Journal of Scientific Computing, Volumes 22 and 23, June 2005 (© 2005)
DOI: 10.1007/s10915-004-4141-9

Two-Level Non-Overlapping Schwarz Preconditioners for a Discontinuous Galerkin Approximation of the Biharmonic Equation

Xiaobing Feng[1] **and Ohannes A. Karakashian**[2]

Received July 18, 2003; accepted (in revised form) May 6, 2004

We present some two-level non-overlapping additive and multiplicative Schwarz methods for a discontinuous Galerkin method for solving the biharmonic equation. We show that the condition numbers of the preconditioned systems are of the order $O(H^3/h^3)$ for the non-overlapping Schwarz methods, where h and H stand for the fine mesh size and the coarse mesh size, respectively. The analysis requires establishing an interpolation result for Sobolev norms and Poincaré–Friedrichs type inequalities for totally discontinuous piecewise polynomial functions. It also requires showing some approximation properties of the multilevel hierarchy of discontinuous Galerkin finite element spaces.

KEY WORDS: Biharmonic equation; discontinuous Galerkin methods, Schwarz methods; domain decomposition.

AMS Subject Classification: 65N55; 65F10.

1. INTRODUCTION

This paper is the second in a series (cf. [11]) devoted to the development of parallel domain decomposition solution methods for discontinuous Galerkin approximations of elliptic partial differential equations. In [11], non-overlapping and overlapping Schwarz preconditioners were developed for the discontinuous Galerkin approximation to *second* order elliptic partial differential equations that was proposed in [3,4]. It was shown that the condition numbers of the preconditioned systems are of the order $O(H/h)$

[1] Department of Mathematics, The University of Tennessee, Knoxville, TN 37996, USA. E-mail: xfeng@math.utk.edu.
[2] Department of Mathematics, The University of Tennessee, Knoxville, TN 37996, USA. E-mail: ohannes@math.utk.edu

for the non-overlapping Schwarz methods, and of the order $O(H/\delta)$ for the overlapping Schwarz methods, where h and H stand for the fine mesh size and the coarse mesh size respectively, and δ denotes the size of the overlaps between subdomains. These theoretical results were also validated by numerical experiments.

The objective of this paper is to develop Schwarz methods for the discontinuous Galerkin approximations of *fourth* order elliptic partial differential equations proposed in [3]. Specifically, two-level non-overlapping additive and multiplicative Schwarz methods are constructed and analyzed. It is shown that the condition numbers of the preconditioned systems are of the order $O(H^3/h^3)$ for the non-overlapping Schwarz methods. This estimate is comparable to those proved earlier for standard finite element methods for fourth order problems (cf. [7, 13] and references therein). To our knowledge, no result on Schwarz methods for discontinuous Galerkin approximations of fourth order problems were known earlier in the literature, and the results of this paper are sharp in their respective classes.

As expected, the Schwarz methods of this paper preserve all the features of the corresponding methods developed in [11]. For our non-overlapping Schwarz methods, a basic assumption is that the subdomain partition $\mathcal{T}_S$ of the domain Ω is subordinate to and coarser than the coarse mesh partition $\mathcal{T}_H$, which in its turn is subordinate to and coarser than the fine mesh (finite element) partition $\mathcal{T}_h$. Given this, some of the main features of our non-overlapping methods are

- The partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ consist of triangular and tetrahedral elements in 2-D and 3-D respectively. The main reason is that these are actual computational meshes. On the other hand, a great deal of flexibility is allowed in the choice of the subdomain partition $\mathcal{T}_S$ beyond the requirement $\mathcal{T}_S \subseteq \mathcal{T}_H$. However, there are restrictions of a more practical nature such as the requirement that the subdomains contain a nearly equal number of elements of $\mathcal{T}_h$ in order to ensure load balancing on a parallel computer.
- As mentioned earlier, the condition numbers of the preconditioned systems with the two-level Schwarz preconditioners are of the order $O((H/h)^3)$ when $\mathcal{T}_H$ and $\mathcal{T}_h$ are quasi-uniform (cf. Theorem 4.6).
- The algorithm is distinguished by its simplicity and ease of implementation. The additive Schwarz method consists of Block–Jacobi preconditioning together with a coarse-mesh correction. Consequently, the work involved in solving the linear system $B\mathbf{x} = \mathbf{c}$, where B is the preconditioner matrix, permits coarse-grain parallelism (cf. (4.13)).

As in [11], the main difficulties in the analysis of the proposed Schwarz methods arise from (1) the complicated mesh-dependent variational form of the discontinuous Galerkin method, which is more involved than that of [11]; (2) most well-known and useful properties such as trace inequalities, the Poincaré inequality, embedding theorems of Sobolev functions, and Nirenberg–Gagliardo interpolation inequalities are no longer valid for totally discontinuous piecewise polynomial functions, and these properties/inequalities must be modified and rederived; (3) approximation properties of the multilevel hierarchy of discontinuous Galerkin finite element spaces do not hold in the usual sense, and therefore, they have to be re-established with suitable modifications. All required technical machineries are collected and shown in Sec. 3. Clearly, besides their usefulness for analyzing the Schwarz methods, these technical results are of independent interest.

We remark that overlapping counterparts of the non-overlapping Schwarz methods of this paper have also been developed by the authors in [12]. However, due to some technical difficulties, only suboptimal order convergence result have been obtained thus far.

The paper is organized as follows. In Sec. 2, the discontinuous Galerkin method and some known facts about the method are recalled. In Sec. 3, we establish some needed technical results for totally discontinuous piecewise polynomial functions as described above. These include a trace inequality and a generalized Poincaré inequality for piecewise H^1 functions as well as an approximation property of piecewise constant functions. These machineries play a crucial role in our convergence analysis. In Sec. 4, two-level non-overlapping additive and multiplicative Schwarz methods are proposed and analyzed for the discontinuous Galerkin method.

We conclude this section by pointing out that while discontinuous Galerkin methods were introduced in the early seventies, they had been less popular due to the disadvantage of a relatively larger number of degrees of freedom per element. One way of offsetting this is at the level of solution of the systems of algebraic equations. Also, for a variety of reasons, one being their flexibility in the use of highly nonuniform and unstructured meshes, discontinuous Galerkin methods have attracted considerable amount of renewed interest in the past decade. They have been used successfully in handling complicated flow problems in heterogeneous porous media and nonlinear hyperbolic problems in fluid dynamics and semiconductor applications. We refer to [1–4,9,11,16,18,21] and the references therein for detailed expositions and recent developments on discontinuous Galerkin methods.

2. PRELIMINARIES

Let $\Omega \subset \mathbf{R}^d$, $d=2,3$ be a bounded convex polygonal domain. We consider the following biharmonic problem:

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

In the case $d=2$, the above problem describes the bending of a clamped elastic plate subject to the external load f, and u stands for the vertical displacement of the plate.

Throughout this paper, the standard space, norm and inner product notation are adopted. Their definitions can be found in [4,8,17]. Also, c is used to denote a generic positive mesh-independent constant.

The discontinuous Galerkin method considered in this paper for discretizing problem (2.1) and (2.2) is the one proposed in [3]. We emphasize that the discontinuous Galerkin method and the results of this paper are valid for both $d=2$ and $d=3$.

To formulate the method, we first need to introduce some notation. Let $\mathcal{T}_h = \{K_i :\ i=1,2,\ldots,m_h\}$ be a family of triangulations of the domain Ω parametrized by $0<h\leqslant 1$. We define

$$\partial K_i := \text{the boundary of } K_i, \qquad e_{ij} := \partial K_i \cap \partial K_j, \qquad e_i := \partial K_i \cap \partial\Omega,$$
$$h_{K_i} := \operatorname{diam}(K_i), \qquad h_{e_{ij}} := \operatorname{diam}(e_{ij}), \qquad h_{e_i} := \operatorname{diam}(e_i)\,.$$

We shall refer to $\mathcal{T}_h$ as the "fine" mesh and assume that it satisfies the following assumptions:

(i) the elements of $\mathcal{T}_h$ satisfy the minimal angle condition;
(ii) $\mathcal{T}_h$ is locally quasi-uniform, that is if two elements K_j and K_ℓ are adjacent ($\operatorname{meas}(\partial K_{j\ell})>0$), then $h_{K_j} \approx h_{K_\ell}$.

