Journal of Scientific Computing, Vol. 30, No. 3, March 2007 (© 2006)
DOI: 10.1007/s10915-006-9100-1

hp-Version *a priori* Error Analysis of Interior Penalty Discontinuous Galerkin Finite Element Approximations to the Biharmonic Equation

Igor Mozolevski,[1] Endre Süli,[2] and Paulo R. Bösing[3]

Received November 9, 2005; accepted (in revised form) May 3, 2006; Published online June 20, 2006

We consider the symmetric formulation of the interior penalty discontinuous Galerkin finite element method for the numerical solution of the biharmonic equation with Dirichlet boundary conditions in a bounded polyhedral domain in $\mathbb{R}^d, d \geqslant 2$. For a shape-regular family of meshes consisting of parallelepipeds, we derive *hp*-version *a priori* bounds on the global error measured in the L^2 norm and in broken Sobolev norms. Using these, we obtain *hp*-version bounds on the error in linear functionals of the solution. The bounds are optimal with respect to the mesh size h and suboptimal with respect to the degree of the piecewise polynomial approximation p. The theoretical results are confirmed by numerical experiments, and some practical applications in Poisson–Kirchhoff thin plate theory are presented.

KEY WORDS: High-order elliptic equations; finite element methods; discontinuous Galerkin methods; *a priori* error analysis; linear functionals.

1. INTRODUCTION

Over the last two decades various Discontinuous Galerkin Finite Element Methods (DGFEMs) have been proposed and studied (see, e.g., the review of DGFEM theory in [11]). The interest in this class of methods has been stimulated by a number of advantages that DGFEMs exhibit over classical conforming finite element methods (FEMs): all of them stem from the

[1]Mathematics Department, Federal University of Santa Catarina, Trindade, Florianópolis, SC- 88040-900, Brazil. E-mail: igor@mtm.ufsc.br
[2]University of Oxford, Computing Laboratory, Wolfson Building, Parks Road, Oxford OX1 3QD, UK.
[3]Applied Mathematics Department, IME, University of São Paulo, Rua do Matão 1010, São Paulo, SP- 05508-090, Brazil.

use of finite element spaces consisting of discontinuous piecewise polynomial functions. One of the most important among these favourable properties is the fact that piecewise polynomials of differing degrees may be used in each element of the mesh without the need for adjusting their values at the interfaces to ensure inter-element continuity; this is particularly helpful from the point of view of implementing p-adaptivity. Similarly, DGFEM approximations facilitate the implementation of various types of h-adaptation strategies through refining and derefining the computational mesh without the need to impose regularity hypotheses on the mesh: hanging nodes are allowed (that is, some mesh elements may be displaced in relation to others). All of these properties make DGFEMs ideal contenders for the design of hp-adaptive algorithms.

Another significant advantage of DGFEMs, which impose the interelement continuity conditions only weakly, in an integral sense, is the easy implementation of finite elements of high order; this is of crucial importance in the numerical solution of boundary-value problems for higher-order elliptic equations, such as the biharmonic equation. In this case, DGFEMs produce efficient and robust algorithms with fewer unknowns, generating systems of equations with positive-definite matrices.

The theory of conforming FEMs for the biharmonic equation is a well-developed area of the theory of finite methods for elliptic equations (see, e.g., [10]). However, the construction of higher-order conforming finite element approximations is a rather complicated and computationally expensive task even in two space-dimensions, since the shape functions and their first partial derivatives need to be continuous. Mixed FEMs are another standard approach to the solution of fourth-order elliptic problems [9]. However, these methods increase the number of unknowns and lead to saddle-point-type problems where the finite element spaces involved have to respect a compatibility condition—the inf-sup condition. A further possibility is to use nonconforming finite element spaces so as to relax the demand for C^1 continuity to C^0 continuity (cf. [8, 10]).

Higher-order FEMs with interior penalties for fourth-order elliptic equations were proposed in [6, 12], as nonstandard finite element techniques for elliptic problems. These methods use classical spaces of conforming finite elements in tandem with penalty terms for the weak enforcement of the continuity of higher-order derivatives at the element interfaces. Hence the task of identifying suitable finite element spaces is much simplified. In connection with this, we note that in [13] a FEM was presented for the numerical solution of problems in computational mechanics (from the Euler–Bernoulli beam theory and the Poisson–Kirchhoff theory of thin-plate-bending), which combines concepts from the realm of continuous Galerkin methods with those from the framework of discontinuous Galerkin methods and stabilization techniques.

Recently, in [22] we considered nonsymmetric *hp*-version DGFEMs with interior penalties for the numerical solution of the biharmonic equation. In that paper, *a priori* error bounds were established in the energy norm which were optimal with respect to h (mesh diameter) and suboptimal in p (the polynomial degree of the approximation). It was also shown through numerical experiments that for problems whose solutions are analytic over the closure of the computational domain the method exhibits an exponential convergence rate under p-refinement.

A more detailed analysis of DGFEMs with interior penalties for the biharmonic equation was presented in [26]. In that paper, symmetric and nonsymmetric formulations as well as their combinations (called semi-symmetric formulations) were considered. An h-optimal *a priori* error bound was proved for each of the methods discussed.

While, in principle, in the energy norm the error analyses of the symmetric and nonsymmetric formulations of DGFEM can be conducted by using the same technique, the situation concerning the derivation of error bounds in the L^2 norm and in broken Sobolev norms (other than the energy norm) is quite different. Due to the adjoint-consistency of symmetric DGFEMs, h-optimal bounds in the L^2 norm may be obtained by using a standard Aubin–Nitsche duality argument (see [1] for the case of second-order elliptic equations). However, the nonsymmetric formulation is not adjoint–consistent and, therefore, the method does not exhibit an optimal order of convergence in the L^2 norm in either h or in p (see, e.g., [23]). Indeed, the error analysis of the nonsymmetric formulation of DGFEM in the L^2 norm is much more delicate and has only been carried out in some particular cases. See, for example, the superpenalty technique in [24] for the Neumann problem for second-order elliptic equations, as well as the work of Larson and Niklasson [21] on the error analysis of interior penalty DGFEMs in the L^2 norm in one dimension.

In many areas of practical interest the ultimate objective of the numerical simulation is not the approximation of the actual solution to the problem in a given norm but, rather, the approximation of functionals of the solution ([2–4]). Examples include the calculation of the stress–intensity factor and the computation of the moments of a plate or its displacement at its center of mass. In the case of second-order elliptic equations, the *a priori* and *a posteriori* error analyses of DGFEMs for linear functionals of the solution can be found in [17–19]. As was demonstrated in [18], only the symmetric formulation of the method is amenable to the derivation of optimal rates of convergence for linear functionals of the solution; in addition, when the solution to the associated dual problem is sufficiently smooth, the convergence rates exhibited by symmetric DGFEM approximations to certain linear functionals of the solution, such as the weighted mean-value of the solution, are twice the rate observed in the energy norm.

The aim of this work is to continue our investigations into *a priori* error estimation for interior penalty DGFEMs for higher-order elliptic equations initiated in [22, 26]. Motivated by the results for second-order elliptic equations mentioned above (particularly the L^2 norm error analysis and the approximation of linear functionals by symmetric DGFEMs), we consider symmetric DGFEMs with interior penalties for the biharmonic equation subject to nonhomogeneous Dirichlet boundary conditions. We present *a priori* error analyses for this formulation in the L^2 norm and in broken Sobolev norms. In addition, we derive an error bound for a linear functional of the solution.

The paper is structured as follows. Section 2 is dedicated to the construction of finite element spaces. In Sec. 3, we state our model problem and its discontinuous Galerkin finite element approximation with interior penalties. Following the ideas of [26], in Sec. 4, we consider the stability analysis of this method. In Sec. 5, we recall an hp-version error bound in the energy norm; then, using an Aubin–Nitsche duality argument, we derive hp-version error bounds in the L^2 norm and in broken Sobolev norms. These bounds are optimal with respect to the mesh size h and are suboptimal with respect to the polynomial degree p. We then carry out an error analysis for a linear functional of the solution, deriving a bound which is, again, optimal with respect to h and suboptimal with respect to p. Finally, in Sec. 6, we present some numerical results, which confirm the theoretical convergence rates and show some applications arising from the Poisson–Kirchhoff theory of thin plates.

2. FINITE ELEMENT SPACES

Let Ω be a bounded open polyhedral domain in $\mathbb{R}^d$, $d \geqslant 2$, with Lipschitz-continuous boundary; let $\partial\Omega$ denote the union of all $(d-1)$-dimensional open faces of Ω. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of partitions of Ω into elements κ, where the κ are open, convex, pairwise disjoint, and such that

$$\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}_h} \overline{\kappa}.$$

For the sake of simplicity, we shall assume that each element $\kappa \in \mathcal{T}_h$ is the affine image of a fixed master-element $\widehat{\kappa}$—i.e., that for each $\kappa \in \mathcal{T}_h$

there exists an affine mapping $\mathcal{F}_\kappa$, such that $\kappa = \mathcal{F}_\kappa(\widehat{\kappa})$—where the master-element is the unit hypercube or the unit simplex in $\mathbb{R}^d$. On the partition $\mathcal{T}_h$, we consider a piecewise constant function $h_{\mathcal{T}_h}$ defined by

$$h_{\mathcal{T}_h}(x) = h_\kappa = \mathrm{diam}(\kappa), \quad x \in \kappa, \ \kappa \in \mathcal{T}_h$$

and we denote by h the maximum of h_κ, $\kappa \in \mathcal{T}_h$. Since the partition $\mathcal{T}_h$ may be irregular (i.e. it may contain hanging nodes), we shall assume that each element $\kappa \in \mathcal{T}_h$ has no more than a fixed number c_d of immediate neighbours κ' which share a face with κ.

