*Journal of Scientific Computing, Vol. 28, Nos. 2/3, September 2006 (© 2006)*
DOI: 10.1007/s10915-006-9082-z

# Numerical Methods and Nature

**E. Turkel**[1]

*Received January 23, 2005; accepted (in revised form) June 13, 2005; Published online March 13, 2006*

In many numerical procedures one wishes to improve the basic approach either to improve efficiency or else to improve accuracy. Frequently this is based on an analysis of the properties of the discrete system being solved. Using a linear algebra approach one then improves the algorithm. We review methods that instead use a continuous analysis and properties of the differential equation rather than the algebraic system. We shall see that frequently one wishes to develop methods that destroy the physical significance of intermediate results. We present cases where this procedure works and others where it fails. Finally we present the opposite case where the physical intuition can be used to develop improved algorithms.

**KEY WORDS:** Finite differences; preconditioning; regularization.

## 1. INTRODUCTION

We consider preconditioning and regularization methods for improving the numerical solution to both hyperbolic and elliptic partial differential equations. The purpose of these techniques is to improve either the convergence rate of an iterative method or else to improve the accuracy of the numerical approximation. Frequently preconditioners are devised by utilizing the structure of the algebraic system, e.g., ADI, ILU, Krylov space methods etc. This approach uses no knowledge of the physical problem other than basic algebraic properties like symmetry and positivity. As an alternative we present two different preconditioners for approximations to the Helmholtz equation that use properties of the differential equation.

Such an approach has both advantages and disadvantages compared to a strictly algebraic approach. One immediate disadvantage is that the

[1] School of Mathematical Sciences, Tel Aviv University Ramat Aviv, Tel Aviv 69978, Israel. E-mail: turkel@post.tau.ac.il

analysis needs to be re-done for different equations. Furthermore, even for a given equation the general properties may depend on other factors, e.g., whether it is an interior or exterior problem. One advantage of such an approach is that it does not depend on the discretization used. Thus, for example, the low speed preconditioner for the compressible fluid dynamic equations, to be presented, has been successfully used on both structured and unstructured grids, for explicit and implicit time algorithms and both centered and upwind spatial discretizations. The Jacobi preconditioner is usually viewed as a discrete preconditioner. However, it can also be viewed as a matrix time step.

The preconditioner for the fluid dynamic equations destroys the intermediate physical meaning of the iterative procedure. However, by achieving new eigenvalues and eigenvectors for the iteration matrix one vastly improves both the convergence rate and the accuracy of the scheme for low speed flow. This is valid for steady state problems. For time dependent problems we discretize the physical time by an implicit scheme. This results in a "pseudo-steady" problem for the solution at the next time step. This is solved by techniques similar to the steady state problem including preconditioning. This is referred to as dual time-stepping. Another approach to change the characteristics of the equations is to introduce a lower order regularization term. There are some free parameters and one is tempted to choose them to allow lower speeds and so larger time steps and a faster convergence rate. For a scalar equation it is well known that the subcharacteristic condition says that this does not work. We shall analyze a new regularization geared specifically for the Euler equations. We shall show that even though it can be written as a symmetric hyperbolic system nevertheless a subcharacteristic condition states that one can not choose larger time steps. Hence, "cheating" nature works nicely for the low speed preconditioning (though there are still robustness questions) however, it does not work for regularization techniques.

In summary, there are occasions when preconditioners are based on the physical properties of the system and on other occasions the preconditioners are chosen to overcome physical properties of the system. In both cases we use properties of the continuous system, i.e., the "physics" of the problem rather than algebraic properties of the discretization.

## 2. LOW SPEED PRECONDITIONING

Consider the hyperbolic system of the unsteady Euler equations appended with pseudo-time derivatives. Let $t$ denote the physical time, while $\tau$ denotes the pseudo-time used to drive each physical time step to

a pseudo-steady state. In quasilinear form we have

$$\frac{\partial w}{\partial \tau} + \xi \frac{\partial w}{\partial t} + A \frac{\partial w}{\partial x} + B \frac{\partial w}{\partial y} + C \frac{\partial w}{\partial z} = 0 \tag{1}$$

$w = (\rho, \rho u, \rho v, \rho w, \rho E)$. The flux Jacobian matrices $A, B, C$ are simultaneously symmetrizable. We are interested in $\tau \to \infty$. For physically steady state problems $\xi = 0$, while for time dependent problems $\xi = 1$. Thus, when $\xi = 1$ we advance the solution to the next physical time step with an implicit scheme. This results in a nonlinear set of equations which can be solved in the same manner as the steady state fluid equations. This is known as dual time stepping since we have both a physical and an artificial time step. We discretize the physical time derivative with a backward difference formula (BDF)

$$\frac{\partial w}{\partial t} \sim \frac{c_t w^{n+1} - E(w^n, w^{n-1}, \dots)}{\Delta t} \tag{2}$$

where $c_t$ is a constant that depends on the order of the temporal scheme. Including the approximation of Eq. (2) and preconditioning the system we replace (1) by

$$\mathbf{P}^{-1} \frac{\partial w}{\partial \tau} + \xi \frac{c_t w^{n+1} - E(w^n, w^{n-1}, \dots)}{\Delta t} + A \frac{\partial w}{\partial x} + B + C \frac{\partial w}{\partial z} = 0 \tag{3}$$

We can express (3) in conservation form as

$$\frac{\mathbf{P}^{-1} \partial w}{\partial \tau} + \xi \frac{c_t w^{n+1} - E(w^n, w^{n-1}, \dots)}{\Delta t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0 \tag{4}$$

The equations are advanced in pseudo-time by a multistage Runge-Kutta (RK) scheme. Let superscript 0 denote the last artificial time and $k$ be the most recent stage of RK. Let $n$ be the last physical time step calculated and $n+1$ the next physical time step. $R^k$ denotes the spatial derivative terms of the residual at the last stage, $k$. Denote the total pseudo-residual by

$$(R^*)^k = R^k + \frac{c_t w^k - E(w^n, \dots)}{\Delta t}$$

Within each subiteration we march the pseudo-time $\tau$ to infinity until $R^*$ is zero. We call this solution at the end of a physical time step the pseudo-steady state. A typical stage of the RK scheme takes the form

$$w^{k+1} = w^0 - \alpha_k \Delta \tau \mathbf{P} (R^*)^k - \alpha_k c_t \Delta \tau \mathbf{P} \left( \frac{w^{k+1} - w^k}{\Delta t} \right) \tag{5}$$