For any two elements K_i, K_j of the partition $\mathcal{T}_h$, we call $e_{ij} := \partial K_i \cap \partial K_j$ an *interior edge (face* when $d=3$)* of $\mathcal{T}_h$ if $\operatorname{meas}(e_{ij}) > 0$. Where $\operatorname{meas}(e_{ij})$ stands for $(d-1)$-dimensional measure of e_{ij}. Note that e_{ij} could be portion of a side/face of the element K_i or K_j in the case of a geometrically nonconforming partition. Also, for any element K_i of $\mathcal{T}_h$, we call $e_i := \partial K_i \cap \partial\Omega$ a *boundary edge* if $\operatorname{meas}(e_i)>0$. Then we define

$$\mathcal{E}^I := \text{ set of interior edges of } \mathcal{T}_h, \qquad \mathcal{E}^B := \text{ set of boundary edges of } \mathcal{T}_h,$$

and let $\mathcal{E} := \mathcal{E}^I \cup \mathcal{E}^B$. In the sequel, $v^{(i)}$ will denote the restriction of v to K_i. Now let $e=e_{ij}\in\mathcal{E}^I$ where $i>j$. We define the jump $[v]$ of v across e by $[v]|_e := v^{(i)}\big|_e - v^{(j)}\big|_e$. If $e=e_i\in\mathcal{E}^B$, we set $[v]|_e = v^{(i)}|_e$. Similarly, we define the jump of the normal derivative of v across an interior edge/face $e=e_{ij},\ i>j$ by

$$\left[\frac{\partial v}{\partial n}\right]\bigg|_e := \frac{\partial v^{(i)}}{\partial n}\bigg|_e - \frac{\partial v^{(j)}}{\partial n}\bigg|_e,$$

where n is the unit normal outward to K_i. Again, if $e=e_i\in\mathcal{E}^B$, we set $[\partial v/\partial n]\big|_e := \partial v^{(i)}/\partial n\big|_e$. Another convention adopted in this paper is the following: For $e=e_{ij}\in\mathcal{E}^I,\ i>j,\ \{v\}|_e := v^{(i)}\big|_e$. For $e=e_i\in\mathcal{E}^B,\ \{v\}|_e := v^{(i)}\big|_e$. We remark that the results of this paper should also be valid if the above convention is replaced by

$$\{v\}|_e := \frac{1}{2}\left(v^{(i)}\big|_e + v^{(j)}\big|_e\right), \qquad e=e_{ij}\in\mathcal{E}^I.$$

Now define the "energy space" E by $E = H^4(K_1)\times H^4(K_2)\times\cdots\times H^4(K_{m_h})$ and the bilinear form $a_h(\cdot,\cdot)$ on $E\times E$ as follows: for $u,v\in E$,

$$\begin{aligned} a_h(u,v) = \sum_{K\in\mathcal{T}_h}(\Delta u,\Delta v)_K + \sum_{e\in\mathcal{E}}\Bigg(&\left\langle [u], \left\{\frac{\partial\Delta v}{\partial n}\right\}\right\rangle_e - \left\langle\left[\frac{\partial u}{\partial n}\right], \{\Delta v\}\right\rangle_e \\ &+\left\langle [v], \left\{\frac{\partial\Delta u}{\partial n}\right\}\right\rangle_e - \left\langle\left[\frac{\partial v}{\partial n}\right], \{\Delta u\}\right\rangle_e + \gamma h_e^{-3}\langle [u],[v]\rangle_e \\ &+\gamma h_e^{-1}\left\langle\left[\frac{\partial u}{\partial n}\right],\left[\frac{\partial v}{\partial n}\right]\right\rangle_e\Bigg). \end{aligned} \tag{2.3}$$

Here $(\cdot,\cdot)_K$ denotes the L^2 integral over K, $\langle\cdot,\cdot\rangle_e$ stands for the L^2 integral over the edge e; γ is a positive constant independent of h, and the terms including γ are the so-called penalty terms.

The bilinear form $a_h(\cdot,\cdot)$ induces the following norm on the space E:

$$\begin{aligned} \|v\|_{2,h} = \Bigg(\sum_{K\in\mathcal{T}_h}\|\Delta v\|_{0,K}^2 + \sum_{e\in\mathcal{E}}\Bigg(&h_e^{-3}|[v]|_{0,e}^2 + h_e^{-1}\left|\left[\frac{\partial v}{\partial n}\right]\right|_{0,e}^2 \\ &+h_e|\{\Delta v\}|_{0,e}^2 + h_e^3\left|\left\{\frac{\partial\Delta v}{\partial n}\right\}\right|_{0,e}^2\Bigg)\Bigg)^{1/2}, \end{aligned} \tag{2.4}$$

where $\|\cdot\|_{0,K}$ and $|\cdot|_{0,e}$ denote the L^2-norm on K and on e, respectively.

We also recall the H^1-like norm $\|\cdot\|_{1,h}$ which is defined as [4,11]

$$\|v\|_{1,h} = \left(\sum_{K\in\mathcal{T}_h}\|\nabla v\|_{0,K}^2 + \sum_{e\in\mathcal{E}}\left(h_e^{-1}|[v]|_{0,e}^2 + h_e\left|\left\{\frac{\partial v}{\partial n}\right\}\right|_{0,e}^2\right)\right)^{1/2}. \tag{2.5}$$

The above norm is the energy norm for second order elliptic problems, which also is useful for us to analyze the properties of V^h in the next section.

The weak formulation of (2.1) and (2.2) is defined as seeking $u \in E \cap H^2(\Omega) \cap H^4_{\rm loc}(\Omega)$ such that

$$a_h(u, v) = (f, v)_\Omega, \quad \forall v \in E \cap H^1(\Omega) \cap H^2_{\rm loc}(\Omega). \tag{2.6}$$

This formulation is indeed consistent with the boundary value problem (2.1) and (2.2).

For any subdomain D of Ω and integer $r \geqslant 3$ (see the remark after Theorem 2.2), let $P_{r-1}(D)$ denote the set of all polynomials of degree less than or equal to $r-1$ on D. The finite element space V^h is defined by

$$V^h = P_{r-1}(K_1) \times P_{r-1}(K_2) \times \cdots \times P_{r-1}(K_{m_h}).$$

Clearly, $V^h \subset E \subset L^2(\Omega)$. But $V^h \not\subset H^2(\Omega)$. In fact, $V^h \not\subset H^1(\Omega)$.

The discontinuous Galerkin method based on the weak formulation (2.6) is defined as follows: find $u_h \in V^h$ such that

$$a_h(u_h, v_h) = (f, v_h)_\Omega \qquad \forall v_h \in V^h. \tag{2.7}$$

The following continuity and coercivity properties of the bilinear form $a_h(\cdot,\cdot)$ were established in [3].

Lemma 2.1. (i)

$$|a_h(u, v)| \leqslant (1+\gamma)\|u\|_{2,h}\|v\|_{2,h} \qquad \forall u, v \in E. \tag{2.8}$$

(ii) There exist positive constants γ_0 and c_a such that for $\gamma \geqslant \gamma_0$

$$a_h(v, v) \geqslant c_a \|v\|^2_{2,h} \qquad \forall v \in V^h. \tag{2.9}$$

We remark that γ_0 depends only on r and the minimum angles of the elements of $\mathcal{T}_h$.

The following error estimates were also established in [3].

Theorem 2.2. Suppose $\partial\Omega$ is smooth so that the solution u of (2.6) is in the Sobolev space $H^s(\Omega)$ for some $s \geqslant 4$. Also, let $u_h \in V^h$, $r \geqslant 4$ denote the solution of (2.7). Then, there exists a constant c independent of h and u, such that

$$\|u-u_h\|_{2,h} \leqslant ch^{s-2}\| u \|_{s,\Omega} \quad \text{for } 4\leqslant s\leqslant r, \tag{2.10}$$

$$\| u-u_h \|_{-\ell,\Omega} \leqslant ch^{s+\ell}\| u \|_{s,\Omega} \quad \text{for } 0\leqslant \ell\leqslant r-4,\ 4\leqslant s\leqslant r, \tag{2.11}$$

$$\Big(\sum_{K\in\mathcal{T}_h} \| D^{\alpha}(u-u_h) \|_{0,K}^2\Big)^{1/2} \leqslant ch^{s-|\alpha|}\| u \|_{s,\Omega} \quad \text{for } 0\leqslant |\alpha|\leqslant s, \quad 4\leqslant s\leqslant r. \tag{2.12}$$

Remark 2.1. The above error estimates were shown in [3] under the assumption that $r\geqslant 4$, that is, for each $K\in\mathcal{T}_h$, $V^h|_K$ contains all cubic polynomials. However, it is not hard to verify that the proofs in [3] also work in the case $r=3$. Specifically, there hold

$$\|u-u_h\|_{2,h} \leqslant ch\| u \|_{s,\Omega} \qquad \text{for } s\geqslant 3,$$
$$\| u-u_h \|_{0,\Omega} \leqslant ch^2\| u \|_{s,\Omega} \qquad \text{for } s\geqslant 3.$$

Note that the L^2-estimate is suboptimal.

With the help of the basis functions of V^h, Eq. (2.7) can be transformed into an $N\times N$ linear system

$$A\mathbf{x}=\mathbf{b}, \tag{2.13}$$

where N denotes the dimension of V^h and the coefficient matrix $A\in \mathbf{R}^{N\times N}$, called the stiffness matrix, is symmetric and positive definite.

