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A Matrix Decomposition MFS Algorithm for Problems in Hollow Axisymmetric Domains

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In this work we apply the Method of Fundamental Solutions (MFS) with fixed singularities and boundary collocation to certain axisymmetric harmonic and biharmonic problems. By exploiting the block circulant structure of the coefficient matrices appearing when the MFS is applied to such problems, we develop efficient matrix decomposition algorithms for their solution. The algorithms are tested on several examples.

KEY WORDS: Method of fundamental solutions; Laplace equation; biharmonic equation; hollow axisymmetric domains; circulant matrices; fast Fourier transform.

AMS SUBJECT CLASSIFICATION: Primary 65N12; 65N38; Secondary 65N15; 65T50; 35J25.

1. INTRODUCTION

In this paper, we investigate the application of the Method of Fundamental Solutions (MFS) to certain axisymmetric harmonic and biharmonic problems. In particular, we consider the MFS with fixed singularities for harmonic and biharmonic problems in axisymmetric hollow domains. We extend the ideas developed in [14], where the MFS is applied to harmonic problems in axisymmetric simply-connected domains, and [8], where the MFS is applied to the corresponding biharmonic problems. In the problems examined in this study, the MFS discretization leads to linear systems the coefficient matrices of which have block circulant structures. Matrix decomposition algorithms are developed for the efficient

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solution of these systems. These algorithms also make use of fast Fourier transforms (FFT). Comprehensive reviews of the recent developments and applications of the MFS and related methods may be found in the survey papers [2, 6, 7, 10]. Also, the books [4, 9, 11] provide useful information concerning various implementational and theoretical aspects of the MFS. In domain–discretization techniques such as finite element and finite difference methods, the reduction of the three-dimensional axisymmetric problem to a two-dimensional problem governed by the axisymmetric version of the governing equation is important because of the complications involved in the discretization of three-dimensional domains [5]. This difficulty is not as pronounced in the MFS as it is a meshless method. Further, the fundamental solutions of these (two-dimensional) equations are complicated and involve complete elliptic integrals. Finally, when the boundary conditions of the problem are not axisymmetric, the two-dimensional approach requires the solution of a sequence of boundary value problems. The approach that we are suggesting in this study avoids these complications.

2. THE HARMONIC CASE

2.1. MFS Formulation

We consider the three-dimensional boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega_1, \\ u &= g \quad \text{on } \partial\Omega_2, \end{aligned}$$

where Δ denotes the Laplace operator and f is a given function. The region $\Omega \subset \mathbb{R}^3$ is axisymmetric, which means that it is formed by rotating a region $\Omega' \in \mathbb{R}^2$ about the z-axis. The boundary of Ω is $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and the boundary of Ω' is defined by the two boundary segments $\partial\Omega_1'$ and $\partial\Omega_2'$, which generate $\partial\Omega_1$ and $\partial\Omega_2$, respectively. In the MFS [6, 14], the solution u is approximated by

$$\begin{aligned} u_{MN}(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{R}, \boldsymbol{S}; P) = &\sum_{m=1}^{M}\sum_{n=1}^{N} c_{m,n}\, k_1(P, R_{m,n}) \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{N} d_{m,n}\, k_1(P, S_{m,n}), \qquad P \in \overline{\Omega}, \end{aligned}$$

where $\boldsymbol{c} = (c_{11}, c_{12}, \ldots, c_{1N}, \ldots, c_{M1}, \ldots, c_{MN})^T$, $\boldsymbol{d} = (d_{11}, d_{12}, \ldots, d_{1N}, \ldots, d_{M1}, \ldots, d_{MN})^T$ and $\boldsymbol{R}$, $\boldsymbol{S}$ are $3MN$-vectors containing the coordinates of the singularities (sources) $R_{m,n},\ S_{m,n},\ m=1,\dots,M,\ n=1,\dots,N$, which lie outside $\overline{\Omega}$. The function $k_1(P,R)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^3$ given by

$$k_1(P,R)=\frac{1}{4\pi\,|P-R|}$$

with $|P-R|$ denoting the distance between the points P and R. The singularities $R_{m,n},\ S_{m,n}$ are fixed on the boundary $\partial\tilde{\Omega}=\partial\tilde{\Omega}_1\cup\partial\tilde{\Omega}_2$ of a solid $\tilde{\Omega}$ surrounding Ω. The solid $\tilde{\Omega}$ is generated by the rotation of the planar domain $\tilde{\Omega}'$ which is similar to Ω'. Clearly $\partial\tilde{\Omega}_1$ and $\partial\tilde{\Omega}_2$ are similar to $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Also, the boundary of $\tilde{\Omega}'$ is defined by the segments $\partial\tilde{\Omega}_1'$ and $\partial\tilde{\Omega}_2'$, which generate $\partial\tilde{\Omega}_1$ and $\partial\tilde{\Omega}_2$, respectively. A set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on $\partial\Omega_1$ and a set of MN collocation points $\{Q_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on $\partial\Omega_2$ in the following way: We first choose N points $\{P_j\}_{j=1}^{N}$ on the boundary segment $\partial\Omega_1'$ and N points $\{Q_j\}_{j=1}^{N}$ on $\partial\Omega_2'$. These can be described by their polar coordinates $(r_{P_j},z_{P_j}),\ (r_{Q_j},z_{Q_j}),\ j=1,\dots,N$, where $r_{P_j},\ r_{Q_j}$ denotes the vertical distance of the points $P_j,\ Q_j$ from the z-axis and $z_{P_j},\ z_{Q_j}$ denotes the z-coordinate of the points $P_j,\ Q_j$-respectively. The points on $\partial\Omega_1$ are, taken to be

$$x_{P_{i,j}}=r_{P_j}\cos\varphi_i,\qquad y_{P_{i,j}}=r_{P_j}\sin\varphi_i,\qquad z_{P_{i,j}}=z_{P_j}$$

and the points on $\partial\Omega_2$ are

$$x_{Q_{i,j}}=r_{Q_j}\cos\varphi_i,\qquad y_{Q_{i,j}}=r_{Q_j}\sin\varphi_i,\qquad z_{Q_{i,j}}=z_{Q_j},$$

where $\varphi_i=2(i-1)\pi/M,\ i=1,\dots,M$. Similarly, we choose a set of MN singularities $\{R_{m,n}\}_{m=1,n=1}^{M,N}$ on $\partial\tilde{\Omega}_1$ and a set of MN singularities $\{S_{m,n}\}_{m=1,n=1}^{M,N}$ on $\partial\tilde{\Omega}_2$ by taking $R_{m,n}=(x_{R_{m,n}},y_{R_{m,n}},z_{R_{m,n}}),\ S_{m,n}=(x_{S_{m,n}},y_{S_{m,n}},z_{S_{m,n}})$, and

$$\begin{aligned}x_{R_{m,n}}&=r_{R_n}\cos\psi_m, & y_{Q_{m,n}}&=r_{R_n}\sin\psi_m, & z_{R_{m,n}}&=z_{R_n},\\ x_{S_{m,n}}&=r_{S_n}\cos\psi_m, & y_{S_{m,n}}&=r_{S_n}\sin\psi_m, & z_{S_{m,n}}&=z_{S_n},\end{aligned}$$

where $\psi_i=2(\alpha+i-1)\pi/M,\ i=1,\dots,M$. The parameter $\alpha\in[-1/2,1/2]$ describes the rotation of the singularities in the azimuthal direction. The N points R_j are chosen on $\partial\tilde{\Omega}_1'$ whereas the N points S_j are chosen on $\partial\tilde{\Omega}_2'$. The coefficients $\boldsymbol{c}$ and $\boldsymbol{d}$ are determined so that the boundary condition is satisfied at the boundary points $\{P_{i,j}\}_{i=1,j=1}^{M,N},\ \{Q_{i,j}\}_{i=1,j=1}^{M,N}$:

$$u_{MN}(\boldsymbol{c},\boldsymbol{d},\boldsymbol{R},\boldsymbol{S};P_{i,j})=f_1(P_{i,j}),\qquad u_{MN}(\boldsymbol{c},\boldsymbol{d},\boldsymbol{R},\boldsymbol{S};Q_{i,j})=f_2(P_{i,j}),$$