Collecting terms, we have

$$\left(I+\alpha_k c_t \frac{\Delta\tau}{\Delta t}\mathbf{P}\right) w^{k+1} = w^0 - \alpha_k \Delta\tau \mathbf{P}(R^*)^k + \alpha_k c_t \frac{\Delta\tau}{\Delta t}\mathbf{P}w^k$$

Assuming constant coefficients we Fourier transform (3) in space. Define $\zeta = \frac{\xi c_t}{\Delta t}$ and

$$\mathbf{G}(\omega_1, \omega_2, \omega_3) = \mathbf{P}[-\zeta I + i(A\omega_1 + B\omega_2 + C\omega_3)] \tag{6}$$

with $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. The condition number is defined as

$$cond\# = \max_{\omega_i} \left| \frac{\lambda_{\max}(\mathbf{G})}{\lambda_{\min}(\mathbf{G})} \right|.$$

where $\lambda$ denotes an eigenvalue of the matrix. Note that the eigenvalues of $i(A\omega_1 + B\omega_2 + C\omega_3)$ are pure imaginary since the matrices are symmetrizable. Physically, the condition number (with $\xi = 0$) can be interpreted as the ratio of the fastest speed to the slowest speed in any direction. If viscous terms are included, then we have additional negative real matrices in (6). We stress that if $\Delta t$ is sufficiently small, then the condition number is close to 1. The methods proposed here will not improve the condition number of such problems and, hence, will not improve the convergence rate to a pseudo-steady state. For the Euler equations, the condition number is approximately inversely proportional to the Mach number.

With a local preconditioner we change the discrete equations at individual grid nodes without introducing new coupling between neighboring nodes. Hence, this technique makes sense only for a system of equations. For a scalar equation, local preconditioning is simply a rescaling of the time variable and has no effect on the numerical solution. This approach is distinct from incomplete LU (ILU) technique which couples all the nodes together. We choose the matrix $\mathbf{P}$ so as to improve the condition number of the equations at the node point. The assumption is that the better the system is conditioned, the faster the iteration process will approach a steady state.

We consider two different local preconditioners, Jacobi and low speed, to alleviate the stiffness of the system. We then formulate a composite preconditioner that combines the complementary properties of the Jacobi and low speed preconditioners to achieve an efficient scheme for solving flows with embedded low speed flows. For time dependent problems where the physical time scale is sufficiently small, and so $\zeta$ is large, preconditioning can harm the convergence rate. For such problems the preconditioning in the update stage should be turned off and should only affect the artificial

viscosity or the upwinding. In summary, local preconditioning has been very successful for improving both the convergence rate and accuracy for low speed steady state flows. For time dependent flows it has been moderately successful in improving the accuracy and less successful in improving the converge rate. Part of the difficulty is that for dual time stepping we wish to perform only a small number of subiterations. Hence, the asymptotic convergence rate is less important than the initial convergence rate. While preconditioning improves the asymptotic rate, it is less successful in improving the initial decrease in the residuals.

### 2.1. Implementation

For low Mach number flows, the ratio of acoustic to convective speeds is large, which results in an ill-conditioned and stiff system. Hence, we introduce a preconditioner to alleviate this stiffness. Because **P** based on conservation variables is a full matrix, we make use of entropy variables $w_0 = \left(\frac{dp}{\rho c}, du, dv, dw, dp - c^2 d\rho\right)$ for which the energy equation decouples from the rest of the governing equations. Furthermore, the Jacobian matrix is sparse in these variables. The simplest preconditioner in $w_0$ variables is given by (see [15, 16, 19])

$$\mathbf{P_0}^{-1} = \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$\beta$ is a parameter which is of the order of the Mach number to approximately equalize all the eigenvalues of $P_0 A_0$.

We need to choose $\beta^2$ and the pseudo-time step. When we ignore the term $\frac{w^{k+1} - w^k}{\Delta t}$ in (5), the RK scheme is explicit for the physical time derivative; however, it requires that the pseudo-time step also include a physical time step contribution. The precise form of this term is given in (10). The present analysis is done on the continuous level, except for the source term that arises from discretization of the physical time derivative by a BDF formula. The amplification matrix for a RK scheme is a polynomial in a stage amplification matrix. The total scheme is stable when all the eigenvalues of the stage amplification matrix lie within the stability region of the particular RK scheme. The stage amplification matrix in pseudo-time

for the Euler equations given by

$$\mathbf{G}(\omega_1,\omega_2,\omega_3)=\mathbf{P}_0\left[-\frac{c_t Vol}{\Delta t}+i(\omega_1 A+\omega_2 B+\omega_3 C)\right] \tag{8}$$

where *Vol* is the cell volume and $A$,$B$,$C$ are the Jacobian matrices of the inviscid flux vectors. The matrices $A$,$B$,$C$ are symmetrizable and so this is a symmetric hyperbolic system.

In Cartesian coordinates the amplification matrix is given by

$$G(\omega_1,\omega_2,\omega_3)=-\frac{c_t\mathbf{P}_0}{\Delta t}+i\frac{\omega_1}{\Delta x}\begin{pmatrix}\beta^2 u & \beta^2 c & 0 & 0\\ c & u & 0 & 0\\ 0 & 0 & u & 0\\ 0 & 0 & 0 & u\end{pmatrix}$$
$$+i\frac{\omega_2}{\Delta y}\begin{pmatrix}\beta^2 v & 0 & \beta^2 c & 0\\ 0 & v & 0 & 0\\ c & 0 & v & 0\\ 0 & 0 & 0 & v\end{pmatrix}+i\frac{\omega_3}{\Delta z}\begin{pmatrix}\beta^2 w & 0 & 0 & \beta^2 c\\ 0 & w & 0 & 0\\ 0 & 0 & w & 0\\ c & 0 & 0 & w\end{pmatrix}$$

Define

$$q=\frac{u\omega_1}{\Delta x}+\frac{v\omega_2}{\Delta y}+\frac{w}{\omega_3}\Delta z \qquad D=-\frac{c_t}{\Delta t}+iq$$

Then

$$G=\begin{pmatrix}\beta^2 D & \beta^2 c\frac{\omega_1}{\Delta x} & \beta^2 c\frac{\omega_2}{\Delta y} & \beta^2 c\frac{\omega_3}{\Delta z}\\ c\frac{\omega_1}{\Delta x} & D & 0 & 0\\ c\frac{\omega_2}{\Delta y} & 0 & D & 0\\ c\frac{\omega_3}{\Delta z} & 0 & 0 & D\end{pmatrix}$$

The eigenvalues of $G$ are $\lambda_0=D$ and

$$\lambda_\pm=\frac{\beta^2+1}{2}D\pm\sqrt{\left(\frac{\beta^2-1}{2}\right)^2 D^2+\beta^2 c^2\left(\left(\frac{\omega_1}{\Delta x}\right)^2+\left(\frac{\omega_2}{\Delta y}\right)^2+\left(\frac{\omega_3}{\Delta z}\right)^2\right)} \tag{9}$$