It is not hard to show that the condition number of A is of the order $O(\underline{h}^{-4})$ where $\underline{h}=\min_{K\in\mathcal{T}_h} h_K$. So the system (2.13) becomes ill-conditioned for small $\underline{h}$. In addition, the size of the linear system becomes large. Consequently, it is not efficient to solve it directly using the classical iterative methods. On the other hand, if one can find a symmetric positive definite $N\times N$ matrix B such that BA is well-conditioned, then any of the classical iterative methods, in particular, the conjugate gradient (CG) method, works effectively on the preconditioned system

$$BA\mathbf{x}=B\mathbf{b}. \tag{2.14}$$

The goal of this paper is to develop some additive and multiplicative Schwarz preconditioners, based on domain decomposition, for the linear system (2.13) and to solve the preconditioned systems using the CG method. For background knowledge and a general theory on the Schwarz method, we refer to [14,19,20,22].

3. Some properties of totally discontinuous piecewise polynomial functions

In this section, we shall first prove an interpolation result between the three norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{1,h}$ and $\|\cdot\|_{2,h}$ for the functions of the discontinuous Galerkin finite element space V^h. Using this interpolation result, we shall then show some Poincaré-Friedrichs type inequalities for functions in V^h. We shall also show some approximation properties between the hierarchy of the discontinuous finite element spaces on multilevel grids. Finally, we shall recall a trace inequality and a generalized Poincaré inequality which were established by the authors in [11] for discontinuous piecewise H^1 functions. These results are the key technical tools upon which our results for the two-level Schwarz methods are based.

We like to point out that all above mentioned results hold not only for piecewise polynomial functions, but also for piecewise H^1 and H^2 functions. However, for the sake of simplicity and the purpose of this paper, we shall only consider totally discontinuous piecewise polynomial functions. Also, since all these results have their counterparts in the standard Sobolev space H^1 and H^2, hence, they can be regarded as generalizations of those well-known properties to piecewise H^1 and H^2 functions. We also refer to [6] for related results for piecewise H^1 functions.

Let D be a bounded convex polygonal domain with diameter H in $\mathbf{R}^d$, $d=2,3$. Let $\mathcal{T}_D$ be a family of triangulations of D parameterized by $0<h_D\leqslant H$. We shall think of D as an element of the coarse mesh $\mathcal{T}_H$ and consider it as being a union of some elements of the fine mesh $\mathcal{T}_h$ corresponding to the collection $\mathcal{T}_D$. Let V_D^h denote the discontinuous finite element space of degree $r-1\,(r\geqslant 3)$ associated with $\mathcal{T}_D$, $\mathcal{E}_D^I$ and $\mathcal{E}_D^B$ denote the set of interior edges and the set of boundary edges of $\mathcal{T}_D$, respectively.

We recall that for any two adjacent triangles $K, K'\in\mathcal{T}_D$ which share the edge e, there hold the trace inequalities (cf. [1,3,8])

$$|\{u\}|_{0,e}^2 \leqslant ch_e^{-1}\Big(\|u\|_{0,K}^2+\|u\|_{0,K'}^2\Big), \tag{3.1}$$

$$\left|\left\{\frac{\partial u}{\partial n}\right\}\right|_{0,e}^2 \leqslant ch_e^{-3}\Big(\|u\|_{0,K}^2+\|u\|_{0,K'}^2\Big). \tag{3.2}$$

The above two inequalities will be used several times later in this paper.

The following lemma establishes an interpolation result which bounds $\|u\|_{1,h}$ in terms of $\|u\|_{L^2}$ and $\|u\|_{2,h}$ for totally discontinuous functions $u\in V_D^h$.

Lemma 3.1. There exists a constant $c>0$, which is independent of h_D, such that for any $u \in V_D^h$ and any $\eta > 0$,

$$\sum_{K\in\mathcal{T}_D} \|\nabla u\|_{0,K}^2 \leqslant c\left(\eta^{-1}\|u\|_{0,D}^2+\eta\|u\|_{2,h,D}^2\right). \tag{3.3}$$

Proof. For any $u \in V_D^h$ and $K \in \mathcal{T}_D$, integration by parts gives

$$\|\nabla u\|_{0,K}^2 = \left\langle \frac{\partial u}{\partial n}, u\right\rangle_{\partial K} - \langle \Delta u, u\rangle_K .$$

Summing the above equality over all $K \in \mathcal{T}_D$ and using Schwarz's inequality on the right-hand side we get for any $\eta > 0$

$$\begin{aligned}\sum_{K\in\mathcal{T}_D} \|\nabla u\|_{0,K}^2 &\leqslant \eta \sum_{K\in\mathcal{T}_D} \|\Delta u\|_{0,K}^2 + \eta^{-1}\|u\|_{0,D}^2 \\ &\quad + \sum_{e\in\mathcal{E}_D^I}\left[\left\langle \frac{\partial u}{\partial n}, u\right\rangle_e\right] + \sum_{e\in\mathcal{E}_D^B}\left\langle \frac{\partial u}{\partial n}, u\right\rangle_e .\end{aligned} \tag{3.4}$$

For the third term on the right-hand side of (3.4) we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_D^I}\left[\left\langle \frac{\partial u}{\partial n}, u\right\rangle_e\right] &= \sum_{e\in\mathcal{E}_D^I}\left(\left\langle\left[\frac{\partial u}{\partial n}\right],\{u\}\right\rangle_e + \left\langle\left\{\frac{\partial u}{\partial n}\right\},[u]\right\rangle_e\right) \\ &\leqslant \sum_{e\in\mathcal{E}_D^I}\left(\eta h_e^{-1}\left|\left[\frac{\partial u}{\partial n}\right]\right|_{0,e}^2 + \eta^{-1}h_e|\{u\}|_{0,e}^2 + \eta h_e^{-3}|[u]|_{0,e}^2\right. \\ &\quad \left. + \eta^{-1}h_e^3\left|\left\{\frac{\partial u}{\partial n}\right\}\right|_{0,e}^2\right) \leqslant \eta^{-1}\|u\|_{0,D}^2 + \eta\|u\|_{2,h,D}^2,\end{aligned} \tag{3.5}$$

having used (3.1) and (3.2). Similarly, for the fourth term on the right hand side of (3.4), we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_D^B}\left\langle \frac{\partial u}{\partial n}, u\right\rangle_e &\leqslant \sum_{e\in\mathcal{E}_D^B}\left(\eta h_e^{-1}\left|\frac{\partial u}{\partial n}\right|_{0,e}^2 + \eta^{-1}h_e|u|_{0,e}^2\right) \\ &\leqslant c\eta^{-1}\|u\|_{0,D}^2 + c\eta\|u\|_{2,h,D}^2 .\end{aligned} \tag{3.6}$$

Finally, the desired estimate (3.3) follows from (3.4)–(3.6). □

In Lemma 3.2 we establish two Poincaré–Friedrichs type inequalities for functions in V_D^h. The proof of the second inequality is carried out with the help of Lemma 3.1.

Lemma 3.2. Suppose that D is a bounded convex polygonal domain, then there exists a constant $c>0$ such that for any $u \in V_D^h$

$$\| u \|_{0,D} \leqslant c\Big(| u |_{1,h,D}^2 + \sum_{e \in \mathcal{E}_D^B} h_e^{-1} | u |_{0,e}^2 \Big)^{1/2}, \tag{3.7}$$

$$| u |_{1,h,D} \leqslant c \| u \|_{2,h,D}, \tag{3.8}$$

where

$$| u |_{1,h,D} = \Big(\sum_{K \in \mathcal{T}_D} \| \nabla u \|_{0,K}^2 + \sum_{e \in \mathcal{E}_D^I} h_e^{-1} | [u] |_{0,e}^2 \Big)^{1/2}. \tag{3.9}$$

Proof. Inequality (3.7) was proved in Lemma 2.1 of [1] using a duality argument. To show (3.8), from (3.9), (3.3) and (3.7), we have

$$|u|_{1,h,D}^2 \leqslant c\eta^{-1} |u|_{1,h,D}^2 + c\eta \|u\|_{2,h,D}^2 + c\eta^{-1} \sum_{e \in \mathcal{E}_D^B} h_e^{-1} |u|_{0,e}^2 + \sum_{e \in \mathcal{E}_D^I} h_e^{-1} | [u] |_{0,e}^2 . \tag{3.10}$$

Now choosing $\eta > c$, we obtain (3.8) by observing that since $h_e < 1$, the third and fourth terms on the right side of (3.10) are bounded by $c\|u\|_{2,h,D}^2$. □

We now recall the following approximation property of the average value of a function in V_D^h, which was shown in [11].

Lemma 3.3. (cf. Lemma 3.2 of [11]). Suppose D is a bounded convex polygonal domain. For any $u \in V_D^h$, let $\overline{u} = 1/\text{meas}(D) \int_D u\,dx$ be the average value of u over D. Then

$$\|u - \overline{u}\|_{0,D} \leqslant c\,H |u|_{1,h,D}. \tag{3.11}$$

In Lemma 3.4 we extend the above first-order approximation property to a second order estimate which holds for linear functions.