The Sobolev space of real order (or index) t of real-valued functions defined on Ω, will be labelled by $\mathrm{H}^t(\Omega)$. Its inner product, norm and semi-norm will be denoted by $(\cdot,\cdot)_{t,\Omega}$, $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$, respectively. For $\mathrm{L}^2(\partial\Omega) = \mathrm{H}^0(\partial\Omega)$ the inner product and the induced norm are

$$\langle \phi, \psi \rangle_{\partial\Omega} = \int_{\partial\Omega} \phi\,\psi\,ds \quad \text{and} \quad \|\psi\|_{\partial\Omega} = \langle \psi, \psi \rangle_{\partial\Omega}^{1/2},$$

respectively.

For a positive integer m, $Q_m(\widehat{\kappa})$ will denote the linear space of tensor-product polynomials of degree m or less in each co-ordinate direction, defined on the master element $\widehat{\kappa}$. Each element κ being the image, under an affine mapping $\mathcal{F}_\kappa$, of the master element $\widehat{\kappa}$, we denote by p_κ the maximum degree m of the elements of $Q_m(\widehat{\kappa})$. Thus, to each $\kappa \in \mathcal{T}_h$ we associate the degree p_κ of local polynomial approximation and a local Sobolev index s_κ. Collecting the p_κ, s_κ and $\mathcal{F}_\kappa$ into the vectors $\mathbf{p} = (p_\kappa : \kappa \in \mathcal{T}_h)$, $\mathbf{s} = (s_\kappa : \kappa \in \mathcal{T}_h)$ and $\mathcal{F} = (\mathcal{F}_\kappa : \kappa \in \mathcal{T}_h)$, we define the following linear spaces:

$$S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F}) = \left\{ v \in \mathrm{L}^2(\Omega) : \ v_{|\kappa} \circ \mathcal{F}_\kappa \in Q_{p_\kappa}(\widehat{\kappa}) \qquad \forall \kappa \ \in \mathcal{T}_h \right\}$$

and

$$\mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h) = \{ v \in \mathrm{L}^2(\Omega) : \ v_{|\kappa} \in \mathrm{H}^{s_\kappa}(\kappa) \qquad \forall \kappa \in \mathcal{T}_h \}.$$

The space $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F})$ will be referred to as the finite elements space. $\mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$ will be called a broken Sobolev space of composite index $\mathbf{s}$ (depending on the mesh $\mathcal{T}_h$); it will be equipped with the following norm and semi-norm:

$$\|v\|_{\mathbf{s},\mathcal{T}_h} = \left(\sum_{\kappa \in \mathcal{T}_h} \|v\|^2_{\mathrm{H}^{s_\kappa}(\kappa)} \right)^{1/2}, \qquad |v|_{\mathbf{s},\mathcal{T}_h} = \left(\sum_{\kappa \in \mathcal{T}_h} |v|^2_{\mathrm{H}^{s_\kappa}(\kappa)} \right)^{1/2},$$

when $\mathbf{s}_\kappa = s$ for all $\kappa \in \mathcal{T}_h$, we shall simply write $\mathrm{H}^s(\Omega, \mathcal{T}_h)$.

Let $\mathcal{E}_h$ be the set of all $(d-1)$-dimensional open faces e of all the elements $\kappa \in \mathcal{T}_h$. On this set we consider the piecewise constant function $h_{\mathcal{E}_h}$ defined by

$$h_{\mathcal{E}_h}(x) = h_e = \mathrm{diam}(e), \quad x \in e, \ \ e \in \mathcal{E}_h.$$

The family $\{\mathcal{T}_h\}$ being shape-regular, there exists a positive constant $c \in (0, 1]$, independent of h, such that

$$ch_\kappa \leqslant h_e \leqslant h_\kappa, \quad \forall e \in \partial\kappa, \ \ \forall \kappa \in \bigcup_{h>0} \mathcal{T}_h.$$

The set $\mathcal{E}_h$ will be divided into two subsets, $\mathcal{E}_h^\circ$ and $\mathcal{E}_h^\partial$, defined by

$$\mathcal{E}_h^\circ = \{e \in \mathcal{E}_h : \ e \subset \Omega\},$$
$$\mathcal{E}_h^\partial = \{e \in \mathcal{E}_h : \ e \subset \partial\Omega\}.$$

In addition, we define

$$\Gamma^\circ = \{x \in \Omega : \ x \in e \quad \text{for} \quad e \in \mathcal{E}_h^\circ\}$$

and we put $\Gamma = \Gamma^\circ \cup \partial\Omega$.

For any face $e \in \mathcal{E}_h^\circ$ there exist exactly two elements κ_i and $\kappa_j (i > j)$ such that $\overline{\kappa_i} \cap \overline{\kappa_j} = \overline{e}$. Thus, for any integer m and $x \in e$, we define

$$\{p^m\}_{\mathcal{E}_h}(x) = \{p^m\}_e = \begin{cases} \frac{p^m_{\kappa_i} + p^m_{\kappa_j}}{2}, & \text{if} \quad e \in \mathcal{E}_h^\circ, \\ p^m_\kappa \ , & \text{if} \quad e \in \mathcal{E}_h^\partial \end{cases}$$

and, for any function $v \in \mathrm{H}^s(\Omega, \mathcal{T}_h), s > 1/2$, we introduce the mean-value and the jump (depending on the enumeration of the elements) of v on e by

$$\{v\} = \begin{cases} \frac{1}{2}\left(v_{|\kappa_i}\right)_{|e} + \frac{1}{2}\left(v_{|\kappa_j}\right)_{|e}, & \text{if} \quad e \in \mathcal{E}_h^\circ, \\ v_{|e} \ , & \text{if} \quad e \in \mathcal{E}_h^\partial, \end{cases}$$

$$[v] = \begin{cases} \left(v_{|\kappa_i}\right)_{|e} - \left(v_{|\kappa_j}\right)_{|e}, & \text{if} \quad e \in \mathcal{E}_h^\circ, \\ v_{|e} \ , & \text{if} \quad e \in \mathcal{E}_h^\partial. \end{cases}$$

To each face $e \in \mathcal{E}_h^\circ$ we assign the unit vector $\nu = \mathbf{n}_{\kappa_i}$ normal to e which points from κ_i to κ_j, and to each face $e \in \mathcal{E}_h^\partial$, $e \subset \partial\kappa$, we associate the exterior unit normal vector $\nu = \mathbf{n}_\kappa$.

3. MODEL PROBLEM AND ITS BROKEN WEAK FORMULATION

We now consider the following boundary value problem for the biharmonic equation subject to non-homogeneous Dirichlet boundary conditions:

find $u \in \mathrm{H}^4(\Omega)$ such that

$$\begin{aligned} \Delta^2 u &= f \quad \text{in} \quad \Omega, \\ u &= g_0 \quad \text{on} \quad \partial\Omega, \\ \mathbf{n}\cdot\nabla u &= g_1 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta^2 u = \Delta(\Delta u)$, $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$, $f \in \mathrm{L}^2(\Omega)$ and g_0 and g_1 are suitably smooth boundary data; for the purposes of our discussion here it will suffice to assume that g_0 and g_1 both belong to $\mathrm{L}^2(\partial\Omega)$. Additional indirect regularity assumptions will be imposed on the data in our error analysis of the DGFEM approximation of this problem, through hypothesising that u has appropriate Sobolev regularity.

This problem describes, for example, the displacement of a thin and isotropic homogeneous plate [27]; it also defines the streamlines of a slow, bidimensional flow of an incompressible fluid [15]. The existence and uniqueness of solutions to this boundary value problem in a polygonal domain has been studied in [7].

Using integration by parts in $\int_\kappa(\Delta^2 u)v\,dx$, summing over the elements of $\mathcal{T}_h$, and applying traditional techniques for the decomposition of numerical fluxes (see, e.g. [26]), we obtain the following identity, valid for all $u \in \mathrm{H}^4(\Omega, \mathcal{T}_h)$ and $v \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$:

$$\begin{aligned} \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \Delta u \Delta v\,dx &= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa(\Delta^2 u)v\,dx - \int_{\Gamma^\circ}(\{\nu\cdot\nabla\Delta u\}[v] + [\nu\cdot\nabla\Delta u]\{v\})ds \\ &\quad + \int_{\Gamma^\circ}(\{\Delta u\}[\nu\cdot\nabla v] + [\Delta u]\{\nu\cdot\nabla v\})ds \\ &\quad + \int_{\partial\Omega}(\Delta u(\nu\cdot\nabla v) - \nu\cdot\nabla(\Delta u)v)ds. \end{aligned}$$

We further observe that using the definition of the jump and the mean value of a function on $\mathcal{E}_h^\partial$, the integrals over Γ° and $\partial\Omega$ may be merged into an integral over Γ. Based on this observation, we introduce the following bilinear form, for $u, v \in \mathrm{H}^4(\Omega, \mathcal{T}_h)$:

$$\mathcal{B}(u,v) = \mathcal{B}_{\mathcal{T}_h}(u,v) + \mathcal{J}_1(u,v) + \mathcal{J}_1(v,u) - \mathcal{J}_2(u,v) - \mathcal{J}_2(v,u) + \mathcal{B}_s(u,v),$$

where

$$\begin{aligned}
B_{\mathcal{T}_h}(u,v) &= \sum_{\kappa\in\mathcal{T}_h} (\Delta u, \Delta v)_{\mathrm{L}^2(\kappa)}\,,\\
J_1(u,v) &= \langle\{\nu\cdot\nabla\Delta u\},[v]\rangle_{\mathrm{L}^2(\Gamma)},\\
J_2(u,v) &= \langle\{\Delta u\},[\nu\cdot\nabla v]\rangle_{\mathrm{L}^2(\Gamma)},\\
B_s(u,v) &= \langle\alpha[u],[v]\rangle_{\mathrm{L}^2(\Gamma)} + \langle\beta[\nu\cdot\nabla u],[\nu\cdot\nabla v]\rangle_{\mathrm{L}^2(\Gamma)}.
\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Gamma)}$ is defined analogously to $\langle\cdot,\cdot\rangle_{\mathrm{L}^2(\partial\Omega)}$. The functions α and β involved in the stabilization term $\mathcal{B}_s(\cdot,\cdot)$ are known as *discontinuity–penalization parameters* and are defined by

$$\alpha_{|e}=\alpha_e, \qquad \beta_{|e},=\beta_e, \quad \forall e\in\mathcal{E}_h,$$

where α_e and β_e are parameters which depend on h and p, and will be specified later on.

