Journal of Scientific Computing, Vol. 27, Nos. 1–3, June 2006 (© 2006)
DOI: 10.1007/s10915-005-9044-x

Dispersive and Dissipative Properties of Discontinuous Galerkin Finite Element Methods for the Second-Order Wave Equation

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Received October 15, 2004; accepted (in revised form) August 9, 2005; Published online January 18, 2006

Discontinuous Galerkin finite element methods (DGFEM) offer certain advantages over standard continuous finite element methods when applied to the spatial discretisation of the acoustic wave equation. For instance, the mass matrix has a block diagonal structure which, used in conjunction with an explicit time stepping scheme, gives an extremely economical scheme for time domain simulation. This feature is ubiquitous and extends to other time-dependent wave problems such as Maxwell's equations. An important consideration in computational wave propagation is the dispersive and dissipative properties of the discretisation scheme in comparison with those of the original system. We investigate these properties for two popular DGFEM schemes: the interior penalty discontinuous Galerkin finite element method applied to the second-order wave equation and a more general family of schemes applied to the corresponding first order system. We show how the analysis of the multi-dimensional case may be reduced to consideration of one-dimensional problems. We derive the dispersion error for various schemes and conjecture on the generalisation to higher order approximation in space.

KEY WORDS: Discontinuous Galerkin finite element method; numerical dispersion; numerical dissipation; wave propagation.

1. INTRODUCTION

Discontinuous Galerkin finite element methods (DGFEM) were originally devised to solve scalar first-order hyperbolic problems [16] but were later generalised to apply to first-order hyperbolic systems [7], elliptic problems

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[3–5, 11, 20] and second-order hyperbolic problems such as Maxwell's equations and the acoustic wave equation [17]. The dispersive and dissipative behaviour of a discretisation scheme plays an important role in the simulation of wave propagation phenomena [18]. Typically, one finds that unless the initial mesh is sufficiently fine numerical dispersion may well produce an approximation that looks plausible, but in fact is not even qualitatively correct due to phase lag or lead arising from the numerical scheme propagating waves at an incorrect speed. For this reason, one often finds various rules of thumb guiding the choice of initial mesh-size. Despite the importance of such criteria and the insights provided by the supporting analysis, relatively little is known concerning the dispersive and dissipative behaviour of general DGFEM schemes, particularly in the case of second-order problems (and the associated first-order systems). A fairly complete analysis of the dispersion and dissipative behaviour of a class of DGFEM methods for a scalar linear conservation law was given in [2] in terms of both the order of the method and the mesh-size and, as a special case, provides a proof of conjectures of Hu and Atkins [13, 14].

In this paper we investigate the dispersive behaviour of a variety of DGFEM methods for the acoustic wave equation based on either the interior penalty discontinuous Galerkin method (IP-DG) or a general DGFEM method based on writing the wave equation as a first-order system and using the fluxes from [4]. In order to focus attention on the DGFEM discretisation we restrict ourselves to semi-discretisation in space. A dispersion analysis of the continuous problem shows that the dispersion relation for the wave equation in two space dimensions separates into independent one-dimensional dispersion relations corresponding to each coordinate direction. Many standard finite element and finite difference methods on Cartesian grids inherit this property which facilitates the dispersion analysis of the discrete problem. We restrict our attention to a uniform grid of squares since we are interested in wave propagation. Away from scatterers it is usual to employ a uniform grid to propagate the wave as accurately as possible. Of course a dispersion analysis can be carried out on more general elements (for example triangles and general quadrilaterals) provided the mesh has suitable translational invariance. However, the computations are more complex and are not considered here.

For the wave equation [10], the standard finite element method using tensor product elements and a tensor product grid has the following features:

1. The multi-dimensional discrete dispersion relation can be written as a sum of one-dimensional discrete dispersion relations in the two coordinate directions.

2. The one-dimensional dispersion relations can be obtained by applying the same finite element method to a one-dimensional wave equation.

In this paper we investigate to what extent such a decomposition is possible for certain DGFEM schemes. We find

1. For all the DG methods considered here we can perform the decomposition mentioned in item (1) above.
2. For the IP-DG method the relevant one-dimensional discrete dispersion relations are obtained by applying the same interior penalty method to the one-dimensional wave equation. However, for the general DGFEM based on discretising a first-order system, the relevant one-dimensional problem is not obtained by directly applying the same DG scheme to a one-dimensional wave equation and analysing the dispersion relation in this simpler case.

The last point above means that a standard one-dimensional dispersion analysis does not give the one-dimensional dispersion relation relevant to building the dispersion relation for a general multi-dimensional DGFEM system and hence cannot be used to account for the dispersive behaviour of multi-dimensional systems. We regard this as a significant result in its own right.

For both interior penalty DGFEM and general schemes based on first-order systems, we provide a dispersion analysis for some low order schemes and conjecture on how the results extend to arbitrary order elements. In particular we explicitly compute the relevant one-dimensional dispersion relations which we show can then be combined to obtain the multi-dimensional discrete dispersion relation. This multi-dimensional dispersion relation shows grid orientation effects as is to be expected, but we have not explicitly investigated that aspect here since it is well understood.

In a previous paper, Ainsworth [2] investigated the dispersion relation of an upwind type DG method applied to the transport equation (one way wave equation). We hoped that this analysis would be applicable to the study in this paper. However, it is not relevant to the IP-DG scheme since the IP-DG scheme is not based on upwinded fluxes and, and as we shall see, gives a different dispersion behaviour. For the general DGFEM based on the first-order system, the relevant one-dimensional problems are generally non-standard and so again Ainsworth's theory is not generally applicable. However, in the case of fully "centred fluxes" that theory is applicable and can be used to obtain a complete description of the one-dimensional dispersion relation, and hence of the multi-dimensional problem, as we shall show.

The plan of the paper is as follows. In Sec. 2, we study an interior penalty discontinuous Galerkin (IP-DG) scheme restricting our attention to tensor product basis functions and show that the dispersion relation does, in general, decompose into one-dimensional relations. We then analyse the method for piecewise polynomials of degree one through three. We also derive explicit bounds for the IP-DG stability parameter in order to avoid pollution by spurious modes. In Sec. 3, we rewrite the second-order wave equation as a first-order system and discretise in space using the general class of DGFEM methods described in [4]. We show that this general scheme includes the standard "upwind" and "centred" DGFEM schemes that have appeared in the literature. We then present a decomposition of the dispersion relation into special one-dimensional problems and comment on why this decomposition is non-standard. In Sec. 3.5, we derive dispersion relations for the scheme in the case of low fixed polynomial degree and in the limit of small mesh size. These results show that there is considerable scope for modifying the performance of the general DGFEM scheme by a suitable choice of parameters. In Sec. 3.4, we consider the special case of fully centred fluxes. In this case a complete description of the dispersion relation for all orders is possible using the results of Ainsworth [2]. In Sec. 4 some indicative numerical examples are presented and we summarise the conclusions in Sec. 5.

2. INTERIOR PENALTY DISCONTINUOUS GALERKIN

In this section we analyse the dispersion relation for a semi-discrete symmetric interior penalty discontinuous Galerkin (IP-DG) scheme (this is a classical method for elliptic problems [20]). The method differs from the IP-DG scheme of Riviere and Wheeler in [17] in that their scheme is based on a non-symmetric formulation and has a different penalty term. A possible advantage of the scheme we propose is that it is non-dissipative (due to the symmetric formulation).

In preparation for the discrete dispersion analysis, we first briefly elaborate on how the dispersion relation for the wave equation may be derived in a way that will extend to the discrete problem and allow us to reduce the analysis to the consideration of one dimensional problems. Consider the wave equation

$$\frac{1}{c^2}u_{tt} = \Delta u \text{ in } \mathbb{R}^2, \tag{1}$$

where $c > 0$ is the prescribed wave speed. We seek a separable solution in the form

$$u(x_1, x_2, t) = u_1(x_1)u_2(x_2)\exp(-\mathrm{i}\omega t),$$

where ω is the temporal frequency. Substituting into the wave equation and using the separation of variables hypothesis we see that u_1 and u_2 satisfy

$$-k^2 = \frac{u_1''}{u_1} + \frac{u_2''}{u_2},$$

where $k = \omega/c$ is the wave number. We choose wave numbers k_1 and k_2 such that $k^2 = k_1^2 + k_2^2$, and

$$\frac{u_j''}{u_j} = -k_j^2.$$

Thus the original multi-dimensional problem is reduced to a pair of canonical one-dimensional problems. If we now seek a wave solution of this problem of the form $u_j(x_j) = \exp(\mathrm{i}\kappa_j x_j)$ we see that $\kappa_j = k_j$ (or $\kappa_j = -k_j$). This type of argument will be extended to the analysis of the numerical scheme where we compute a numerical wave number $k_{j,h} \approx k_j$. The difference between the numerical wave number and the true wave number quantifies the phase error for the method.

To perform a dispersion analysis of the numerical scheme, we assume that a uniform mesh τ_h of square elements of side h are used to cover the entire plane. The unit outward normal on the boundary of an element $K \in \tau_h$ is denoted by $\boldsymbol{n}_K$. A neighbouring element K' meeting K at an edge e will have unit outward normal $\boldsymbol{n}_{K'} = -\boldsymbol{n}_K$ on e. The edges of the element K are denoted by N, E, W and S in the obvious way (see Fig. 1).

It will be necessary to consider jumps and averages of the discontinuous functions across element interfaces in the mesh and we adopt the fairly standard notation of [4]. If u and $\boldsymbol{v}$ are respectively piecewise smooth scalar and vector functions on the mesh (i.e. smooth on each element) then, on an edge e between elements K and K', let

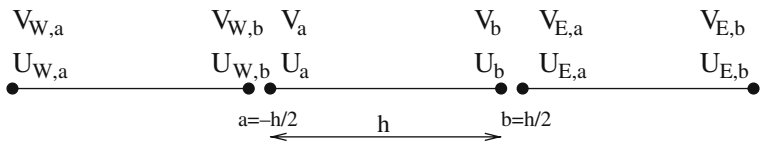

Fig. 1. Notation for the $p = 1$ dispersion analysis for systems. The degrees of freedom of segment number e are denoted by (U_a, V_a) at the left end and by (U_b, V_b) at the right end. On adjacent elements the relevant degrees of freedom are $(U_{W,b}, V_{W,b})$ and $(U_{E,a}, V_{E,a})$.

$$\{\{u\}\} = \frac{1}{2}(u|_K + u|_{K'}), \qquad \{\{\boldsymbol{v}\}\} = \frac{1}{2}(\boldsymbol{v}|_K + \boldsymbol{v}|_{K'}) \tag{2}$$

$$[\![u]\!] = (u|_K \boldsymbol{n}_K + u|_{K'} \boldsymbol{n}_{K'}), \qquad [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_K \cdot \boldsymbol{n}_K + \boldsymbol{v}|_{K'} \cdot \boldsymbol{n}_{K'}. \tag{3}$$

Let $\mathbb{Q}_p$ denote the set of polynomials of degree at most p in x and y, and $\mathbb{P}_p$ denote the set of polynomials of degree at most p in a single variable (so $\mathbb{Q}_p = \mathbb{P}_p \times \mathbb{P}_p$). The symmetric IP-DG scheme for the wave equation gives rise to the following semi-discrete variational problem of finding $u(\boldsymbol{x}, t)$ such that for each time $t \geqslant 0\, u(., t)|_K \in \mathbb{Q}_p$ for each element K in the mesh together with

$$\sum_{K\in\tau_h} \frac{1}{c^2}(u_{tt}, \phi)_K + \sum_{K\in\tau_h} (\nabla u, \nabla\phi)_K$$
$$- \sum_{e\in\tau_h} [\langle \{\{\nabla u\}\}, [\![\phi]\!]\rangle_e + \langle \{\{\nabla\phi\}\}, [\![u]\!]\rangle_e] + \sum_{e\in\tau_h} \alpha \langle [\![u]\!], [\![\phi]\!]\rangle_e = 0$$

for all piecewise smooth test functions ϕ with $\phi|_K \in \mathbb{Q}_p$. Here

$$(u, v)_K = \int_K uv\, dA \quad \text{and} \quad \langle u, v\rangle_e = \int_e uv\, ds.$$