$i = 1, \dots, M,\ j = 1, \dots, N$. This yields an $2MN \times 2MN$ linear system of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c} \boldsymbol{c} \\ \hline \boldsymbol{d} \end{array}\right) = \left(\begin{array}{c} \boldsymbol{f} \\ \hline \boldsymbol{g} \end{array}\right), \tag{2.1}$$

where the matrices $A,\ B,\ C$ and D are block circulant [3] $MN \times MN$ matrices, that is

$$A = \operatorname{circ}(A_1, A_2, \dots, A_N), \qquad B = \operatorname{circ}(B_1, B_2, \dots, B_N),$$
$$C = \operatorname{circ}(C_1, C_2, \dots, C_N), \qquad D = \operatorname{circ}(D_1, D_2, \dots, D_N).$$

The matrices $A_\ell, B_\ell, C_\ell, D_\ell, \ell = 1, \dots, M$, are $N \times N$ matrices defined by

$$(A_\ell)_{j,n} = \frac{1}{4\pi\,|P_{1,j} - R_{\ell,n}|}, \qquad (B_\ell)_{j,n} = \frac{1}{4\pi\,|P_{1,j} - S_{\ell,n}|},$$
$$(C_\ell)_{j,n} = \frac{1}{4\pi\,|Q_{1,j} - R_{\ell,n}|}, \qquad (D_\ell)_{j,n} = \frac{1}{4\pi\,|Q_{1,j} - S_{\ell,n}|},$$

$\ell = 1, \dots, M\ j, n = 1, \dots, N$. The system (2.1) can therefore be written as

$$\left(\sum_{\ell=1}^{M} \mathcal{P}^{\ell-1} \otimes A_\ell\right)\boldsymbol{c} + \left(\sum_{\ell=1}^{M} \mathcal{P}^{\ell-1} \otimes B_\ell\right)\boldsymbol{d} = \boldsymbol{f},$$
$$\left(\sum_{\ell=1}^{M} \mathcal{P}^{\ell-1} \otimes C_\ell\right)\boldsymbol{c} + \left(\sum_{\ell=1}^{M} \mathcal{P}^{\ell-1} \otimes D_\ell\right)\boldsymbol{d} = \boldsymbol{g},$$

where the matrix $\mathcal{P}$ is the $M \times M$ permutation matrix $\mathcal{P} = \operatorname{circ}(0, 1, 0, \dots, 0)$ and $\otimes$ denotes the matrix tensor product [12].

2.2. Matrix Decomposition Algorithm

In the case we are examining, a Matrix Decomposition Algorithm [1] involves the reduction of the $2MN \times 2MN$ system (2.1) to M decoupled $2N \times 2N$ systems. This is achieved by exploiting the block circulant structure of the matrices $A,\ B,\ C$ and D. If U is the unitary $M \times M$ Fourier matrix, it is well-known [3, 13] that circulant matrices are diagonalized in the following way. If $C = \operatorname{circ}(c_1, \dots, c_M)$, then $C = U^* D U$, where $D = \operatorname{diag}(\hat{c}_1, \dots, \hat{c}_M)$, and

$$\hat{c}_j = \sum_{k=1}^{M} c_k\, \omega^{(k-1)(j-1)}.$$

In particular, the permutation matrix $\mathcal{P} = \text{circ}(0, 1, 0, \dots, 0)$ is diagonalized as $\mathcal{P} = U^* T U$, where

$$T = \text{diag}\big(1, \omega, \omega^2, \dots, \omega^{M-1}\big), \quad \omega = \mathrm{e}^{2\pi \mathrm{i}/M}. \tag{2.2}$$

Next we simplify system (2.1). Let

$$\left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right) = \left(\begin{array}{c|c} U \otimes \mathcal{I}_N & 0 \\ \hline 0 & U \otimes \mathcal{I}_N \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} U^* \otimes \mathcal{I}_N & 0 \\ \hline 0 & U^* \otimes \mathcal{I}_N \end{array}\right)$$

and

$$\left(\begin{array}{c} \tilde{\boldsymbol{c}} \\ \hline \tilde{\boldsymbol{d}} \end{array}\right) = \left(\begin{array}{c|c} U \otimes \mathcal{I}_N & 0 \\ \hline 0 & U \otimes \mathcal{I}_N \end{array}\right) \left(\begin{array}{c} \boldsymbol{c} \\ \hline \boldsymbol{d} \end{array}\right), \qquad \left(\begin{array}{c} \tilde{\boldsymbol{f}} \\ \hline \tilde{\boldsymbol{g}} \end{array}\right) = \left(\begin{array}{c|c} U \otimes \mathcal{I}_N & 0 \\ \hline 0 & U \otimes \mathcal{I}_N \end{array}\right) \left(\begin{array}{c} \boldsymbol{f} \\ \hline \boldsymbol{g} \end{array}\right).$$

Then, after pre-multiplication by $\mathcal{I}_2 \otimes U \otimes \mathcal{I}_N$, (2.1) becomes

$$\left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right) \left(\begin{array}{c} \tilde{\boldsymbol{c}} \\ \hline \tilde{\boldsymbol{d}} \end{array}\right) = \left(\begin{array}{c} \tilde{\boldsymbol{f}} \\ \hline \tilde{\boldsymbol{g}} \end{array}\right). \tag{2.3}$$

Since

$$A = \sum_{k=1}^{M} \mathcal{P}^{k-1} \otimes A_k,$$

then

$$\begin{aligned} \tilde{A} &= (U \otimes \mathcal{I}_N)\bigg(\sum_{k=1}^{M} \mathcal{P}^{k-1} \otimes A_k\bigg)(U^* \otimes \mathcal{I}_N) \\ &= \sum_{k=1}^{M} \big(U\, \mathcal{P}^{k-1}\, U^*\big) \otimes A_k = \sum_{k=1}^{M} T^{k-1} \otimes A_k \end{aligned}$$

and similarly

$$\tilde{B} = \sum_{k=1}^{M} T^{k-1} \otimes B_k, \qquad \tilde{C} = \sum_{k=1}^{M} T^{k-1} \otimes C_k \quad \text{and} \quad \tilde{D} = \sum_{k=1}^{M} T^{k-1} \otimes D_k.$$

The system (2.3) can therefore be decomposed into M *decoupled* $2N \times 2N$ systems

$$\boldsymbol{G}_m \left(\frac{\tilde{\boldsymbol{c}}_m}{\tilde{\boldsymbol{d}}_m} \right) = \left(\frac{\tilde{\boldsymbol{f}}_m}{\tilde{\boldsymbol{g}}_m} \right), \quad m = 1, \ldots, M,$$

where $\tilde{\boldsymbol{f}}_m = \left(\tilde{f}_{m,1}, \ldots, \tilde{f}_{m,N}\right)^T$, $\tilde{\boldsymbol{g}}_m = \left(\tilde{g}_{m,1}, \ldots, \tilde{g}_{m,N}\right)^T$,

$$\boldsymbol{G}_m = \left(\begin{array}{c|c} \hat{A}_m & \hat{B}_m \\ \hline \hat{C}_m & \hat{D}_m \end{array} \right) \tag{2.4}$$

and

$$\begin{aligned} \hat{A}_m &= \sum_{j=1}^{M} \omega^{(m-1)(j-1)} A_j, & \hat{B}_m &= \sum_{j=1}^{M} \omega^{(m-1)(j-1)} B_j, \\ \hat{C}_m &= \sum_{j=1}^{M} \omega^{(m-1)(j-1)} C_j, & \hat{D}_m &= \sum_{j=1}^{M} \omega^{(m-1)(j-1)} D_j, \end{aligned} \tag{2.5}$$

$m = 1, \ldots, M$. We thus have the following efficient algorithm for the solution of system (2.3).

