	4.61-03	9
10	9.08-05	9	9.08-05	10
12	1.27-06	10	1.27-06	11
14	1.33-08	10	1.33-08	12
16	1.09-10	11	1.09-10	12
18	9.52-13	11	9.52-13	13

The maximum errors e_N and the numbers of PCG iterations for Problems 1 and 2 are presented in Tables I and II.
For the rectangle Ω with a cross-point, we considered the problems:

Problem 3. The exact solution $u(x,y)=\cos(3x+4y)$ and Ω defined by $a_1=-1$, $b_1=1$, $c_1=2$, $a_2=0$, $b_2=0.5$, $c_2=2$.

Problem 4. The exact solution $u(x,y)=\cosh(3x+4y)$ and Ω defined by $a_1=-1$, $b_1=0$, $c_1=1$, $a_2=-1$, $b_2=0.5$, $c_2=1$.

The maximum errors e_N and the numbers of PCG iterations for Problems 3 and 4 are presented in Tables III and IV.

With the number of PCG iterations equal to $3N\log_2 N$, CPU times for Problem 2, shown in Table V, confirm that the cost of our method is $O(N^3)$.

Table III. Errors and Numbers of PCG Iterations for Problem 3 in Rectangle with Cross-point

N	Error e_N	Iterations with stopping criterion	Error e_N	$3N\log_2 N$ iterations
6	3.17-02	10	3.17-02	8
8	1.11-03	12	1.11-03	9
10	2.38-05	11	2.38-05	10
12	3.47-07	12	3.47-07	11
14	3.75-09	12	3.75-09	12
16	3.11-11	13	3.11-11	12
18	3.61-13	13	3.61-13	13

Table IV. Errors and Numbers of PCG Iterations for Problem 4 in Rectangle with Cross-point

N	Error e_N	Iterations with stopping criterion	Error e_N	$3N\log_2 N$ iterations
6	6.99-01	10	6.99-01	8
8	2.08-02	11	2.08-02	9
10	4.10-04	12	4.10-04	10
12	5.74-06	12	5.74-06	11
14	6.02-08	13	6.02-08	12
16	4.90-10	13	4.90-10	12
18	4.76-12	13	4.76-12	13

In our last example, we solved the problem

$$\Delta u = -1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is the L-shaped region defined by $a_1 = a_2 = -1, b_1 = b_2 = 0, c_1 = c_2 = 1$. It is well known (see, for example, Chapter 8 in [18]) that the solution of this problem has singularities at the corners. For $N = 8, 16, 32, 64, 128, 256$, with the number of PCG iterations equal to $3N\log_2 N$, we calculated the maximum norm errors $e_{i,N}$, $i = 1, 2, 3$, using, in place of the exact solution u, the approximate solution corresponding to $N = 384$. As expected, the errors presented in Table VI exhibit slow convergence due to the corner singularities.

Table V. CPU Times for Problem 2

N	CPU time (seconds)
6	0.01
12	0.04
24	0.19
48	1.26
96	8.94

Table VI. Errors for Singular Problem on L-shaped Region

N	Error e_N
8	6.56-3
16	2.18-3
32	4.23-4
64	1.37-4
128	4.28-5
256	1.30-5

10. CONCLUSIONS

We present a spectral collocation method, with the collocation points being the Legendre–Gauss nodes, for the solution of Poisson's equation on a rectangular polygon partitioned into rectangular subregions. In contrast to other spectral collocation approaches, our method does not rely on the variational formulation of the continuous problem and, in particular, involves a novel and simple way of treating cross-points. The implementational simplicity of our method is based on the solution of decoupled spectral collocation problems on the rectangular subregions and on the solution of the discrete Steklov–Poincaré equation associated with the normal derivative equations at the interface collocation points; the cross point equations do not enter explicitly into the definition of the discrete Steklov–Poincaré equation. The discrete Steklov–Poincaré equation is solved using the preconditioned conjugate gradient method with a preconditioner obtained from the discrete Steklov–Poincaré operators corresponding to pairs of adjacent rectangular subregions. The use of appropriate basis functions along with separation of variables renders our method more efficient than other spectral collocation methods. In the future, we will study the dependence of our preconditioner on the polynomial degree and the number of rectangular subregions, and, if necessary, we will introduce a coarse grid modification of the preconditioner.