We confine attention to tensor product meshes and it will be convenient to define new jump and average operators associated with the four edges of K using the $NEWS$ labelling (see Fig. 1). In particular, on the Eastern (E) edge

$$\{\{u_1\}\}_E = \frac{1}{2} \lim_{\epsilon\to 0,\, \epsilon>0} u_1(h/2-\epsilon) + \frac{1}{2} \lim_{\epsilon\to 0,\, \epsilon>0} u_1(h/2+\epsilon), \tag{4}$$

$$[\![u_1]\!]_E = \lim_{\epsilon\to 0,\, \epsilon>0} u_1(h/2-\epsilon) - \lim_{\epsilon\to 0,\, \epsilon>0} u_1(h/2+\epsilon). \tag{5}$$

and on the Western (W) edge

$$\{\{u_1\}\}_W = \frac{1}{2} \lim_{\epsilon\to 0,\, \epsilon>0} u_1(-h/2-\epsilon) + \frac{1}{2} \lim_{\epsilon\to 0,\, \epsilon>0} u_1(-h/2+\epsilon), \tag{6}$$

$$[\![u_1]\!]_W = \lim_{\epsilon\to 0,\, \epsilon>0} u_1(-h/2-\epsilon) - \lim_{\epsilon\to 0,\, \epsilon>0} u_1(-h/2+\epsilon). \tag{7}$$

Note that the jump is now defined as left limit minus right limit rather than in terms of normals as in (3). The operators on the North and South edges are defined similarly (we only give the jump terms since the averages $\{\{.\}\}_N$ and $\{\{.\}\}_S$ are obvious):

$$[\![u_2]\!]_N = \lim_{\epsilon\to 0,\,\epsilon>0} u_2(h/2-\epsilon) - \lim_{\epsilon\to 0,\,\epsilon>0} u_2(h/2+\epsilon), \tag{8}$$

$$[\![u_2]\!]_S = \lim_{\epsilon\to 0,\,\epsilon>0} u_2(-h/2-\epsilon) - \lim_{\epsilon\to 0,\,\epsilon>0} u_2(-h/2+\epsilon). \tag{9}$$

The analysis will be reduced to the problem on one square centred at the origin so we also define $\phi_{1,E}=\phi_1(h/2)$, $\phi_{1,W}=\phi_1(-h/2)$ and similarly for $\phi_{2,N}$ and $\phi_{2,S}$. In addition let

$$(u, v) = \int_{-h/2}^{h/2} uv\, ds.$$

We seek a discrete separable solution of the form

$$u = u_1(x)u_2(y)\exp(-\mathrm{i}\omega t), \tag{10}$$

where $u_j \in \mathbb{P}_p$, $j=1,2$. Choosing a separable test function $\phi=\phi_1(x)\phi_2(y)$, $\phi_j \in \mathbb{P}_p$, $j=1,2$ supported on the element K centred at the origin in the variational statement and simplifying, we obtain

$$\begin{aligned}
&-\frac{\omega^2}{c^2}(u_1,\phi_1)(u_2,\phi_2) + (u_1',\phi_1')(u_2,\phi_2) + (u_1,\phi_1)(u_2',\phi_2') \\
&\quad -\left(\{\{u_1'\}\}_E\phi_{1,E} + (1/2)\phi_{1,E}'[\![u_1]\!]_E\right)(u_2,\phi_2) \\
&\quad +\left(\{\{u_1'\}\}_W\phi_{1,W} - (1/2)\phi_{1,W}'[\![u_1]\!]_W\right)(u_2,\phi_2) \\
&\quad -\left(\{\{u_2'\}\}_N\phi_{2,N} + (1/2)\phi_{2,N}'[\![u_2]\!]_N\right)(u_1,\phi_1) \\
&\quad +\left(\{\{u_2'\}\}_S\phi_{2,S} - (1/2)\phi_{2,S}'[\![u_2]\!]_W\right)(u_1,\phi_1) \\
&\quad +\alpha\left(\left([\![u_1]\!]_E\phi_{1,E} - [\![u_1]\!]_W\phi_{1,W}\right)(u_2,\phi_2)\right. \\
&\quad \left.+\left([\![u_2]\!]_N\phi_{2,N} - [\![u_2]\!]_S\phi_{2,S}\right)(u_1,\phi_1)\right) = 0.
\end{aligned}$$

Note that in deriving this equation we have made critical use of the separability assumption that implies, for example, that u_2 does not jump across vertical edges since it depends only on the y-variable whose value may only jump across horizontal element edges.

Dividing through by $(u_1,\phi_1)(u_2,\phi_2)$ and using the fact that the test functions ϕ_1 and ϕ_2 may be varied independently, we see that u_1 satisfies a one-dimensional equation with wave number k_1 given by

$$\begin{aligned}
&-k_1^2(u_1,\phi_1) + (u_1',\phi_1') - \{\{u_1'\}\}_E\phi_{1,E} - (1/2)\phi_{1,E}'[\![u_1]\!]_E \\
&\quad +\{\{u_1'\}\}_W\phi_{1,W} - (1/2)\phi_{1,W}'[\![u_1]\!]_W \\
&\quad +\alpha\left([\![u_1]\!]_E\phi_{1,E} - [\![u_1]\!]_W\phi_{1,W}\right) = 0,
\end{aligned} \tag{11}$$

with a similar equation for u_2 in terms of the wave number k_2. These wave numbers satisfy the dispersion relation

$$k_1^2 + k_2^2 = k^2.$$

We find it convenient to return to a bilinear form involving the one-dimensional Helmholtz equation by integrating by parts using the fact that

$$(u_1', \phi_1') = u_{1,E}'\phi_{1,W} - u_{1,W}'\phi_{1,W} - (u_1'', \phi_1)$$

to obtain an alternative form of (11)

$$\begin{aligned}(u_1'' + k_1^2 u_1, \phi_1) + \frac{1}{2}\left(\phi_{1,E}'[\![u_1]\!]_E + \phi_{1,W}[\![u_1]\!]_W\right) \\ -\phi_{1,E}\left(\frac{1}{2}[\![u_1']\!]_E + \alpha[\![u_1]\!]_E\right) + \phi_{1,W}\left(\frac{1}{2}[\![u_1']\!]_W - \alpha[\![u_1]\!]_W\right) = 0.\end{aligned} \tag{12}$$

Now we scale to the interval $I = [-1, 1]$ using the transformation $\hat{x} = 2x/h$ and write $U(s) = u_1(x)$ and $\phi(s) = \phi_1(x)$. Defining in addition the dimensionless wave number $\kappa = k_1 h/2$ and dimensionless penalty parameter $\gamma = \alpha h/2$ we obtain

$$\begin{aligned}(U'' + \kappa^2 U, \phi)_I + \frac{1}{2}\left(\phi_E'[\![U]\!]_E + \phi_W'[\![U]\!]_W\right) \\ -\phi_E\left(\frac{1}{2}[\![U']\!]_E + \gamma[\![U]\!]_E\right) + \phi_W\left(\frac{1}{2}[\![U]\!]_W - \gamma[\![U]\!]_W\right) = 0,\end{aligned} \tag{13}$$

where $\phi_E = \phi(1)$, $\phi_W = \phi(-1)$ and $(u, v)_I = \int_I uv\, ds$.

We can now use (13) to perform the dispersion analysis. We seek Bloch wave solutions and hence assume that there is a factor λ for which

$$\begin{aligned} U_E^+ &= \lambda U_W^+, \qquad U_W^- = \lambda^{-1} U_E^-, \\ U_E'^+ &= \lambda U_W'^+, \qquad U_W'^- = \lambda^{-1} U_E'^-. \end{aligned}$$

Using these definitions in (13) we obtain the eigenvalue problem of computing λ and a finite element function $U \in \mathbb{P}_p$ such that, for all finite element functions $\phi \in \mathbb{P}_p$ on I

$$\begin{aligned}(U'' + \kappa^2 U, \phi)_I + \frac{1}{2}\left(\phi_E'(U_E - \lambda U_W) + \phi_W(\lambda^{-1}U_E - U_W\right) \\ -\phi_E\left(\frac{1}{2}(U_E' - \lambda U_W') + \gamma(U_E - \lambda U_W)\right) \\ -\phi_W\left(\frac{1}{2}(\lambda^{-1}U_E' - U_W') - \gamma(\lambda^{-1}U_E - U_W)\right) = 0.\end{aligned} \tag{14}$$

In this analysis we can write $\lambda = \exp(\mathrm{i}\theta)$ or, to facilitate comparison with the exact dispersion relation, in dimensional variables $\lambda = \exp(\mathrm{i}hk_{1,h})$. The problem takes the form of an eigenvalue problem where λ is the eigenvalue. Once we have computed λ we can obtain $k_{1,h}$ and comparing it to k_1 gives a measure of the dispersion error.

The following observation will be used throughout to simplify the treatment of problems of the type given in equation (14). Firstly, observe that non-trivial modes $U \in \mathbb{P}_p$ may be split into what may be dubbed as *propagating* and *trapped* modes. A trapped mode refers to a non-trivial solution that vanishes at the endpoints E and W of the element, and satisfies the equation over the subspace of test functions that all vanish at the endpoints. This accounts for $\dim \mathbb{P}_p - 2$ modes. The remaining modes correspond to modes that are non-zero at the endpoints, and for which $(U'' + \kappa^2 U, \phi)_I$ vanishes for all test functions satisfying homogeneous boundary conditions. In view of the fact that equation (14) then reduces to a quadratic algebraic equation for λ, we expect to (and do) find that there are two such modes giving $\dim \mathbb{P}_p$ modes in total. By forming tensor products of these modes on the reference cell as in equation (10), we obtain a full complement of solutions for the space $\mathbb{Q}_p$ in two dimensions on the reference cell. Furthermore, by extending the solution to the entire space as a Bloch wave and taking advantage of the completeness result of Oden and Keller [15], we obtain a full complement of solutions for the original problem.

2.1. Dispersion Error for Various Elements

In this section we shall apply the results of the previous section to compute explicit dispersion relations for various piecewise polynomials.

2.1.1. Piecewise Constant Elements

In principle, the IP-DG method can be applied with piecewise constant finite elements. In that case U is constant and $\phi = 1$ so $\phi' = U' = 0$ and the eigenvalue problem (14) simplifies greatly to

$$\left(-2\kappa^2 + \gamma(2 - \lambda - \lambda^{-1})\right) U = 0.$$

Thus if $\lambda = \exp(\mathrm{i}\theta)$ we see that the above equation implies $\cos\theta = (1 - \kappa^2/\gamma)$, or

$$\cos\theta - \cos 2\kappa = \left(2 - \frac{1}{\gamma}\right)\kappa^2 + O(\kappa^4).$$

For small κ we can expand this as a series and compare with the exact wave number k_1 to get

$$k_{1,h} - k_1 = -\frac{1}{2}\left(2 - \sqrt{\frac{2}{\gamma}}\right) k_1 + O(k_1^3 h^2)$$

where numerical wave number is given by $k_{1,h} = \theta/h$.

If the method is to be consistent, then we are obliged to choose $\gamma = 1/2$ and this corresponds to a penalty parameter of $\alpha = 1/h$. Although we would not advocate using piecewise constant approximation (e.g. it is only consistent with a careful choice of α), it is nevertheless illuminating to consider this case and, with the correct choice of α, the method has a second-order relative error in the wave number.

2.1.2. Piecewise Linear Elements

Using (14) with $U = U_W(1-s)/2 + U_E(1+s)/2$ and selecting $\phi = (1-s)/2$ gives the equation

$$\left(\frac{\kappa^2}{3} + \frac{\gamma}{\lambda} - \frac{1}{2\lambda}\right) U_E + \left(\frac{2\kappa^2}{3} + \frac{\lambda}{4} + \frac{1}{4\lambda} - \gamma\right) U_W = 0$$

and using $\phi = (1+s)/2$ gives

$$\left(\frac{2\kappa^2}{3} + \frac{\lambda}{4} + \frac{1}{4\lambda} - \gamma\right) U_E + \left(\frac{\kappa^2}{3} + \gamma\lambda - \frac{1}{2}\lambda\right) U_W = 0.$$

These equations have a non-trivial solution if and only if

$$\cos^2\theta + 4\left(\kappa^2 - \frac{2}{3}\gamma\kappa^2 - \gamma\right)\cos\theta - \left(1 + \frac{16}{3}\gamma\kappa^2 - \frac{4}{3}\kappa^4 - 4\gamma\right) = 0.$$

There are two roots to this equation. To understand their nature, consider the limiting case as $\kappa \to 0$ leading to the equation

$$\cos^2\theta - 4\gamma\cos\theta - (1 - 4\gamma) = 0$$

which has the roots $\cos\theta = 1$ or $\cos\theta = 4\gamma - 1$. In view of the fact that $\kappa = k_1 h/2$, the first of these corresponds to the physical mode. The second is a spurious mode, and in order for this to be damped out we need θ to be imaginary so we must demand $4\gamma - 1 > 1$ (recall γ is positive). Thus

we obtain the stability requirement that $\gamma > 1/2$ or, in terms of the original parameters, $\alpha > 1/h$. This type of stability condition is typical of IP-DG schemes but we have not seen quantitative bounds derived in this way before.