Algorithm

Step 1. `Compute` $\tilde{\boldsymbol{f}} = (U \otimes \mathcal{I}_N)\, \boldsymbol{f}, \quad \tilde{\boldsymbol{g}} = (U \otimes \mathcal{I}_N)\, \boldsymbol{g}$.
Step 2. `Construct the matrices` $\boldsymbol{G}_m, \quad m = 1, \ldots, M$.
Step 3. `Solve the` M `systems` (2.4).
Step 4. `Compute` $\boldsymbol{c} = (U^* \otimes \mathcal{I}_N)\, \tilde{\boldsymbol{c}}, \ \boldsymbol{d} = (U^* \otimes \mathcal{I}_N)\, \tilde{\boldsymbol{d}}$.

Remarks. In Step 1, because of the form of the matrix U, the operation can be carried out a cost of $\mathcal{O}(NM \log M)$ via an appropriate FFT algorithm. Similarly, in Step 4, because of the form of the matrix U^*, the operation can be carried out via inverse FFTs at a cost of order $\mathcal{O}(NM \log M)$ operations. In Step 2 we need to perform an M-dimensional inverse FFT, in order to compute the entries of the matrices $\hat{A}_j, \hat{B}_j, \hat{C}_j, \hat{D}_j$, $j = 1, \ldots, M$, from (2.5). which done at a cost of $\mathcal{O}(N^2 M \log M)$ operations. In Step 3, we need to solve M complex linear systems of order N, which is done using an LU-factorization with partial pivoting at a cost of $\mathcal{O}(MN^3)$ operations. The FFT and inverse FFT operations are performed using the NAG[1] routines C06EAF, C06FPF, C06FQF and C06FRF.

3. THE BIHARMONIC CASE

3.1. MFS Formulation

We now consider the three-dimensional boundary value problem

$$\begin{aligned} \Delta^2 u &= 0 && \text{in } \Omega, \\ u = f_1 \quad &\text{and } \frac{\partial u}{\partial n} = g_1 && \text{on } \partial\Omega_1, \\ u = f_2 \quad &\text{and } \frac{\partial u}{\partial n} = g_2 && \text{on } \partial\Omega_2. \end{aligned} \tag{3.1}$$

The region $\Omega \in \mathbb{R}^3$ is axisymmetric and, as in the harmonic case, formed by rotating a region $\Omega' \in \mathbb{R}^2$ about the z-axis. We keep the same notation as in the harmonic case for the boundaries of Ω and Ω'. In the MFS [6, 8], the solution u is approximated by

$$\begin{aligned} u_{MN}(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{R}, \boldsymbol{S}; P) = &\sum_{m=1}^{M}\sum_{n=1}^{N} c_{m,n}^1\, k_1(P, R_{m,n}) \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{N} c_{m,n}^2\, k_2(P, R_{m,n}) \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{N} d_{m,n}^1\, k_1(P, S_{m,n}) \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{N} d_{m,n}^2\, k_2(P, S_{m,n}), \quad P \in \overline{\Omega}, \end{aligned} \tag{3.2}$$

where $\boldsymbol{c}_1 = (c_{11}^1, c_{12}^1, \ldots, c_{1N}^1, \ldots, c_{M1}^1, \ldots, c_{MN}^1)^T, \boldsymbol{c}_2 = (c_{11}^2, c_{12}^2, \ldots, c_{1N}^2, \ldots, c_{M1}^2, \ldots, c_{MN}^2)^T, \boldsymbol{d}_1 = (d_{11}^1, d_{12}^1, \ldots, d_{1N}^1, \ldots, d_{M1}^1, \ldots, d_{MN}^1)^T, \boldsymbol{d}_2 = (d_{11}^2, d_{12}^2, \ldots, d_{1N}^2, \ldots, d_{M1}^2, \ldots, d_{MN}^2)^T$, and $\boldsymbol{R}$, $\boldsymbol{S}$ are $3MN$-vectors containing the coordinates of the singularities $R_{m,n}$, $S_{m,n}$, $m = 1, \ldots, M$, $n = 1, \ldots, N$, which lie outside $\overline{\Omega}$. The function $k_2(P, S)$ is the fundamental solution of the the biharmonic equation in $\mathbb{R}^3$ given by

[1] *Numerical Algorithms Group Library Mark 20*, NAG Ltd, Wilkinson House, Jordan Hill Road, Oxford, UK, 2001.

$$k_2(P, R) = \frac{1}{8\pi} |P - R|.$$

The $2MN$ collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$, $\{Q_{i,j}\}_{i=1,j=1}^{M,N}$ and the $2MN$ singularities $\{P_{m,n}\}_{m=1,n=1}^{M,N}$, $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ are chosen in exactly the same way as in the harmonic case. The coefficients $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are determined so that the boundary conditions are satisfied at the boundary points:

$$\begin{aligned}
&u_{MN}(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{R}, \boldsymbol{S}; P_{i,j}) = f_1(P_{i,j}),\\
&u_{MN}(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{R}, \boldsymbol{S}; Q_{i,j}) = f_2(P_{i,j}),\\
&\frac{\partial}{\partial n} u_{MN}(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{R}, \boldsymbol{S}; P_{i,j}) = g_1(P_{i,j}),\\
&\frac{\partial}{\partial n} u_{MN}(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{R}, \boldsymbol{S}; Q_{i,j}) = g_2(P_{i,j}),
\end{aligned} \tag{3.3}$$

$i = 1, \dots, M,\ j = 1, \dots, N$. This yields an $4MN \times 4MN$ linear system of the form

$$\left(\begin{array}{c|c|c|c} A^{11} & A^{12} & A^{13} & A^{14} \\ \hline A^{21} & A^{22} & A^{23} & A^{24} \\ \hline A^{31} & A^{32} & A^{33} & A^{34} \\ \hline A^{41} & A^{42} & A^{43} & A^{44} \end{array}\right) \left(\begin{array}{c} \boldsymbol{c}_1 \\ \hline \boldsymbol{c}_2 \\ \hline \boldsymbol{d}_1 \\ \hline \boldsymbol{d}_2 \end{array}\right) = \left(\begin{array}{c} \boldsymbol{f}_1 \\ \hline \boldsymbol{f}_2 \\ \hline \boldsymbol{g}_1 \\ \hline \boldsymbol{g}_2 \end{array}\right), \tag{3.4}$$

where the matrices A^{rs}, $r, s = 1, 2, 3, 4$ are *block circulant* $MN \times MN$ matrices, that is

$$A^{rs} = \mathrm{circ}\big(A_1^{rs}, A_2^{rs}, \dots, A_M^{rs}\big), \quad r, s = 1, 2, 3, 4.$$

The matrices A^{rs}, $r, s = 1, 2, 3, 4$, can be written as

$$A^{rs} = \Big(\mathcal{I}_M \otimes A_1^{rs} + \mathcal{P} \otimes A_2^{rs} + \mathcal{P}^2 \otimes A_3^{rs} + \cdots + \mathcal{P}^{M-1} \otimes A_M^{rs}\Big).$$