Lemma 3.4. Suppose D is a bounded convex polygonal domain. For any $u \in V_D^h$, let $I_D u \in P_1(D)$ denote the "elliptic projection" of u defined by

$$b_h(I_D u, v)_D = b_h(u, v)_D \qquad \forall v \in P_1(D), \tag{3.12}$$

$$(I_D u, 1)_D = (u, 1)_D, \tag{3.13}$$

where

$$b_h(u, v)_D = \sum_{K \in \mathcal{T}_D} \big(\nabla u, \nabla v\big)_K - \sum_{e \in \mathcal{E}_D^I} \left(\left\langle [u], \left\{ \frac{\partial v}{\partial n} \right\} \right\rangle_e + \left\langle \left\{ \frac{\partial u}{\partial n} \right\}, [v] \right\rangle_e \right). \tag{3.14}$$

Then there exist constants $c_i > 0\,(i = 1, 2, 3, 4)$ such that

$$| I_D u |_{1,h,D} \leqslant c_1 | u |_{1,h,D}, \tag{3.15}$$

$$\| u - I_D u \|_{0,D} \leqslant c_2 H | u - I_D u |_{1,h,D}, \tag{3.16}$$

$$| u - I_D u |_{1,h,D} \leqslant c_3 H \| u \|_{2,h,D}, \tag{3.17}$$

$$\| u - I_D u \|_{0,D} \leqslant c_4 H^2 \| u \|_{2,h,D}. \tag{3.18}$$

Proof. First, we remark that the existence of $I_D u$ is trivial by observing the fact that $b_h(v, w)_D = (\nabla v, \nabla w)_D$, $\forall v, w \in P_1(D)$.

Second, (3.15) follows from the definition of I_D and Schwarz's inequality. (3.16) is an immediate consequence of (3.11), and (3.18) follows trivially from (3.16) and (3.17). Hence, we only need to give a proof for (3.17).

Now, let $u_0 = I_D u$ and $w = u - u_0$. Since u_0 is continuous in D, from the definition of $| \cdot |_{1,h,D}$ and $\| \cdot \|_{2,h,D}$ we know that (3.17) holds if the following inequality holds

$$\sum_{K \in \mathcal{T}_D} \| \nabla w \|_{0,K}^2 \leqslant c H^2 \| u \|_{2,h,D}^2 + \frac{1}{2H^2 c_2^2} \| w \|_{0,D}^2. \tag{3.19}$$

The proof of (3.19) follows the same line as the proof of (3.3). However, here we need to handle additional terms due to the contributions of $\partial u_0 / \partial n$ on the element interfaces, and to eliminate that effect on the right-hand side of the inequality. Notice that

$$\big(\nabla u_0, \nabla v\big)_D = b_h(u_0, v)_D \quad \forall v \in P_1(D),$$

and set $v=u_0$ in (3.12) and use the above identity to get

$$\sum_{K\in\mathcal{T}_D}\left(\nabla w,\nabla u_0\right)_K-\sum_{e\in\mathcal{E}_D^I}\left\langle [u],\left\{\frac{\partial u_0}{\partial n}\right\}\right\rangle_e=0. \tag{3.20}$$

Now applying (3.4) and (3.5) to $w=u-u_0$ we get

$$\sum_{K\in\mathcal{T}_D}\|\nabla w\|_{0,K}^2\leqslant\eta\sum_{K\in\mathcal{T}_D}\|\Delta u\|_{0,K}^2+\eta^{-1}\|w\|_{0,D}^2+\sum_{K\in\mathcal{T}_D}\left\langle\frac{\partial w}{\partial n},w\right\rangle_{\partial K} \tag{3.21}$$

for any $\eta>0$. Here we have used the fact that $\Delta u_0\equiv 0$.

Since $u_0\in P_1(D)$, using the identity (3.20) the third term on the right-hand side of (3.21) can be bounded as follows (note that $\Delta u_0\equiv 0$)

$$\begin{aligned}
\sum_{K\in\mathcal{T}_D}\left\langle\frac{\partial w}{\partial n},w\right\rangle_{\partial K} &= \sum_{K\in\mathcal{T}_D}\left(\left\langle\frac{\partial u}{\partial n},w\right\rangle_{\partial K}-\left(\nabla u_0,\nabla w\right)_K\right)\\
&= \sum_{e\in\mathcal{E}_D^I}\left(\left\langle\left[\frac{\partial u}{\partial n}\right],\{w\}\right\rangle_e+\left\langle\left\{\frac{\partial u}{\partial n}\right\},[u]\right\rangle_e\right)\\
&\quad+\sum_{e\in\mathcal{E}_D^B}\left\langle\frac{\partial u}{\partial n},w\right\rangle_e-\sum_{e\in\mathcal{E}_D^I}\left\langle\left\{\frac{\partial u_0}{\partial n}\right\},[u]\right\rangle_e\\
&= \sum_{e\in\mathcal{E}_D^I}\left(\left\langle\left[\frac{\partial u}{\partial n}\right],\{w\}\right\rangle_e+\left\langle\left\{\frac{\partial w}{\partial n}\right\},[u]\right\rangle_e\right)+\sum_{e\in\mathcal{E}_D^B}\left\langle\frac{\partial u}{\partial n},w\right\rangle_e\\
&\leqslant \eta^{-1}\left(\sum_{e\in\mathcal{E}_D^I}\left(h_e|\{w\}|_{0,e}^2+h_e^3\left|\left\{\frac{\partial w}{\partial n}\right\}\right|_{0,e}^2\right)\right.\\
&\quad\left.+\sum_{e\in\mathcal{E}_D^B}h_e|w|_{0,e}^2\right)+\eta\|u\|_{2,h,D}^2\\
&\leqslant c\eta^{-1}\|w\|_{0,D}^2+\eta\|u\|_{2,h,D}^2.
\end{aligned} \tag{3.22}$$

Here we have used the trace inequalities (3.1) and (3.2) to get the final inequality.

Substituting (3.22) into (3.21) yields

$$\sum_{K\in\mathcal{T}_D}\|\nabla w\|_{0,K}^2\leqslant c\eta^{-1}\|w\|_{0,D}^2+2\eta\|u\|_{2,h,D}^2. \tag{3.23}$$

Hence, (3.19) follows from (3.23) upon choosing $\eta=2c_2^2cH^2$. □

Corollary 3.5. For any $u \in V_D^h$, let $L_D u \in P_1(D)$ denote the L^2-projection of u defined by

$$\left(L_D u, v\right) = (u, v) \qquad \forall v \in P_1(D). \tag{3.24}$$

Then there exist a constant $c > 0$ such that

$$\| u - L_D u \|_{0,D} \leqslant cH | u |_{1,h,D}, \tag{3.25}$$

$$\| u - L_D u \|_{0,D} \leqslant cH | u - L_D u |_{1,h,D}, \tag{3.26}$$

$$\| u - L_D u \|_{0,D} \leqslant cH^2 \| u \|_{2,h,D}. \tag{3.27}$$

Proof. (3.25) follows from the definition of L_D and the estimate (3.11). (3.26) is given immediately by (3.11). Finally, (3.27) follows from the fact that

$$\| u - L_D u \|_{0,D} \leqslant \| u - I_D u \|_{0,D}$$

and (3.18). □

Remark 3.1. (a). Since an inequality similar to (3.17) does not hold for L_D in general, we did not work out a direct proof for (3.27).

(b). Although the results of Lemma 3.4 and Corollary 3.5 will not be needed for establishing the main results of this paper in Sec. 4, they are interesting results on their own.

We conclude this section with two lemmas. The first lemma is a trace inequality for piecewise functions, the second one is a generalized Poincaré inequality. Both lemmas will be needed in the next section for analyzing non-overlapping Schwarz methods. The proofs make use of the *starlike* property: A domain D is called starlike if there exist a point $\mathbf{x}_0 \in D$ and a constant $c > 0$ such that

$$(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n} \geqslant c \operatorname{diam}(D), \text{ for almost all } \mathbf{x} \in \partial D.$$

Lemma 3.6. (cf. Lemma 3.1 of [11]). Suppose D is a bounded polygonal domain (not necessarily convex) which is also starlike. Then for any $u \in V_D^h$, there holds the following trace inequality

$$|u|_{0,\partial D}^2 \leqslant c\left(H^{-1} \|u\|_{0,D}^2 + H |u|_{1,h,D}^2\right). \tag{3.28}$$

Lemma 3.7. Suppose the domain D is as in Lemma 3.6. Let D' be a starlike polygonal domain contained in D and suppose that $\operatorname{dist}(x, \partial D') \leqslant \rho$ $\forall x \in \partial D$ for some $0 < \rho < H$. Let $B_\rho = D \setminus D'$ denote the boundary layer with width ρ between D and D'. Then for any $u \in V_D^h$, there holds the following generalized Poincaré inequality

$$\|u\|_{0,B_\rho}^2 \leqslant c\rho\Big(H^{-1}\|u\|_{0,D}^2 + H|u|_{1,h,D}^2\Big). \tag{3.29}$$

Proof. We construct a vector field $\mathbf{w}$ on B_ρ possessing the following properties.

(i) $\mathbf{w}(\mathbf{x})=0$ on $\partial D'$;
(ii) $|\mathbf{w}(\mathbf{x})| \leqslant c\,d(\mathbf{x}) \leqslant c\,\rho$ for $x\in B_\rho$.
(iii) $\mathbf{w}\in \mathrm{H}(\mathrm{div}, B_\rho)$ and div $\mathbf{w}(\mathbf{x})\geqslant 2$ on B_ρ.