We now assume $\gamma > 1/2$, focus attention on the physical mode and consider the case where κ small (i.e. h small). The physical mode is then given by

$$\cos\theta_p - \cos 2\kappa = \frac{2}{3}\kappa^4 + \frac{4}{45}\frac{1+18\gamma}{1-2\gamma}\kappa^6 + O(\kappa^8)$$

where θ_p denotes the value of θ corresponding to the physical choice of the two roots. The first difference compared with the case of piecewise constant approximation, is that there is no choice of γ giving a higher order dispersion relation. Writing the dispersion error directly in terms of the wave number k_1 and the mesh size h, we find that

$$k_{1,h} - k_1 = -\frac{1}{24}k_1^3 h^2 + O(k_1^5 h^4).$$

Thus the piecewise linear scheme gives second-order accuracy in the relative error in the wave number which is typical for standard continuous piecewise linear finite elements or second-order finite difference methods. In particular, this is the best order possible and coincides with that of piecewise constants with a judicious choice of the penalty parameter, or that of piecewise linear approximation using a standard Galerkin formulation.

2.1.3. Quadratic Elements

The analysis proceeds along the same lines as for piecewise linear elements. In this case we write

$$U = U_W(1-s)/2 + U_E(1+s)/2 + U_C(1-s^2)$$

and choosing ϕ to be successively the three basis functions gives an eigenvalue problem for the linear system in the unknowns U_E, U_C and U_W. The characteristic equation for $\lambda = \exp(i\theta)$ turns out to be

$$\frac{4}{3}\left(1+\frac{\kappa^2}{5}\right)\cos^2\theta + \left(\frac{4}{3}\kappa^2 - 4 + \frac{8}{3}\gamma - \frac{32}{45}\kappa^4 + \frac{8}{45}\gamma\kappa^4 + \frac{32}{45}\kappa^2\gamma\right)\cos\theta + \left(\frac{8}{3} - \frac{8}{3}\gamma + \frac{8}{135}\kappa^6 - \frac{64}{15}\kappa^2 - \frac{8}{15}\gamma\kappa^4 + \frac{208}{45}\gamma\kappa^2\right) = 0.$$

Note that this equation remains quadratic in $\cos\theta$. As before, we distinguish two roots. In the limit as $\kappa \to 0$ the equation becomes

$$\frac{4}{3}(\cos\theta - 1)(\cos\theta - 2 + 2\gamma) = 0$$

so we again have the physical root $\cos\theta = 1$ as well as a root corresponding to a spurious mode $\cos\theta = 2 - 2\gamma$. For stability we need $2 - 2\gamma < -1$ and this implies $\gamma > 3/2$ or $\alpha > 3/h$.

Therefore, assuming that $\alpha > 3/h$, we concentrate on the physical mode and find that

$$\cos\theta_p - \cos 2\kappa = \frac{4}{45}\frac{\gamma - 3}{2\gamma - 1}\kappa^6 - \frac{2}{4725}\frac{368\gamma^2 - 2048\gamma + 2157}{(2\gamma - 1)^2}\kappa^8 + O(\kappa^{10}),$$

and

$$k_{1,h} - k_1 = -\frac{1}{720}\frac{\gamma - 3}{2\gamma - 1}k_1^5 h^4 + \frac{1}{604800}\frac{88\gamma^2 - 1068\gamma + 1737}{(2\gamma - 1)^2}k_1^7 h^6 + O(k_1^9 h^8).$$

As with piecewise constant approximation, we find that a particular choice of the penalty parameter, in this case $\gamma = 3$ or $\alpha = 6/h$ gives

$$k_{1,h} - k_1 = -\frac{1}{22400}k_1^7 h^6 + O(k_1^9 h^8)$$

and produces a higher order of accuracy in the dispersion behaviour for quadratic elements. The order of the method for a general parameter agrees with what one would expect for standard quadratic finite element approximation. However, choosing an optimal value of γ gives a higher order than can be achieved using the standard formulation.

2.1.4. Cubic Elements

The computation of the dispersion relation for cubic elements follows the same steps as for linear or quadratic elements. Expanding the field U in terms of cubic basis functions and then selecting ϕ in (14) successively to be each one of the basis functions, we obtain a 4×4 eigenvalue problem one again resulting in only a quadratic equation for $\cos\theta$, whereas one might have expected a quartic. The complexity of the computations

is such that there is little to be gained by presenting full details here. In the limit as $\kappa \to 0$ the quadratic simplifies to

$$(\cos\theta - 1)(\cos\theta + 3 - 4\gamma/3) = 0$$

and find that the spurious mode can be controlled by choosing $-3 + 4\gamma/3 > 1$ or $\gamma > 3$ so that $\alpha > 6/h$. The physical mode has the expansion

$$\cos\theta_p - \cos 2\kappa = \frac{2}{1575}\kappa^8 + \frac{4}{496125}\frac{127\gamma + 24}{3 - \gamma}\kappa^{10} + O(\kappa^{12})$$

and this time there is no optimal choice of the parameter γ that can improve the order of approximation in the dispersion relation. In fact

$$k_{1,h} - k_1 = -\frac{1}{201600}k_1^7 h^6 + O(k_1^9 h^8)$$

so we can see that the method is sixth order accurate for any α provided α is chosen to guarantee stability.

2.2. General Order Approximation

The analysis in the cases of elements of order $p = 0, 1, 2, 3$ can be generalised to arbitrary order of approximation but becomes rather complicated and we shall confine ourselves to stating conjectures on the possible generalisations.

We conjecture that the bound on the value of the parameter γ needed to eliminate the spurious mode is given by

$$\gamma > \frac{1}{4}p(p+1)$$

or

$$\alpha > \frac{1}{2h}p(p+1)$$

which agrees with the earlier results in the cases $p = 1, \ldots, 3$.

The expression for the discrete dispersion relation is given by

$$\cos\theta_p - \cos 2\kappa = \frac{2^{2p+1}}{2p+1}\left[\frac{p!}{(2p)!}\right]^2 \kappa^{2p+2} + O(\kappa^{2p+4})$$

for odd order approximation p, while for even order the corresponding expression is given by

$$\cos\theta_p - \cos 2\kappa = \frac{2^{2p+1}}{2p+1}\left[\frac{p!}{(2p)!}\right]^2 \frac{4\gamma - (p+1)(p+2)}{4\gamma - p(p-1)}\kappa^{2p+2} + O(\kappa^{2p+4}).$$

These expressions agree with those derived earlier in the special cases $p = 0, \ldots, 3$. Likewise, we obtain estimates for the accuracy of the discrete wave number

$$k - k_1 = -\frac{1}{2(2p+1)}\left[\frac{p!}{(2p)!}\right]^2 k_1^{2p+1} h^{2p} + O(k_1^{2p+3} h^{2p+2})$$

for odd order p, while for even order

$$k - k_1 = -\frac{1}{2(2p+1)}\left[\frac{p!}{(2p)!}\right]^2 \frac{4\gamma - (p+1)(p+2)}{4\gamma - p(p-1)} k_1^{2p+1} h^{2p} + O\left(k_1^{2p+3} h^{2p+2}\right)$$

which agree with the previous expressions in the cases $p = 1, 2, 3$. (The case $p = 0$ is different owing to the fact that it is inconsistent without the correct choice of penalty parameter). For an even order approximation, the optimal choice for the penalty parameter is

$$\gamma = \frac{1}{4}(p+1)(p+2)$$

or

$$\alpha = \frac{1}{2h}(p+1)(p+2)$$

which agrees with our earlier results in the special cases $p = 0$ and $p = 2$. We note that the value of the penalty parameter needed for stability grows as $O(p^2/h)$, in agreement with the growth estimates assumed in the*a priori* error analyses. In the situation considered here, we obtain a finer result including the value of the constant in the growth estimate.

A rather lengthy proof of the bounds on the value of the penalty parameter needed to eliminate the spurious mode is known, but a proof of the error estimates remains open. Nevertheless, we are reasonably confident that the conjectures are true based on comparison with results obtained using computer algebra up to $p = 100$.

3. GENERALISED DGFEM METHODS

An important general class of discontinuous Galerkin methods for elliptic problems is described in Arnold *et al.* [4] and consists of writing the equation as equivalent first-order system and then discretising using DGFEM. We now follow this approach to derive a DGFEM for the wave

equation (1) expressed as a first-order system as follows

$$\frac{1}{c}u_t = v_x + w_y, \tag{15}$$

$$\frac{1}{c}v_t = u_x, \tag{16}$$

$$\frac{1}{c}w_t = u_y. \tag{17}$$

The splitting is by no means unique. For instance, we might have left two time derivatives on u and not included time derivatives on v and w. The formulation described in the remainder of this section is a special case of the non-linear hyperbolic system solver proposed by Cockburn and Shu (see for example [6, 9]) and is often employed in the scientific computing literature. For the linear case it is used, for example, by the USEMe [12] family of codes.

The continuous dispersion analysis of the system follows the usual steps. However, we choose to spell out the ideas since it forms the foundation of the less standard discrete dispersion analysis of the DGFEM. As before, we seek a separable time harmonic solution of the form

$$u(x, y, t) = u_1(x)u_2(y)\exp(-\mathrm{i}\omega t)$$

with similar expressions for v and w. Using these expressions, the differential equation becomes

$$-\mathrm{i}\frac{\omega}{c}u_1u_2 = v_1'v_2 + w_1w_2', \tag{18}$$

$$-\mathrm{i}\frac{\omega}{c}v_1v_2 = u_1'u_2, \tag{19}$$

$$-\mathrm{i}\frac{\omega}{c}w_1w_2 = u_1u_2'. \tag{20}$$

The second two equations imply that there are constants γ_1 and γ_2 such that (using also the wave number k)

$$\frac{u_1'}{v_1} = -\mathrm{i}k\gamma_1 \quad \text{and} \quad \frac{v_2}{u_2} = \gamma_1, \tag{21}$$

$$\frac{u_2'}{w_2} = -\mathrm{i}k\gamma_2 \quad \text{and} \quad \frac{w_1}{u_1} = \gamma_2. \tag{22}$$

Using these expressions in equation (18) shows that

$$-\mathrm{i}k = \gamma_1\frac{v_1'}{u_1} + \gamma_2\frac{w_2'}{u_2}.$$

Again using a separation of variables argument shows that there are constants γ_3 and γ_4 such that

$$\frac{v_1'}{u_1} = -\mathrm{i}k\gamma_3 \quad \text{and} \quad \frac{w_2'}{u_2} = -\mathrm{i}k\gamma_4. \tag{23}$$

and $\gamma_1\gamma_3 + \gamma_2\gamma_4 = 1$. The system satisfied by the functions u_1 and v_1 is the one-dimensional mixed system

$$u_1' = -\mathrm{i}k\gamma_1 v_1, \tag{24}$$

$$v_1' = -\mathrm{i}k\gamma_3 u_1. \tag{25}$$

Thus $u_1'' = -k^2\gamma_1\gamma_3 u_1$ and we see that if $u_1 = \exp(\mathrm{i}k_1 x)$ and $v_1 = -k_1 \exp(\mathrm{i}k_1 x)/(k\gamma_1))$ then

$$k_1^2 = \gamma_1\gamma_3 k^2$$

and similarly if $u_2 = \exp(\mathrm{i}k_2 y)$, then $k_2^2 = \gamma_2\gamma_4 k^2$ so we obtain the dispersion relation

$$k^2 = k_1^2 + k_2^2$$

where $u = \exp(\mathrm{i}(k_1 x + k_2 y - \omega t))$ as expected.