Because $D$ is a complex number, so is $\lambda_\pm$. We define $\lambda_{inv}=\max(|\lambda_+|,|\lambda_-|)$. The artificial time step is determined by demanding that $\lambda_{inv}$ be within the stability domain of the RK scheme for all $\omega_i$ with $\omega_1^2+\omega_2^2+\omega_3^2=1$. Since $\lambda_\pm$ is a complex number, this leads to a condition on the time step that depends on the details of the stability curve. Hence, we replace this by a condition on the real and imaginary parts separately. The formula we use for calculating the artificial time step, $\Delta\tau\sim\frac{1}{\lambda}$, is given by

$$\frac{1}{\Delta\tau}=\frac{1}{\Delta\tau_{ss}}+\frac{K_\tau}{c_t\Delta t} \tag{10}$$

where $\Delta\tau_{ss}$ is the steady state (without dual time-stepping) artificial time step which is a sum of inviscid and viscous contributions.

The difficulty in determining $\beta$ is that $D$ is a complex number. Let $\beta_{ss}^2$ denote $\beta$ for the steady state problem. We choose

$$\begin{aligned} \beta_{inv}^2 &= K_1 M^2 + K_2 M_{ref}^2 \\ \beta_{ss}^2 &= K_3(Re_\Delta)\beta_{inv}^2 \end{aligned} \tag{11}$$

The formula for $\beta^2$ with dual time stepping is then given by

$$\tilde{\beta}^2 = \beta_{ss}^2 + K_\beta \left( \frac{c_t}{c\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 + \left(\frac{1}{\Delta z}\right)^2}\,\Delta t} \right)^p \tag{12}$$

$$\beta^2 = \max(\tilde{\beta}^2, 1) \tag{13}$$

where $K_1, K_2, K_\beta$ are constants. The cell Reynolds number, $Re_\Delta$, is the ratio, for each cell, of the inviscid to viscous fluxes. It can also be defined in terms of a ratio of the inviscid to viscous time step. $M$ is the local Mach number and $M_{ref}$ is a reference Mach number, representative of the free stream Mach number. Based on numerical experimentation, $p = \frac{1}{2}$ in (12) yields the most consistent results (see [21] for more details). Because we do not let $\beta^2$ exceed unity, the preconditioning is turned off locally in the farfield for external problems, where the cell volumes are large. Similarly, when $\Delta t$ is small enough, then preconditioning is turned off globally. When the contribution of the physical time step is large enough, then the preconditioning does not improve the convergence of the subiterations, but it is still useful for improving the accuracy of the numerical solution. We note that the low speed preconditioner was constructed to improve the convergence rate to a steady state. It was later found [17,19] that it is necessary for accuracy at low Mach numbers.

## 3. JACOBI PRECONDITIONING

The Jacobi preconditioning [1,13] is based on adding a matrix-based artificial viscosity and then choosing $P^{-1}$ as the terms on the diagonal (i.e., the coefficient of $w_{ij}$). The result for a central difference scheme is

$$P_J^{-1} = \zeta I + |A| + |B| + |C| \tag{14}$$

The good high frequency damping characteristics of the Jacobi preconditioner make it an ideal candidate for coupling with a multigrid scheme.

Because this formulation connects the preconditioning with the artificial viscosity (or upwinding), the matrix $P$ is affected by the details of the discretization. However, Eq. (14) has also been used with other artificial viscosities such as CUSP (see Caughey and Jameson [5]). We prefer to view the preconditioner represented by Eq. (14) as a matrix or characteristic inverse time step (see [24] for a similar view). A multistage, non-preconditioned RK scheme uses an artificial time step given by

$$\Delta\tau = \frac{CFL}{\zeta + \rho(A) + \rho(B) + \rho(C)} \tag{15}$$

where $\rho$ is the spectral radius and $CFL$ is a number chosen to achieve stability. A matrix time step for the Jacobi preconditioner replaces this by

$$\Delta\tau = CFL\,(\zeta I + |A| + |B| + |C|)^{-1} \tag{16}$$

In calculating the absolute value of the matrices, one needs to cutoff the eigenvalues to prevent them from becoming too small (see [23] for more details).

## 4. PRECONDITIONING SQUARED

Consider **P** given by (7) with $\beta$ given by (11, 12). We combine the low speed preconditioning with Jacobi preconditioning by starting with an artificial viscosity based on the low speed preconditioning for increased accuracy and then forming the Jacobi preconditioning for better convergence rates [18]. Let $P$ be the low speed preconditioning, and let the physical time derivative be represented by (2). Then the preconditioned scheme (showing only the second-order dissipation) is given by

$$\begin{aligned} P_J^{-1} &= {P_0}^{-1}\,(\zeta P_0 + |P_0 A| + |P_0 B| + |P_0 C|) \\ P_J^{-1}\Delta w &= \frac{c_t w^{n+1} - E(w^n, w^{n-1}, \ldots)}{\Delta t} + F_x + G_y + H_z \\ &\quad - \frac{\Delta x}{2}\left[({P_0}^{-1}|P_0 A| w_x)_x + ({P_0}^{-1}|P_0 B| w_y)_y + ({P_0}^{-1}|P_0 C| w_z)_z\right] \\ &\equiv \text{Res} \end{aligned} \tag{17}$$

One can simultaneously symmetrize the matrices $P_0, |P_0 A|, |P_0 B|, |P_0 C|$. Hence, the artificial viscosity and accuracy is based on the low speed preconditioner $P_0$. However, the acceleration based on equalizing the eigenvalues uses a combination of low speed and Jacobi preconditioning.

## 5. RESULTS

The results are computed using TLNSD, a finite volume central difference code augmented by a matrix artificial viscosity. The equations are advanced in time with a dual time stepping scheme, as described above. A five stage RK scheme, accelerated by residual smoothing and multigrid is used for advancing the solutions in pseudo-time, see [9,25].

We examine the use of both Jacobi and low speed preconditioners for steady flow. We consider a RAE2822 airfoil using a $320 \times 64$ C grid. In Figures 1a and 1b, we present results for an inflow Mach number of 0.05. We see that the Jacobi preconditioning by itself helps relatively little for this low Mach number flow. In contrast, using the low speed preconditioning gives a large improvement in the residual convergence. The combined low speed and Jacobi preconditioning gives a dramatic improvement, yielding 8 orders of magnitude decrease in the residual in 300 multigrid cycles. The residual is reduced by about 11 orders of magnitude. For a five stage RK formula, this is equivalent to 1500 explicit sweeps through the grid. Such rapid convergence represents a significant improvement for low-speed, viscous, turbulent flow computations on high aspect ratio grids.