We set $\mathbf{w}=\nabla d^2(\mathbf{x})$ where $d(\mathbf{x})$ is the distance function to $\partial D'$.

We shall consider only the case $d=2$. A similar argument can also be worked out when D is a three-dimensional starlike, bounded polygonal domain. Each point $\mathbf{x}\in B_\rho$ is either a type-I point, or a type-II point, or a type-III point which lies on the dotted lines (see Fig. 1).

Now $d^2(\mathbf{x})$ is a piecewise quadratic polynomial. Indeed, in local coordinate systems, $d^2(\mathbf{x})=x_2^2$ in a type-II region and $d^2(\mathbf{x})=x_1^2+x_2^2$ in a type-I region. Thus (i) and (ii) are easily seen to hold. As for (iii), elementary calculations reveal that the normal component of $\mathbf{w}$ is continuous across the interfaces of these regions. Thus $\mathbf{w}\in \mathrm{H}(\mathrm{div}, B_\rho)$. Also, since the Laplacian Δ is invariant under orthogonal transformations, we see that div $\mathbf{w}(\mathbf{x})=\Delta d^2(\mathbf{x})=2$ in a type-II region while div $\mathbf{w}(\mathbf{x})=\Delta d^2(\mathbf{x})=4$ in a type-I region. Since the two-dimensional Lebesgue measure of the set of all type-III points is zero, (iii) holds as well.

Let $\mathcal{T}_{B_\rho}=\{K\in\mathcal{T}_D : K\cap B_\rho\neq\emptyset\}$. Using (iii) and the divergence theorem we have

$$\begin{aligned} 2\int_{B_\rho} u^2\,dx &\leqslant \int_{B_\rho} u^2 \,\mathrm{div}\;\mathbf{w}\,dx \\ &= \sum_{K\in\mathcal{T}_{B_\rho}} \left\{\int_{\partial(K\cap B_\rho)} u^2\mathbf{w}\cdot\mathbf{n}\,ds - 2\int_{K\cap B_\rho} u\nabla u\cdot\mathbf{w}\,dx\right\}. \end{aligned} \tag{3.30}$$

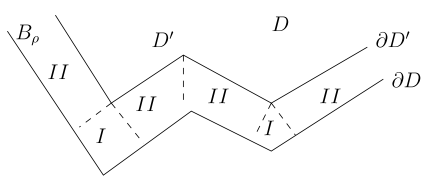

Fig. 1. Illustration of three types of points in B_ρ

Making use of (ii), it follows from the Cauchy–Shwarz and arithmetic-geometric mean inequalities that,

$$\Big|\sum_{K\in\mathcal{T}_{B_\rho}}\int_{K\cap B_\rho} u\nabla u\cdot\mathbf{w}\,dx\Big| \leqslant c\rho\sum_{K\in\mathcal{T}_{B_\rho}}\int_{K\cap B_\rho}|u\nabla u|\,dx$$
$$\leqslant \tfrac{1}{2}\|u\|^2_{0,B_\rho}+c\rho^2\textstyle\sum_{K\in\mathcal{T}_{B_\rho}}\|\nabla u\|^2_{0,K}.$$

Thus, using this in (3.30), we get

$$\|u\|^2_{B_\rho}\,dx \leqslant \sum_{K\in\mathcal{T}_{B_\rho}}\int_{\partial(K\cap B_\rho)} u^2\mathbf{w}\cdot\mathbf{n}\,ds + c\rho^2|u|^2_{1,h,D}. \tag{3.31}$$

Let $\mathcal{E}^I_{B_\rho}=\{e\in\mathcal{E}^I_D : e\cap\overline{B}_\rho\neq\emptyset\}$. Then, from (i) it follows that

$$\sum_{K\in\mathcal{T}_{B_\rho}}\int_{\partial(K\cap B_\rho)} u^2\mathbf{w}\cdot\mathbf{n}\,ds=\int_{\partial D}u^2\mathbf{w}\cdot\mathbf{n}\,ds+\sum_{e\in\mathcal{E}^I_{B_\rho}}\int_{e\cap B_\rho}\Big((u^{(i)})^2-(u^{(j)})^2\Big)\mathbf{w}\cdot\mathbf{n}\,ds.$$

Applying Lemma 3.6, we obtain

$$\int_{\partial D}u^2\mathbf{w}\cdot\mathbf{n}\,ds\leqslant c\rho\Big(H^{-1}\|u\|^2_D+H|u|^2_{1,h,D}\Big). \tag{3.32}$$

Finally, as was done in [11], we obtain the inequality

$$\sum_{e\in\mathcal{E}^I_{B_\rho}}\int_{e\cap B_\rho}\big((u^{(i)})^2-(u^{(j)})^2\big)\mathbf{w}\cdot\mathbf{n}\,ds\leqslant\tfrac{1}{2}\|u\|^2_{0,B_\rho}+c\rho^2|u|^2_{1,h,B_\rho}. \tag{3.33}$$

Using (3.33) and (3.32) in (3.31) we obtain (3.29). □

4. THE NON-OVERLAPPING SCHWARZ METHODS

In this section, we shall develop two-level non-overlapping additive and multiplicative Schwarz preconditioners for the discontinuous Galerkin method.

4.1. Formulation of the Additive Schwarz Preconditioner

Let $\mathcal{T}_S$ denote a partition of Ω into p non-overlapping subdomains $\Omega_i,\ i=1,\dots,p$ and let

$$\Gamma_j=\partial\Omega_j\cap\partial\Omega,\qquad \Gamma_{jk}=\partial\Omega_j\cap\partial\Omega_k.$$

Further, let $\mathcal{T}_H$ denote a (coarse) partition (triangulation) of Ω with mesh size $H>0$. We shall assume that $\mathcal{T}_H$ possesses the same minimal angle and local quasi-uniformity properties of $\mathcal{T}_h$. Furthermore, we shall always assume that $\mathcal{T}_h, \mathcal{T}_H$ and $\mathcal{T}_S$ are related by

$$\mathcal{T}_S \subseteq \mathcal{T}_H \subseteq \mathcal{T}_h,$$

i.e., each Ω_i is a union of some elements of $\mathcal{T}_H$ each of which is a union of elements of $\mathcal{T}_h$. Let us note again that very general subdomains Ω_i can be used without affecting the bound $O(H^3/h^3)$.

It is well known (cf. [20,22]) that the first step towards constructing the additive Schwarz preconditioner is to have a valid subspace decomposition of the finite element space V^h. For the discontinuous Galerkin method considered in this paper, since $V^h \subset L^2(\Omega)$ and no continuity constraints are imposed on the functions in V^h, it is easy to construct such a space decomposition. We remark that the space decomposition is much more complicated for well-known conforming and nonconforming plate elements (cf. [7,13] and reference therein).

We define the subspaces $\{V_j^h\}_{j=1}^p$ associated with the subdomains $\{\Omega_j\}_{j=1}^p$ by

$$V_j^h = \{v \in V^h | v=0 \text{ in } \Omega \setminus \overline{\Omega}_j\}, \quad j=1,2,\cdots,p.$$

By construction, the following space decomposition result holds

Lemma 4.1. V^h is the direct sum of the subspaces $\{V_j^h\}_{j=1}^p$, that is,

$$V_i^h \cap V_j^h = \{0\}, \qquad i \neq j, \quad i,j=1,\ldots,p, \tag{4.1}$$

$$V^h = V_1^h + V_2^h + \cdots + V_p^h. \tag{4.2}$$

Having obtained the above space decomposition, the second step requires the construction of a subdomain bilinear form (or a subdomain solver) on each subdomain. To this end, we define $a_i(\cdot,\cdot)$ on $V_i^h \times V_i^h$ to be the restriction of $a_h(\cdot,\cdot)$ to $V_i^h \times V_i^h$ in the sense that

$$a_i(u,v) = a_h(u,v), \qquad \forall u,v \in V_i^h, \quad i=1,\ldots,p. \tag{4.3}$$

Let $\mathcal{E}_i^I$ and $\mathcal{E}_i^B$ denote the set of interior edges and the set of boundary edges in Ω_i and $\mathcal{E}_i = \mathcal{E}_i^I \cup \mathcal{E}_i^B$ for $i=1,2,\ldots,p$. Applying this principle to $a_h(\cdot,\cdot)$, we see that for $i=1,\ldots,p$,

$$\begin{aligned} a_i(u,v) = & \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\Omega_i}} (\Delta u,\Delta v)_K + \sum_{e\in\mathcal{E}_i}\Bigg(\left\langle [u], \left\{\frac{\partial \Delta v}{\partial n}\right\}\right\rangle_e - \left\langle \left[\frac{\partial u}{\partial n}\right], \{\Delta v\}\right\rangle_e \\ & + \left\langle [v], \left\{\frac{\partial \Delta u}{\partial n}\right\}\right\rangle_e - \left\langle \left[\frac{\partial v}{\partial n}\right], \{\Delta u\}\right\rangle_e + \gamma h_e^{-3}\langle [u],[v]\rangle_e \\ & + \gamma h_e^{-1}\left\langle \left[\frac{\partial u}{\partial n}\right], \left[\frac{\partial v}{\partial n}\right]\right\rangle_e \Bigg). \end{aligned} \tag{4.4}$$