3.1. General DGFEM for the First-Order System

In this section we summarise the general DGFEM method of [4] applied to the first-order system. As for the IP-DG method, let K denote a square in an infinite uniform mesh of squares covering $\mathbb{R}^2$ with edges parallel to the coordinate axis and lengths of size h (see Fig. 1). On an element K in the mesh, $(u, v, w) \in (H^1(K))^3$ satisfies (15)–(17) and a general DGFEM method is derived by multiplying each equation by a test function and integrating by parts. Then we obtain

$$\begin{aligned}
\frac{1}{c}(u_t, \phi)_K + (v, \phi_x)_K + (w, \phi_y)_K &= \langle \hat{\boldsymbol{v}} \cdot \boldsymbol{n}_K, \phi \rangle_{\partial K}, \\
\frac{1}{c}(v_t, \psi)_K + (u, \psi_x)_K &= \langle \hat{u}\, n_{K,1}, \psi \rangle_{\partial K}, \\
\frac{1}{c}(w_t, \xi)_K + (u, \xi_y)_K &= \langle \hat{u}\, n_{K,2}, \xi \rangle_{\partial K},
\end{aligned}$$

where $\boldsymbol{n}_K = (n_{K,1}, n_{K,2})^T$ is the unit outward normal to K and $\hat{\boldsymbol{v}}$ and $\hat{u}$ are "numerical fluxes" computed from the values of u, v and w on elements

adjacent to the relevant edge. Let $\boldsymbol{v} = (v, w)^T$. Then following [4] we use the notation in (2) and (3) to define the fluxes by

$$\hat{u} = \{\{u\}\} + \boldsymbol{C}_{1,2} \cdot [\![u]\!] - C_{2,2}[\![\boldsymbol{v}]\!] \tag{26}$$

$$\hat{\boldsymbol{v}} = \{\{\boldsymbol{v}\}\} - C_{1,1}[\![u]\!] - \boldsymbol{C}_{1,2}[\![\boldsymbol{v}]\!]. \tag{27}$$

In practice, for the uniform mesh used here, it suffices to choose $\boldsymbol{C}_{1,2} = C_{1,2}\boldsymbol{e}_1$ on vertical edges and $\boldsymbol{C}_{1,2} = C_{1,2}\boldsymbol{e}_2$ on horizontal edges where $C_{1,2}$ is a scalar and $\boldsymbol{e}_i, i = 1, 2$ are the unit vectors in the coordinate directions.

3.2. Upwind and Centred Schemes

An obvious question is the relationship between the general choice of the numerical flux (26)–(27) in the previous section and the standard "upwind" choice. We show that the upwind choice is a special case of the general one. To do this we start by describing the upwind flux. Let D denote the partitioned 3×3 matrix given by

$$D = \left(\begin{array}{c|c} 0 & \boldsymbol{n}_K^T \\ \hline \boldsymbol{n}_K & 0 \end{array}\right).$$

Then D may be diagonalised and we find that $D = P\Lambda P^T$ where

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \boldsymbol{n}_K/\sqrt{2} & -\boldsymbol{n}_K/\sqrt{2} & \boldsymbol{n}_K^{\perp} \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\boldsymbol{n}_K^{\perp}$ is a unit vector orthogonal to $\boldsymbol{n}_K = (n_{K,1}, n_{K,2})^T$ given by $\boldsymbol{n}_K^{\perp} = (-n_{K,2}, n_{K,1})^T$. Then we define

$$|\Lambda| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $|D| = P|\Lambda|P^T$. For any α with $0 < \alpha \leqslant 1$ we set $D^+ = (D + \alpha|D|)/2$ and $D^- = (D - \alpha|D|)/2$. Now let $\hat{\boldsymbol{\sigma}}_{\text{upwind}}$ denote the upwind numerical flux so that

$$\hat{\boldsymbol{\sigma}}_{\text{upwind}} = \begin{pmatrix} \hat{\boldsymbol{v}} \cdot \boldsymbol{n}_K \\ \hat{u} n_{1,K} \\ \hat{u} n_{2,K} \end{pmatrix}.$$

Let $\boldsymbol{u}$ denote the vector $(u, \boldsymbol{v}^T)^T$ on K and let $\boldsymbol{u}'$ be the corresponding vector on an adjacent element K'. Then recalling that $\{\{\boldsymbol{u}\}\} = (\boldsymbol{u} + \boldsymbol{u}')/2$, we see that the upwind flux is given as follows

$$\begin{aligned}\hat{\boldsymbol{\sigma}}_{\text{upwind}} &= D^+\boldsymbol{u}' + D^-\boldsymbol{u} \\ &= D\{\{\boldsymbol{u}\}\} + \frac{\alpha}{2}|D|(\boldsymbol{u}' - \boldsymbol{u}) \\ &= D\{\{\boldsymbol{u}\}\} + \begin{pmatrix} -\alpha/2 & 0 \\ 0 & -\frac{\alpha}{2}\boldsymbol{n}_K\boldsymbol{n}_K^T \end{pmatrix}\begin{pmatrix} u - u' \\ \boldsymbol{v} - \boldsymbol{v}' \end{pmatrix}. \end{aligned} \tag{28}$$

The fully upwinded flux corresponds to $\alpha = 1$. The value $\alpha = 0$ is also used sometimes and in this case the flux is said to be centred and we define $\hat{\boldsymbol{\sigma}}_{\text{centred}} = D\{\{\boldsymbol{u}\}\}$.

Now let us rewrite the numerical flux (26)–(27) in the same way. Denoting this flux by $\hat{\boldsymbol{\sigma}}_{\text{DG}}$ we have,

$$\begin{aligned}\hat{\boldsymbol{\sigma}}_{\text{DG}} &= D\begin{pmatrix} \{\{u\}\} + \boldsymbol{C}_{1,2}\cdot\boldsymbol{n}_K(u - u') - C_{2,2}(\boldsymbol{v}\cdot\boldsymbol{n}_K - \boldsymbol{v}'\cdot\boldsymbol{n}_K) \\ \{\{\boldsymbol{v}\}\} - C_{1,1}\boldsymbol{n}_K(u - u') - \boldsymbol{C}_{1,2}(\boldsymbol{v}\cdot\boldsymbol{n}_K - \boldsymbol{v}'\cdot\boldsymbol{n}_K) \end{pmatrix} \\ &= D\left(\{\{\boldsymbol{u}\}\} + \begin{pmatrix} \boldsymbol{C}_{1,2}\cdot\boldsymbol{n}_K & -C_{22}\boldsymbol{n}_K^T \\ -C_{1,1}\boldsymbol{n}_K & -\boldsymbol{C}_{1,2}\boldsymbol{n}_K^T \end{pmatrix}\begin{pmatrix} u - u' \\ \boldsymbol{v} - \boldsymbol{v}' \end{pmatrix}\right) \\ &= D\{\{\boldsymbol{u}\}\} + \begin{pmatrix} -C_{1,1} & -(\boldsymbol{n}_K^T\boldsymbol{C}_{1,2})\boldsymbol{n}_K^T \\ (\boldsymbol{C}_{1,2}\cdot\boldsymbol{n}_K)\boldsymbol{n}_K & -C_{2,2}\boldsymbol{n}_K\boldsymbol{n}_K^T \end{pmatrix}\begin{pmatrix} u - u' \\ \boldsymbol{v} - \boldsymbol{v}' \end{pmatrix}. \end{aligned}$$

Comparing this expression to (28) we see that the expressions can be made equal by choosing $\boldsymbol{C}_{1,2} = 0$ and $C_{1,1} = C_{2,2} = \alpha/2$. This holds for $0 \leqslant \alpha \leqslant 1$. The fully upwinded case is $\alpha = 1$ or $C_{1,1} = C_{2,2} = 1/2C_{1,2} = 0$ and the centred case is $\alpha = 0$ or $C_{1,1} = C_{2,2} = 0$.

3.3. Discrete Separation of Variables

Now we shall show that the dispersion analysis of the first-order system in two space dimensions can be reduced to considering a pair of one-dimensional problems. However, these problems are non-standard and differ from what one would obtain by simply applying the DGFEM scheme to the one-dimensional first-order system. This contrasts with the situation identified for IP-DG scheme considered earlier, where it was found that the analysis could be reduced to the consideration of a one-dimensional scheme that did coincide what one would obtain by applying IP-DG to a one-dimensional wave equation.

The semi-discrete problem consists of seeking $(u_h, v_h, w_h) \in (\mathbb{Q}_p)^3$ such that

$$\begin{aligned}
\frac{1}{c}(u_{h,t}, \phi_h)_K + (v_h, \phi_{h,x})_K + (w_h, \phi_{h,y})_K &= \langle \hat{\boldsymbol{v}}_h \cdot \boldsymbol{n}_K, \phi_h \rangle_{\partial K}, \\
\frac{1}{c}(v_{h,t}, \psi_h)_K + (u_h, \psi_{h,x})_K &= \langle \hat{u}_h n_{K,1}, \psi_h \rangle_{\partial K}, \\
\frac{1}{c}(w_{h,t}, \xi_h)_K + (u_h, \xi_{h,y})_K &= \langle \hat{u}_h n_{K,2}, \xi_h \rangle_{\partial K},
\end{aligned}$$

for all $(\phi_h, \psi_h, \xi_h) \in (\mathbb{Q}_p)^3$.

We seek a separable solution on K of the form

$$\begin{aligned}
u_h(x, y, t) &= u_1(x) u_2(y) \exp(-\mathrm{i}\omega t), \\
v_h(x, y, t) &= v_1(x) v_2(y) \exp(-\mathrm{i}\omega t), \\
w_h(x, y, t) &= w_1(x) w_2(y) \exp(-\mathrm{i}\omega t),
\end{aligned}$$

where $u_i, v_i, w_i \in \mathbb{P}_p$, $i = 1, 2$. Selecting test functions $\phi(x, y) = \phi_1(x)\phi_2(y)$, $\phi_i \in \mathbb{P}_p$, $i = 1, 2$ (and similarly for the other test functions), we obtain

$$\begin{aligned}
-\mathrm{i}k(u_1, \phi_1)(u_2, \phi_2) + (v_1, \phi_1')(v_2, \phi_2) + (w_1, \phi_1)(w_2, \phi_2') \\
= \langle \hat{\boldsymbol{v}}_h \cdot \boldsymbol{n}_K, \phi_1 \phi_2 \rangle_{\partial K} \qquad (29) \\
-\mathrm{i}k(v_1, \psi_1)(v_2, \psi_2) + (u_1, \psi_1')(u_2, \psi_2) = \langle \hat{u}_h n_{K,1}, \psi_1 \psi_2 \rangle_{\partial K} \qquad (30) \\
-\mathrm{i}k(w_1, \xi_1)(w_2, \xi_2) + (u_1, \xi_1)(u_2, \xi_2') = \langle \hat{u}_h n_{K,2}, \xi_1 \xi_2 \rangle_{\partial K} \qquad (31)
\end{aligned}$$

Now we need to examine the flux terms in detail. Prompted by the dispersion analysis for the continuous problem, we first consider the second two equations above. This implies that we need to examine $\hat{u}_h$. Labelling the edges of K as N, S, E and W as for the IP-DG analysis we obtain

$$\langle \hat{u}_h n_{K,1}, \psi_1 \psi_2 \rangle = (\hat{u}_{E,h}, \psi_{1,E} \psi_2) - (\hat{u}_{W,h}, \psi_{1,W} \psi_2),$$

where $\hat{u}_{E,h}$ is the "flux" on the Eastern edge at $x = h/2$ and similarly $\hat{u}_{W,h}$ is the flux at the edge $x = -h/2$. Using (26) on the Eastern edge we see that

$$\hat{u}_{E,h} = \{\{u_1\}\}_E u_2 + C_{1,2} [\![u_1]\!]_E u_2 - C_{2,2} [\![v_1]\!]_E v_2,$$

where the operators are given by (4)–(5). Thus

$$(\hat{u}_{E,h}, \psi_{1,E} \psi_2) = \left[\left(\{\{u_1\}\}_E + C_{1,2} [\![u_1]\!]_E \right) (u_2, \psi_2) - C_{2,2} [\![v_1]\!]_E (v_2, \psi_2) \right] \psi_{1,E}.$$