For $\ell = 1, \dots, M$ the $N \times N$ submatrices $A_\ell^{rs} = \big((A_\ell^{rs})_{j,n}\big)$, are defined by

$$\begin{aligned}
\left(A_\ell^{11}\right)_{j,n} &= \frac{1}{4\pi} \frac{1}{|P_{1,j} - R_{\ell,n}|}, & \left(A_\ell^{12}\right)_{j,n} &= \frac{1}{4\pi} \frac{1}{|P_{1,j} - S_{\ell,n}|},\\
\left(A_\ell^{13}\right)_{j,n} &= \frac{1}{8\pi} |P_{1,j} - R_{\ell,n}|, & \left(A_\ell^{14}\right)_{j,n} &= \frac{1}{8\pi} |P_{1,j} - S_{\ell,n}|,\\
\left(A_\ell^{21}\right)_{j,n} &= \frac{1}{4\pi} \frac{1}{|Q_{1,j} - R_{\ell,n}|}, & \left(A_\ell^{22}\right)_{j,n} &= \frac{1}{4\pi} \frac{1}{|Q_{1,j} - S_{\ell,n}|},\\
\left(A_\ell^{23}\right)_{j,n} &= \frac{1}{8\pi} |Q_{1,j} - R_{\ell,n}|, & \left(A_\ell^{24}\right)_{j,n} &= \frac{1}{8\pi} |Q_{1,j} - S_{\ell,n}|,
\end{aligned}$$

$$\left(A_\ell^{31}\right)_{j,n} = \frac{1}{4\pi}\frac{\partial}{\partial n}\left[\frac{1}{|P_{1,j}-R_{\ell,n}|}\right], \qquad \left(A_\ell^{32}\right)_{j,n} = \frac{1}{4\pi}\frac{\partial}{\partial n}\left[\frac{1}{|P_{1,j}-S_{\ell,n}|}\right],$$

$$\left(A_\ell^{33}\right)_{j,n} = \frac{1}{8\pi}\frac{\partial}{\partial n}|P_{1,j}-R_{\ell,n}|, \qquad \left(A_\ell^{34}\right)_{j,n} = \frac{1}{8\pi}\frac{\partial}{\partial n}|P_{1,j}-S_{\ell,n}|,$$

$$\left(A_\ell^{41}\right)_{j,n} = \frac{1}{4\pi}\frac{\partial}{\partial n}\frac{1}{|Q_{1,j}-R_{\ell,n}|}, \qquad \left(A_\ell^{42}\right)_{j,n} = \frac{1}{4\pi}\frac{\partial}{\partial n}\frac{1}{|Q_{1,j}-S_{\ell,n}|},$$

$$\left(A_\ell^{43}\right)_{j,n} = \frac{1}{8\pi}\frac{\partial}{\partial n}|Q_{1,j}-R_{\ell,n}|, \qquad \left(A_\ell^{44}\right)_{j,n} = \frac{1}{8\pi}\frac{\partial}{\partial n}|Q_{1,j}-S_{\ell,n}|.$$

3.2. Matrix Decomposition Algorithm

In this case, a Matrix Decomposition Algorithm involves the reduction of the $4MN \times 4MN$ system (3.4) to M decoupled $4N \times 4N$ systems. Let us denote by H the $4MN \times 4MN$ matrix

$$H = \left(\begin{array}{c|c|c|c} A^{11} & A^{12} & A^{13} & A^{14} \\ \hline A^{21} & A^{22} & A^{23} & A^{24} \\ \hline A^{31} & A^{32} & A^{33} & A^{34} \\ \hline A^{41} & A^{42} & A^{43} & A^{44} \end{array}\right)$$

$$= \sum_{k=0}^{M-1} \left(\begin{array}{c|c|c|c} \mathcal{P}^k \otimes A_{k+1}^{11} & \mathcal{P}^k \otimes A_{k+1}^{12} & \mathcal{P}^k \otimes A_{k+1}^{13} & \mathcal{P}^k \otimes A_{k+1}^{14} \\ \hline \mathcal{P}^k \otimes A_{k+1}^{21} & \mathcal{P}^k \otimes A_{k+1}^{22} & \mathcal{P}^k \otimes A_{k+1}^{23} & \mathcal{P}^k \otimes A_{k+1}^{24} \\ \hline \mathcal{P}^k \otimes A_{k+1}^{31} & \mathcal{P}^k \otimes A_{k+1}^{32} & \mathcal{P}^k \otimes A_{k+1}^{33} & \mathcal{P}^k \otimes A_{k+1}^{34} \\ \hline \mathcal{P}^k \otimes A_{k+1}^{41} & \mathcal{P}^k \otimes A_{k+1}^{42} & \mathcal{P}^k \otimes A_{k+1}^{43} & \mathcal{P}^k \otimes A_{k+1}^{44} \end{array}\right).$$

Clearly,

$$(U \otimes \mathcal{I}_N)\,(\mathcal{P}^{k-1} \otimes A_k^{rs})\,(U^* \otimes \mathcal{I}_N) = (U\mathcal{P}^{k-1}U^*) \otimes A_k^{rs} = T^{k-1} \otimes A_k^{rs}$$

for $k = 1, \ldots, M$ and $r, s = 1, 2, 3, 4$. Pre-multiplication of system (3.4) by $\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N$ yields

$$(\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N)\, H\, (\mathcal{I}_4 \otimes U^* \otimes \mathcal{I}_N)\,(\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N)\,\boldsymbol{s} = (\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N)\,\boldsymbol{t}, \quad (3.5)$$

where $\boldsymbol{s} = [\boldsymbol{c}_1|\boldsymbol{c}_2|\boldsymbol{d}_1|\boldsymbol{d}_2]^T$, $\boldsymbol{t} = [\boldsymbol{f}_1|\boldsymbol{f}_2|\boldsymbol{g}_1|\boldsymbol{g}_2]^T$, since $(U^* \otimes \mathcal{I}_N)(U \otimes \mathcal{I}_N) = \mathcal{I}_{MN}$. The system (3.5) can be written alternatively

$$\hat{H}\hat{\boldsymbol{s}} = \hat{\boldsymbol{t}}, \qquad (3.6)$$

where

$$\hat{\boldsymbol{s}} = (\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N)\, \boldsymbol{s} = \left(\begin{array}{c} (U \otimes \mathcal{I}_N)\, \boldsymbol{c}_1 \\ \hline (U \otimes \mathcal{I}_N)\, \boldsymbol{c}_2 \\ \hline (U \otimes \mathcal{I}_N)\, \boldsymbol{d}_1 \\ \hline (U \otimes \mathcal{I}_N)\, \boldsymbol{d}_2 \end{array} \right) = \left(\begin{array}{c} \tilde{\boldsymbol{c}}_1 \\ \hline \tilde{\boldsymbol{c}}_2 \\ \hline \tilde{\boldsymbol{d}}_1 \\ \hline \tilde{\boldsymbol{d}}_2 \end{array} \right),$$

$$\hat{\boldsymbol{t}} = (\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N)\, \boldsymbol{t} = \left(\begin{array}{c} (U \otimes \mathcal{I}_N)\, \boldsymbol{f}_1 \\ \hline (U \otimes \mathcal{I}_N)\, \boldsymbol{f}_2 \\ \hline (U \otimes \mathcal{I}_N)\, \boldsymbol{g}_1 \\ \hline (U \otimes \mathcal{I}_N)\, \boldsymbol{g}_2 \end{array} \right) = \left(\begin{array}{c} \tilde{\boldsymbol{f}}_1 \\ \hline \tilde{\boldsymbol{f}}_2 \\ \hline \tilde{\boldsymbol{g}}_1 \\ \hline \tilde{\boldsymbol{g}}_2 \end{array} \right)$$

and

$$\begin{aligned} \hat{H} &= (\mathcal{I}_4 \otimes U \otimes \mathcal{I}_N)\, H\, (\mathcal{I}_4 \otimes U^* \otimes \mathcal{I}_N) \\ &= \sum_{k=0}^{M-1} \left(\begin{array}{c|c|c|c} T^k \otimes A_{k+1}^{11} & T^k \otimes A_{k+1}^{12} & T^k \otimes A_{k+1}^{13} & T^k \otimes A_{k+1}^{14} \\ \hline T^k \otimes A_{k+1}^{21} & T^k \otimes A_{k+1}^{22} & T^k \otimes A_{k+1}^{23} & T^k \otimes A_{k+1}^{24} \\ \hline T^k \otimes A_{k+1}^{31} & T^k \otimes A_{k+1}^{32} & T^k \otimes A_{k+1}^{33} & T^k \otimes A_{k+1}^{34} \\ \hline T^k \otimes A_{k+1}^{41} & T^k \otimes A_{k+1}^{42} & T^k \otimes A_{k+1}^{43} & T^k \otimes A_{k+1}^{44} \end{array} \right), \end{aligned}$$

