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ORIGINAL PAPER

Amplitude equation for a diffusion–reaction system in presence of complexing reaction with the activator species: the Brusselator model

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Abstract
For Brusselator diffusion–reaction model involving complex forming reaction with the activator species, an amplitude equation has been derived in the framework of a weakly nonlinear theory. Complexing reaction with the activator species strongly influences the time-dependent amplitudes such as in Hopf-wave bifurcations, whereas time-independent amplitudes such as in Turing—bifurcations, are independent of complexing reaction with the activator species. Complexing reaction arrests the arrival of Hopf—bifurcations and the domain of excitable non-oscillations such created may be used effectively for Turing-structure generation by inducing inhomogeneous perturbations of nonzero wavenumber mode. Any major complexing interaction with the activator species in a biological oscillatory network is bound to alter the domains of Hopf/Turing bifurcations affecting the course of physiological self-organization processes.

Keywords Hopf-wave bifurcation · Turing patterns · Diffusion–reaction systems · Amplitude equation

1 Introduction

Instabilities breaking temporal and spatial symmetries [1] have been observed in diverse fields of non-equilibrium systems including nonlinear optics [2], hydrodynamics [3], and oscillating chemical reactions [4–7]. Nonlinear diffusion–reaction (D–R) systems exhibit self-organized concentration patterns, when the steady states

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become unstable to inhomogeneous perturbations, but remain stable to homogeneous perturbations. These patterns, originated from the coupling between nonlinear chemical reaction and molecular diffusion, are called Turing structures [7]—these patterns are periodic in space with an intrinsic wavelength, and stationary in time. Diffusion coefficient of the inhibitor species must be sufficiently larger than that of the activator species, is the necessary condition for Turing instability to occur. This requirement for Turing structure formation is termed [8] as long range inhibition and short range activation.

The discovery of Turing structures [4–6] in CIMA reaction and in its variant CDIMA reaction seems to be the result of fortunate coincidence that starch and poly-acrylamide (PAA) gel were also present in the system. In agarose gel, which does not interact chemically with iodine species, the presence of starch is necessary [9, 10] for generating Turing structures. At low starch concentration, the system is shifted to oscillatory state producing travelling waves rather than Turing structures.

Color indicators, as for example, starch having large molecular weight and low diffusivity (~zero in gels) are often used for controlling and visualizing these patterns. A color indicator binds itself with the activator species, thereby reducing its effective mobility [11]. Hopf-bifurcating waves may be produced at low indicator concentrations (for the details of Hopf/Turing bifurcations see Sect. 3), but Turing structures overtake them at high indicator concentrations because of reduced free concentration of the activator species due to its complexing reaction with the color indicator. Thus, color indicators can effectively control the distance between the thresholds of Hopf and Turing bifurcations, which may coincide at a co-dimension-two Turing–Hopf bifurcation point (CTHP) [12, 13]. The co-dimension of a bifurcation is the actual number of parameters, which must be varied for the bifurcation to occur. A CTHP is characterized by two bifurcating situations and generally appears on co-dimension-two manifolds in the parameter space. Topaz et al. [14] have investigated time-periodic forcing of spatially extended systems near a CTHP. Biancalani et al. [15] have investigated stochastic Turing patterns in a microscopic model of the Brusselator [16].

Although Turing structures may be computed directly from D–R equations by setting the parameter values appropriately, the structural transitions and the stability of various forms of Turing structures can be interpreted effectively by an amplitude equation (AE), which may be derived in the framework of a weakly nonlinear theory. AE [1, 8, 17] for Turing pattern selection [4–6, 9, 10, 18] has been derived in the past for a few D–R systems including Brusselator [19, 20], Swift–Hohenberg model and others [12, 21–24]. AE derivation for Swift–Hohenberg model [25] has been widely used for qualitative understanding of convective structures originated from Benard–Marangoni instability or non-Boussinesq Benard convection [26, 27]. Yansu et al. [28] have derived an amplitude equation of the Brusselator D–R system with time delay based on Friedholm solvability condition and multi-scale analysis method near the critical point of phase transition including a discussion on its stability analysis.