## 6. RELAXATION

We consider a relaxation scheme for the Euler equations developed recently [26]. In conservation form we have

$$\begin{aligned}
\rho_t + (u\rho)_x + (v\rho)_y &= 0 \\
(\rho u)_t + (\rho u^2 + ap + b\pi)_x + (\rho uv)_y &= 0 \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + ap + b\pi)_y &= 0 \\
(\rho E)_t + (u(\rho E + dp + e\pi))_x + (v(\rho E + dp + e\pi))_y &= 0 \\
(\rho\pi)_t + (\rho\pi u)_x + (\rho\pi v)_y &= -\frac{\rho}{\varepsilon}(p - \pi)
\end{aligned} \tag{18}$$

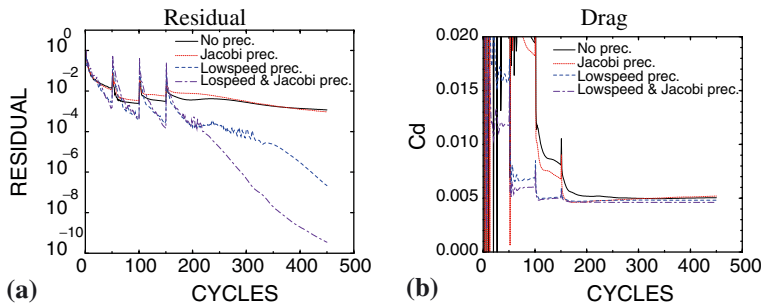


**Fig. 1.** RAE2822 convergence history, $M_\infty = 0.05$.

where $a, b, d, e$ are constants with $a+b=d+e=1$ and $\varepsilon$ is the relaxation parameter which approaches zero. As usual

$$\rho E = \frac{p}{\gamma-1} + \rho\frac{(u^2+v^2)}{2} \qquad dp = (\gamma-1)\left[dE - \frac{u^2+v^2}{2}d\rho - \rho\,(udu+vdv)\right]$$

Converting (18) to primitive variables we get

$$\begin{aligned}
\rho_t + u\rho_x + v\rho_y + \rho\left(u_x+v_y\right) &= 0\\
\rho u_t + \rho u u_x + \rho v u_y + a p_x + b\pi_x &= 0\\
\rho v_t + \rho u v_x + \rho v v_y + a p_y + b\pi_y &= 0\\
p_t + z_1(up_x + vp_y) + z_2(u_x+v_y) + z_3(u\pi_x + v\pi_y) &= 0\\
\pi_t + u\pi_x + v\pi_y &= -\frac{1}{\varepsilon}(p-\pi)
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
z_1 &= 1-(\gamma-1)(a-d)\\
z_2 &= (1+(\gamma-1)d\ )\,p + e(\gamma-1)\pi\\
z_3 &= (\gamma-1)(e-b) = (\gamma-1)(a-d)
\end{aligned}$$

We introduce a pseudo entropy by

$$dS = dp - \frac{z_2}{\rho}d\rho \tag{20}$$

Then

$$S_t + uS_x + vS_y = 0$$

We next calculate the eigenvalues and eigenvectors. We eliminate $\rho$ in (19) in favor of the entropy variable. We then get

$$\begin{aligned}
p_t + z_1(up_x + vp_y) + z_2(u_x+v_y) + z_3(u\pi_x + v\pi_y) &= 0\\
u_t + uu_x + vu_y + \frac{ap_x + b\pi_x}{\rho} &= 0\\
v_t + uv_x + vv_y + \frac{ap_y + b\pi_y}{\rho} &= 0\\
S_t + uS_x + vS_y &= 0\\
\pi_t + u\pi_x + v\pi_y &= -\frac{\rho}{\varepsilon}(p-\pi)
\end{aligned} \tag{21}$$

or in matrix form

$$w_t + Aw_x + Bw_y = F$$

where

$$w = \begin{pmatrix} p \\ u \\ v \\ S \\ \pi \end{pmatrix}, \quad A = \begin{pmatrix} z_1 u & z_2 & 0 & 0 & z_3 u \\ \frac{a}{\rho} & u & 0 & 0 & \frac{b}{\rho} \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix} \quad B = \begin{pmatrix} z_1 v & 0 & z_2 & 0 & z_3 v \\ 0 & v & 0 & 0 & 0 \\ \frac{a}{\rho} & 0 & v & 0 & \frac{b}{\rho} \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix}$$

Let $U = u\omega_1 + v\omega_2$. Then

$$A\omega_1 + B\omega_2 = \begin{pmatrix} z_1 U & z_2\omega_1 & z_2\omega_2 & 0 & z_3 U \\ \frac{a\omega_1}{\rho} & U & 0 & 0 & \frac{b\omega_1}{\rho} \\ \frac{a\omega_2}{\rho} & 0 & U & 0 & \frac{b\omega_2}{\rho} \\ 0 & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & U \end{pmatrix} \qquad \omega_1^2 + \omega_2^2 = 1$$

$A\omega_1 + B\omega_2$ has a double convective eigenvalue of $U$. The eigenvalues are given by

$$\begin{aligned} \lambda_0 &= U \quad \text{triple} \\ \lambda_\pm &= \frac{(z_1+1)U \pm \sqrt{(z_1-1)^2U^2 + 4\frac{az_2}{\rho}}}{2}. \end{aligned} \tag{22}$$

The original Euler equations correspond to

$$a = 1 \qquad b = 0 \qquad d = 1 \qquad e = 0.$$

The opposite extreme is given by

$$a = 0 \qquad b = 1 \qquad d = 0 \qquad e = 1.$$

In this case we have

$$z_1 = 1 \qquad z_2 = p + (\gamma - 1)\pi \qquad \lambda_\pm = U.$$

Thus, all the eigenvalues degenerate to $U$. Two of the eigenvalues correspond to convection. However, the other eigenspace degenerates into a single eigenvector. Hence, the system is no longer strictly hyperbolic since we do not have distinct eigenvectors. Therefore, we shall choose $a \neq 0$. By choosing $a$ small we can keep the acoustic eigenvalues close to $U$. This

corresponds to the effect of low speed preconditioning. We note that for low speed preconditioning choosing the optimal parameters leads to problems. Hence, for robustness one chooses parameters slightly less than optimal.