where T is given by (2.2). The solution of system (3.6) can therefore be decomposed into the solution of the M independent $4N \times 4N$ systems

$$H_m \left(\begin{array}{c} \tilde{\boldsymbol{c}}_1^m \\ \hline \tilde{\boldsymbol{c}}_2^m \\ \hline \tilde{\boldsymbol{d}}_1^m \\ \hline \tilde{\boldsymbol{d}}_2^m \end{array} \right) = \left(\begin{array}{c} \tilde{\boldsymbol{f}}_1^m \\ \hline \tilde{\boldsymbol{f}}_2^m \\ \hline \tilde{\boldsymbol{g}}_1^m \\ \hline \tilde{\boldsymbol{g}}_2^m \end{array} \right), \quad m = 1, \ldots, M, \tag{3.7}$$

where

$$H_m = \left(\begin{array}{c|c|c|c} \hat{A}_j^{11} & \hat{A}_m^{12} & \hat{A}_m^{13} & \hat{A}_m^{14} \\ \hline \hat{A}_m^{21} & \hat{A}_m^{22} & \hat{A}_m^{23} & \hat{A}_m^{24} \\ \hline \hat{A}_m^{31} & \hat{A}_m^{32} & \hat{A}_m^{33} & \hat{A}_m^{34} \\ \hline \hat{A}_m^{41} & \hat{A}_m^{42} & \hat{A}_m^{43} & \hat{A}_m^{44} \end{array} \right) \quad \text{and} \quad \hat{A}_m^{k\ell} = \sum_{j=1}^{M} \omega^{(m-1)(j-1)}\, A_j^{k\ell}, \tag{3.8}$$

$k, \ell = 1, 2, 3, 4,\ m = 1, \ldots, M$. We thus have the following efficient algorithm

Algorithm

Step 1. `Compute` $\tilde{\boldsymbol{f}}_1 = (U \otimes \mathcal{I}_N)\boldsymbol{f}_1, \tilde{\boldsymbol{f}}_2 = (U \otimes \mathcal{I}_N)\boldsymbol{f}_2, \tilde{\boldsymbol{g}}_1 = (U \otimes \mathcal{I}_N)\boldsymbol{g}_1,\ \tilde{\boldsymbol{g}}_2 = (U \otimes \mathcal{I}_N)\boldsymbol{g}_2$.
Step 2. `Construct the matrices` $\boldsymbol{H}_m\ \ m = 1, \dots, M$.
Step 3. `Solve(3.7).`
Step 4. `Compute` $\boldsymbol{c}_1 = (U^* \otimes \mathcal{I}_N)\tilde{\boldsymbol{c}}_1, \boldsymbol{c}_2 = (U^* \otimes \mathcal{I}_N)\tilde{\boldsymbol{c}}_2, \boldsymbol{d}_1 = (U^* \otimes \mathcal{I}_N)\tilde{\boldsymbol{d}}_1,\ \boldsymbol{d}_2 = (U^* \otimes \mathcal{I}_N)\tilde{\boldsymbol{d}}_2$.

Remarks. In Step 1 the operation can be carried out at a cost of $\mathcal{O}(NM \log M)$ using FFTs. Similarly, Step 4 can be carried out at a cost of order $\mathcal{O}(NM \log M)$. In Step 2, we need to perform an M-dimensional inverse FFT, in order to compute the matrices $\hat{A}_k^{rs}, r, s = 1, 2, 3, 4,\ \ k = 1, \dots, M$, from (3.8). This can be done at a cost of $\mathcal{O}(N^2 M \log M)$ operations. In Step 3, we need to solve M complex linear systems of order N. This is done using an LU-factorization with partial pivoting at a cost of $\mathcal{O}(M\,N^3)$ operations.

4. EXAMPLES OF AXISYMMETRIC SOLIDS

4.1. Case I: Thick Spherical Shell

We first consider the case where the domain $\Omega \subset \mathbb{R}^3$ is the 3-dimensional domain defined by $\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \varrho_1 < |\boldsymbol{x}| < \varrho_2\right\}$. In this case, the $2MN$ singularities $R_{m,n}$ and $S_{m,n}$ are fixed on the boundary $\partial\tilde{\Omega} = \partial\tilde{\Omega}_1 \cup \partial\tilde{\Omega}_2$ of the 3-dimensional domain defined by $\tilde{\Omega} = \{\boldsymbol{x} \in \mathbb{R}^3 : R_1 < |\boldsymbol{x}| < R_2\}$ where $R_1 < \varrho_1 < \varrho_2 < R_2$. A set of $M\,N$ collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on the boundary $\partial\Omega_1$ of Ω (i.e. the surface of a sphere of radius ϱ_1) and a set of $M\,N$ collocation points $\{Q_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on the boundary $\partial\Omega_2$ (i.e. the surface of a sphere of radius ϱ_2) so that if $P_{i,j} = (x_{P_{i,j}}, y_{P_{i,j}}, z_{P_{i,j}})$ and $Q_{i,j} = (x_{Q_{i,j}}, y_{Q_{i,j}}, z_{Q_{i,j}})$, then

$$
\begin{aligned}
x_{P_{i,j}} &= \varrho_1 \sin\vartheta_j \cos\varphi_i, & y_{P_{i,j}} &= \varrho_1 \sin\vartheta_j \sin\varphi_i, & z_{P_{i,j}} &= \varrho_1 \cos\vartheta_j,\\
x_{Q_{i,j}} &= \varrho_2 \sin\vartheta_j \cos\varphi_i, & y_{Q_{i,j}} &= \varrho_2 \sin\vartheta_j \sin\varphi_i, & z_{Q_{i,j}} &= \varrho_2 \cos\vartheta_j,
\end{aligned}
$$

where $\varphi_i = 2(i-1)\pi/M,\ i = 1, \dots, M$ and $\vartheta_j = j\pi/(N+1),\ j = 1, \dots, N$. Note that we avoid the points corresponding to $\vartheta_j = 0$ and $\vartheta_j = \pi$ as these remain invariant under rotation in the φ-direction and hence lead to singular matrices. Similarly, we choose a set of $M\,N$ singularities $\{R_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_1$ (i.e., the surface of a sphere of radius R_1) and a set of $M\,N$ singularities $\{S_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_2$ (i.e., the surface of a sphere of radius R_2), by taking $R_{i,j} = (x_{R_{i,j}}, y_{R_{i,j}}, z_{R_{i,j}})$, $S_{i,j} = (x_{S_{i,j}}, y_{S_{i,j}}, z_{S_{i,j}})$, with

$$x_{R_{i,j}} = R_1 \sin\vartheta_j \cos\psi_i, \quad y_{R_{i,j}} = R_1 \sin\vartheta_j \sin\psi_i, \quad z_{R_{i,j}} = R_1 \cos\vartheta_j,$$
$$x_{S_{i,j}} = R_2 \sin\vartheta_j \cos\psi_i, \quad y_{S_{i,j}} = R_2 \sin\vartheta_j \sin\psi_i, \quad z_{S_{i,j}} = R_2 \cos\vartheta_j,$$

where $\psi_i = 2(\alpha + i - 1)\pi/M$, $i = 1, \dots, M$, $0 \leqslant \alpha < 1$.