Using the corresponding result for the Western edge, equation (30) may be written explicitly as

$$\begin{aligned}
&-\mathrm{i}k(v_1,\psi_1)(v_2,\psi_2)+(u_1,\psi_1')(u_2,\psi_2)\\
&\quad=\left[\left(\{\{u_1\}\}_E+C_{1,2}[\![u_1]\!]_E\right)(u_2,\psi_2)-C_{2,2}[\![v_1]\!]_E(v_2,\psi_2)\right]\psi_{1,E}\\
&\qquad-\left[\left(\{\{u_1\}\}_W+C_{1,2}[\![u_1]\!]_W\right)(u_2,\psi_2)-C_{2,2}[\![v_1]\!]_W(v_2,\psi_2)\right]\psi_{1,W}.
\end{aligned}$$

This equation can be rewritten as

$$\begin{aligned}
&\left[-\mathrm{i}k(v_1,\psi_1)+C_{2,2}[\![v_1]\!]_E\psi_{1,E}-C_{2,2}[\![v_1]\!]_W\psi_{1,W}\right](v_2,\psi_2)\\
&\quad+\left[(u_1,\psi_1')-\left(\{\{u_1\}\}_E+C_{1,2}[\![u_1]\!]_E\right)\psi_{1,E}\right.\\
&\quad\left.+\left(\{\{u_1\}\}_W+C_{1,2}[\![u_1]\!]_W\right)\psi_{1,W}\right](u_2,\psi_2)=0.
\end{aligned}$$

We see that there is a constant γ_1 such that

$$\frac{(u_1,\psi_1')-\left(\{\{u_1\}\}_E+C_{1,2}[\![u_1]\!]_E\right)\psi_{1,E}+\left(\{\{u_1\}\}_W+C_{1,2}[\![u_1]\!]_W\right)\psi_{1,W}}{-\mathrm{i}k(v_1,\psi_1)+C_{2,2}[\![v_1]\!]_E\psi_{1,E}-C_{2,2}[\![v_1]\!]_W\psi_{1,W}}=-\gamma_1, \quad (32)$$

$$\frac{(v_2,\psi_2)}{(u_2,\psi_2)}=\gamma_1. \quad (33)$$

Similarly from equation (31) we obtain that there is a constant γ_2 such that

$$\frac{(u_2,\xi_2')-\left(\{\{u_2\}\}_N+C_{1,2}[\![u_2]\!]_N\right)\psi_{2,N}+\left(\{\{u_2\}\}_S+C_{1,2}[\![u_2]\!]_S\right)\psi_{2,S}}{-\mathrm{i}k(w_2,\xi_2)+C_{2,2}[\![w_2]\!]_N\psi_{2,N}-C_{2,2}[\![w_2]\!]_S\psi_{2,S}}=-\gamma_2 \quad (34)$$

$$\frac{(w_1,\xi_1)}{(u_1,\xi_1)}=\gamma_2. \quad (35)$$

Now turning to equation (29) and writing out the numerical fluxes explicitly we have

$$\begin{aligned}
&-\mathrm{i}k(u_1,\phi_1)(u_2,\phi_2)+(v_1,\phi_1')(v_2,\phi_2)+(w_1,\phi_1)(w_2,\phi_2')\\
&\quad=\left[\left(\{\{v_1\}\}_E-C_{1,2}[\![v_1]\!]_E\right)(v_2,\phi_2)-C_{1,1}[\![u_1]\!]_E(u_2,\phi_2)\right]\phi_{1,E}\\
&\qquad+\left[\left(\{\{w_2\}\}_N-C_{1,2}[\![w_2]\!]_N\right)(w_1,\phi_1)-C_{1,1}[\![u_2]\!]_N(u_1,\phi_1)\right]\phi_{2,N}\\
&\qquad-\left[\left(\{\{v_1\}\}_W-C_{1,2}[\![v_1]\!]_W\right)(v_2,\phi_2)-C_{1,1}[\![u_1]\!]_W(u_2,\phi_2)\right]\phi_{1,W}\\
&\qquad-\left[\left(\{\{w_2\}\}_S-C_{1,2}[\![w_2]\!]_S\right)(w_1,\phi_1)-C_{1,1}[\![u_2]\!]_S(u_1,\phi_1)\right]\phi_{2,S}.
\end{aligned}$$

Dividing both sides by $(u_1, \phi_1)(u_2, \phi_2)$ and collecting terms, this implies that

$$
\begin{aligned}
&-\mathrm{i}k + \frac{(v_1, \phi_1') - \left(\{\{v_1\}\}_E - C_{1,2}[\![v_1]\!]_E\right)\phi_{1,E} + \left(\{\{v_1\}\}_W - C_{1,2}[\![v_1]\!]_W\right)\phi_{1,W}}{(u_1, \phi_1)} \frac{(v_2, \phi_2)}{(u_2, \phi_2)} \\
&+ \frac{(w_1, \phi_1)}{(u_1, \phi_1)} \frac{(w_2, \phi_2') - \left(\{\{w_2\}\}_W - C_{1,2}[\![w_2]\!]_N\right)\phi_{2,N} + \left(\{\{w_2\}\}_S - C_{1,2}[\![w_2]\!]_S\right)\phi_{2,S}}{(u_2, \phi_2)} \\
&\quad = -C_{1,1}\left(\frac{[\![u_1]\!]_E\phi_{1,E}}{(u_1, \phi_1)} + \frac{[\![u_2]\!]_N\phi_{2,N}}{(u_2, \phi_2)} - \frac{[\![u_1]\!]_W\phi_{1,W}}{(u_1, \phi_1)} - \frac{[\![u_2]\!]_S\phi_{2,S}}{(u_2, \phi_2)}\right).
\end{aligned}
$$

Now using (33) and (35) we obtain

$$
\begin{aligned}
&-\mathrm{i}k + \gamma_1 \frac{(v_1, \phi_1') - \left(\{\{v_1\}\}_E - C_{1,2}[\![v_1]\!]_E\right)\phi_{1,E} + \left(\{\{v_1\}\}_W - C_{1,2}[\![v_1]\!]_W\right)\phi_{1,W}}{(u_1, \phi_1)} \\
&+ \gamma_2 \frac{(w_2, \phi_2') - \left(\{\{w_2\}\}_N - C_{1,2}[\![w_2]\!]_N\right)\phi_{2,N} + \left(\{\{w_2\}\}_S - C_{1,2}[\![w_2]\!]_S\right)\phi_{2,S}}{(u_2, \phi_2)} \\
&\quad = -C_{1,1}\left(\frac{[\![u_1]\!]_E\phi_{1,E}}{(u_1, \phi_1)} + \frac{[\![u_2]\!]_N\phi_{2,N}}{(u_2, \phi_2)} - \frac{[\![u_1]\!]_W\phi_{1,W}}{(u_1, \phi_1)} - \frac{[\![u_2]\!]_S\phi_{2,S}}{(u_2, \phi_2)}\right).
\end{aligned}
$$

By the usual separation of variables argument there are constants γ_3 and γ_4 such that the following analog of (23) holds

$$
\begin{aligned}
&\frac{(v_1, \phi_1') - \left(\{\{v_1\}\}_E - C_{1,2}[\![v_1]\!]_E\right)\phi_{1,E} + \left(\{\{v_1\}\}_W - C_{1,2}[\![v_1]\!]_W\right)\phi_{1,W}}{(u_1, \phi_1)} \\
&\quad + \frac{C_{1,1}}{\gamma_1}\left(\frac{[\![u_1]\!]_E\phi_{1,E}}{(u_1, \phi_1)} - \frac{[\![u_1]\!]_W\phi_{1,W}}{(u_1, \phi_1)}\right) = \mathrm{i}k\gamma_3 \\
&\frac{(w_2, \phi_2') - \left(\{\{w_2\}\}_N - C_{1,2}[\![w_2]\!]_N\right)\phi_{2,N} + \left(\{\{w_2\}\}_S - C_{1,2}[\![w_2]\!]_S\right)\phi_{2,S}}{(u_2, \phi_2)} \\
&\quad + \frac{C_{1,1}}{\gamma_2}\left(+\frac{[\![u_2]\!]_N\phi_{2,N}}{(u_2, \phi_2)} - \frac{[\![u_2]\!]_S\phi_{2,S}}{(u_2, \phi_2)}\right) = \mathrm{i}k\gamma_4 \\
&\gamma_1\gamma_3 + \gamma_2\gamma_4 = 1.
\end{aligned}
$$

Thus we see that the reduced one-dimensional problems that determine the dispersion relation are as follows:

$$
\begin{aligned}
&(u_1, \psi_1') - \left(\{\{u_1\}\}_E + C_{1,2}[\![u_1]\!]_E\right)\psi_{1,E} + \left(\{\{u_1\}\}_W + C_{1,2}[\![u_1]\!]_W\right)\psi_{1,W} \\
&\quad = -\gamma_1\left(-\mathrm{i}k(v_1, \psi_1) + C_{2,2}[\![v_1]\!]_E\psi_{1,E} - C_{2,2}[\![v_1]\!]_W\psi_{1,W}\right), \qquad (36) \\
&(v_1, \phi_1') - \left(\{\{v_1\}\}_E - C_{1,2}[\![v_1]\!]_E\right)\phi_{1,E} + \left(\{\{v_1\}\}_W - C_{1,2}[\![v_1]\!]_W\right)\phi_{1,W} \\
&\quad + \frac{C_{1,1}}{\gamma_1}\left([\![u_1]\!]_E\phi_{1,E} - [\![u_1]\!]_W\phi_{1,W}\right) = \mathrm{i}k\gamma_3(u_1, \phi_1) \qquad (37)
\end{aligned}
$$

together with a similar problem for (u_2, v_2).

This problem is a little non-standard. But we note that if we multiply (37) by γ_1 and define $\tilde{v}_1 = \gamma_1 v_1$ then these equations may be rewritten as (where we have used the fact that $\gamma_1\gamma_3 = k_1^2/k^2$),

$$\begin{aligned}
&(u_1, \psi_1') - \left(\{\{u_1\}\}_E + C_{1,2}[\![u_1]\!]_E\right)\psi_{1,E} + \left(\{\{u_1\}\}_W + C_{1,2}[\![u_1]\!]_W\right)\psi_{1,W} \\
&\quad = \mathrm{i}k(\tilde{v}_1, \psi_1) - C_{2,2}\left([\![\tilde{v}_1]\!]_E\psi_{1,E} - [\![\tilde{v}_1]\!]_W\psi_{1,W}\right), \qquad (38)\\
&(\tilde{v}_1, \phi_1') - \left(\{\{\tilde{v}_1\}\}_E - C_{1,2}[\![\tilde{v}_1]\!]_E\right)\phi_{1,E} + \left(\{\{\tilde{v}_1\}\}_W - C_{1,2}[\![\tilde{v}_1]\!]_W\right)\phi_{1,W} \\
&\quad = \mathrm{i}\frac{k_1^2}{k}(u_1, \phi_1) - C_{1,1}\left([\![u_1]\!]_E\phi_{1,E} - [\![u_1]\!]_W\phi_{1,W}\right). \qquad (39)
\end{aligned}$$

These are the discontinuous Galerkin analogs of the first equations in (21) and (23), respectively. We have thus shown that the dispersion analysis of the two-dimensional discontinuous Galerkin method can be reduced to the study of one-dimensional discontinuous Galerkin problems. However, the presence of both k_1 and k in (38) and (39) shows that they are not a direct DGFEM discretisation of (21) and (23).