We wish to check if we can simultaneously symmetrize both matrices. If this can be done then the system is symmetric hyperbolic. One way to do check this is to diagonalize the matrix $A$. We can then easily check under which conditions $B$ is symmetrizable. It is relatively easy to show that this is a necessary and sufficient condition to simultaneously symmetrize two matrices.

One verifies that there is symmetry if and only if

$$a = d \qquad a \neq 0 \qquad b = e \tag{23}$$

and so

$$\begin{aligned} z_1 &= 1 \\ z_3 &= 0 \\ z_2 &= p + (\gamma - 1)(ap + b\pi) \end{aligned}$$

To complete the symmetrization we define

$$p = X\hat{p} \qquad \pi = -Y\hat{p} + Z\hat{\pi}$$

or equivalently

$$\hat{p} = \frac{p}{X} \qquad \hat{\pi} = \frac{\pi}{Z} + \frac{Yp}{XZ}$$

where

$$aX^2 - bXY = \rho z_2 \qquad \frac{XZ^2}{Y} = \rho z_2 \tag{24}$$

From the two equations we get $aX^2 - bXY = b\frac{XZ^2}{Y}$. As always $a + b = 1$ and $z_2 = p + (\gamma - 1)(ap + b\pi)$.

We consider the extreme cases:

$a = 1 \quad b = 0$

Then $X = \sqrt{\rho z_2}$, $Y = 0$ Z nonzero and we recover the symmetrization of the Euler equations in entropy variables.

$a = 0 \quad b = 1$

So $aX^2 - bXY = \frac{XZ^2}{Y}$ implies $-XY = \frac{XZ^2}{Y}$ or $-Y^2 = Z^2$ which is impossible. So, as we knew, in this extreme case the matrix cannot be symmetrized.

This approach has several advantages compared with the classical preconditioning algorithm (4). The preconditioning matrix $P$ depends on a parameter $\beta$ that usually depends on a Mach number (local or reference). Hence, the condition number of the preconditioned system does not depend on the speed of sound, $c$, for low speed flow but it does depend on $\beta$ and so the Mach number. Thus, we have reduced the condition number from $\frac{1}{M}$ to one that is independent of the Mach number but depends on $\frac{v}{u}$ . This is less than optimal when $u$ and $v$ are considerably different. However, in the pressure relaxation approach the matrix $A$ has eigenvalues that depend only on $u$ (in Cartesian coordinates) and $B$ depends only on $v$. This will carry over to the artificial viscosity/upwinding and so behaves similar to a CUSP scheme.

A disadvantage of the method is that one requires the storage of one new variable, $\pi$. This of course, is still much less than the standard relaxation approach but is still is 20% for three dimensional flow. If one is also storing several turbulence variables then the percentage increase is less, i.e. for a two equation turbulence model in 3D we go from 7 to 8 variables, an increase of 15% in the storage.

## 7. SUBCHARACTERISTIC CONDITION

To further study the properties of the relaxtion we consider the growth of the equations in time as described by Natalini [21] for a scalar equation. If there is growth then the solutions to the regularized equations cannot approach the solution of the original system as $\varepsilon \to 0$. We linearize (21) about a zero flow with $\rho_0 = 1$. We further only consider the one dimensional case. We then have

$$\begin{aligned} p_t + \bar{z}_2 u_x &= 0 \\ u_t + a p_x + b\pi_x &= 0 \\ S_t &= 0 \\ \pi_t &= \frac{1}{\varepsilon}(p - \pi) \end{aligned} \tag{25}$$

where $\bar{z}_2 = p_0 + (\gamma - 1)(a p_0 + b\pi_0)$ is constant. Now $S$ decouples from the system. We Fourier transform the equations in space and assume

$$\begin{aligned} p(x,t) &= A(t) e^{i\xi \frac{x}{\varepsilon}} \\ \pi(x,t) &= B(t) e^{i\xi \frac{x}{\varepsilon}} \\ u(x,t) &= C(t) e^{i\xi \frac{x}{\varepsilon}} \end{aligned} \tag{26}$$

Then

$$\frac{dW}{dt} = \frac{d}{dt}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i\xi\bar{z}_2 \\ 1 & -1 & 0 \\ -i\xi a & -i\xi b & 0 \end{pmatrix} W = QW$$

The characteristic polynomial becomes

$$\lambda^2(\lambda+1) + (a\lambda+1)\bar{z}_2\xi^2 = 0 \tag{27}$$

This is a cubic equation with real positive coefficients. If $a=1$ then the roots are $\lambda_0 = -1$ and $\lambda_\pm = \pm i\sqrt{\bar{z}_2}\xi$. So the real part of the eigenvalues are negative for $\lambda_0$ and are zero for the acoustic eigenvalues $\lambda_\pm$. We calculate the general formula for the solution of a cubic equation

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0$$

$$a_2 = 1 \qquad a_1 = aa_0$$

$a_0 = \bar{z}_2\xi^2$ is an arbitrary real positive number. We wish to find the solution as a function of $a$. Define

$$Q = \frac{3aa_0 - 1}{9}$$
$$R = \frac{9aa_0 - 27a_0 - 2}{54} = -\left(\frac{3-a}{6}a_0 + \frac{1}{27}\right)$$

Then

$$D = Q^3 + R^2 = \frac{a_0}{27}\left(a^3 a_0^2 + \left(\frac{3}{4}(3-a)^2 - a^2\right)a_0 + 1\right)$$

and

$$\sqrt{D} = \frac{(a_0 a)^{3/2}}{\sqrt{27}}\sqrt{1 + \frac{\frac{3}{4}(3-a)^2 - a^2)}{a_0 a^3} + \frac{1}{a^3 a_0^2}}$$

To simplify the arithmetic we assume that $a_0$ is large (i.e., $\xi \gg 1$) and expand the square root. Then

$$\sqrt{D} \sim \left(\frac{a_0 a}{3}\right)^{3/2}\left(1 + \frac{1}{2}\frac{\frac{3}{4}(3-a)^2 - a^2}{a_0 a^3}\right)$$

It then follows that

$$\sqrt{D}+R=\left(\frac{a_0 a}{3}\right)^{3/2}-\frac{3-a}{6}a_0+O(1)\sim\left(\frac{a_0 a}{3}\right)^{3/2}\left(1-\frac{3-a}{6}\left(\frac{3}{a_0 a}\right)^{3/2}a_0\right)$$

$$\sqrt{D}-R=\left(\frac{a_0 a}{3}\right)^{3/2}+\frac{3-a}{6}a_0+O(1)\sim\left(\frac{a_0 a}{3}\right)^{3/2}\left(1+\frac{3-a}{6}\left(\frac{3}{a_0 a}\right)^{3/2}a_0\right)$$

Therefore

$$S\sim(\sqrt{D}+R)^{1/3}\sim\left(\frac{a_0 a}{3}\right)^{3/2}\left(1-\frac{3-a}{18}\left(\frac{3}{a}\right)^{3/2}a_0^{-1/2}\right)$$

$$T\sim(\sqrt{D}-R)^{1/3}\sim-\left(\frac{a_0 a}{3}\right)^{3/2}\left(1+\frac{3-a}{18}\left(\frac{3}{a}\right)^{3/2}a_0^{-1/2}\right)$$

It follows that

$$S+T=-\frac{3-a}{3a}$$

Finally, from Cartan's formula for the cubic equation we get the real part of the acoustic eigenvalues

$$Re(\lambda_\pm)=-\frac{1}{3}-\frac{S+T}{2}\sim\frac{1-a}{2a}$$

So, $Re(\lambda_\pm)=0$ for $a=1$ but for $Re(\lambda_\pm)<0$ we require that $a>1$.