We also define the linear functional

$$l(v)=l_{\mathcal{T}_h}(v)+l_s(v),$$

where

$$\begin{aligned}
l_{\mathcal{T}_h}(v) &= (f,v)_{\mathrm{L}^2(\Omega)} - \langle g_0,\mathbf{n}\cdot\nabla\Delta v\rangle_{\mathrm{L}^2(\partial\Omega)} + \langle g_1,\Delta v\rangle_{\mathrm{L}^2(\partial\Omega)},\\
l_s(v) &= \langle\alpha g_0, v\rangle_{\mathrm{L}^2(\partial\Omega)} + \langle\beta g_1,\mathbf{n}\cdot\nabla v\rangle_{\mathrm{L}^2(\partial\Omega)}.
\end{aligned}$$

We emphasise here our notational convention that $\partial\Omega$ is the union of all $(d-1)$-dimensional open faces of the polyhedron Ω (rather than the boundary of the domain Ω).

Thus, we consider the following broken weak formulation of the boundary value problem (1): find $u\in\mathrm{H}^4(\Omega,\mathcal{T}_h)$ such that,

$$\mathcal{B}(u,v)=l(v), \quad \forall v\in\mathrm{H}^4(\Omega,\mathcal{T}_h). \tag{2}$$

The hp-version of the symmetric DGFEM with interior penalty, associated with the formulation (2) is: find $u_h\in S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathcal{F})$ such that,

$$\mathcal{B}(u_h,v)=l(v), \quad \forall v\in S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathcal{F}). \tag{3}$$

4. STABILITY ANALYSIS

Our aim is to obtain *a priori* error bounds for the *hp*-version discontinuous Galerkin method (3). We begin by establishing some preliminary results. First, we associate with the bilinear form $\mathcal{B}(\cdot,\cdot)$ the norm $|||\cdot|||$ on $\mathrm{H}^4(\Omega,\mathcal{T}_h)$ (see [26] for more details), defined by

$$|||u|||^2 = \sum_{\kappa\in\mathcal{T}_h} \|\Delta u\|^2_{\mathrm{L}^2(\kappa)} + \|\sqrt{\alpha}\,[u]\|^2_{\mathrm{L}^2(\Gamma)} + \|\sqrt{\beta}\,[\nu\cdot\nabla u]\|^2_{\mathrm{L}^2(\Gamma)}$$

$$+ \left\|\frac{1}{\sqrt{\alpha}}\{\nu\cdot\nabla\Delta u\}\right\|^2_{\mathrm{L}^2(\Gamma)} + \left\|\frac{1}{\sqrt{\beta}}\{\Delta u\}\right\|^2_{\mathrm{L}^2(\Gamma)},$$

where $\|\cdot\|_{\mathrm{L}^2(\Gamma)}$ is defined analogously to $\|\cdot\|_{\mathrm{L}^2(\partial\Omega)}$.

The next Lemma, proved in [26], establishes the consistency of the broken weak formulation (2) stated above.

Lemma 1. The broken weak formulation (2) of the boundary problem (1) is consistent in the space $\mathrm{H}^4(\Omega)$, in the sense that any solution u to the boundary value problem, such that $u\in\mathrm{H}^4(\Omega)$, is also a solution to (2).

One of the consequences of the above lemma is the validity of the Galerkin orthogonality property

$$\mathcal{B}(u-u_h,v)=0,\quad \forall v\in S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathcal{F}), \tag{4}$$

whenever $u\in\mathrm{H}^4(\Omega)$. This property will play a key role in the error analysis; thus, in what follows, we shall assume that the solution to the boundary problem (1) satisfies this regularity hypothesis, i.e. $u\in\mathrm{H}^4(\Omega)$.

The continuity of the bilinear form $\mathcal{B}(\cdot,\cdot)$ on $\mathrm{H}^4(\Omega,\mathcal{T}_h)$ with respect to the norm $|||\cdot|||$ is stated in the next lemma.

Lemma 2. There exists a positive constant c, such that,

$$|\mathcal{B}(u,v)|\leqslant c|||u|||\,|||v|||,\quad \forall u,v\in\mathrm{H}^4(\Omega,\mathcal{T}_h), \tag{5}$$

where c is independent of $h_\kappa,\kappa\in\mathcal{T}_h$.

Following the ideas in [26], in the next lemma, we present an alternative proof (slightly different from that in [26]) of the coercivity of the bilinear form $\mathcal{B}(\cdot,\cdot)$ on the finite element space $S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathcal{F})$.

Lemma 3. Let

$$\alpha_e = \sigma_\alpha \frac{\{p^6\}_e}{h_e^3} \quad \text{and} \quad \beta_e = \sigma_\beta \frac{\{p^2\}_e}{h_e} \tag{6}$$

on $e \in \mathcal{E}_h$; then, there exist positive constants $c_\alpha > 0$ and $c_\beta > 0$, such that for any $\sigma_\alpha \geqslant \underline{\sigma_\alpha} > 0$ and $\sigma_\beta \geqslant \underline{\sigma_\beta} > 0$, where $\underline{\sigma_\alpha}$ and $\underline{\sigma_\beta}$ are chosen so that

$$\frac{c_\alpha}{\underline{\sigma_\alpha}} + \frac{c_\beta}{\underline{\sigma_\beta}} < 1 \tag{7}$$

there exists a positive constant $\theta = \theta(c_\alpha, c_\beta, \underline{\sigma}_\alpha, \underline{\sigma}_\beta)$, such that,

$$\mathcal{B}(u, u) \geqslant \theta |||u|||^2, \quad \forall u \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F}).$$

Proof. For $u \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F})$, using the definitions of $\mathcal{B}(\cdot, \cdot)$ and $|||\cdot|||$, we have,

$$\begin{aligned}\mathcal{B}(u,u) - \theta |||u|||^2 &= (1-\theta)\mathcal{B}_{\mathcal{T}_h}(u,u) + 2\mathcal{J}_1(u,u) - 2\mathcal{J}_2(u,u) + (1-\theta)\mathcal{B}_s(u,u) \\ &\quad - \theta\left(\left\|\frac{1}{\sqrt{\alpha}}\{\nu \cdot \nabla \Delta u\}\right\|^2_{\mathrm{L}^2(\Gamma)} + \left\|\frac{1}{\sqrt{\beta}}\{\Delta u\}\right\|^2_{\mathrm{L}^2(\Gamma)}\right).\end{aligned}$$

Using the inequality $ab \leqslant \frac{1}{2}\left(\varepsilon a^2 + b^2/\varepsilon\right)$, which is valid for any real numbers a and b, and for any $\varepsilon > 0$, the terms $\mathcal{J}_1(\cdot,\cdot)$ and $\mathcal{J}_2(\cdot,\cdot)$ are bounded by

$$|\mathcal{J}_1(u,u)| \leqslant \frac{1}{2}\left(\epsilon_1 \left\|\frac{1}{\sqrt{\alpha}}\{\nu \cdot \nabla \Delta u\}\right\|^2_{\mathrm{L}^2(\Gamma)} + \frac{1}{\epsilon_1}\left\|\sqrt{\alpha}\,[u]\right\|^2_{\mathrm{L}^2(\Gamma)}\right)$$

and

$$|\mathcal{J}_2(u,u)| \leqslant \frac{1}{2}\left(\epsilon_2 \left\|\frac{1}{\sqrt{\beta}}\{\Delta u\}\right\|^2_{\mathrm{L}^2(\Gamma)} + \frac{1}{\epsilon_2}\left\|\sqrt{\beta}\,[\nu \cdot \nabla u]\right\|^2_{\mathrm{L}^2(\Gamma)}\right).$$

For a face $e \in \mathcal{E}_h^\circ$, which is the boundary of elements κ_i and κ_j, using the inverse inequalities

$$\|\xi\|^2_{\mathrm{L}^2(\partial\kappa)} \leqslant c_0 \frac{p_\kappa^2}{h_\kappa} \|\xi\|^2_{\mathrm{L}^2(\kappa)}, \quad \text{and} \quad \|\nabla\xi\|^2_{\mathrm{L}^2(\partial\kappa)} \leqslant c_1 \frac{p_\kappa^6}{h_\kappa^3} \|\xi\|^2_{\mathrm{L}^2(\kappa)}, \quad \forall \xi \in Q_{p_\kappa}(\kappa),$$

we obtain,

$$\begin{aligned}\left\| \frac{1}{\sqrt{\beta}}\{\Delta u\}\right\|^2_{L^2(e)} &\leqslant \frac{c_\beta}{\sigma_\beta}\frac{h_e}{\{p^2\}_e}\left(\frac{p^2_{\kappa_i}}{2h_{\kappa_i}}\|\Delta u\|^2_{L^2(\kappa_i)}+\frac{p^2_{\kappa_j}}{2h_{\kappa_j}}\|\Delta u\|^2_{L^2(\kappa_j)}\right)\\ &\leqslant \frac{c_\beta}{\sigma_\beta}\left(\|\Delta u\|^2_{L^2(\kappa_i)}+\|\Delta u\|^2_{L^2(\kappa_j)}\right)\end{aligned}$$

with c_β dependent on c_0. This also being valid for $e\in\mathcal{E}_h^\partial$, we obtain,

$$\left\| \frac{1}{\sqrt{\beta}}\{\Delta u\}\right\|^2_{L^2(\Gamma)} \leqslant \frac{c_\beta}{\sigma_\beta}\mathcal{B}_{\mathcal{T}_h}(u,u).$$

Similarly,

$$\left\| \frac{1}{\sqrt{\alpha}}\{\nu\cdot\nabla\Delta u\}\right\|^2_{L^2(\Gamma)} \leqslant \frac{c_\alpha}{\sigma_\alpha}\mathcal{B}_{\mathcal{T}_h}(u,u),$$

where c_α depends on c_1.