We introduce the "interface bilinear form" $\mathcal{I}(\cdot,\cdot): V^h\times V^h\to R$

$$\begin{aligned} \mathcal{I}(u,v) = \sum_{\substack{\Gamma_{mk}\\ m>k}} \sum_{\substack{e_{j\ell}\subset\Gamma_{mk}\\ j>\ell}} \Bigg\{ & -\left\langle \frac{\partial \Delta u^{(j)}}{\partial n}, v^{(\ell)}\right\rangle_{e_{j\ell}} - \left\langle \frac{\partial \Delta v^{(j)}}{\partial n}, u^{(\ell)}\right\rangle_{e_{j\ell}} \\ & + \left\langle \frac{\partial u^{(\ell)}}{\partial n}, \Delta v^{(j)}\right\rangle_{e_{j\ell}} + \left\langle \frac{\partial v^{(\ell)}}{\partial n}, \Delta u^{(j)}\right\rangle_{e_{j\ell}} \\ & - \gamma h_{j\ell}^{-3}\left(\langle u^{(j)}, v^{(\ell)}\rangle_{e_{j\ell}} + \langle u^{(\ell)}, v^{(j)}\rangle_{e_{j\ell}}\right) \\ & - \gamma h_{j\ell}^{-1}\left(\left\langle \frac{\partial u^{(j)}}{\partial n}, \frac{\partial v^{(\ell)}}{\partial n}\right\rangle_{e_{j\ell}} + \left\langle \frac{\partial u^{(\ell)}}{\partial n}, \frac{\partial v^{(j)}}{\partial n}\right\rangle_{e_{j\ell}}\right)\Bigg\}. \end{aligned} \tag{4.5}$$

Lemma 4.2. For $u,v\in V^h$, let $u_i,v_i\in V_i^h, i=1,\ldots,p$ be given (uniquely) by $u=\sum_{i=1}^p u_i,\quad v=\sum_{i=1}^p v_i$. Then, there holds

$$a_h(u,v)=\sum_{i=1}^p a_i(u_i,v_i)+\mathcal{I}(u,v). \tag{4.6}$$

Proof. The assertion follows from direct calculations. □

We now introduce, in addition to the subspaces V_i^h, $i=1,\ldots,p$, a coarse mesh subspace V_0^h of V^h corresponding to the partition $\mathcal{T}_H$ and a corresponding bilinear form $a_0(\cdot,\cdot): V_0^h\times V_0^h\to R$. It is well known that this construction is crucial in obtaining a good preconditioner. Let the integer r_H be chosen satisfying $2 \leqslant r_H \leqslant r$. Let

$$V_0^h = \prod_{D \in \mathcal{T}_H} P_{r_H - 1}(D), \qquad \hat{V}_0^h = \prod_{D \in \mathcal{T}_H} P_1(D). \tag{4.7}$$

Clearly, $\hat{V}_0^h \subset V_0^h \subset V^h$. Also, our (theoretical) estimates are valid independent of the choice of r_H. Also, we define $a_0(\cdot,\cdot)$ by

$$a_0(u, v) = a_h(u, v), \quad \forall u, v \in V_0^h. \tag{4.8}$$

Now we are ready to define the additive operator

$$T = T_0 + T_1 + \cdots + T_p, \tag{4.9}$$

where T_j is a projection operator from V^h to V_j^h defined by

$$a_j(T_j u, v) = a_h(u, v) \qquad \forall v \in V_j^h, \qquad j = 0, 1, 2, \ldots, p. \tag{4.10}$$

It follows from (4.3) and (4.8) that the forms $a_j(\cdot,\cdot)$ are symmetric and coercive. Thus the operators T_j are well defined. Following the framework given in [10,20,22], the additive Schwarz method consists in replacing the discrete problem (2.7) by the equation

$$Tu = g, \qquad g = \sum_{j=0}^{p} g_j, \tag{4.11}$$

where $g_j = T_j u$ is defined as the solution of

$$a_j(g_j, v) = F(v) \qquad \forall v \in V_j^h, \quad j = 0, 1, 2, \ldots, p. \tag{4.12}$$

In matrix notation, the additive Schwarz preconditioner corresponds to choosing the matrix B in (2.12) as

$$B = R_0^T A_0^{-1} R_0 + (R_1^T A_1 R_1 + \cdots + R_p^T A_p R_p)^{-1}, \tag{4.13}$$

where A_j is the stiffness matrix corresponding to $a_j(\cdot,\cdot)$ and R_j^T is the matrix representation of the embedding operator $i : V_j^h \to V^h$, $j = 0, \ldots, p$.

Now the question is whether the preconditioned system (4.11) is well-conditioned, in particular, whether the condition number of T, or equivalently that of the matrix BA, depends "favorably" on the mesh sizes h and H. These questions will be addressed in the next subsection.

4.2. Condition Number Estimate for the Additive Schwarz Method

To estimate the condition number of T, we shall use the general abstract convergence theory of Schwarz methods given in [20]. We shall do so by verifying that a set of three *Assumptions* are satisfied and by estimating the constants $C_0^2, \rho(\mathcal{E})$ and ω appearing there in terms of the parameters of our method (cf. p. 155 of [20]).

The verification of the first assumption requires showing that for all $u \in V^h$

$$\sum_{i=0}^{p} a_i(u_i, u_i) \leqslant C_0^2 a_h(u, u) \tag{4.14}$$

for *some* representation $u = \sum_{i=0}^{p} u_i$. In view of (4.6) we shall need to obtain a bound for the interface form $\mathcal{I}(u, u)$.

Lemma 4.3. There exists a constant $c > 0$ such that for any $w \in V^h$

$$|\mathcal{I}(w, w)| \leqslant \| w \|_{2,h}^2 + c \sum_{D\in\mathcal{T}_H} \left(H^{-1} h_D^{-3} \| w \|_{0,D}^2 + H h_D^{-3} | w |_{1,h,D}^2 \right). \tag{4.15}$$

Proof. For any $w \in V^h$, from the definitions of $\mathcal{I}(\cdot, \cdot)$ and $\| \cdot \|_{2,h,\Omega}$, and Schwarz's inequality we have

$$\begin{aligned}
\mathcal{I}(w, w) = 2 \sum_{\substack{\Gamma_{mk}\\ m>k}} \sum_{\substack{e_{j\ell}\subset\Gamma_{mk}\\ j>\ell}} & \left\{ -\left\langle \frac{\partial \Delta w^{(j)}}{\partial n}, w^{(\ell)} \right\rangle_{e_{j\ell}} + \left\langle \frac{\partial w^{(\ell)}}{\partial n}, \Delta w^{(j)} \right\rangle_{e_{j\ell}} \right. \\
& \left. - \gamma h_{j\ell}^{-3} \langle w^{(j)}, w^{(\ell)} \rangle_{e_{j\ell}} - \gamma h_{j\ell}^{-1} \left\langle \frac{\partial w^{(j)}}{\partial n}, \frac{\partial w^{(\ell)}}{\partial n} \right\rangle_{e_{j\ell}} \right\} \\
= 2 \sum_{\substack{\Gamma_{mk}\\ m>k}} \sum_{e\subset\Gamma_{mk}} & \left\{ \left\langle \left\{ \frac{\partial \Delta w}{\partial n} \right\}, [w] \right\rangle_e - \left\langle \left\{ \frac{\partial \Delta w}{\partial n} \right\}, \{w\} \right\rangle_e \right. \\
& - \left\langle \left[\frac{\partial w}{\partial n} \right], \{\Delta w\} \right\rangle_e + \left\langle \left\{ \frac{\partial w}{\partial n} \right\}, \{\Delta w\} \right\rangle_e \\
& + \gamma h_e^{-3} \left(\langle \{w\}, [w] \rangle_e - | \{w\} |_{0,e}^2 \right) \\
& \left. + \gamma h_e^{-1} \left(\left\langle \left\{ \frac{\partial w}{\partial n} \right\}, \left[\frac{\partial w}{\partial n} \right] \right\rangle_e - \left| \left\{ \frac{\partial w}{\partial n} \right\} \right|_{0,e}^2 \right) \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|w\|_{2,h}^2 + c(\gamma) \sum_{\substack{\Gamma_{mk}\\ m>k}} \sum_{e\subset\Gamma_{mk}} \left(h_e^{-3} |\{w\}|_{0,e}^2 + h_e^{-1} \left| \left\{ \frac{\partial w}{\partial n} \right\} \right|_{0,e}^2 \right) \\
&\leqslant \|w\|_{2,h}^2 + c(\gamma) \sum_{D\in\mathcal{T}_H} \left(h_D^{-3} |\{w\}|_{0,\partial D}^2 + h_D^{-1} \left| \left\{ \frac{\partial w}{\partial n} \right\} \right|_{0,\partial D}^2 \right) \\
&\leqslant \|w\|_{2,h}^2 + c(\gamma) \sum_{D\in\mathcal{T}_H} \left(h_D^{-3} H^{-1} \|w\|_{0,D}^2 + h_D^{-3} H |w|_{1,h,D}^2 \right) \\
&\quad + c(\gamma) \sum_{D\in\mathcal{T}_H} h_D^{-1} \left| \left\{ \frac{\partial w}{\partial n} \right\} \right|_{0,\partial D}^2 . \qquad (4.16)
\end{aligned}
$$

Here we have used the trace inequality (3.28) to get the final step.