In conclusion even though the homogenous equation allows for slower eigenvalues than $u+c$ the subcharacteristic condition requires that $a>1$ i.e., that the speeds are all greater than the original Euler equations. This was to be expected by a physical interpretation of the CFL condition. However, now it results from the subcharacteristic condition rather than a stability argument.

The remaining advantage of the regularized equation is that by indeed increasing the eigenvalues of the system we can make the real part negative. i.e., if we are going to a steady state then the solution decays for large time while the original Euler equation only allows for traveling waves.

## 8. HELMHOLTZ EQUATION

Though the Helmholtz equation is elliptic it is considerably harder to solve than the Laplace equation [20]. Because of the wave propagation, accuracy requires that the number of points per wave length increases

as the frequency increases [3]. The equation may involve complex quantities and the matrix to be inverted is nonsymmetric, indefinite and ill-conditioned. The basic solution is oscillatory which makes it difficult to use multigrid methods since the error is not well represented on coarse grids. The most difficult grids are not the very coarse grids but rather the intermediate levels. The methods must be able to handle variable and even discontinuous coefficients. In conclusion, the main difficulty in solving indefinite problems is the possible lack of robustness.

In particular we consider preconditioners, for Krylov space methods, based on properties of the differential equation itself. Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \tag{28}$$

If we consider acoustic scattering about a body then we append to the differential equation boundary conditions. Along the body we specify either Dirichlet or Neumann conditions depending on whether the scatterer is hard or soft. At infinity we impose the Sommerfeld radiation boundary condition. In a finite difference or finite element setting the Sommerfeld condition is replaced by an absorbing boundary condition at a finite outer surface e.g., [4]. Hence, the solution of the Helmholtz is complex.

To improve the convergence rate of a Krylov method we introduce a preconditioning. One possibility is to base the preconditioning on the structure of the resultant finite discretization matrix. Instead we consider a preconditioning based on continuous approximations to the Helmholtz equation. Only at the end do we replace this by a discretized version. The simplest approximation to (28) is to set $k = 0$ both in the differential equation and in the boundary conditions. This is not particularly good when the Sommerfeld radiation condition is replaced by a PML. Hence, a slightly better preconditioner is to replace $k$ in (28) by $k_0$ where $k_0$ is below the first eigenvalue and so the operator is still positive definite [2, 14]. A better approximation is to replace the real valued function $k^2$ by a complex valued function $k_0^2(1 + i\alpha)$ [6, 7]. When $k_0$ is sufficiently small or else $\alpha$ is $O(1)$ then one can solve the preconditioned equations efficiently by multigrid. In fact one iteration of a FMG algorithm is usually best [8].

## 9. SCATTERING

We consider the three dimensional scattering around a body. Define

$$B(\theta, \varphi)v = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} \tag{29}$$

The Helmholtz equation in spherical coordinates is given by

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+B(\theta,\varphi)u+r^2k^2u=0 \tag{30}$$

$$B_1u=\frac{\partial u}{\partial r}-\left(ik-\frac{1}{r}\right)u=O\left(\frac{1}{r^3}\right)\quad \text{as}\quad r\to\infty$$

We have expressed the Sommerfeld radiation condition in terms of the first BGT approximation [4]. Defining a new variable $s=kr$ then both the Helmholtz equation and the Sommerfeld radiation condition are independent of $k$ in the $s$ coordinate, i.e., $k$ directly affects the solution only through the $r$ variable. Another way that $k$ can enter the solution is through the boundary conditions. Define a new variable given by the fundamental solution

$$u=v\frac{e^{ikr}}{kr} \tag{31}$$

This can be considered a preconditioning based on spherical waves rather than plane waves. For general configurations one can replace the spherical distance $r$ by an ellipsoidal or spheroidal coordinate. Consider a plane wave $u^{\text{inc}}(x)=e^{ikx\cos\theta}$ impacting on a sphere (either hard or soft) of radius $a$. The solution is

$$u=-\sum_{n=0}^{\infty}i^n(2n+1)a_nh_n^{(1)}(kr)P_n(\cos\theta) \tag{32}$$

for a soft sphere $a_n=\dfrac{j_n(ka)}{h_n^{(1)}(ka)}$ for a hard sphere $a_n=\dfrac{j_n'(ka)}{h_n^{(1)\prime}(ka)}$

The spherical Hankel functions $j_n(kr)$ and $h_n(kr)$ both have the form

$$\frac{e^{ikr}}{kr}\sum_{m=0}^{n}\frac{b_{nm}}{(kr)^m}$$

$v$ satisfies the equation (see (29))

$$\triangle v+2\left(ik-\frac{1}{r}\right)\frac{\partial v}{\partial r}=\frac{\partial^2 v}{\partial r^2}+2ik\frac{\partial v}{\partial r}+\frac{1}{r^2}B(\theta,\varphi)v=0 \tag{33}$$

$$\widetilde{B}_1v=\frac{\partial v}{\partial r}=0\quad \text{at outer boundary}$$

For the function $v$ we get the functional form

$$v=\frac{1}{k}\sum_{n=0}^{\infty}\frac{b_n}{(kr)^n}F_n(\theta,\varphi)\ =\frac{b_0}{k}F_0(\theta,\varphi)+\ldots\qquad F_n \text{ independent of } k \tag{34}$$

i.e., $v$ has no oscillatory part that depends explicitly on $k$ just terms that decay in $r$. The number of significant terms depends linearly on the wavenumber $k$. Hence, we hope that a numerical approximation to this problem for $v$ should require a grid that only depends on the points per wavelength. For other bodies there might be some dependence greater than linear in $k$ in the $\theta$ and $\varphi$ directions in the near field but not the far field. Outside a sphere surrounding the scatterer one can prove

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} h_n^{(1)}(kr) Y_n^m(\hat{x})$$

converges absolutely and uniformly on compact subsets.