Combining these results, we have

$$\begin{aligned}\mathcal{B}(u,u)-\theta|||u|||^2 \geqslant &\left(1-\theta-\frac{c_\alpha}{\sigma_\alpha}(\theta+\epsilon_1)-\frac{c_\beta}{\sigma_\beta}(\theta+\epsilon_2)\right)\mathcal{B}_{\mathcal{T}_h}(u,u)\\ &+\left(1-\theta-\frac{1}{\epsilon_1}\right)\left\|\sqrt{\alpha}\,[u]\right\|^2_{L^2(\Gamma)}\\ &+\left(1-\theta-\frac{1}{\epsilon_2}\right)\left\|\sqrt{\beta}\,[\nu\cdot\nabla u]\right\|^2_{L^2(\Gamma)}. \qquad (8)\end{aligned}$$

It follows from (7) that there exists $\epsilon_1>0$ such that,

$$1<\epsilon_1<\left(\frac{c_\alpha}{\sigma_\alpha}+\frac{c_\beta}{\sigma_\beta}\right)^{-1};$$

hence,

$$1-\frac{1}{\epsilon_1}>0 \quad\text{and}\quad 1-\epsilon_1\left(\frac{c_\alpha}{\sigma_\alpha}+\frac{c_\beta}{\sigma_\beta}\right)>0. \qquad (9)$$

In addition, there exists ϵ_2 such that $1<\epsilon_2<\epsilon_1$. Hence,

$$1-\frac{1}{\epsilon_2}>0$$

and, from the second inequality in (9), we deduce that

$$0 < 1 - \epsilon_1 \frac{c_\alpha}{\sigma_\alpha} - \epsilon_2 \frac{c_\beta}{\sigma_\beta}.$$

We now choose θ in such a way that,

$$1 - \frac{1}{\epsilon_1} > \theta > 0, \qquad 1 - \frac{1}{\epsilon_2} > \theta > 0 \qquad \text{and} \qquad \frac{1 - \epsilon_1 \frac{c_\alpha}{\sigma_\alpha} - \epsilon_2 \frac{c_\beta}{\sigma_\beta}}{1 + \frac{c_\alpha}{\sigma_\alpha} + \frac{c_\beta}{\sigma_\beta}} > \theta > 0.$$

Thus,

$$1 - \theta - \frac{1}{\epsilon_1} > 0, \quad 1 - \theta - \frac{1}{\epsilon_2} > 0 \quad \text{and} \quad 1 - \theta - \frac{c_\alpha}{\sigma_\alpha}(\theta + \epsilon_1) - \frac{c_\beta}{\sigma_\beta}(\theta + \epsilon_2) > 0.$$

With this, we have ensured that the coefficients which multiply the terms on the right-hand side of (8) are all positive, and thereby

$$\mathcal{B}(u, u) - \theta |||u|||^2 \geqslant 0, \quad \forall u \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F}).$$

5. *A PRIORI* ERROR BOUNDS

Before embarking on the *a priori* error analysis of the method, we formulate an *hp*-version approximation property by discontinuous piecewise polynomial functions [5]. In the sequel, we shall confine ourselves to shape-regular subdivisons $\mathcal{T}_h$ consisting of parallelepipeds; shape-regular partitions consisting of simplices or, indeed, of a mixture of simplices and parallelepipeds, are handled in exactly the same manner.

Lemma 4. Let us suppose that $\mathcal{T}_h$ is a partition of Ω consisting of shape-regular d-parallelepipeds. Then, for any $u \in \mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$, with $\mathbf{s} = (s_\kappa,\ \kappa \in \mathcal{T}_h)$, and for each $\mathbf{p} = (p_\kappa,\ \kappa \in \mathcal{T}_h)$, $p_\kappa \in \mathbb{N}_0$, there exists a projector

$$\pi_p^h\colon \mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h) \to S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F}), \quad (\pi_p^h u)_{|\kappa} = \pi_p^h(u_{|\kappa})$$

such that, for $0 \leqslant q \leqslant s_k$, $s_\kappa \geqslant 0$,

$$\|u - \pi_h^p u\|_{q,\kappa} \leqslant C \frac{h_\kappa^{\mu_\kappa - q}}{p_\kappa^{s_\kappa - q}} \|u\|_{s_\kappa, \kappa}, \quad \forall \kappa \in \mathcal{T}_h, \tag{10}$$

where $\mu_\kappa = \min(p_\kappa + 1, s_\kappa)$, $h_\kappa = \mathrm{diam}(\kappa)$ and $e \subset \partial\kappa$.

The following theorem, provides an *hp*-version *a priori* bound on the error of the method in the energy norm.

Theorem 1. We suppose that $\Omega \subset \mathbb{R}^d$ is a bounded polyhedral domain and $\{\mathcal{T}_h\}_{h>0}$ is a shape-regular family of partitions formed by d-parallelepipeds. Let $\mathbf{p} = (p_\kappa,\ \kappa \in \mathcal{T}_h)$, with $p_\kappa \in \mathbb{N}$, $p_\kappa \geqslant 2$, $\kappa \in \mathcal{T}_h$, be the vector of local polynomial degrees which we assume to have bounded local variation (see [26] for details). To each face $e \in \mathcal{E}_h$ we assign the values $\alpha_e = \sigma_\alpha \frac{\{p^6\}_e}{h_e^3}$ and $\beta_e = \sigma_\beta \frac{\{p^4\}_e}{h_e}$, and suppose that σ_α and σ_β are such that the bilinear form $\mathcal{B}(\cdot,\cdot)$ is coercive (cf. Lemma 3). If the solution u to the boundary value problem (2) belongs to $\mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$, with $\mathbf{s} = (s_\kappa,\ \kappa \in \mathcal{T}_h)$, $s_\kappa \geqslant 4$, $\kappa \in \mathcal{T}_h$, then the corresponding numerical solution $u_h \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F})$, defined by (3), satisfies the following error bound:

$$|||u - u_h|||^2 \leqslant C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2\mu_\kappa - 4}}{p_\kappa^{2s_\kappa - 7}} \|u\|_{s_\kappa,\kappa}^2, \tag{11}$$

where $2 \leqslant \mu_\kappa \leqslant \min(p_\kappa + 1, s_\kappa)$, and C is a constant dependent only on the space-dimension d, the regularity constant c, the constant from the bounded local variation condition of the polynomial degree vector $\mathbf{p}$, and $s = \max_{\kappa \in \mathcal{T}_h} s_\kappa$.

Proof. This theorem was established in [26] for the case of $p_\kappa \geqslant 3$, $\kappa \in \mathcal{T}_h$. We note that the theorem is still valid when $p_\kappa = 2$ for all $\kappa \in \mathcal{T}_h$; indeed, in that case we have that $\nabla \cdot (\Delta \eta) = 0$ for all $\eta \in S^2(\Omega, \mathcal{T}_h, \mathcal{F})$, and therefore, the argument is considerably simplified as there is then no constraint on the choice of α_e from the point of view of coercivity. These observations lead to the following result: if $p_\kappa = 2$ for all $\kappa \in \mathcal{T}_h$ and the solution u to boundary value problem (2) belongs to $\mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$, with $\mathbf{s} = (s_\kappa,\ \kappa \in \mathcal{T}_h)$, $s_\kappa \geqslant 4$, $\kappa \in \mathcal{T}_h$, then the corresponding numerical solution $u_h \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F})$, defined by (3), satisfies the following error bound:

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}}^2 \leqslant C \sum_{\kappa \in \mathcal{T}_h} h_\kappa^{2\mu_\kappa - 4} \|u\|_{s_\kappa,\kappa}^2,$$

where $2 \leqslant \mu_\kappa \leqslant \min(3, s_K)$, and C is a constant dependent only on the space dimension d, the shape-regularity constant c and on $s = \max_{\kappa \in \mathcal{T}_h} s_\kappa$. Hence the stated result.

The error analysis of the method in norms other than the energy norm will be conducted as follows: by using the hp-version error bound in the energy norm, we shall obtain an hp-version error bound in the L^2 norm which will, in turn, be used to establish error bounds in broken Sobolev norms. The error bound in the L^2 norm will be obtained through an Aubin–Nitsche duality argument; for this purpose, we shall require the following elliptic regularity hypothesis for the *dual problem* which, in our setting of a self-adjoint fourth-order elliptic problem, is simply the counterpart of the original problem with homogeneous boundary conditions.