An important simplification occurs when $C_{1,1} = C_{2,2} = 0$ in the case of centred fluxes. Then

$$\begin{aligned}
&(u_1, \psi_1') - \left(\{\{u_1\}\}_E + C_{1,2}[\![u_1]\!]_E\right)\psi_{1,E} + \left(\{\{u_1\}\}_W + C_{1,2}[\![u_1]\!]_W\right)\psi_{1,W} \\
&\quad = \mathrm{i}k(\tilde{v}_1, \psi_1), \\
&(\tilde{v}_1, \phi_1') - \left(\{\{\tilde{v}_1\}\}_E - C_{1,2}[\![\tilde{v}_1]\!]_E\right)\phi_{1,E} + \left(\{\{\tilde{v}_1\}\}_W - C_{1,2}[\![\tilde{v}_1]\!]_W\right)\phi_{1,W} \\
&\quad = \mathrm{i}\frac{k_1^2}{k}(u_1, \phi_1).
\end{aligned}$$

Now defining $\tilde{u}_1 = k_1 u_1/k$ we obtain

$$\begin{aligned}
&(\tilde{u}_1, \psi_1') - \left(\{\{\tilde{u}_1\}\}_E + C_{1,2}[\![\tilde{u}_1]\!]_E\right)\psi_{1,E} + \left(\{\{\tilde{u}_1\}\}_W + C_{1,2}[\![\tilde{u}_1]\!]_W\right)\psi_{1,W} \\
&\quad = \mathrm{i}k_1(\tilde{v}_1, \psi_1), \qquad (40)\\
&(\tilde{v}_1, \phi_1') - \left(\{\{\tilde{v}_1\}\}_E - C_{1,2}[\![\tilde{v}_1]\!]_E\right)\phi_{1,E} + \left(\{\{\tilde{v}_1\}\}_W - C_{1,2}[\![\tilde{v}_1]\!]_W\right)\phi_{1,W} \\
&\quad = \mathrm{i}k_1(\tilde{u}_1, \phi_1). \qquad (41)
\end{aligned}$$

Thus in the case of generalised centred fluxes (i.e. $C_{1,1} = C_{2,2} = 0$ but perhaps $C_{1,2} \neq 0$), the decomposition of the two-dimensional problem into one-dimensional problems gives a clean separation (no factors of k in the one-dimensional problem). This scheme will turn out to be non-dissipative, and it is possible to choose $C_{1,2}$ to obtain an improved dispersion relation.

3.4. Relationship with Schemes for One-Way Wave Equation

We now explore the possibility of relating the discrete dispersion relation of the one-dimensional problem (38)–(39) to the work of Ainsworth [2]

on the one way wave equation. In order to do this, we shall attempt to decompose the system (38)–(39) into an appropriate pair of discrete one way problems. We first dilate the line using the map $\hat{x} = 2x/h$ and denote the reference interval by $\hat{K} = [-1, 1]$. Then abusing notation by defining $u(\hat{x}) = u_1(x)$ and $v(\hat{x}) = \tilde{v}_1(x)$ where $\hat{x} = 2x/h$ and similarly for the test functions we obtain

$$\begin{aligned}
&(u, \psi')_{\hat{K}} - \left(\{\{u\}\}_E + C_{1,2}[\![u]\!]_E\right)\psi_E + \left(\{\{u\}\}_W + C_{1,2}[\![u]\!]_W\right)\psi_W \\
&\quad = \mathrm{i}\frac{kh}{2}(v, \psi)_{\hat{K}} - C_{2,2}\left([\![v]\!]_E\psi_E - [\![v]\!]_W\psi_W\right), \qquad (42)\\
&(v, \phi')_{\hat{K}} - \left(\{\{v\}\}_E - C_{1,2}[\![v]\!]_E\right)\phi_E + \left(\{\{v\}\}_W - C_{1,2}[\![v]\!]_W\right)\phi_W \\
&\quad = \mathrm{i}\frac{k_1^2 h}{2k}(u, \phi)_{\hat{K}} - C_{1,1}\left([\![u]\!]_E\phi_E - [\![u]\!]_W\phi_W\right), \qquad (43)
\end{aligned}$$

where $(\cdot, \cdot)_{\hat{K}}$ denotes the inner product on $[-1, 1]$ and the subscript E and W now refer to the reference interval $\hat{K}$. Integrating by parts and rearranging we obtain:

$$\begin{aligned}
&-\mathrm{i}\frac{kh}{2}(v, \psi)_{\hat{K}} - (u', \psi)_{\hat{K}} + \left(1/2 - C_{1,2}\right)[\![u]\!]_E\psi_E \\
&\quad + \left(1/2 + C_{1,2}\right)[\![u]\!]_W\psi_W + C_{2,2}\left([\![v]\!]_E\psi_E - [\![v]\!]_W\psi_W\right) = 0, \\
&-\mathrm{i}\frac{k_1^2 h}{2k}(u, \phi)_{\hat{K}} - (v', \phi)_{\hat{K}} + \left(1/2 + C_{1,2}\right)[\![v]\!]_E\phi_E \\
&\quad + \left(1/2 - C_{1,2}\right)[\![v]\!]_W\phi_W + C_{1,1}\left([\![u]\!]_E\phi_E - [\![u]\!]_W\phi_W\right) = 0.
\end{aligned}$$

Now we make the Bloch wave (or plane wave) assumption that the solution varies by the same phase factor λ across each element so that $u(1^+) = \lambda u(-1^+)$ and $u(-1^-) = (1/\lambda)u(1^-)$ (and similarly for v) so that the equations may be reduced to a single element $\hat{K}$ and becomes

$$\begin{aligned}
&-\mathrm{i}\frac{kh}{2}(v, \psi)_{\hat{K}} - (u', \psi)_{\hat{K}} + \left(1/2 - C_{1,2}\right)(u(1^-) - \lambda u(-1+))\psi(1) \\
&\quad + \left(1/2 + C_{1,2}\right)((1/\lambda)u(1^-) - u(-1^+))\psi(-1) \\
&\quad + C_{2,2}\left((v(1^-) - \lambda v(-1^+))\psi(1) - ((1/\lambda)v(1^-) - v(-1^+))\psi(-1)\right) = 0, \\
&-\mathrm{i}\frac{k_1^2 h}{2k}(u, \phi)_{\hat{K}} - (v', \phi)_{\hat{K}} + \left(1/2 + C_{1,2}\right)(v(1^-) - \lambda v(-1+))\phi(1) \\
&\quad + \left(1/2 - C_{1,2}\right)((1/\lambda)v(1^-) - v(-1^+))\phi(-1) \\
&\quad + C_{1,1}\left((u(1^-) - \lambda u(-1^+))\phi(1) - ((1/\lambda)u(1^-) - u(-1^+))\phi(-1)\right) = 0.
\end{aligned}$$

Next we make the assumption that $u = u_0 w$, $v = v_0 w$ (i.e. the solution is a discrete plane wave within each element) and choose $\phi = \phi_0\xi$ and $\psi = \psi_0\xi$

where the subscripted quantities are constants and the functions ξ and w are polynomials of degree k. Introducing the vectors $V=(v_0,u_0)^T$ and $\Psi=(\psi_0,\phi_0)$ we may write the above equation as

$$-\Psi^T M V(w,\xi)_{\hat{K}} - \Psi^T A V(w',\xi)_{\hat{K}} + \Psi^T B V(w(1^-)-\lambda w(-1^+))\xi(1) + \Psi^T C V((1/\lambda)w(1^-)-w(-1^+))\xi(-1)=0$$

where

$$M = i\frac{h}{2k}\begin{pmatrix} k^2 & 0 \\ 0 & k_1^2 \end{pmatrix}, \qquad A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$B=\begin{pmatrix} C_{2,2} & \frac{1}{2}-C_{1,2} \\ \frac{1}{2}+C_{1,2} & C_{1,1} \end{pmatrix} \quad \text{and} \quad C=\begin{pmatrix} -C_{2,2} & \frac{1}{2}+C_{1,2} \\ \frac{1}{2}-C_{1,2} & -C_{1,1} \end{pmatrix}.$$

Since the above equation must hold for all vectors Ψ, this equation has a non-trivial solution if and only if the matrix

$$-M(w,\xi)_{\hat{K}} - A(w',\xi)_{\hat{K}} + B(w(1^-)-\lambda w(-1^+))\xi(1) + C((1/\lambda)w(1^-)-w(-1^+))\xi(-1) \tag{44}$$

is singular. To try split into left and right going waves, we can diagonalise $A=E^T DE$ where

$$E=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

But then

$$EBE^T=\frac{1}{2}\begin{pmatrix} C_{1,1}+C_{2,2}+1 & C_{1,2}+C_{2,2}-C_{1,1} \\ C_{2,2}-C_{1,2}-C_{1,1} & C_{2,2}+C_{1,1}-1 \end{pmatrix}.$$

Thus the matrix B is simultaneously diagonalisable with A if $C_{1,2}=0$ and $C_{1,1}=C_{2,2}$. When $C_{2,2}>0$ this is precisely the choice for the upwind DG scheme as we have seen in Sec. 3.2. Unfortunately even with this choice

$$EME^T=\frac{ih}{4k}\begin{pmatrix} k^2+k_1^2 & k^2-k_1^2 \\ k^2-k_1^2 & k^2+k_1^2 \end{pmatrix}.$$

Since this is not generally diagonal (except when $k=k_1$, i.e. for a wave travelling along the x axis), we can not apply the results of [2] to this problem (except for wave travelling along the axis). There is one case however where the results of Ainsworth are applicable and this is when $C_{1,1}=C_{2,2}=0$ and $C_{1,2}=0$. Using the scaling of variables used in the development of (40)–(41) we can eliminate k in this special case. Effectively, we

can take $k = k_1$ and all matrices are simultaneously diagonalisable. We then obtain the scalar "one way" wave equation governing w and λ:

$$i\frac{k_1 h}{2}(w,\xi)_{\hat{K}} - (w',\xi)_{\hat{K}} + \frac{1}{2}(\lambda w(-1^+) - w(1^-))\xi(1) + \frac{1}{2}(w(-1^+) - (1/\lambda)w(1))\xi(-1) = 0$$

which is equivalent to equation (35) considered in [2] in the case of centred fluxes. Various limiting cases were considered in [2], but we shall restrict our attention to the limiting case $hk \ll 1$. Applying Theorem 2 of the above reference, we obtain the following expression for the error in the discrete wave-number for a p-th order DGFEM,

$$k_p - k_{h,p} = \frac{1}{2}\left(\frac{p!}{(2p+1)!}\right)^2 \begin{cases} -\frac{p+1}{2p+1}k^{2p+3}h^{2p+2} & p \text{ even} \\ \frac{2p+1}{p+1}k^{2p+1}h^{2p} & p \text{ odd.} \end{cases}$$

In particular, for the zeroth order scheme $p = 0$, we obtain

$$k_{h,0} = k_0 + \frac{1}{6}k^3h^2$$

while for the first-order scheme $p = 1$, we obtain

$$k_{h,1} = k_1 - \frac{1}{180}k^5h^4.$$

The attempt to reduce the analysis to that of the one-way wave equation for more general schemes, by attempting to diagonalise the matrix in (44) directly, has not proved fruitful in the sense that the results are difficult to interpret and do not compare directly to the work in [2].

3.5. Dispersion Error for General DGFEM Methods

Here we derive the dispersion relation for piecewise constant and piecewise linear elements.

3.5.1. Dispersion Error for Constant Elements

DGFEM schemes using piecewise constant elements are often referred to as "finite volume" schemes [19], and are an important class from the point of view of practical calculations. In this case u_1 and $\tilde{v}_1$ are piecewise constant. On the central element let $(u_1, \tilde{v}_1) = (U, V)$ and on the East and West

elements the vector of unknowns is (U_E, V_E) and (U_W, V_W), respectively. We select $\psi_1 = \phi_1 = 1$ in (38) and (39) and obtain the following equations

$$
\begin{aligned}
&-\left(\frac{U_E+U}{2}+C_{1,2}(U-U_E)\right)+\left(\frac{U_W+U}{2}+C_{1,2}(U_W-U)\right)\\
&\quad = \mathrm{i}khV - C_{2,2}(2V-V_E-V_W)\\
&-\left(\frac{V_E+V}{2}-C_{1,2}(V-V_E)\right)+\left(\frac{V_W+V}{2}-C_{1,2}(V_W-V)\right)\\
&\quad = \mathrm{i}\frac{k_1^2}{k}hU - C_{1,1}(2U-U_E-U_W).
\end{aligned}
$$

Now we write these equations as a matrix system. Let $\boldsymbol{w} = (U, V)^T$, $\boldsymbol{w}_E = (U_E, V_E)^T$ and $\boldsymbol{w}_W = (U_W, .V_W)^T$. Then the above system can be written as

$$C_W \boldsymbol{w}_W + (S+C)\boldsymbol{w} + C_E \boldsymbol{w}_E = 0, \tag{45}$$

where the matrices are given as follows:

$$C_E = \begin{pmatrix} -1/2+C_{1,2} & -C_{2,2} \\ -C_{1,1} & -1/2-C_{1,2} \end{pmatrix} \quad \text{and} \quad C_W = \begin{pmatrix} 1/2+C_{1,2} & -C_{2,2} \\ -C_{1,1} & 1/2-C_{1,2} \end{pmatrix}.$$

$$C = \begin{pmatrix} -2C_{1,2} & 2C_{2,2} \\ 2C_{1,1} & 2C_{1,2} \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & -\mathrm{i}kh \\ -\mathrm{i}k_1^2 h/k & 0 \end{pmatrix}.$$