4.2. Case II: Sphere with Interior Cylinder Removed

We next consider the domain

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < \varrho_2 \right\} \setminus \left\{ (x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 < \varrho_1^2,\ -h < z < h \right\}, \quad \varrho_1, h < \varrho_2.$$

In this case a set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on the boundary $\partial\Omega_1$ of Ω (i.e., the surface of a cylinder of radius ϱ_1 and height h) and a set of MN collocation points $\{Q_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on the boundary $\partial\Omega_2$ (i.e., the surface of a sphere of radius ϱ_2) so that if $P_{i,j} = (x_{P_{i,j}}, y_{P_{i,j}}, z_{P_{i,j}})$ and $Q_{i,j} = (x_{Q_{i,j}}, y_{Q_{i,j}}, z_{Q_{i,j}})$, then

$$x_{P_{i,j}} = r_{P_j} \cos\varphi_i,\ y_{P_{i,j}} = r_{P_j} \sin\varphi_i,\ z_{P_{i,j}} = z_{P_j},$$
$$x_{Q_{i,j}} = \varrho_2 \sin\vartheta_j \cos\varphi_i,\ y_{Q_{i,j}} = \varrho_2 \sin\vartheta_j \sin\varphi_i,\ z_{Q_{i,j}} = \varrho_2 \cos\vartheta_j,$$

where $\varphi_i = 2(i-1)\pi/M$, $i = 1, \dots, M$ and $\vartheta_j = j\pi/(N+1)$, $j = 1, \dots, N$. The polar coordinates $\left(r_{P_j}, z_{P_j}\right)$, $j = 1, \dots, N$, represent N points on the boundary of the rectangle $(0, \varrho_1) \times (-h, h)$. Similarly, we choose a set of MN singularities $\{R_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_1$ (i.e., the surface of a cylinder of radius R_1 and height $2H$) and a set of MN singularities $\{S_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_2$ (i.e. the surface of a sphere of radius R_2), by taking $R_{i,j} = (x_{R_{i,j}}, y_{R_{i,j}}, z_{R_{i,j}})$, $S_{i,j} = (x_{S_{i,j}}, y_{S_{i,j}}, z_{S_{i,j}})$, with

$$x_{R_{i,j}} = \tilde{r}_{R_j} \cos\psi_i,\ y_{R_{i,j}} = \tilde{r}_{R_j} \sin\psi_i,\ z_{R_{i,j}} = \tilde{z}_{R_j},$$
$$x_{S_{i,j}} = R_2 \sin\vartheta_j \cos\psi_i,\ y_{S_{i,j}} = R_2 \sin\vartheta_j \sin\psi_i, z_{S_{i,j}} = R_2 \cos\vartheta_j,$$

where $\psi_i = 2(\alpha + i - 1)\pi/M$, $i = 1, \dots, M$. The polar coordinates $(\tilde{r}_{Q_j}, \tilde{z}_{Q_j})$, $j = 1, \dots, N$ describe N points on the boundary of the rectangle $(0, R_1) \times (-H, H)$ with $R_1 < \varrho_1 < \varrho_2 < R_2$ and $H < h$.

4.3. Case III: Cylinder with Interior Sphere Removed

We next consider the following domain

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < \varrho_2^2,\ -h < z < h \right\} \setminus \left\{ \boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < \varrho_1 \right\}, \quad \varrho_1 < h, \varrho_2. \tag{4.1}$$

In this case a set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on the boundary $\partial\Omega_1$ of Ω (i.e. the surface of a sphere of radius ϱ_1) and a set of MN collocation points $\{Q_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on the boundary $\partial\Omega_2$ (i.e. the surface of a cylinder of radius ϱ_2 and height h) so that if $P_{i,j} = (x_{P_{i,j}}, y_{P_{i,j}}, z_{P_{i,j}})$ and $Q_{i,j} = (x_{Q_{i,j}}, y_{Q_{i,j}}, z_{Q_{i,j}})$, then

$$x_{P_{i,j}} = \varrho_1 \sin\vartheta_j \cos\varphi_i,\ y_{P_{i,j}} = \varrho_1 \sin\vartheta_j \sin\varphi_i,\ z_{P_{i,j}} = \varrho_1 \cos\vartheta_j,$$
$$x_{Q_{i,j}} = r_{Q_j} \cos\varphi_i,\ y_{P_{i,j}} = r_{Q_j} \sin\varphi_i,\ z_{Q_{i,j}} = z_{Q_j},$$

where $\varphi_i = 2(i-1)\pi/M,\ i = 1,\dots,M$ and $\vartheta_j = j\pi/(N+1),\ j = 1,\dots,N$. The polar coordinates $(r_{Q_j}, z_{Q_j}),\ j = 1,\dots,N$, represent N points on the boundary of the rectangle $(0,\varrho_2)\times(-h,h)$. Similarly, we choose a set of MN singularities $\{R_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_1$ (i.e. the surface of a sphere of radius R_1) and a set of MN singularities $\{S_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_2$ (i.e. the surface of a cylinder of radius R_2 and height $2H$), by taking $R_{i,j} = (x_{R_{i,j}}, y_{R_{i,j}}, z_{R_{i,j}})$, $S_{i,j} = (x_{S_{i,j}}, y_{S_{i,j}}, z_{S_{i,j}})$, with

$$x_{R_{i,j}} = R_1 \sin\vartheta_j \cos\psi_i,\ y_{R_{i,j}} = R_1 \sin\vartheta_j \sin\psi_i, z_{R_{i,j}} = R_1 \cos\vartheta_j,$$
$$x_{S_{i,j}} = \tilde{r}_{S_j} \cos\psi_i,\ y_{S_{i,j}} = \tilde{r}_{S_j} \sin\psi_i,\ z_{S_{i,j}} = \tilde{z}_{S_j},$$

where $\psi_i = 2(\alpha + i - 1)\pi/M,\ i = 1,\dots,M$. The polar coordinates $(\tilde{r}_{S_j}, \tilde{z}_{S_j}),\ j = 1,\dots,N$, describe N points on the boundary of the rectangle $(0, R_2)\times(-H, H)$ with $R_1 < \varrho_1 < \varrho_2 < R_2$ and $H > h$.

4.4. Case IV: Torus with Interior Torus Removed

We finally consider the case where the domain $\Omega \subset \mathbb{R}^3$ is defined by

$$\Omega = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : \varrho_1^2 < \left(\sqrt{x^2+y^2} - \varrho_3\right)^2 + z^2 < \varrho_2^2\right\}, \tag{4.2}$$

$\varrho_1 < \varrho_2 < \varrho_3$, where its boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ can be described by the parametric equations

$$x_1 = \varrho_3 \cos\varphi + \varrho_1 \cos\varphi \cos\vartheta,\ y_1 = \varrho_3 \sin\varphi + \varrho_1 \sin\varphi \cos\vartheta,\ z_1 = \varrho_1 \sin\vartheta,$$
$$x_2 = \varrho_3 \cos\varphi + \varrho_2 \cos\varphi \cos\vartheta,\ y_2 = \varrho_3 \sin\varphi + \varrho_2 \sin\varphi \cos\vartheta,\ z_2 = \varrho_2 \sin\vartheta,$$

where $0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant \vartheta \leqslant 2\pi$ with $(x_1, y_1, z_1) \in \partial\Omega_1$ and $(x_2, y_2, z_2) \in \partial\Omega_2$.