Hypothesis 5. Let us suppose that, whenever $\phi \in \mathrm{L}^2(\Omega)$, the weak solution Φ of the problem

$$\Delta^2\Phi = \phi \quad \text{on} \quad \Omega, \qquad (12)$$
$$\Phi = 0 \quad \text{on} \quad \partial\Omega,$$
$$\mathbf{n}\cdot\nabla\Phi = 0 \quad \text{on} \quad \partial\Omega$$

belongs to the space $\mathrm{H}^4(\Omega)$, and there exists a positive constant C, dependent only on Ω and d, such that,

$$\|\Phi\|_{4,\Omega} \leqslant C\|\phi\|_{0,\Omega}. \qquad (13)$$

The validity of this hypothesis in two-dimensional polygonal domains depends on the size of the largest internal angle in Ω (see, e.g., Chapter 3 in the book of Grisvard [7, 16]). In particular, Hypothesis 5 holds when Ω is a bounded convex polygon in $\mathbb{R}^2$. For an analogous result in a three-dimensional bounded convex polyhedron, we refer to Theorem 6 on page 182 of [20].

Theorem 2. Let us suppose that besides the hypotheses of the previous theorem, the elliptic regularity Hypothesis 5 holds. Then, if $u \in \mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$, with $s_\kappa \geqslant 4$, $\kappa \in \mathcal{T}_h$, is the solution to (2) and $u_h \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F})$ is the solution to the discontinuous Galerkin approximation (3) to problem (2), then the following error bound holds:

$$\|u-u_h\|_{0,\Omega} \leqslant C\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|_{s_\kappa,\kappa}^2\right)^{1/2}, \qquad (14)$$

where $\mu_\kappa = \min(p_\kappa+1, s_\kappa)$ and C is independent of h_κ and p_κ.

Proof. The regularity hypothesis imposed on the exact solution u of the boundary-value problem ensures that u is the solution to (2) and is, therefore, also the solution to (3). Thus, the following Galerkin orthogonality property holds:

$$B(u-u_h, v) = 0, \qquad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F}).$$

By considering the broken weak formulation of the dual problem (12), noting that $\Phi \in \mathrm{H}^4(\Omega)$, $\Phi = \mathbf{n}\cdot\nabla\Phi = 0$ on $\partial\Omega$ and making use of symmetry of the bilinear form, we can deduce that

$$(u-u_h,\phi)_{0,\Omega} = l(u-u_h) = B(\Phi, u-u_h) = B(u-u_h,\Phi). \tag{15}$$

Let π_h^p be the projector in $\mathrm{L}^2(\Omega)$ onto $S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathcal{F})$ defined by Lemma 4. Then, upon subtracting $B(u-u_h,\pi_h^p\Phi)=0$ from (15) and applying Lemma 2, we find that

$$(u-u_h,\phi)_{0,\Omega} \leqslant |||u-u_h|||\,|||\Phi-\pi_h^p\Phi|||. \tag{16}$$

Our objective is to suitably bound each of the two terms on the right-hand side of (16). The bound on the first term follows from Theorem 1. For the second term, according to the definition of the norm $|||\cdot|||$, we have,

$$\begin{aligned}
|||\Phi-\pi_h^p\Phi|||^2 &\equiv \sum_{\kappa\in\mathcal{T}_h}\|\Delta(\Phi-\pi_h^p\Phi)\|^2_{\mathrm{L}^2(\kappa)} + \|\sqrt{\alpha}\,[\Phi-\pi_h^p\Phi]\|^2_{\mathrm{L}^2(\Gamma)} \\
&\quad + \|\sqrt{\beta}\,[\mathbf{n}\cdot\nabla(\Phi-\pi_h^p\Phi)]\|^2_{\mathrm{L}^2(\Gamma)} + \|\sqrt{\beta^{-1}}\,\{\Delta(\Phi-\pi_h^p\Phi)\}\|^2_{\mathrm{L}^2(\Gamma)} \\
&\quad + \|\sqrt{\alpha^{-1}}\,\{\mathbf{n}\cdot\nabla(\Delta(\Phi-\pi_h^p\Phi))\}\|^2_{\mathrm{L}^2(\Gamma)} \\
&\equiv I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Therefore, supposing that $p_\kappa \geqslant 2$, using that $\Phi\in\mathrm{H}^4(\Omega)$ and combining (10) with the multiplicative trace inequalities, we have,

$$I_1 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa^4}\|\Phi\|^2_{4,\kappa},\quad I_2 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\|\Phi\|^2_{4,\kappa},\quad I_3 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\|\Phi\|^2_{4,\kappa},$$

$$I_4 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa^7}\|\Phi\|^2_{4,\kappa},\qquad I_5 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa^7}\|\Phi\|^2_{4,\kappa}.$$

Thus, we have the following bound on the second term on the right-hand side of (16):

$$|||\Phi-\pi_h^p\Phi|||^2 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\|\Phi\|^2_{4,\kappa} \leqslant C\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\|\Phi\|^2_{4,\Omega}. \tag{17}$$

Substituting (17) and (11) into (16) and using Hypothesis 5,

$$(u-u_h,\phi)_{0,\Omega} \leqslant C\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\right)^{1/2}\|\Phi\|_{4,\Omega}\left(\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|^2_{s_\kappa,\kappa}\right)^{1/2}$$

$$\leqslant C\|\phi\|_{0,\Omega}\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|^2_{s_\kappa,\kappa}\right)^{1/2}.$$

Thus, for $\phi\neq 0$, we have that

$$\frac{(u-u_h,\phi)_{0,\Omega}}{\|\phi\|_{0,\Omega}}\leqslant C\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|^2_{s_\kappa,\kappa}\right)^{1/2};$$

taking the supremum over $\phi\in \mathrm{L}^2(\Omega)$ we complete the proof.

We observe that if $s_\kappa=s\geqslant 4$, $h_\kappa\leqslant h$ and $p_\kappa\geqslant p\geqslant 2$ for all $\kappa\in\mathcal{T}_h$, then

$$\|u-u_h\|_{0,\Omega}\leqslant C\frac{h^\mu}{p^{s-3}}\|u\|_{s,\mathcal{T}_h} \tag{18}$$

with $\mu=\min(p+1,s)$. This bound is optimal in h, and suboptimal in p, by p^3.

In the next theorem, we present an hp-version *a priori* error bound for the method in the norm of the broken Sobolev space $\mathrm{H}^{\mathbf{q}}(\Omega,\mathcal{T}_h)$.

Theorem 3. Let $u\in\mathrm{H}^{\mathbf{s}}(\mathcal{T}_h,\Omega)$, with $s_\kappa\geqslant 4$ for all $\kappa\in\mathcal{T}_h$, be the solution to (2), and let $u_h\in S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathcal{F})$ be the solution to the discontinuous Galerkin approximation (3) to problem (2). Then, assuming the hypotheses of the previous theorem, the following bound holds:

$$\sum_{\kappa\in\mathcal{T}_h}\|u-u_h\|^2_{q_\kappa,\kappa}\leqslant C\max_{\kappa\in\mathcal{T}_h}\frac{p_\kappa^{4q_\kappa}}{h_\kappa^{2q_\kappa}}\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|^2_{s_\kappa,\kappa}\right)$$

for $0\leqslant q_\kappa\leqslant\frac{s_\kappa-3}{2}$ and $\mu_\kappa=\min(p_\kappa+1,s_\kappa)$.

Proof. By applying the triangle inequality and the bound (10) on the projection error, we obtain

$$\sum_{\kappa\in\mathcal{T}_h}\|u-u_h\|^2_{q_\kappa,\kappa}\leqslant 2\sum_{\kappa\in\mathcal{T}_h}\|u-\pi_h^p u\|^2_{q_\kappa,\kappa}+2\sum_{\kappa\in\mathcal{T}_h}\|\pi_h^p u-u_h\|^2_{q_\kappa,\kappa}$$

$$\leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(\mu_\kappa-q_\kappa)}}{p_\kappa^{2(s_\kappa-q_\kappa)}}\|u\|^2_{s_\kappa,\kappa}+2\sum_{\kappa\in\mathcal{T}_h}\|\pi_h^p u-u_h\|^2_{q_\kappa,\kappa}.$$

Using inverse estimates (see Theorem 4.76 in [25]), we deduce that

$$\sum_{\kappa\in\mathcal{T}_h}\|u-u_h\|_{q_\kappa,\kappa}^2 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(\mu_\kappa-q_\kappa)}}{p_\kappa^{2(s_\kappa-q_\kappa)}}\|u\|_{s_\kappa,\kappa}^2 + C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{-2q_\kappa}}{p_\kappa^{-4q_\kappa}}\|\pi_h^p u-u_h\|_{0,\kappa}^2$$

$$\leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(\mu_\kappa-q_\kappa)}}{p_\kappa^{2(s_\kappa-q_\kappa)}}\|u\|_{s_\kappa,\kappa}^2 + C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{-2q_\kappa}}{p_\kappa^{-4q_\kappa}}\|\pi_h^p u-u\|_{0,\kappa}^2$$

$$+\ C\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{-2q_\kappa}}{p_\kappa^{-4q_\kappa}}\|u-u_h\|_{0,\Omega}^2.$$

Applying (10) again together with (14), we have

$$\sum_{\kappa\in\mathcal{T}_h}\|u-u_h\|_{q_\kappa,\kappa}^2 \leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(\mu_\kappa-q_\kappa)}}{p_\kappa^{2(s_\kappa-q_\kappa)}}\|u\|_{s_\kappa,\kappa}^2 + C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(\mu_\kappa-q_\kappa)}}{p_\kappa^{2(s_\kappa-2q_\kappa)}}\|u\|_{s_\kappa,\kappa}^2$$

$$+\ C\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{-2q_\kappa}}{p_\kappa^{-4q_\kappa}}\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|_{s_\kappa,\kappa}^2\right)$$

$$\leqslant C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(\mu_\kappa-q_\kappa)}}{p_\kappa^{2(s_\kappa-2q_\kappa)}}\|u\|_{s_\kappa,\kappa}^2$$

$$+\ C\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{-2q_\kappa}}{p_\kappa^{-4q_\kappa}}\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^4}{p_\kappa}\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2\mu_\kappa-4}}{p_\kappa^{2s_\kappa-7}}\|u\|_{s_\kappa,\kappa}^2\right),$$

which concludes the proof.