As usual for a dispersion analysis we assume that the solution on adjacent elements differs by a multiplicative factor λ so that

$$\boldsymbol{w}_W = \boldsymbol{w}/\lambda \quad \text{and} \quad \boldsymbol{w}_E = \lambda \boldsymbol{w}.$$

Then the discrete plane wave solutions are the non-trivial solutions of

$$(\lambda^{-1} C_W + (S+C) + C_E \lambda)\boldsymbol{w} = 0.$$

The determinant of the matrix in this equation must vanish and we find that

$$a(\lambda^2 + \lambda^{-2}) + b(\lambda + \lambda^{-1}) + c = 0,$$

where

$$
\begin{aligned}
a &= \frac{1}{4} - C_{1,2}^2 - C_{1,1}C_{2,2},\\
b &= 4C_{1,2}^2 + 4C_{2,2}C_{1,1} - \mathrm{i}\frac{C_{2,2}k_1^2 h}{k} - \mathrm{i}khC_{1,1},\\
c &= -\frac{1}{2} + \mathrm{i}\frac{2C_{2,2} * k_1 h}{k} + (hk_1)^2 - 6C_{1,2}^2 - 6C_{2,2}C_{1,1} + \mathrm{i}2khC_{1,1}.
\end{aligned}
$$

As we shall see it is then useful to write $\lambda = \exp(i\theta)$ for some phase factor θ so

$$\lambda^2 + \lambda^{-2} = 2\cos(2\theta) \quad \text{and} \quad \lambda + \lambda^{-1} = 2\cos(\theta).$$

Using standard trigonometric identities the determinant condition simplifies to

$$4a\cos^2\theta + 2b\cos\theta + c - 2a = 0. \tag{46}$$

This gives, obviously, θ in terms of a, b and c.

The limit of (46) as $h \to 0$ offers useful information about the method. In the limit as $h \to 0$ equation (46) becomes

$$\alpha\cos^2\theta + (1/2 - 2\alpha)\cos\theta - \alpha - 1/2 = 0,$$

where $\alpha = 1/4 - C_{1,1}C_{2,2} - C_{1,2}^2$. Assuming $\alpha \neq 0$, the roots of this equation are $\cos\theta = 1$ or $\cos\theta = (2\alpha - 1)/(2\alpha)$. As in the case of the IP-DG method, to suppress the spurious mode we want the phase factor θ corresponding to the spurious mode to be complex and so require that $(2\alpha - 1)/(2\alpha) < -1$ which implies $\alpha < 1/4$ or equivalently $C_{1,1}C_{2,2} + C_{1,2}^2 > 0$. In the special case of $\alpha = 0$, the above equation has just one root and there is no need for a stability condition.

We now consider the physical root of the dispersion relation (the root θ_p such that $\theta_p \to 0$ as $h \to 0$). In this case $\theta_p = k_{1,h}h$ where $k_{1,h}$ is the discrete wave number. Using MAPLE we find that

$$\begin{aligned} k_{1,h} = k_1 + i\frac{k_1(k^2C_{1,1} + k_1^2C_{2,2})}{2k}h \\ - \frac{k_1(12k^2k_1^2C_{1,2}^2 + 9k_1^4C_{2,2}^2 - 4k^2k_1^2 + 30k_1^2k^2C_{1,1}C_{2,2} + 9k^4C_{1,1}^2)}{24k^2}h^2 \\ + O(h^3). \end{aligned}$$

The first term (after k_1) on the right-hand side is complex and this implies that the method is dissipative. The first real term after k_1 on the right-hand side gives the phase error of the method and is second-order. Thus the physical dispersion relation generally exhibits a first-order dissipation term and second-order phase error. We now consider some special cases.

LDG Scheme: This method was proposed by [8] (see also [4]). It corresponds to the choice $C_{2,2} = 0$. In this case the physical dispersion relation is (assuming $C_{1,2} \neq 0$)

$$k_{1,h} = k_1 + \mathrm{i}\frac{C_{1,1}kk_1}{2}h - \frac{(12C_{1,2}^2k_1^2 + 9C_{1,1}^2k^2 - 4k_1^2)k_1}{24}h^2 + O(h^3).$$

As in the general case, the scheme is first-order dissipative and second-order dispersive. There does not seem to be a particularly good choice of the remaining parameters $C_{1,2}$ and $C_{1,1}$ (within the stability requirement above) and a standard choice might be $C_{1,1} = 1/2$ and $C_{1,2} = 0$ (although our stability result would need $C_{1,2} > 0$).

Centred Scheme: For the centred scheme we choose $C_{1,1} = C_{2,2} = 0$. In this case the dispersion relation for the general scheme is

$$k_{1,h} = k_1 - \frac{(3C_{1,2}^2 - 1)k_1^3}{6}h^2 + O(h^4).$$

As we expect for the centred scheme, the method is not dissipative, and is second-order dispersive. Making the special choice $C_{1,2} = 1/\sqrt{3}$ we now find the improved dispersion relation

$$k_{1,h} = k_1 + \frac{k_1^5}{180}h^4 + O(h^6),$$

which is non-dispersive, fourth order and satisfies the limiting stability requirement that $C_{1,2} \neq 0$.

Upwind Scheme: The fully upwind scheme corresponds to the choice $C_{1,1} = C_{2,2} = 1/2$ and has the dispersion relation

$$k_{1,h} = k_1 + \mathrm{i}\frac{(k_1^2 + k^2)k_1}{4k}h - \frac{k_1(14k^2k_1^2 + 9k^4 + 48k^2k_1^2C_{1,2}^2 + 9k_1^4)}{96k^2}h^2 + O(h^3).$$

As for the LDG scheme the method is first-order dissipative and second-order dispersive. There is no obvious advantage to choosing $C_{1,2}$ other than zero, which is the standard choice for upwind finite volume schemes.

3.5.2. *Dispersion Error for Linear Elements*

We now follow the same procedure as for piecewise constant elements to analyse the case of linear elements ($p = 1$ in (38) and (39)). Using the notation in Fig. 3 and selecting and $\psi_1 = \phi_1 = (h/2 - x)/h$ in (38) and (39) we obtain

$$
\begin{aligned}
&-\frac{(U_b+U_a)}{2}+\left(\frac{U_a+U_{W,b}}{2}+C_{1,2}(U_{W,b}-U_a)\right)\\
&\quad=\mathrm{i}k\left(\frac{V_a}{3}+\frac{V_b}{6}\right)h+C_{2,2}(V_{W,b}-V_a)\\
&-\frac{(V_b+V_a)}{2}+\left(\frac{V_a+V_{W,b}}{2}-C_{1,2}(V_{W,b}-V_a)\right)\\
&\quad -C_{1,1}(U_{W,b}-U_a)=\mathrm{i}\frac{k_1^2}{k}\left(\frac{U_a}{3}+\frac{U_b}{6}\right)h
\end{aligned}
$$

In the same way choosing $\phi_1=\psi_1=(x+h/2)/h$ we obtain

$$
\begin{aligned}
&\frac{(U_b+U_a)}{2}-\left(\frac{U_b+U_{E,a}}{2}+C_{1,2}(U_b-U_{E,a})\right)\\
&\quad=\mathrm{i}k\left(\frac{V_a}{6}+\frac{V_b}{3}\right)h-C_{2,2}(V_b-V_{E,a})\\
&\frac{(V_b+V_a)}{2}-\left(\frac{V_b+V_a}{2}-C_{1,2}(V_b-V_{E,a})\right)\\
&\quad +C_{1,1}(U_b-U_{E,a})=\mathrm{i}\frac{k_1^2}{k}\left(\frac{U_a}{6}+\frac{U_b}{3}\right)h
\end{aligned}
$$

Now we write these equations as a matrix system. Let $w=(U_a, U_b, V_a, V_b)^T$ (similarly for w_W and w_E). Then the above system can be written as

$$
C_W w_W+(S+C)w+C_E w_E=0, \tag{47}
$$

where the matrices are defined as follows. The West exterior coupling matrix is

$$
C_W=\begin{pmatrix}
0 & 1/2+C_{1,2} & 0 & -C_{2,2}\\
0 & 0 & 0 & 0\\
0 & -C_{1,1} & 0 & 1/2-C_{1,2}\\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and the East exterior coupling matrix is given by

$$
C_E=\begin{pmatrix}
0 & 0 & 0 & 0\\
-1/2+C_{1,2} & 0 & -C_{2,2} & 0\\
0 & 0 & 0 & 0\\
-C_{1,1} & 0 & -1/2-C_{1,2} & 0
\end{pmatrix}.
$$

On the central element the coupling matrix arising from the flux contributions at either end of the interval is

$$C = \begin{pmatrix} 1/2 - C_{1,2} & 0 & C_{2,2} & 0 \\ 0 & -1/2 - C_{1,2} & 0 & C_{2,2} \\ C_{1,1} & 0 & 1/2 + C_{1,2} & 0 \\ 0 & C_{1,1} & 0 & -1/2 + C_{1,2} \end{pmatrix}$$

Finally, the integral terms contribute to the following matrix on element e

$$S = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\mathrm{i}\frac{kh}{3} & -\mathrm{i}\frac{kh}{6} \\ \frac{1}{2} & \frac{1}{2} & -\mathrm{i}\frac{kh}{6} & -\mathrm{i}\frac{kh}{3} \\ -\mathrm{i}\frac{k_1^2 h}{3k} & -\mathrm{i}\frac{k_1^2 h}{6k} & -\frac{1}{2} & -\frac{1}{2} \\ -\mathrm{i}\frac{k_1^2 h}{6k} & -\mathrm{i}\frac{k_1^2 h}{3k} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Note that at higher order (i.e. $p > 1$) the coupling matrices are essentially unchanged (except for more rows and columns of zeros) and only the interior matrix is modified.

Following the arguments for the piecewise constant case, we find that the phase factor θ must satisfy equation (46) but now with the following coefficients

$$\begin{aligned} a &= -\frac{1}{4}C_{2,2}C_{1,1} - \frac{1}{36}C_{2,2}k_1^2h^2C_{1,1} + \frac{1}{144}h^2k_1^2 - \frac{1}{4}C_{1,2}^2 \\ &\quad -\frac{1}{36}C_{1,2}^2k_1^2h^2 + \frac{1}{16} \\ b &= \frac{1}{72}\mathrm{i}kk_1^2h^3\,C_{1,1} - \frac{1}{4} - \frac{1}{12}\mathrm{i}C_{2,2}\frac{k_1^2}{k}h - \frac{1}{12}\,\mathrm{i}khC_{1,1} + \frac{1}{24}k_1^2h^2 \\ &\quad -\frac{1}{72}\mathrm{i}\frac{k_1^4}{k}h^3C_{2,2} - \frac{2}{9}C_{1,2}^2k_1^2h^2 - \frac{2}{9}C_{2,2}k_1^2h^2C_{1,1} \\ c &= \frac{1}{18}\mathrm{i}kk_1^2h^3C_{1,1} - \frac{7}{72}k_1^2h^2 + \frac{1}{6}\mathrm{i}hC_{1,1} + \frac{1}{2}\,C_{2,2}\,C_{1,1} + \frac{1}{2}\,{C_{1,2}}^2 \\ &\quad +\frac{1}{144}k_1^4h^4 + \frac{1}{6}\mathrm{i}C_{2,2}\frac{k_1^2}{k}h - \frac{1}{2}\,C_{2,2}k_1^2h^2C_{1,1} - \frac{1}{2}C_{1,2}^2k_1^2h^2 \\ &\quad +\frac{3}{8} - \frac{1}{18}\mathrm{i}\frac{k_1^4}{k}h^3C_{2,2} \end{aligned}$$

At this stage it is useful to consider (46) (but with the coefficients above) in the limit as $h \to 0$. The equation becomes

$$(\cos\theta - 1)(\alpha\cos\theta + \alpha - 1/2) = 0,$$

where $\alpha = 1/4 - C_{1,1}C_{2,2} - C_{1,2}^2$. As for the piecewise constant case, the root $\cos\theta = 1$ is the physical root and the second root (assuming $\alpha \neq 0$) is spurious. If $\alpha = 0$ (this occurs when $C_{1,1} = C_{2,2} = 1/2$ and $C_{1,2} = 0$ which is the parameter choice corresponding to full upwinding) there is no spurious mode and no need for a stability condition. If $\alpha \neq 0$ then the spurious root can be made complex if $\alpha < 1/4$ or if $C_{1,1}C_{2,2} + C_{1,2}^2 > 0$. Interestingly this is the same condition as for the piecewise constant case. Note that energy arguments show that we need to choose $C_{1,1} \geqslant 0$ and $C_{2,2} \geqslant 0$ for stability so that the limiting analysis here does not completely determine the stability properties of the semi-discrete scheme.