We choose a set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ on the boundary $\partial\Omega_1$ of Ω (i.e. the surface of a torus with radii ϱ_1, ϱ_3) and a set of MN collocation points $\{Q_{i,j}\}_{i=1,j=1}^{M,N}$ on the boundary $\partial\Omega_2$ (i.e. the surface of a torus with radii ϱ_2, ϱ_3) so that if $P_{i,j} = (x_{P_{i,j}}, y_{P_{i,j}}, z_{P_{i,j}})$ and $Q_{i,j} = (x_{Q_{i,j}}, y_{Q_{i,j}}, z_{Q_{i,j}})$, then

$$\begin{aligned}
x_{P_{m,n}} &= \varrho_3 \cos\varphi_n + \varrho_1 \cos\varphi_n \cos\vartheta_m,\\
y_{P_{m,n}} &= \varrho_3 \sin\varphi_n + \varrho_1 \sin\varphi_n \cos\vartheta_m,\\
z_{P_{m,n}} &= \varrho_1 \sin\vartheta_m, x_{Q_{m,n}} = \varrho_3 \cos\varphi_n + \varrho_2 \cos\varphi_n \cos\vartheta_m,\\
y_{Q_{m,n}} &= \varrho_3 \sin\varphi_n + \varrho_2 \sin\varphi_n \cos\vartheta_m, \;\; z_{Q_{m,n}} = \varrho_2 \sin\vartheta_m,
\end{aligned}$$

where $\vartheta_m = 2(m-1)\pi/N$, $m = 1, \ldots, M$ and $\varphi_n = 2(n-1)\pi/N, n = 1, \ldots, N$. Similarly, we choose a set of MN singularities $\{R_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_1$ (i.e. the surface of a torus with radii R_1, ϱ_3) and a set of MN singularities $\{S_{i,j}\}_{i=1,j=1}^{M,N}$ on $\partial\tilde{\Omega}_2$ (i.e. the surface of a torus with radii R_2, ϱ_3), by taking $R_{i,j} = (x_{R_{i,j}}, y_{R_{i,j}}, z_{R_{i,j}})$, $S_{i,j} = (x_{S_{i,j}}, y_{S_{i,j}}, z_{S_{i,j}})$, where

$$\begin{aligned}
x_{R_{m,n}} &= R_3 \cos\psi_n + R_1 \cos\psi_n \cos\vartheta_m,\\
y_{R_{m,n}} &= R_3 \sin\psi_n + R_1 \sin\psi_n \cos\vartheta_m, \qquad z_{R_{m,n}} = R_1 \sin\vartheta_m,\\
x_{S_{m,n}} &= R_3 \cos\psi_n + R_2 \cos\psi_n \cos\vartheta_m,\\
y_{S_{m,n}} &= R_3 \sin\psi_n + R_2 \sin\psi_n \cos\vartheta_m, \qquad z_{S_{m,n}} = R_2 \sin\vartheta_m,
\end{aligned}$$

where $\psi_n = 2(\alpha + n - 1)\pi/N$, $n = 1, \ldots, N$ with $0 \leqslant \alpha < 1$.

5. NUMERICAL RESULTS

The algorithms described in Sec. 2.2 and 3.2 were tested in regions defined by the solids described in Sec. 4, for both harmonic and biharmonic problems.

5.1. Harmonic Case

We considered two examples with boundary conditions corresponding to the exact solutions:

Example (a). $u = \cosh(0.3x)\cosh(0.4y)\cos(0.5z)$.

Example (b). $u = x^2 - 2y^2 + z^2$.

In the description of the numerical results we shall be referring to, say, Example (b) in the solid described in Case III, as Example 3b. In these examples, the maximum relative error was calculated on a uniform grid on the boundary (since all the functions involved are harmonic and the maximum principle applies). In the cases of the spherical shell (Case

I) and the toroidal domain (Case IV) the maximum relative error was calculated at $2 \times 23 \times 23$ points on the boundary, whereas in the Cases II and III the maximum relative error was calculated at $2 \times 20 \times 20$ points on the corresponding boundaries. For Case I we present a full set of results for Example (a). The results for Cases II–IV are similar to those of Case I for both examples. The full set of results for all cases and both examples can be found in [15].

Case I. We considered Example (a) in a thick spherical shell with $\varrho_1 = 1$, $\varrho_2 = 2$. We varied the angular parameter α and examined how this affected the accuracy of the MFS approximation for various values of N and for different $\varepsilon = \varrho_1 - R_1 = R_2 - \varrho_2$ (Fig. 1). Because of the symmetry of the problem about $\alpha = 1/2$, we only considered $0 \leqslant \alpha \leqslant 1/2$. We present six cases for $\varepsilon = 0.1, 0.2, 0.8$ for $N(=M) = 8, 12, 16, 24, 32, 48$ and 64. From these results we see that for the smaller values of ε the error appears to have a minimum value for $\alpha \approx 1/4$. This is consistent with the observations reported in [8, 14]. We also varied the radius R_2 of the external sphere when $\alpha = 0$, while keeping R_1 fixed and equal to 0.5, and examined how this affected the accuracy of the approximation for different values of N (Fig. 2a). As can be seen from the figures, the error decreases exponentially as we increase R_2 up to a certain point, beyond which it starts increasing again. This is due to the ill-conditioning of the corresponding matrices for large R_2 (and large N) and was also reported in [8, 14]. In addition, we varied both radii R_1 and R_2 (of the inner and external spheres, respectively) simultaneously and examined how this affected the accuracy of the approximation for various N (Fig. 2b). Again, it was observed that the error decreases up to a certain value of ε, beyond which it starts increasing.

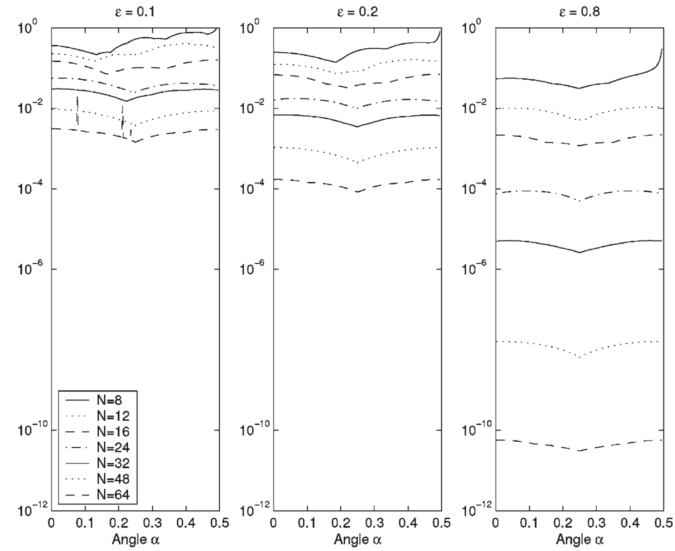
5.2. Biharmonic Case

We considered two examples with boundary conditions corresponding to the exact solutions:

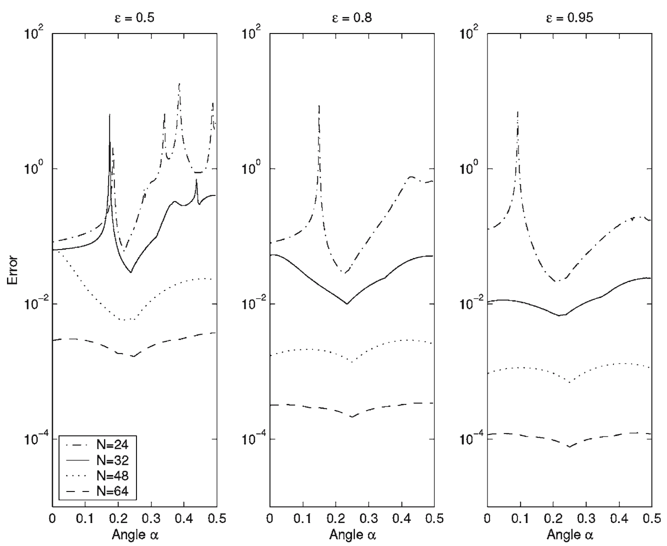
Example (c). $u = (x^2 + y^2 + z^2)\cosh(0.3x)\cosh(0.4y)\cos(0.5z).$

Example (d). $u = x^4 - 2y^4 + z^4.$

In the description of the numerical results we shall be referring to, say, Example (c) in the solid described in Case III, as Example 3c. In these examples, the maximum relative error was calculated on a uniform grid in the interior of the domain. In the cases of the spherical shell (Case I)