We observe that if $s_\kappa=s$, $q_\kappa=q$, $h_\kappa=h$ and $p_\kappa=p$ for all $\kappa\in\mathcal{T}_h$, then the following error bound holds:

$$\sum_{\kappa\in\mathcal{T}_h}\|u-u_h\|_{q,\kappa}^2 \leqslant C\frac{h^{2(\mu-q)}}{p^{2(s-3-2q)}}\|u\|_{s,\mathcal{T}_h}^2 \tag{19}$$

for $0\leqslant q\leqslant\frac{s-3}{2}$ and $\mu=\min(p+1,s)$. Again, this bound is optimal in h, and suboptimal in p.

Finally, we present an error analysis of the symmetric DGFEM (3) for linear functionals of the solution. Let us denote by $J(\cdot)$ an arbitrary linear functional of the solution to problem (2). The question we wish to investigate here is how well $J(u_h)$ approximates $J(u)$. For this purpose, we consider the following *dual problem*:

find $z\in\mathrm{H}^4(\Omega,\mathcal{T}_h)$, such that,

$$\mathcal{B}(w,z)=J(w),\quad \forall w\in\mathrm{H}^4(\Omega,\mathcal{T}_h). \tag{20}$$

In the sequel we will assume that (20) has a unique solution; this implicitly presupposes that J is defined on $\mathrm{H}^4(\Omega, \mathcal{T}_h)$.

Problem (20) is the starting point of the *a priori* error analysis for linear functionals of the solution. Taking $w = u - u_h$ in the last equation and using the Galerkin orthogonality property, we obtain

$$
\begin{aligned}
J(u) - J(u_h) &= J(u - u_h) = \mathcal{B}(u - u_h, z) \\
&= \mathcal{B}(u - u_h, z - z_h), \quad \forall z_h \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F}).
\end{aligned}
$$

Decomposing the error as

$$u - u_h = (u - \pi_h^p u) + (\pi_h^p u - u_h) = \eta + \xi$$

and, using the Galerkin orthogonality (4), the continuity of the bilinear form and the definition of the norm $|||\cdot|||$, we have:

$$
\begin{aligned}
|||\xi|||^2 &= \mathcal{B}(\xi, \xi) = \mathcal{B}(-\eta + u - u_h, \xi) = -\mathcal{B}(\eta, \xi) + \mathcal{B}(u - u_h, \xi) = -\mathcal{B}(\eta, \xi) \\
&\leqslant |||\eta|||\,|||\xi|||
\end{aligned}
$$

resulting in $|||\xi||| \leqslant |||\eta|||$.

On selecting $z_h = \pi_h^p z$ and applying Lemma 2, we obtain

$$
\begin{aligned}
|J(u) - J(u_h)| &= |\mathcal{B}(u - u_h, z - \pi_h^p z)| = |\mathcal{B}(\eta, z - \pi_h^p z) + \mathcal{B}(\xi, z - \pi_h^p z)| \\
&\leqslant |||\eta|||\,|||z - \pi_h^p z||| + |||\xi|||\,|||z - \pi_h^p z||| \leqslant 2|||\eta|||\,|||z - \pi_h^p z|||.
\end{aligned}
$$

Assuming that $u \in \mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$, with $s_\kappa \geqslant 4$ for all $\kappa \in \mathcal{T}_h$, and $z \in \mathrm{H}^{\mathbf{t}}(\Omega, \mathcal{T}_h)$, with $t_\kappa \geqslant 4$ for all $\kappa \in \mathcal{T}_h$, and applying Lemma 4 ([22, 26]), we obtain the following bound on the error in the approximation of the linear functional of the solution:

$$|J(u) - J(u_h)| \leqslant C \left(\sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2\mu_\kappa - 4}}{p_\kappa^{2s_\kappa - 7}} \|u\|_{s_\kappa,\kappa}^2 \right)^{1/2} \times \left(\sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2\gamma_\kappa - 4}}{p_\kappa^{2t_\kappa - 7}} \|z\|_{t_\kappa,\kappa}^2 \right)^{1/2},$$

where $2 \leqslant \mu_\kappa \leqslant \min(p_\kappa + 1, s_\kappa)$, $2 \leqslant \gamma_\kappa \leqslant \min(p_\kappa + 1, t_\kappa)$.

Thus we have proved the following theorem.

Theorem 4. Let us suppose that $\Omega \subset \mathbb{R}^d$ is a bounded polyhedral domain $\{\mathcal{T}_h\}_{h>0}$ a shape-regular family of partitions formed by d-parallelepipeds and $\mathbf{p} = (p_\kappa,\ \kappa \in \mathcal{T}_h)$, with $p_\kappa \in \mathbb{N}$ and $p_\kappa \geqslant 2$ for all $\kappa \in \mathcal{T}_h$, the vector of local polynomial degrees, which we suppose to have bounded local variation. Further, we assume that the solutions u and z of problems

(2) and (20) are such that $u \in \mathrm{H}^{\mathbf{s}}(\Omega, \mathcal{T}_h)$, with $s_\kappa \geqslant 4$ for all $\kappa \in \mathcal{T}_h$, and $z \in \mathrm{H}^{\mathbf{t}}(\Omega, \mathcal{T}_h)$, with $t_\kappa \geqslant 4$ for all $\kappa \in \mathcal{T}_h$. Then, the following error bound holds for the linear functional $J(\cdot)$ of the solution $u_h \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathcal{F})$ to problem (3):

$$|J(u) - J(u_h)|^2 \leqslant C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2\mu_\kappa - 4}}{p_\kappa^{2s_\kappa - 7}} \|u\|_{s_\kappa,\kappa}^2 \times \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2\gamma_\kappa - 4}}{p_\kappa^{2t_\kappa - 7}} \|z\|_{t_\kappa,\kappa}^2, \tag{21}$$

where $2 \leqslant \mu_\kappa \leqslant \min(p_\kappa + 1, s_\kappa)$, $2 \leqslant \gamma_\kappa \leqslant \min(p_\kappa + 1, t_\kappa)$ and C is a constant independent of h_κ and p_κ.

In the particular case, when $p_\kappa = p \geqslant 2$, $s_\kappa = s \geqslant 4$, $t_\kappa = t \geqslant 4$, $\mu_\kappa = \mu$, $\gamma_\kappa = \gamma$, with s, t, μ and γ being integers, and $h_\kappa = h$ for all $\kappa \in \mathcal{T}_h$, Theorem 4 indicates that the error in the approximation of the functional may be bounded as follows:

$$|J(u) - J(u_h)| \leqslant C \frac{h^{\mu+\gamma-4}}{p^{s+t-7}} \|u\|_{s,\Omega} \|z\|_{t,\Omega}, \tag{22}$$

where $2 \leqslant \mu \leqslant \min(p+1, s)$ and $2 \leqslant \gamma \leqslant \min(p+1, t)$.

We observe that this error bound is optimal with respect to h and suboptimal with respect to p by p^3. We also highlight that this convergence rate is twice that of the one in the energy norm.

6. NUMERICAL RESULTS

We shall now present some numerical experiments to confirm the theoretically established error bounds.

6.1. Example 1

Here, we present some numerical experiments in order to confirm the theoretical orders of h-convergence of the method presented above. We solve the boundary value problem (1) in $\Omega = (0,1) \times (0,1)$, subject to homogeneous boundary conditions, and f being chosen so that the exact solution is the following (entire analytic) function

$$u(x, y) = 4x^2(x-1)^2 y^2 (y-1)^2 \exp(0.75(x+y)).$$

As mentioned in Lemma 3, the coefficients σ_α and σ_β of the stabilization parameters (6) need to be greater than $\underline{\sigma_\alpha}$ and $\underline{\sigma_\beta}$, respectively. Determining these constants in practice is not a trivial task, however, the results of our numerical experiments led us to select $\underline{\sigma_\alpha} = \underline{\sigma_\beta} = 10$.

Since the solution $u \in C^{\infty}(\bar{\Omega})$ and the elliptic regularity Hypothesis 5 is satisfied, Theorems 2 and 3 imply that, for $p \geqslant 3$, the DGFEMs presented above must exhibit the following optimal orders of convergence with respect to h: $\mathcal{O}(h^{p+1})$ in $\mathrm{L}^2(\Omega)$; $\mathcal{O}(h^p)$ in $\mathrm{H}^1(\Omega)$ and $\mathcal{O}(h^{p-1})$ in $\mathrm{H}^2(\Omega)$. This may be, in fact, seen in Figs. 1–3, where we plot, for different degrees p of the polynomial approximation, the global error of the DGFEM in the respective norm as a function of the discretization parameter h. We observe that in the $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^2(\Omega)$ norms the optimal order of convergence is also achieved for $p=2$, whereas in the $\mathrm{L}^2(\Omega)$ norm the order is $\mathcal{O}(h^p)$ instead of $\mathcal{O}(h^{p+1})$ (see the dashed line in figures).