The approximation to the equation is Hermitian when finite elements are used. For a Dirichlet condition at the scatterer the total problem for $v$ is Hermitian symmetric. However, the second order absrobing boundary condition leads to a non-symmetric problem.

The definition of the far field pattern is

$$u(x) = \frac{e^{ikr}}{r} \left( u_\infty \left( \frac{x}{r} \right) + O\left( \frac{1}{r} \right) \right) \qquad r \to \infty$$

Since $v = \frac{e^{ikr}}{kr}$ this translates to

$$kv(x) = u_\infty \left( \frac{x}{r} \right) + O\left( \frac{1}{kr} \right) \qquad r \to \infty$$

Using (34) we can improve this to

$$kv + kr \frac{\partial v}{\partial r} = u_\infty \left( \frac{x}{r} \right) + O\left( \frac{1}{k^2 r^2} \right)$$

$$kv + kr \frac{\partial v}{\partial r} + \frac{k}{2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v}{\partial r} \right) = kv + 2kr \frac{\partial v}{\partial r} + \frac{kr^2}{2} \frac{\partial^2 v}{\partial r^2} = u_\infty \left( \frac{x}{r} \right) + O\left( \frac{1}{k^3 r^3} \right)$$

### 9.1. Results for Three Dimensional Scattering

We consider scattering around a sphere. We assume that the solution is independent of $\phi$ and so only two independent variables appear even though the problem is three dimensional. We present the errors in Table I.

We have improvements of a factor of 20 when using the second order boundary condition. When the outer boundary is very close to the scatterer then there is not much improvement but both are very good.

**Table I.** Three Dimensional Scattering Around Sphere $R=1$, $k=3$

| $R$ max | $BC$ | #r | #$\theta$ | helm err($L^2$) | helm relerr | N err($L^2$) | N relerr |
|---|---|---|---|---|---|---|---|
| 1.1 | 1 | 3 | 60 | .023 | .054 | .0236 | .054 |
| 1.1 | 2 | 3 | 60 | .005 | .011 | .005 | .011 |
| 2 | 1 | 10 | 60 | .086 | .067 | .083 | .065 |
| 2 | 2 | 10 | 60 | .010 | .008 | .005 | .004 |
| 3 | 1 | 20 | 60 | .058 | .033 | .049 | .028 |
| 3 | 2 | 20 | 60 | .021 | .012 | .0026 | .0015 |
| 5 | 1 | 40 | 60 | .065 | .026 | .026 | .010 |
| 5 | 2 | 40 | 60 | .057 | .023 | .003 | .001 |

We consider the number of elements needed to obtain a given accuracy as a function of $k$ and $h$ for both the Helmholtz equation and the nonoscillatory version. We use the second order BGT boundary condition. We fix the desired accuracy at 10% relative accuracy measured in both the $L^2$ norm of the area of integration and the $L^2$ norm of the normal derivative on the surface. We also vary the position of the outer boundary. We see, in Table II, that approximations to (33) yield more accurate solutions than those to the Helmholtz equation but the pollution still exists. A rough calculation indicates that for the original Helmholtz equation linear finite elements requires about $N_\theta = 1.56k^{\frac{3}{2}}$ points while the new formulation requires about $N_\theta = 1.33k^{\frac{3}{2}}$ points. It is interesting that the new formulation requires fewer points in the $\theta$ direction even though the transformation is only in the $r$ direction. The original Helmholtz formulation requires that $N_r$ increase as $k^{\frac{3}{2}}$ while the new formulation requires $N_r$ to increase faster than linear in $k$ but less than $k^{\frac{3}{2}}$. Hence, we reduce the severity of the pollution in the $r$ direction. For fixed $k$ the Helmholtz formulation requires the number of points in $r$ to increase faster than linear as we move the artificial boundary further out. However, (31) requires only a linear growth in the number of points in the $r$ direction as a function of $r$. This is because the error does not grow in $r$ for the new formulation while it does grow for approximations to the Helmholtz equation. The number of points required in each direction is difficult to calculate since one can trade off points in $r$ against the number of points needed in $\theta$ to achieve a given accuracy.

## 10. CONCLUSIONS

Frequently choosing a preconditioner based on properties of the continuous equations, i.e., the "physics" of the problem leads to a more efficient and more accurate algorithm. In particular this is true of the local

**Table II.** Relative Error for Three Dimensional Scattering Around Sphere

| $k$ | $k^{\frac{3}{2}}$ | scheme | $Rmax$ | $\#r$ | $\#\ \theta$ | #tot | vol | $\frac{\partial u}{\partial n}$ | FPP |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 11 | $u$ | 1.1 | 3 | 18 | 76 | .012 | .053 | .060 |
| | | $u$ | 2 | 8 | 18 | 171 | .056 | .077 | .088 |
| | | $u$ | 3 | 18 | 18 | 361 | .092 | .086 | .080 |
| | | $u$ | 4 | 35 | 18 | 684 | .096 | .054 | .084 |
| | | $v$ | 1.1 | 3 | 16 | 68 | .011 | .050 | .048 |
| | | $v$ | 2 | 4 | 15 | 80 | .040 | .091 | .060 |
| | | $v$ | 3 | 8 | 15 | 144 | .048 | .091 | .061 |
| | | $v$ | 4 | 12 | 15 | 208 | .055 | .091 | .062 |
| 10 | 32 | $u$ | 1.1 | 3 | 50 | 204 | .031 | .069 | .032 |
| | | $u$ | 2 | 16 | 50 | 867 | .082 | .094 | .058 |
| | | $u$ | 3 | 48 | 50 | 2499 | .099 | .045 | .053 |
| | | $u$ | 4 | 100 | 50 | 5151 | .099 | .029 | .052 |
| | | $v$ | 1.1 | 3 | 40 | 164 | .090 | .077 | .048 |
| | | $v$ | 2 | 8 | 40 | 369 | .048 | .100 | .085 |
| | | $v$ | 3 | 16 | 40 | 697 | .061 | .095 | .084 |
| | | $v$ | 4 | 24 | 40 | 1025 | .070 | .094 | .084 |
| 20 | 89 | $u$ | 1.1 | 3 | 140 | 564 | .067 | .099 | .086 |
| | | $u$ | 2 | 40 | 140 | 5781 | .094 | .070 | .029 |
| | | $u$ | 3 | 140 | 140 | 19881 | .098 | .021 | .031 |
| | | $u$ | 4 | 290 | 140 | 41031 | .100 | .015 | .030 |
| | | $v$ | 1.1 | 5 | 115 | 596 | .067 | .097 | .089 |
| | | $v$ | 2 | 18 | 115 | 2204 | .060 | .094 | .093 |
| | | $v$ | 3 | 36 | 115 | 4292 | .076 | .091 | .090 |
| | | $v$ | 4 | 54 | 115 | 6380 | .085 | .091 | .091 |
| 40 | 256 | $u$ | 1.1 | 15 | 400 | 15708 | .100* | .067 | .065 |
| | | $u$ | 2 | 115 | 400 | 46516 | .096 | .032 | .020 |
| | | $u$ | 3 | 390 | 400 | 156791 | .099 | .012 | .018 |
| | | $u$ | 4 | 815 | 400 | 327216 | .100 | .009 | .017 |
| | | $v$ | 1.1 | 15 | 340 | 5456 | .104* | .075 | .064 |
| | | $v$ | 2 | 40 | 340 | 13981 | .078 | .096 | .093 |
| | | $v$ | 3 | 80 | 340 | 27621 | .093 | .094 | .094 |
| | | $v$ | 4 | 125 | 340 | 42966 | .100 | .088 | .087 |