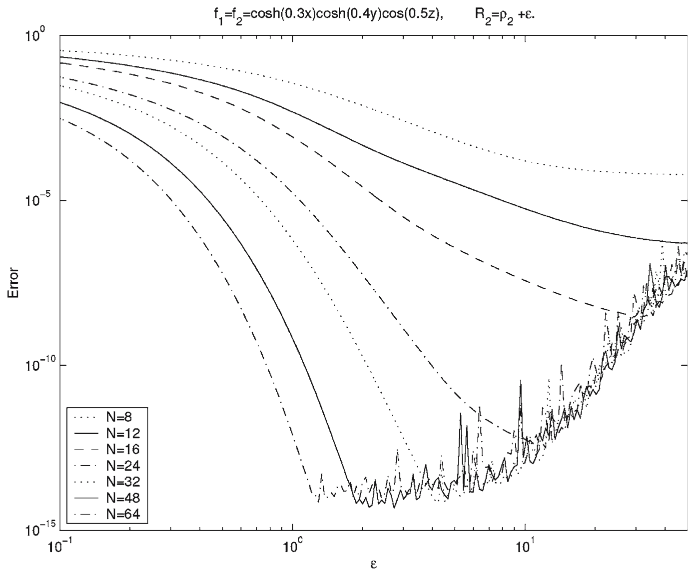


(a) $\varepsilon = .1,\ .2,\ .8$ in Example $1a$

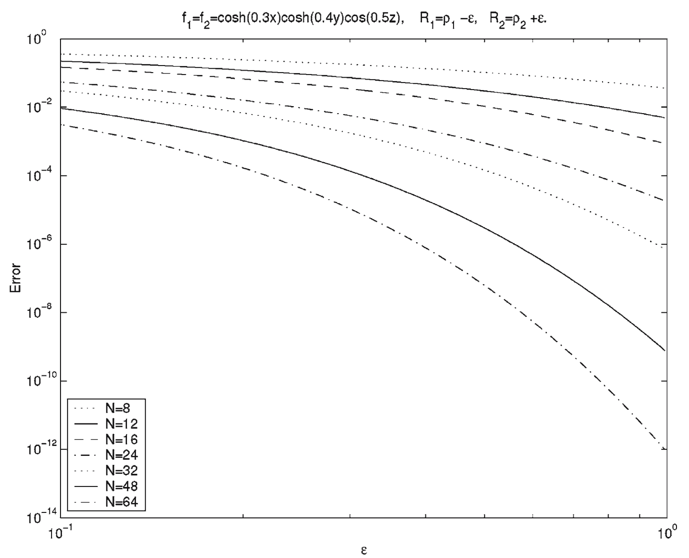


(b) $\varepsilon = .5,\ .8,\ .95$ in Example $4c$

Fig. 1. Log–plot of error versus angular parameter α for different values of N and ε.

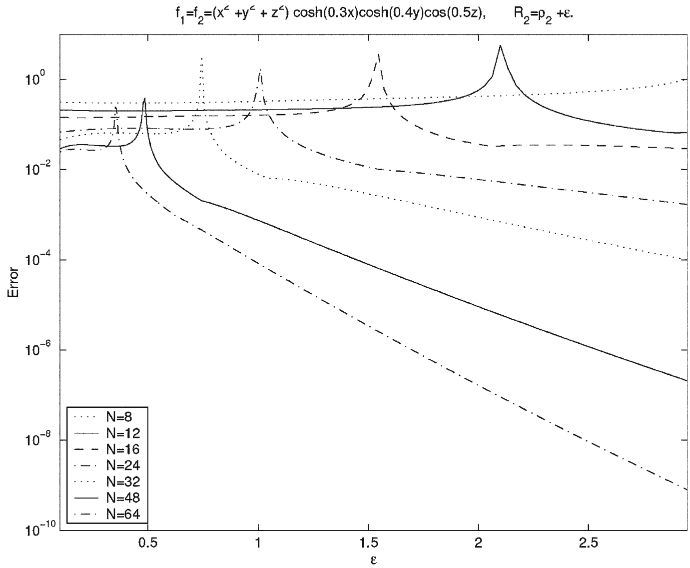


(a) Example 1a

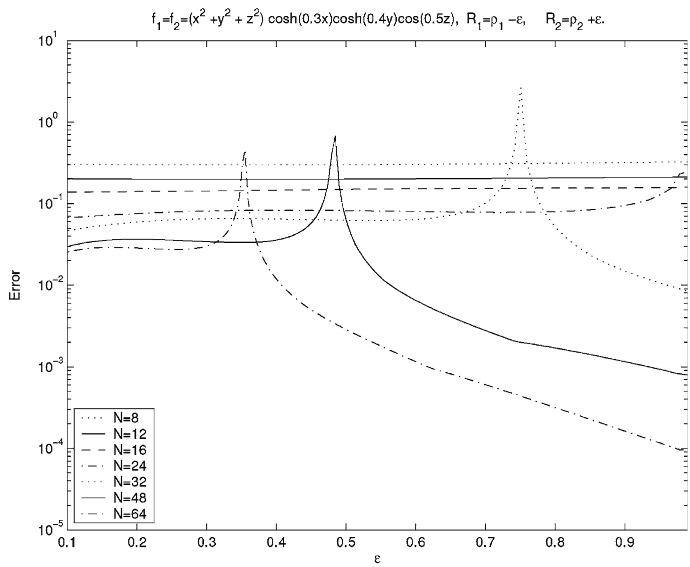


(b) Example 1a

Fig. 2. Log–plot of maximum relative error versus ε.



(c) Example 4c



(d) Example 4c

Fig. 2. continued

and the toroidal domain (Case IV) the maximum relative error was calculated at $20 \times 23 \times 23$ interior points, whereas in the Cases II and III the maximum relative error was calculated at $20 \times 20 \times 20$ interior points. The results for the biharmonic problems are similar to the results for the harmonic case. We therefore only present a full set of results for Case IV for Example (c). The full set of results for all cases and Examples (c) and (d) can be found in [15].

Case IV. We considered Example (c) in a torus with an interior torus removed with $\varrho_1 = 1$, $\varrho_2 = 2$ and $\varrho_3 = 5$. In Fig. 1b, we varied α and examined how this affected the accuracy of the MFS approximation for $N(=M) = 24, 32, 48, 64$ and for different $\varepsilon = \varrho_1 - R_1 = R_2 - \varrho_2 = 0.5, 0.8, 0.95$. For certain values of ε and N there is a minimum at $\alpha \approx 1/4$. In Fig. 2c, we varied R_2 while keeping R_1 fixed and equal to 0.5 and in Fig. 2d we varied both R_1 and R_2. In all these cases we observed that as ε increased the accuracy improved until a certain point beyond which it deteriorated.

6. CONCLUSIONS

In this paper we propose efficient MFS algorithms for the solution of harmonic and biharmonic problems in hollow axisymmetric domains. These algorithms, which rely on matrix decomposition techniques with the use of FFTs, can also be applied to hollow axisymmetric problems governed by other differential equations such as the Helmholtz equation and the Cauchy–Navier equations of elasticity. Further, other boundary method discretizations also lead to the block circulant structures of the matrices arising in the problems examined in this study, thus these algorithms could be employed with these methods. It should be noted that the current algorithm can be applied to different combinations of boundary conditions, provided these conditions are taken uniformly everywhere on that part of the boundary ($\partial\Omega_1$ or $\partial\Omega_2$). This method could also be extended to the solution of inhomogeneous and time-dependent problems using the method of particular solutions [2, 10]. The optimal location of the sources in the MFS is still an open question. This is the reason for which we examined the behaviour of the error as ε varied. When the interior hole is very small, the positioning of the interior pseudo-boundary can be problematic as, on the one hand we cannot take ε to be too small as thus results in poor accuracy and, on the other hand we cannot take ε to be too large as then the problem would become very ill-conditioned because of the very small size of the internal pseudo-boundary. As has been reported in previous studies matrix decomposition algorithms of this type improved (often significantly) the conditioning of the global system.

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