To bound the last term on the right-hand side of (4.16), for each $D \in \mathcal{T}_H$, we let D_h denote the boundary layer of D formed by the elements K of $\mathcal{T}_D$ that are adjacent to ∂D. It is easy to see that the region enclosed by the inner boundary of D_h is polygonal and starlike. Now using the trace inequality (3.2) on each triangle K in D_h, and applying the generalized Poincaré inequality (3.29),we obtain

$$
\begin{aligned}
h_D^{-1} \left| \left\{ \frac{\partial w}{\partial n} \right\} \right|_{0,\partial D}^2 &\leqslant c h_D^{-4} \|w\|_{0,D_h}^2 \qquad (4.17)\\
&\leqslant c\left(h_D^{-3} H^{-1} \|w\|_{0,D}^2 + h_D^{-3} H |w|_{1,h,D}^2 \right).
\end{aligned}
$$

Finally, the proof is completed by combining (4.16) and (4.17). □

In order to show inequality (4.14), we need to construct a suitable decomposition for each $u \in V^h$. To that end, we need some preparation.

Let $W^H \subset H^1(\Omega)$ denote the *continuous* piecewise linear finite element space on $\mathcal{T}_H$. Define the bilinear form on $V^h \times V^h$

$$
b_h(u,v) = \sum_{K\in\mathcal{T}_h} \left(\nabla u, \nabla v\right)_K - \sum_{e\in\mathcal{E}^I} \left(\left\langle [u], \left\{ \frac{\partial v}{\partial n} \right\} \right\rangle_e + \left\langle \left\{ \frac{\partial u}{\partial n} \right\}, [v] \right\rangle_e \right). \qquad (4.18)
$$

Since $b_h(v,w) = (\nabla v, \nabla w)_\Omega$, $\forall v, w \in H^1(\Omega)$, we have $\forall \psi \in H^2(\Omega)$ (cf. [8])

$$
\inf_{v\in W^H} \left(b_h(\psi - v, \psi - v)^{\frac{1}{2}} + H^{-1} \|\psi - v\|_{0,\Omega} \right) \leqslant cH \|\psi\|_{2,\Omega}. \qquad (4.19)
$$

For any $u \in V^h$, define the (global) "elliptic projection" $J_h^H u \in W^H$ of u by

$$b_h(J_h^H u, v) = b_h(u, v) \quad \forall v \in W^H, \tag{4.20}$$

$$(J_h^H u, v) = (u, 1). \tag{4.21}$$

The existence of $J_h^H u$ follows from the fact that $b_h(v, w) = (\nabla v, \nabla w)_\Omega$, $\forall v, w \in H^1(\Omega)$.

The operator J_h^H has the following approximation properties.

Lemma 4.4. Suppose that Ω is a bounded convex polygonal domain, then there exist a constant $c > 0$, which are independent of h, p, δ and H, such that for any $u \in V^h$

$$\| u - J_h^H u \|_{0,\Omega} \leqslant cH | u - J_h^H u |_{1,h,\Omega}, \tag{4.22}$$

$$| u - J_h^H u |_{1,h,\Omega} \leqslant cH \| u \|_{2,h,\Omega}, \tag{4.23}$$

$$\| u - J_h^H u \|_{0,\Omega} \leqslant cH^2 \| u \|_{2,h,\Omega}. \tag{4.24}$$

Proof. (4.21) implies that $w \equiv u - J_h^H u$ has zero mean, then the following auxiliary problem has a solution (unique up to an additive constant) $\varphi \in H^2(\Omega)$ (cf. [15])

$$-\Delta\varphi = w \qquad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

and

$$\| \varphi \|_{2,\Omega} \leqslant c \| w \|_{0,\Omega}. \tag{4.25}$$

Now, (4.19) and (4.25) yield

$$\begin{aligned} \| w \|_{0,\Omega}^2 &= -(\Delta\varphi, w)_\Omega = b_h(\varphi, w) \\ &= \inf_{v \in W^H} b_h(\varphi - v, w) \\ &\leqslant \inf_{v \in W^H} b_h(\varphi - v, \varphi - v)^{1/2} b_h(w, w)^{1/2} \\ &\leqslant cH \| \varphi \|_{2,\Omega} | w |_{1,h,\Omega} \\ &\leqslant cH \| w \|_{0,\Omega} | w |_{1,h,\Omega}, \end{aligned} \tag{4.26}$$

which gives (4.22).

Clearly, (4.24) is a trivial consequence of (4.22) and (4.23). Hence, it remains to show (4.23). Since the proof follows exactly the same lines as the proof of (3.17), we only highlight the key steps. First, setting $v = J_h^H u$ in (4.20) yields

$$\sum_{K\in\mathcal{T}_h} \left(\nabla(u - J_h^H u), \nabla J_h^H u\right)_K - \sum_{e\in\mathcal{E}^I} \left\langle [u], \left\{ \frac{\partial (J_h^H u)}{\partial n} \right\} \right\rangle_e = 0. \tag{4.27}$$

Now proceeding exactly as in Lemma 3.4, we obtain

$$\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{0,K}^2 \leqslant c\eta^{-1} \| w \|_{0,\Omega}^2 + 2\eta \| u \|_{2,h}^2. \tag{4.28}$$

Hence, (4.23) follows from (4.28) after choosing $\eta = 2c^3 H^2$. □

Lemma 4.5. For any $u \in V^h$, let $u = \sum_{j=0}^p u_j,\ u_j \in V_j^h,\ j = 0, \ldots, p$ where $u_0 = J_h^H u \in \hat{V}_0^h \subset V_0^h$, and $u_1, \ldots, u_p$ are determined (uniquely) by $u - u_0 = u_1 + \cdots + u_p$. Then

$$\sum_{j=0}^p a_i(u_j, u_j) \leqslant c\left(\frac{H}{h}\right)^3 a_h(u, u). \tag{4.29}$$

Proof. From (4.6) we have

$$a_h(u - u_0, u - u_0) = \sum_{j=1}^p a_j(u_j, u_j) + I(u - u_0, u - u_0).$$

Using Schwarz's inequality on the bilinear forms and the fact that $a_0(u_0, u_0) = a_h(u_0, u_0)$, we get

$$\begin{aligned}\sum_{j=0}^p a_j(u_j, u_j) &= a_h(u - u_0, u - u_0) + a_h(u_0, u_0) - I(u - u_0, u - u_0)\\ &\leqslant 2a_h(u, u) + 3a_h(u_0, u_0) + |I(u - u_0, u - u_0)|. \end{aligned} \tag{4.30}$$

We next estimate $|I(u - u_0, u - u_0)|$. It follows from (4.15), (4.23), (4.24) and (2.9) that

$$\begin{aligned}|I(u - u_0, u - u_0)| &\leqslant c \sum_{D\in\mathcal{T}_H} \left(Hh_D^{-3} | u - u_0 |_{1,h,D}^2 + H^{-1} h_D^{-3} \| u - u_0 \|_{0,D}^2 \right) \\ &\qquad + a_h(u - u_0, u - u_0) \\ &\leqslant c \left(Hh^{-3} | u - u_0 |_{1,h,\Omega}^2 + H^{-1} h^{-3} \| u - u_0 \|_{0,\Omega}^2 \right. \\ &\qquad \left. + a_h(u - u_0, u - u_0) \right) \\ &\leqslant c \left(H^3 h^{-3} a_h(u, u) + a_h(u_0, u_0) \right). \end{aligned} \tag{4.31}$$