$*$ adding nodal points doesn't improve accuracy.

low Mach preconditioner for the compressible steady state fluid dynamic equations. It is also true for improving the accuracy of approximations to acoustic scattering about bodies. The continuous preconditioning was less helpful for accelerating the rate of convergence for a time dependent problem when using a dual time approach. Finally, for relaxation methods we found that though we could symmetrize the system nevertheless the subcharacterstic condition implied that we could not accelerate the convergence rate to a steady state.

## REFERENCES


1. Allmaras, S. (1993). Analysis of a local matrix preconditioner for the 2-D Navier-Stokes equations. *AIAA Paper 93-3330*.
2. Bayliss, A, Goldstein, C.I., and Turkel, E. (1983). An iterative method for the Helmholtz equation. *J. Comput. Phys.* **49**, 443–457.
3. Bayliss, A., Goldstein, C., and Turkel, E. (1985). On accuracy conditions for the numerical computation of waves. *J. Comput. Phys.* **59**, 396–404.
4. Bayliss, A., Gunzburger, M., and Turkel, E. (1982). Boundary conditions for the numerical solution of elliptic equations in exterior regions. *SIAM J. Appl. Math.* **42**, 430–451.
5. Caughey, D. A., and Jameson, A. (1992). Fast preconditioned multigrid solution of the euler and Navier–Stokes equations for steady compressible flows. *AIAA Paper 2002-0963*.
6. Erlangga, Y. A., Vuik C., and Oosterlee, C. W. (2004). On a class of preconditioners for the Helmholtz equation. *Appl. Numer. Math.* **50**, 409–425.
7. Erlangga, Y. A., Oosterlee, C. W., and Vuik, C. (2006). A novel multigrid based preconditioner for heterogenous Helmholtz problems. *Scient. Comput.* **27**, 1471–1492.
8. Gozani, J., Nachshon, A., and Turkel, E. (1984). Conjugate gradient coupled with multigrid for an indefinite problem. *Adv. Comput. Meth. PDEs V*. 425–427.
9. Jameson, A., Schmidt, W., and Turkel, E. (1981). Numerical solutions of the Euler equations by a finite volume method using Runge–Kutta time-stepping schemes. *AIAA Paper* 81-1259.
10. Jameson, A. (1995). Analysis and design of numerical schemes for gas dynamics I: artificial diffusion, upwind biasing, limiters and their effect on accuracy and multigrid cobergence. *Inter. J. Comput. Fluid D.* **4**, 171–218.
11. Laird, A. L., and Giles, M. (2002). Preconditioned iterative solution of the 2D Helmholtz equation. *Report 02/12* Oxford Computer Lab, Oxford, UK.
12. Natalini, R. (1999). Recent mathematical results on hyperbolic relaxation problems. *Analysis of systems of conservation laws*, Chapman and HallCRC, Boca Raton, FL, 128–198.
13. Pierce, N. A., and Giles, M. B. (1997). Preconditioned multigrid methods for compressible flow codes on stretched meshes. *J. Comput. Phys.* **136**, 425–445.
14. Singer, I., and Turkel, E. (2004). A Perfectly matched layer for the Helmholtz equation in a semi-infinite strip. *J. Comput. Phys.* **201**, 439–465.
15. Turkel, E. (1987). Preconditioned methods for solving the incompressible and low speed compressible equations. *J. Comput. Phys.* **72**, 277–298.
16. Turkel, E. (1993). A review of preconditioning methods for fluid dynamics. *Appl. Numer. Math.* **12**, 257–284.
17. Turkel, E., Fiterman, A., and van Leer, B. (1994). Preconditioning and the limit of the compressible to the incompressible flow equations for finite difference schemes. In Caughey, D. A., and Hafez, M. M. (eds.), *Frontiers of Computational Fluid Dynamics 1994*, John Wiley and Sons, 215–234.
18. Turkel, E. (1997). Preconditioning-squared methods for multidimensional aerodynamics. *AIAA Paper 97-2025*.
19. Turkel, E. (1999). Preconditioning techniques in computational fluid dynamics. *Ann. Rev. Fluid Mech.* **31**, 385–416.
20. Turkel, E. (2001). Numerical difficulties solving time harmonic equations. In Brandt, A., Bernholc, J., and Binder, K. (eds.), *Multiscale Computational Methods in Chemistry and Physics*, IOS Press, Ohmsha, pp. 319–337.
21. Turkel, E. (2002). Robust preconditioning for steady and unsteady viscous flows, 33rd AIAA fluid dynamics conference, Reno NV, *AIAA paper 02-0962*.

22. Turkel, E., Farhat C., and Hetmaniuk, U. (2004). Improved accuracy for the Helmholtz equation in unbounded domains. *Inter. J. Numer. Meth. Eng.* **59**, 1963–1988.
23. Turkel, E., and Vatsa, V. N. (2005). Local preconditioners for steady state and dual time-stepping. *Math. Model. Numer. Anal.* ESAIM: M2AN **39**(3), 515–536.
24. van Leer, B., Lee, W. T., and Roe, P. L. (1991). Characteristic time-stepping or local preconditioning of the Euler equations. *AIAA Paper 91-1552*.
25. Vatsa, V. N., and Wedan, B. W. (1990). Development of a multigrid code for 3-d Navier-stokes equations and its application to a grid-refinement study. *Comput. Fluids* **18**, 391–403
26. Venkata Reddy, P., and Raghurama Rao, S. V. (2005). A new Euler solver with pressure relaxation. Fluid Mechanics Report, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India.