In order to study the behaviour of the DGFEM in the approximation of linear functionals of the solution, we consider the following linear functionals:

$$J_1(u) = \int_{\Omega} u(x, y)\, d\Omega \qquad \text{and} \qquad J_2(u) = u(0.5, 0.5).$$

The convergence history for the functional J_1 corroborates the theoretical predictions of Theorem 4 in terms of the rate of h-convergence: the results presented in Fig. 4 shows the absolute value of the error

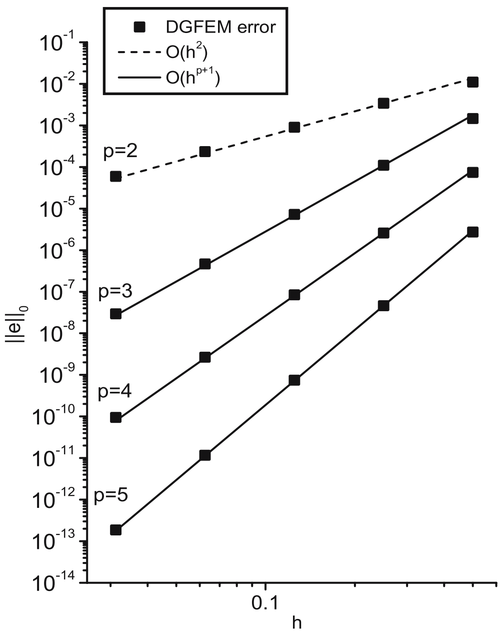

Fig. 1. L^2 norm of the error.

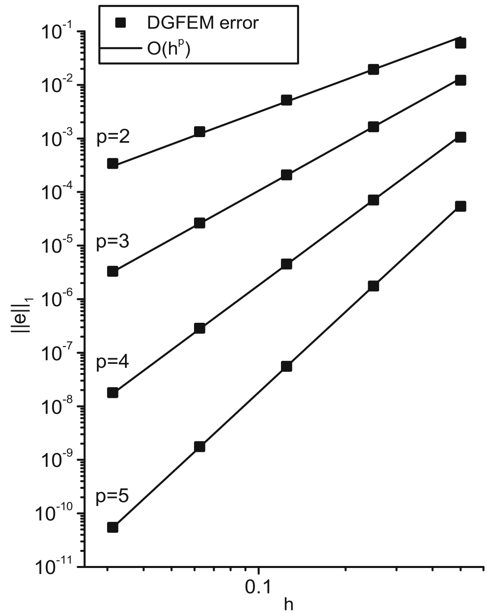

Fig. 2. H^1 norm of the error.

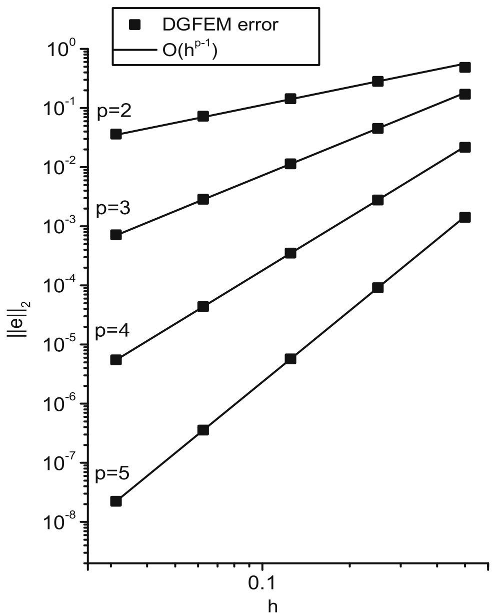

Fig. 3. H^2 norm of the error.

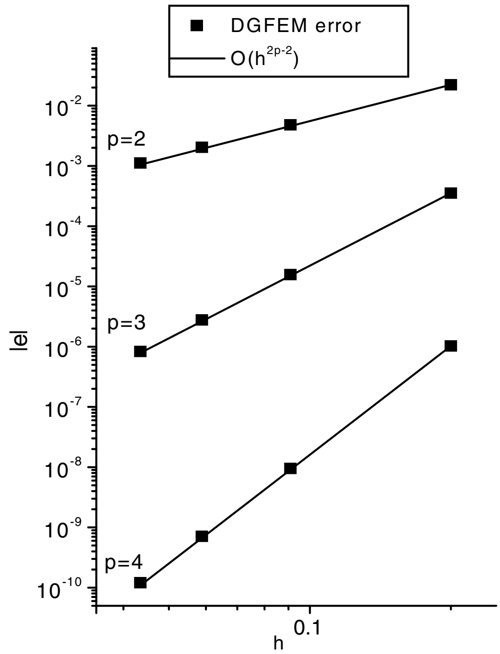

Fig. 4. Error in approximating $J_1(u)$.

$J_1(u) - J_1(u_h)$, as a function of the discretization parameter h. We see that the optimal h-order of convergence, $\mathcal{O}(h^{2p-2})$ (i.e., doubling of the order of convergence observed in the $\mathrm{H}^2(\Omega)$ norm) is achieved.

For J_2 the solution z of the dual problem (20) does not belong to $\mathrm{H}^{p+1}(\Omega, \mathcal{T}_h)$, $p \geqslant 2$; in fact, z lies in $\mathrm{H}^{3-\varepsilon}(\Omega)$ only for any $\varepsilon > 0$, so the predicted rate of h-convergence (upon taking $\mu = p+1$ and $\gamma = 3-\varepsilon$ in the inequality (22)) is $\mathcal{O}(h^{p-\varepsilon})$. The computationally observed decay of the error $|J_2(u) - J_2(u_h)| = \mathcal{O}(h^{2p-2})$ shown in Fig. 5 for $p = 3, 4$, on relatively coarse meshes of size $h = 1/2, 1/4, 1/8, 1/16$, does not seem to corroborate the analytically established convergence rate of $|J_2(u) - J_2(u_h)| = \mathcal{O}(h^{p-\varepsilon})$. Further experiments with $J_2(u) = u(x_0)$ including, in addition to $x_0 = (0.5, 0.5)$, the evaluation points $x_0 = (0.25, 0.75)$ and $x_0 = (0.4817, 0.7316)$, however, showed that the numerically observed $\mathcal{O}(h^{2p-2})$ error decay is preasymptotic behaviour on coarse h-grids. Under further h-refinement, starting from $h = 1/2$ and successively halving the mesh spacing down to $h = 1/128$, we observed that after the preasymptotic $\mathcal{O}(h^{2p-2})$ behaviour the error decay settles to the theoretically predicted decay rate of (approximately) $\mathcal{O}(h^{p-\varepsilon})$; Fig. 6 shows our numerical results for the case of $p = 3$.

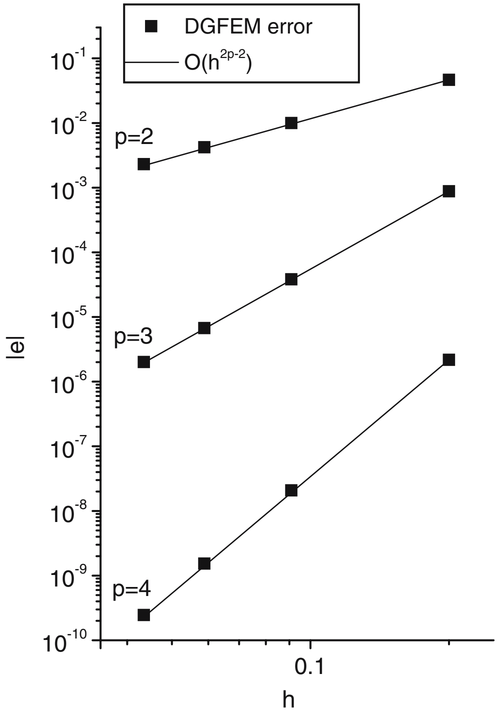

Fig. 5. Error in approximating $J_2(u)$.

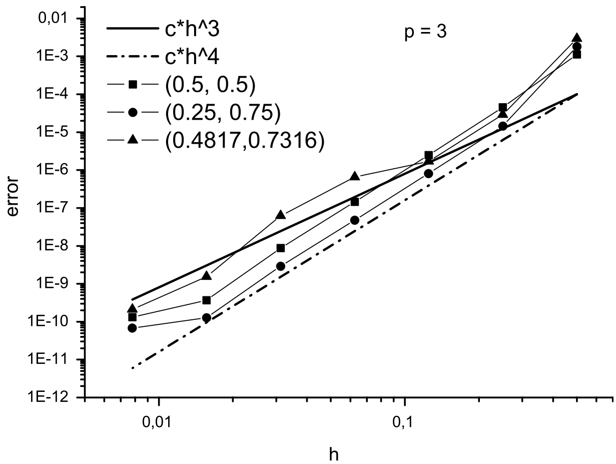

Fig. 6. Error in approximating the functional $J_2(u) = u(x_0)$, with evaluation points $x_0 = (0.5, 0.5), x_0 = (0.25, 0.75)$ and $x_0 = (0.4817, 0.7316)$, on meshes of size $h = 2^{-k}$, $k = 1, \ldots, 7$, and $p = 3$.

6.2. Example 2

In this example, we consider two problems from the theory of bending of thin plates. The first problem concerns a thin clamped plate, bending upon the action of a uniform force, while in the second problem the plate is subjected to the action of a central point-force. Thus, we take $f(x, y) = 1$ and $f(x, y) = \delta(x - \frac{1}{2}, y - \frac{1}{2})$, respectively. In both cases, we take $\Omega = (0, 1) \times (0, 1)$ and $g_0 = g_1 = 0$.

For both loadings, the value $J_2(u) = u(0.5, 0.5)$ at the centre of the plate was obtained by means of a series representation of the exact solution u (cf. [27]). For the case of the singular load $f(x, y) = \delta(x - \frac{1}{2}, y - \frac{1}{2})$, the solution u does not belong to $\mathrm{H}^4(\Omega)$ (in fact, $u \in \mathrm{H}^{3-\varepsilon}(\Omega)$ for any $\varepsilon > 0$), but is nevertheless continuous at $(\frac{1}{2}, \frac{1}{2})$, so $J_2(u)$ is meaningful.