We now compute the small h asymptotics of the phase in some cases. Writing $\theta = k_{1,h}h$ we can compute $k_{1,h}$, the discrete wave number, in terms of the other parameters in the scheme. We now study a few special cases.

1. *LDG Scheme*: Choose $C_{2,2} = 0$ (and assuming $C_{1,2} \neq 0$). We obtain

$$k_{1,h} = k_1 + \frac{1}{288}\frac{\mathrm{i}k{k_1}^3 C_{1,1}}{{C_{1,2}}^2}h^3 + \frac{1}{17280}\frac{\left(96{k_1}^2{C_{1,2}}^4 - 20{C_{1,2}}^2{k_1}^2 - 5C_{1,1}^2k^2\right){k_1}^3}{{C_{1,2}}^4}h^4 + O(h^5).$$

   As in the piecewise constant case, the presence of an imaginary term in the series corresponds to dissipation in the method. The first real term in the series (after k_1) determines the phase error of the method. Therefore the dispersion error is $O(h^4)$ compared to $O(h^2)$ for standard linear elements. Motivated by the work of Warburton and Hesthaven we could choose $C_{1,1} = 1/2$ and then if we choose $C_{1,2} = 1/h$ we obtain:

$$k_{1,h} = k_1 + \frac{k_1^5}{180}h^4 + \frac{1}{576}\mathrm{i}kk_1^5h^5 + O(h^6).$$

2. *Centred Scheme*: Setting $C_{1,1} = C_{2,2} = 0$ we obtain

$$k_{1,h} = k_1 + \frac{1}{4320}\frac{\left(24{C_{1,2}}^2 - 5\right){k_1}^5}{{C_{1,2}}^2}h^4 + \frac{1}{870912}\frac{\left(576{C_{1,2}}^4 - 357{C_{1,2}}^2 + 56\right){k_1}^7}{{C_{1,2}}^4}h^6 + O(h^8).$$

   This scheme is not dissipative. We see that the centred scheme can be optimised to give a high order $O(h^6)$ dispersion error by

selecting $C_{1,2}=\sqrt{5/24}$ to yield

$$k_{1,h}=k_1+\frac{53}{302400}k_1{}^7h^6+O(h^8)$$

3. *Full Upwinding*: The currently most practically significant DGFEM scheme in computational electromagnetics is the method of Hesthaven and Warburton [12]. Applied to the wave equation with $p=1$ their choice of fluxes corresponds to full upwinding and is obtained by taking $C_{1,1}=C_{2,2}=1/2$ and $C_{1,2}=0$ (in fact they use a more general choice depending on the coefficients of the partial differential equation, but this reduces to the one here for a simple wave equation). The resulting error series is then

$$k_{1,h}=k_1+\frac{1}{144}\frac{\mathrm{i}k_1{}^3\left(k_1{}^2+k^2\right)}{k}h^3-\frac{1}{4320}\frac{k_1^3(6k_1^2k^2+5k_1^4+5k^4)}{k^2}h^4+O\left(h^5\right)$$

The method is dissipative and dispersive to the same order as the general LDG scheme.
Choosing $C_{1,1}=C_{1,2}=1/2$ and $C_{1,2}=1/h$ gives

$$k_{1,h}=k_1+\frac{k_1^5}{180}h^4+\mathrm{i}\frac{(k_1^2+k^2)k_1^3}{576k}h^5+O(h^6).$$

4. NUMERICAL EXAMPLES FOR HELMHOLTZ EQUATION IN ONE DIMENSION

In this section we show two numerical examples related to the generalised DGFEM scheme. These illustrate some of the properties of these methods and their relationship to our analysis. We assume $k_1=k$ (so $k_2=0$) and consider the one-dimensional time harmonic problem of finding u and v such that

$$\mathrm{i}kv-u'=0,\quad \mathrm{i}ku-v'=0.$$

In order to study plane wave solutions we impose unusual boundary conditions. We demand u and v satisfy

$$\begin{aligned}\mathrm{i}kv-u'=0,&\qquad \mathrm{i}ku-v'=0,\quad \text{in } (0,1)\\ u(0)=1,&\qquad v(1)-u(1)=0.\end{aligned}$$

which has solution $u(x)=v(x)=e^{\mathrm{i}kx}$.

We can then apply the generalised DGFEM scheme to this problem using linear elements on a uniform grid. Our first example is for the LDG scheme with $C_{2,2} = 0$. The dissipation behaviour of the scheme is governed by the sign of $C_{1,1}$. When $C_{1,1} = 0$ we revert to the centred scheme which is dissipation free. If $C_{1,1} > 0$ the method is dissipative and the solution will decay across the domain (dissipation can be advantageous for avoiding high frequency instability in time domain codes). For $C_{1,1} > 0$ the solution will grow. In Fig. 2 we show this using $k = 12\pi$ and a grid resulting in 10 degrees of freedom per wavelength (i.e. 5 grid cells per wavelength).

Our second result is for the fully upwinded scheme using $C_{1,1} = C_{2,2} = 1/2$ and $C_{1,2} = 0$ (and again linear elements). In this case we have shown that the method is dissipative, but also that is has excellent phase accuracy. This is demonstrated in Fig. 3.

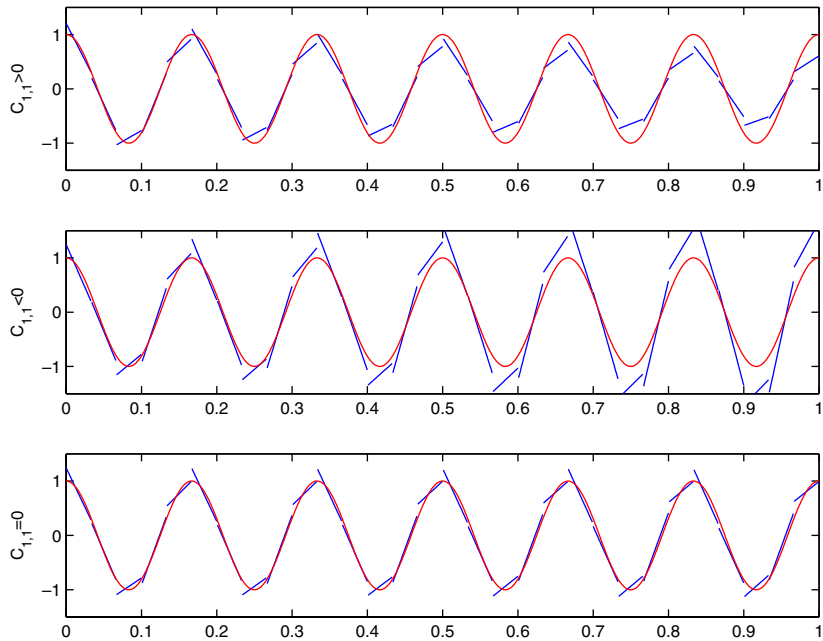

Fig. 2. The real part of u plotted against x for LDG showing decay for $C_{1,1} > 0$, growth for $C_{1,1} < 0$ and dissipation free for $C_{1,1} = 0$. Here $k = 12\pi$ and there are 10 degrees of freedom per wavelength. Note that in each case the solution has excellent phase accuracy maintained across the grid.

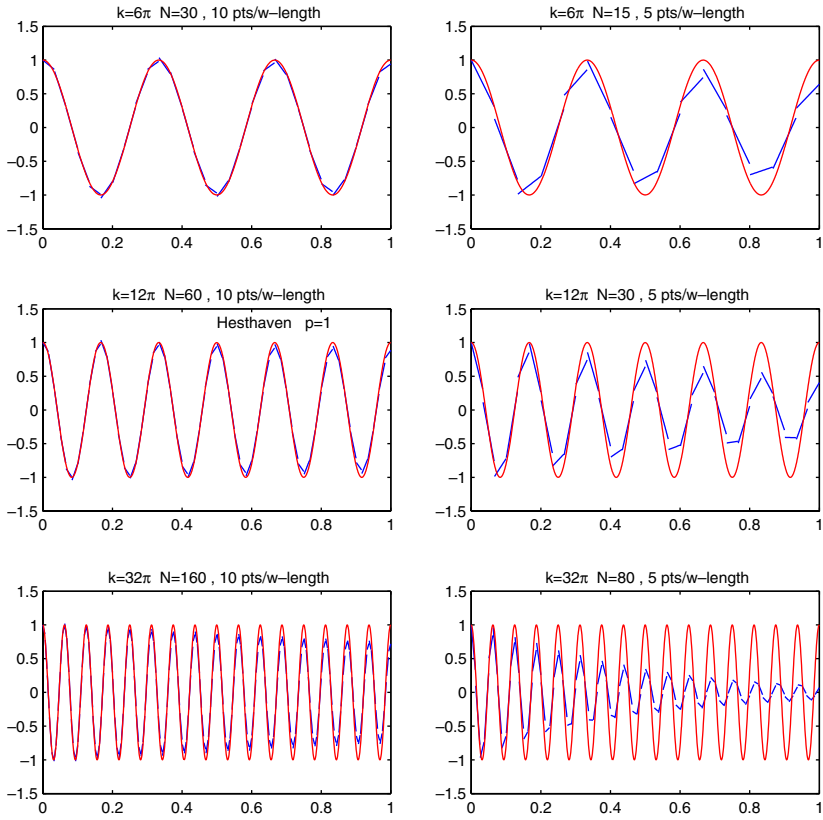

Fig. 3. Results for the fully upwinded scheme. The method is dissipative with dissipation decreasing as the grid is refined. This is illustrated here where we show the approximation to the real part of u when $k = 6\pi$, 12π and 32π and either 5 or 10 degrees of freedom per wavelength. In each case the excellent phase accuracy of the scheme is obvious.

5. CONCLUSION

We have provided a framework for analysing the dispersion relation of various DGFEM schemes and pointed out the peril of assuming that a direct one-dimensional dispersion analysis applies also to the two-dimensional problem.

For the IP-DG scheme considered here we show how to reduce to a pair of one dimensional problems and then have explicitly analysed some cases. Based on the results reported in Sec. 2.1, we conjecture that:

- For pth degree polynomials there is a stable choice of γ and, for all but one exceptional choice of γ, the order of convergence of the

dispersion relation is $O(k(kh)^{2p})$. This is in accordance with results for standard p degree conforming elements [1] and suggests that IP-DG performs asymptotically no better or worse than standard elements from the point of view of phase error.

- When the order p is even there is an optimal choice of the stability parameter given by $\alpha = (p+1)(p+2)/2h$ that offers an increase in the order of accuracy in the dispersion error to $O(k(kh)^{2p+2})$. For odd degree p there is no such choice.
- As the order p increases, the value of the stability parameter needed to eliminate the spurious mode grows as $p(p+1)/2h$ which is consistent with the choice assumed in the *a priori* analyses.

Of course we have analysed just one representative IP-DG scheme. The general approach we have made should be applicable, for example, to the method of [17] but the analysis has yet to be performed.

For general DGFEM we have shown the following

- The dispersion relation does decompose into one-dimensional problems, but these problems are not obtained by the direct application of DGFEM to one-dimensional problems. Thus the straightforward analysis of one-dimensional DGFEM problems does not apply to the multi-dimensional problem (except for wave propagation along coordinate directions).
- There are many possible "good" choices of parameters. The centred scheme is attractive in that it is non-dissipative and can be made to satisfy the limiting stability constraint with a suitable choice of the coupling parameters.
- Much more work needs to be done to fully analyse the dispersion properties of higher order elements, and to test the various methods numerically.

ACKNOWLEDGMENTS

The research of MA was supported in part by a Leverhulme Research Fellowship. The research of PM was supported by a grant from AFOSR and by an EPSRC visitor grant to the University of Reading, UK.

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