It remains to bound $a_h(u_0,u_0)$. Since u_0 is piecewise linear on $\mathcal{T}_H$ and hence also on $\mathcal{T}_h$, from the definition of $a_h(\cdot,\cdot)$, (3.28), (4.17), (4.23), (4.24) and (2.9) we get

$$\begin{aligned}
a_h(u_0,u_0) &= \gamma \sum_{e\in\mathcal{E}} \left(h_e^{-3} |[u_0]|_{0,e}^2 + h_e^{-1} \left| \left[\frac{\partial u_0}{\partial n} \right] \right|_{0,e}^2 \right) \\
&\leqslant \gamma \sum_{D\in\mathcal{T}_H} \sum_{e\subset\partial D} \Bigg(h_e^{-3} |[u]|_{0,e}^2 + h_e^{-3} |[u-u_0]|_{0,e}^2 \\
&\quad + h_e^{-1} \left| \left[\frac{\partial u}{\partial n} \right] \right|_{0,e}^2 + h_e^{-1} \left| \left[\frac{\partial (u-u_0)}{\partial n} \right] \right|_{0,e}^2 \Bigg) \\
&\leqslant \gamma \| u \|_{2,h}^2 + c(\gamma) \sum_{D\in\mathcal{T}_H} \left(h_D^{-3} | u-u_0 |_{0,\partial D}^2 + h_D^{-1} \left| \frac{\partial (u-u_0)}{\partial n} \right|_{0,\partial D}^2 \right) \\
&\leqslant \gamma \| u \|_{2,h}^2 + c(\gamma) \sum_{D\in\mathcal{T}_H} \Big(h_D^{-3} H^{-1} \| u-u_0 \|_{0,D}^2 \\
&\quad + h_D^{-3} H | u-u_0 |_{1,h,D}^2 \Big) \\
&\leqslant \gamma \| u \|_{2,h}^2 + c(\gamma) \left(H^{-1} h^{-3} \| u-u_0 \|_{0,\Omega}^2 + H h^{-3} | u-u_0 |_{1,h,\Omega}^2 \right) \\
&\leqslant c(\gamma) \left(\frac{H}{h} \right)^3 a_h(u,u).
\end{aligned} \tag{4.32}$$

The proof is completed after substituting (4.31) and (4.32) into (4.30). □

Thus far we have shown that *Assumption 1* holds with $C_0^2 = O(H^3/h^3)$. Verifying *Assumption 2* consists in obtaining a bound for the spectral radius $\rho(\mathcal{E})$ of the $p\times p$ matrix $\mathcal{E}$ given as follows: Let $0 \leqslant \mathcal{E}_{ij} \leqslant 1$ be the minimal values such that

$$|a_h(u_i,u_j)| \leqslant \mathcal{E}_{ij} a_h(u_i,u_i)^{1/2} a_h(u_j,u_j)^{1/2}, \quad u_i \in V_i^h,\ \ u_j \in V_j^h,\ \ i,j=1,\ldots,p.$$

That such values exist is a consequence of the Cauchy-Schwarz inequality. The important thing however is to obtain a small bound on ρ. To do so, we observe that

$$a_h(u_i,u_j) = 0 \text{ if } \operatorname{meas}(\Gamma_{ij}) = 0, \quad i,j = 1,\ldots,p.$$

For the remaining cases, we take $\mathcal{E}_{ij} = 1$. If follows at once from Gershgorin's circle theorem that

$$\rho(\mathcal{E}) \leqslant \max_m \operatorname{card}\{k \mid \operatorname{meas}(\Gamma_{mk}) > 0\} + 1, \tag{4.33}$$

i.e., $\rho(\mathcal{E})$ is bounded by one plus the maximum number of adjacent subdomains a given subdomain can have. In practice this number is usually $\leqslant 5$. Even for "unusual" subdomain partitions, this number is not expected to be large.

As for *Assumption 3*, Let $\omega \in (0,1)$ be the minimum constant such that

$$a_h(u_i, u_i) \leqslant \omega a_i(u_i, u_i), \quad \forall u_i \in V_i^h, \quad i = 0, \ldots, p. \tag{4.34}$$

Recall that we defined the subdomain bilinear forms $a_i(\cdot,\cdot)$ precisely by $a_i(u_i, u_i) = a_h(u_i, u_i), \ = 0, \ldots, p$; thus (4.34) holds trivially with $\omega = 1$.

With this, the first main theorem of the paper is at hand:

Theorem 4.6. The condition number $\kappa(T)$ of the operator T of the additive Schwarz method defined in this section satisfies

$$\kappa(T) \leqslant c\big(1 + \rho(\mathcal{E})\big)\left(\frac{H}{h}\right)^3. \tag{4.35}$$

Proof. This is an immediate consequence of Lemma 3 in Chapter 5 of [20] and our estimates (4.29), (4.33) and (4.34). □

4.3. The Non-Overlapping Multiplicative Schwarz Method

In this section, we will briefly present a multiplicative version of the additive Schwarz method developed above. The multiplicative Schwarz preconditioner is constructed following the general setting of Chapter 5 of [20] using the operators $\{T_j\}$ introduced in Sec. 4.1. Multiplicative Schwarz methods can also be constructed as iterative methods, for more discussion in this direction, we refer to [5, 19, 20, 22].

Let T_j be as in Sec. 4.1 and be defined by (4.5) for $j = 0, 1, 2, \ldots, P$. Following [20], the symmetric multiplicative Schwarz preconditioner B can be symbolically written as

$$B = [I - (I - T_0)(I - T_1) \cdots (I - T_P)(I - T_P) \cdots (I - T_1)(I - T_0)]A^{-1}, \tag{4.36}$$

where I denotes the identity operator on V^h.

Therefore, the preconditioned operator $\widehat{T} \equiv BA$ has the form

$$\widehat{T} = I - (I - T_0)(I - T_1) \cdots (I - T_P)(I - T_P) \cdots (I - T_1)(I - T_0). \tag{4.37}$$

We conclude this section by stating the second main theorem of this paper on the convergence estimate for the new operator $\widehat{T}$.

Theorem 4.7. The statement of Theorem 4.6 holds for the operator $\widehat{T}$. That is,

$$\operatorname{cond}(\widehat{T}) \leqslant C\left(1+2\rho(\mathcal{E})^2\right)\left(\frac{H}{h}\right)^3. \tag{4.38}$$

Proof. The estimate (4.38) is an immediate application of the estimates (4.29), (4.33) and (4.34), and Lemma 4 of Chapter 5 of [20] with $C_0^2 = O(H^3/h^3)$, and $\omega = 1$. □

REFERENCES

1. Arnold, D. N. (1982). An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.* **19**, 742–760.
2. Arnold, D. N., Brezzi, F., Cockburn, B., and Marini, L. D. (2002). Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* **39**, 1749–1779.
3. Baker, G. A. Finite element methods for elliptic equations using nonconforming elements. *Math. Comput*. **31**, 1977, pp.45–59.
4. Baker, G. A., Jureidini, W. N., and Karakashian, O. A. (1990). Piecewise solenoidal vector fields and the Stokes problems. *SIAM J. Numer. Anal.* **27**, 1466–1485.
5. Bramble, J. H., Pasciak, J. E., Wang, J., and Xu, J. (1991). Convergence estimates for product iterative methods with applications to domain decomposition. *Math. Comp.* **57**, 1–21.
6. Brenner, S. (2003). Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.* **41**, 306–324.
7. Brenner, S. (1996). Two-level additive Schwarz preconditioners for nonconfroming plate elements. *Numer. Math.* **72**, 419–447.
8. Ciarlet, P. G. (1978). *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam.
9. Douglas, Jr. J., and Dupont, T. (1976). *Interior Penalty Procedures for Elliptic and Parabolic Galerkin Methods*, Lecture Notes In Physics Vol. 58, Springer-Verlag, Berlin.
10. Dryja, M., and Widlund, O. B. (1990). Towards a unified theory of domain decomposition algorithms for elliptic problems. In T. Chan (ed.), *Proceedings of Third International Symposium on Domain Decomposition Methods for Partial Differential Equations SIAM,* Philadelphia, 3–21.
11. Feng, X., and Karakashian, O. A. (2001). Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.* **39**, 1343–1365.
12. Feng, X., and Karakashian, O. A. Two-level overlapping Schwarz methods for a discontinuous Galerkin approximation of fourth order elliptic problems, preprint.
13. Feng, X., and Rahman, M. T. (2002). An additive average Schwarz method for the plate bending problem. *J. Numer. Math.* **10**, 109–125.
14. Griebel, M., and Oswald, P. (1995). On the abstract theory of additive and multiplicative Schwarz algorithms. *Numer. Math.* **70**, 163–180.

15. Grisvard, P. *Singularities in Boundary Value Problems*. Research Notes in Applied Mathematics, Vol. 22, MASSON and Springer-Verlag, Berlin.
16. Houston, P., Süli, E., and Schwab, C. (2002). Discontinuous hp-finite element methods for advection–diffusion problems. Tech. Report # 2000-07, Seminar für Angewandte Mathenatik, ETH, Zürich, *SIAM J. Numer. Anal.* **39**, 2133–2163.
17. Karakashian, O. A., and Jureidini, W. N. (1998). A nonconforming finite element method for the stationary Navier–Stokes Equations. *SIAM J. Numer. Anal.* **35**, 93–120.
18. Oden, J. T., and Baumann, C. E. (2000). A conservative DGM for convection–diffusion and Navier–Stokes problems. In Cockburn, B. karniadakis, G. E. Shu C.-W. (eds.),. *Proceedings of the International Symposium on the Discontinuous Galerkin Method* Springer lecture notes in Computational Science and Eng. Vol. 11, 179–196.
19. Quarteroni, A. and Valli, A. (1999). *Domain Decomposition Methods for Partial Differential Equations*, Oxford University Press, New York.
20. Smith, B. E., Bjørstad, P. E., and Gropp, W. D. (1996). *Domain Decomposition, Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, New York.
21. Wheeler, M. (1978). An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.* **15**, 152–161
22. Xu, J. (1992). Iterative methods by space decomposition and subspace correction. *SIAM Rev.* **34**, 581–613.