In Tables I and II the behavior of the DGFEM error $|\frac{J_2(u) - J_2(u_h)}{J_2(u)}|$, in relation to h-refinement and p-refinement, respectively, are presented. In both cases, we give the relative errors for the DGFEMs in the nonsymmetric formulation (NIPG), considered in [22], and in the symmetric formulation (SIPG) of this study—for both the uniform force and the central point-force. We observe that the method converges rapidly in the case of

Table I. h-convergence at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$, $p = 3$

	Uniformly distributed load		Concentrated central load	
Mesh	Rel. error NIPG	Rel. error SIPG	Rel. error NIPG	Rel. error SIPG
3×3	1.139369e − 02	7.543377e − 03	1.052643e − 01	1.040824e − 01
9×9	8.649072e − 04	1.424936e − 04	1.138004e − 02	1.236476e − 02
17×17	1.955373e − 04	1.334661e − 05	4.106647e − 03	3.474865e − 03
33×33	4.374135e − 05	1.048839e − 06	1.114557e − 03	9.151037e − 04

Table II. p-convergence at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$, mesh 9×9

	Uniformly distributed load		Concentrated central load	
p	Rel. error NIPG	Rel. error SIPG	Rel. error NIPG	Rel. error SIPG
$p = 2$	3.910627e − 03	2.239174e − 02	2.038094e − 02	3.225565e − 02
$p = 3$	8.649072e − 04	1.424936e − 04	1.380046e − 02	1.236476e − 02
$p = 4$	3.066361e − 05	1.076654e − 06	2.633187e − 03	2.546534e − 03
$p = 5$	2.930244e − 07	9.830120e − 08	2.603813e − 03	2.605662e − 03
$p = 6$	3.783422e − 08	4.317365e − 10	1.123051e − 03	1.078158e − 03

the action of a uniform force and it has a slow rate of convergence when it is applied to the model problem with a central point-force (due to the singularity in the solution at the point $(\frac{1}{2}, \frac{1}{2})$ in both the primal and the dual problem).

We also observe that in the case of the model problem with a uniform force, for the SIPG of the DGFEM considered here, there is a significant reduction in the size of the error when compared with the nonsymmetric DGFEM formulation considered in [22], highlighting the advantages of using an adjoint-consistent method in the approximation of functionals.

7. CONCLUSIONS

We derived *a priori* error bounds in the L^2 norm and in broken Sobolev norms, for the symmetric version of the discontinuous Galerkin finite element method applied to the Dirichlet problem for the biharmonic equation. The symmetric method is adjoint-consistent, which allowed us to obtain error bounds which are optimal with respect to the mesh size h and slightly suboptimal with respect to the degree of the polynomial approximation p. The symmetry of the bilinear form of the method was shown to be of crucial importance in the derivation of error bounds for the approximation of linear functionals of the solution; this led to the doubling of the convergence rate compared to that in the energy norm. We confirmed the optimality of the theoretically established h-convergence rates by a series of numerical experiments and also showed the application of the method to some practical problems in elasticity theory.

In numerical experiments which are not reported here, by considering a fixed uniform square mesh and a variety of analytical solutions whose Sobolev index was exactly equal to 4, we attempted to assess the sharpness, or otherwise, of the theoretically established convergence rates under p-refinement. For the set of model problems we had considered, vastly varying orders of p-convergence were observed, which did not permit us to reliably confirm the (sub)optimality of the theoretically predicted rates of p-convergence. Nevertheless, our suspicion (which, as a matter of fact, is reinforced by the findings of the paper [14] concerned with hp-version DGFEMs for second-order elliptic problems) is that our theoretically predicted rates of p-convergence are *not* of optimal order.

ACKNOWLEDGMENTS

We are grateful to the anonymous referee for helpful and constructive comments. We also wish to thank Professor Serge Nicaise (Université de Valenciennes) for drawing to our attention the paper of Kozlov and Maz'ya [20]. Igor Mozolevski was partially supported by CNPq, Brazil and P.R. Bösing received a grant from CNPq, Brazil.

REFERENCES

1. Arnold, D. N., Brezzi, F., Cockburn, B., and Marini, D. (2001). Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* **39**(5), 1749–1779.
2. Babuška, I., and Miller, A. (1984). The post-processing approach in the finite element method—part 1: Calculation of displacements, stresses and other higher derivatives of the displacements. *Int. J. Numer. Meth. Eng.* **20**(6), 1085–1109.
3. Babuška, I., and Miller, A. (1984). The post-processing approach in the finite element method—part 2: The calculation of stress intensity factors. *Int. J. Numer. Meth. Eng.* **20**(6), 1111–1129.
4. Babuška, I., and Miller, A. (1984). The post-processing approach in the finite element method—part 3: A posteriori error estimates and adaptive mesh selection. *Int. J. Num. Meth. Eng.* **20**(6), 2311–2325.
5. Babuška, I., and Suri, M. (1987). The *hp*-version of the finite element method with quasiuniform meshes. *Math. Model. Numer. Anal.* **21**, 199–238.
6. Baker, G. (1977). Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.* **31**, 44–59.
7. Blum, H., and Rannacher, R. (1980). On the boundary value problem of the biharmonic operator on domains with angular corners. *Math. Meth. Appl. Sci.* **2**, 556–581.
8. Brenner, S. C., and Scott, L. R. (2002). *The Mathematical Theory of Finite Element Methods*, 2nd edn. Spinger, Berlin.
9. Brezzi, F., and Fortin, M. (1991). *Mixed and Hybrid Finite Element Methods*. Springer, Berlin.
10. Ciarlet, P. G. (1991). Basic error estimates for elliptic problems. In Ciarlet, P., and Lions, J. (eds.), *Handbook of Numerical Analysis*, Vol. 2. North Holland, Amsterdam.
11. Cockburn, B., Karniadakis, G. E., and Shu, C.-W. (eds.) (2000). *Discontinuous Galerkin Methods*, *LNCSE*, Vol. 11, Springer, Berlin.
12. Douglas, J. J., and Dupont, T. (1976). Interior penalty procedures for elliptic and parabolic Galerkin methods. *Lectures Notes in Physics*, Vol. 58. Springer-Verlag, Berlin.
13. Engel, G., Garikipati, K., Hughes, T., Larson, M., Mazzei, L., and Taylor, R. (2002). Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Comput. Meth. Appl. Mech. Eng.* **191**, 3669–3750.
14. Georgoulis, E. H., and Süli, E. (2005). Optimal error estimates for the *hp*-version interior penalty discontinuous Galerkin finite element method. *IMA J. Numer. Anal.* **25**, 205–220.
15. Girault, V., and Raviart, P. (1986). *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*. Spring-Verlag, Berlin.
16. Grisvard, P. (1992). *Singularities in Boundary Value Problems*. Research Notes in Applied Mathematics. Masson, Paris.
17. Harriman, K., Gavaghan, D. J., and Süli, E. (2004) *The Importance of Adjoint Consistency in the Approximations of Linear Functionals Using the Discontinuous Galerkin Finite Element Method*. Tech. Rep. 04/18, Oxford University Computing Laboratory.
18. Harriman, K., Houston, P., Senior, B., and Süli, E. (2002). *hp*-version discontinuous Galerkin methods with interior penalty for partial differential equations with

nonnegative characteristic form. In Shu, C.-W., Tang, T., and Cheng, S.-Y. (eds.), *Recent Advances in Scientific Computing and Partial Differential Equations, Contemporary Mathematics* Vol. 330, AMS, 2003. pp. 89-119.
19. Kanschat, G., and Rannacher, R. (2002). Local error analysis of the interior penalty discontinuous Galerkin method for second order elliptic problems. *J. Numer. Math.* **10**(4), 249–274.
20. Kozlov, V. A., and Maz'ya, V. G. (1996). Singularities in solutions to mathematical physics problems in non-smooth domains. In Cea, J. et al. (ed.), *Partial Differential Equations and Functional Analysis. Proceedings of the conference held in November 1994 in memory of Pierre Grisvard. Progr. Nonlinear Differential Equations Appl.*, Vol. 22, Birkhäuser Boston, Boston, MA. pp. 174–206.
21. Larson, M., and Niklasson, A. (2004). Analysis of a family of discontinuous Galerkin methods for elliptic problems: the one dimensional case. *Numer. Math.* **99**(1), 113–130.
22. Mozolevski, I., and Süli, E. (2003). A priori error analysis for the hp-version of the discontinuous Galerkin finite element method for the biharmonic equation. *Comput. Meth. Appl. Math.* **3**(4), 596–607.
23. Oden, J. T., Babuška, I., and Baumann, C. (1998). The local discontinuous Galerkin finite element method for diffusion problems. *J. Comput. Phys.* **146**, 491–519.
24. Rivière, B., Wheeler, M. F., and Girault, V. (2001). A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *J. Numer. Anal.* **39**(3), 902–931.
25. Schwab, C. (1998). *p- and hp- Finite Element Methods. Theory and Applications to Solid and Fluid Mechanics*. Oxford University Press, Oxford.
26. Süli, E., and Mozolevski, I. (2004). *hp-version Interior Penalty DGFEMs for the Biharmonic Equation*. Techinical Report 04/05, Oxford University Computing Laboratory. http://web.comlab.ox.ac.uk/oucl/publications/natr/na-04-05.html
27. Timoshenko, S. P., and Woinowsky-Krieger, S. (1984). *Theory of plates and shells*, 25 edn. McGraw-Hill International Book Company, New york.