REFERENCES

1. Bernardi, C., Maday, Y., and Patera, A. T. (1994). A new nonconforming approach to domain decomposition: the mortar element method. In Brezis, H., and Lions, J.-L. (eds.), Nonlinear Partial Differential Equations and Their Applications, College de France Seminar XI, Pitman, London, pp. 13–51.
2. Bernardi, C., and Maday, Y. (2001). Spectral, Spectral Element and Mortar Element Methods, Theory and Numerics of Differential Equations, Universitext, Springer, Berlin, pp. 1–57.
3. Bialecki, B., and Dryja, M. (2003). A nonoverlapping domain decomposition for orthogonal spline collocation problems. *SIAM J. Numer. Anal.* **41**, 1709–1728.
4. Bialecki, B., and Karageorghis, A. (2004). Legendre Gauss spectral collocation for the Helmholtz equation on a rectangle. *Numer. Algorithms* **36**, 203–227.
5. Chan, T. F., and Mathew, T. P. (1994). Domain decomposition algorithms. *Acta Numerica* 61–143.
6. Douglas, J., Jr., and Dupont, T. (1974). *Collocation Methods for Parabolic Equations in a Single Space Variable*. Lecture Notes in Mathematics, Vol. 358, Springer, New York.
7. Funaro, D. (1986). A multidomain spectral approximation of elliptic equations. *Numer. Methods Partial Differ. Equations* **2**, 187–205.
8. Funaro, D., Quarteroni, A., and Zanolli, P. (1988). An iterative procedure with interface relaxation for domain decomposition methods. *SIAM J. Numer. Anal.* **25**, 1213–1236.
9. Gervasio, P., Ovtchinnikov, E., and Quarteroni, A. (1997). The spectral projection decomposition method for elliptic equations in two dimensions. *SIAM J. Numer. Anal.* **34**, 1616–1639.
10. Golub, G., and Van Loan, C. (1996). *Matrix Computations*, John Hopkins University Press, Baltimore, MD.
11. Maday, Y., and Patera, A. T. (1989). Spectral element methods for the Navier–Stokes equations. In Noor, A., and Oden, J. (eds.), State of the Art Surveys in Computational Mechanics, American Society of Mathematical Engineers, New York, pp. 71–143.
12. Manna, M., Vacca, A., and Deville, M. O. (2004). Preconditioned spectral multi-domain discretization of the incompressible Navier–Stokes equations. *J. Comput. Phys.* **201**, 204–223.
13. Orszag, S. A. (1980). Spectral methods for problems in complex geometries. *J. Comput. Phys.* **37**, 70–92.
14. Quarteroni, A., and Sacchi-Landriani, G. (1988). Domain decomposition preconditioners for spectral collocation method. *J. Sci. Comput.* **3**, 45–76.
15. Quarteroni, A., and Valli, A. (1999). *Domain Decomposition Methods for Partial Differential Equations*, Oxford University Press, New York.
16. Shen, J. (1994). Efficient spectral-Galerkin method I. Direct solvers of second- and forthorder equations using Legendre polynomials. *SIAM J. Sci. Comput.* **15**, 1489–1505.
17. Smith, B. F., Bjørstad, P. E., and Gropp, W. D. (1996). *Domain Decomposition. Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge.
18. Strang, G., and Fix, G. J. (1973). *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, NJ.
19. Szabó, B., and Babuskă, I. (1991). *Finite Element Analysis*, Wiley, New York.
20. Toselli, A., and Widlund, O. (2005). *Domain Decomposition Methods—Algorithms and Theory*, Springer, Berlin.