Because of lengthy algebraic calculations only those D–R systems have been chosen for AE derivation, which have simple-looking kinetic steady states from analytical calculations. In this paper we have derived the AE for an irreversible Brusselator model [16] in presence of complex forming reaction with the activator species X (see Eq. 2.2). Such derivations and obtained results are interesting since complexing reaction with the activator species may play a major role in biological oscillatory networks in self-organizing physiological systems. Attempts have been made to interpret [29–33] the structural transitions and stability of various forms of Turing structures.

Section 2 discusses on the irreversible Brusselator [16] D–R model in presence of complexing reaction [11, 34] with the activator species X. Section 3 undertakes a linear stability analysis of the D–R equations (2.6) and (2.7). Section 4 divides the D–R equations into linear and nonlinear parts as given in Eqs. (4.0) to (4.3). Section 5 presents the nonlinear analysis. This is based on (i) expansion of the evolution of the inhomogeneous perturbations, which unfold in terms of the amplitudes of branching solution for critical bifurcation parameter (μ) in the neighborhood of Turing bifurcation point [$\mu_c = 0$; see Eq. (6.5)] and (ii) derivation of two solvability conditions from second and third order terms [see Eqs. (5.13) and (5.18)]. Section 6 incorporates these two solvability conditions into the expanded partial time derivative of amplitude [see Eqs. (5.4), (6.2), and (6.3)], which after neglecting the terms of order ε^3 and higher, generates the AE. The new results are summarized in Sects. 7 and 8.

2 The model

A two variable Brusselator model [16] consisting of two intermediate species X and Y is given by the following four irreversible steps.

$$\begin{aligned} &A \rightarrow X \\ &2X + Y \rightarrow 3X \\ &B + X \rightarrow Y + D \\ &X \rightarrow E \end{aligned} \tag{2.0a}$$

which leads to the overall reaction given by

$$A + B \rightarrow D + E \tag{2.0b}$$

The D-R equations are given by

$$\begin{aligned} \partial X/\partial t &= k_1 a + k_2 X^2 Y - k_3 b X - k_4 X + D_X (\partial^2 X/\partial l^2) \\ \partial Y/\partial t &= -k_2 X^2 Y + k_3 b X + D_Y (\partial^2 Y/\partial l^2) \end{aligned} \tag{2.1a}$$

where $k_i (i = 1 \text{ to } 4)$ is the rate constant of step i; a and b are molar concentrations of the reactants A and B respectively, which are kept constant in time; X and Y are molar concentrations of the intermediate species X and Y respectively; D_X and D_Y are diffusion coefficients (cm^2 s^{-1}) of the species X and Y respectively and l (cm) is the geometrical coordinate. Assuming that $k_1/s^{-1} = k_2/M^{-2}s^{-1} = k_3/M^{-1}s^{-1} = k_4/s^{-1} = 1$, which was originally proposed by Prigogine et al. [16], one gets the D-R equations in the form as given in equations (2.1b)

$$\partial X/\partial t \ = a - (b+1)X + X^2 Y + D_X \left(\partial^2 X/\partial l^2 \right)$$

$$
\begin{aligned}
&= f(X,Y) + D_X\left(\partial^2 X/\partial l^2\right) \\
\partial Y/\partial t &= bX - X^2Y + D_Y\left(\partial^2 Y/\partial l^2\right) \\
&= g(X,Y) + D_Y\left(\partial^2 Y/\partial l^2\right)
\end{aligned}
\tag{2.1b}
$$

If the activator species X is assumed involved in a rapid comlexing equilibrium reaction (2.2a) [11, 34] similar to that reported in the Turing structure experiments and modelling of CIMA oscillatory reaction [4–6, 9–11], the chemical equilibrium reaction is given by

$$X + C \leftrightarrows XC \tag{2.2a}$$

where the activator X is captured partially to produce the complex XC after reacting with the complexing species C. The equilibrium constant K of this complex formation reaction is given by

$$K = \frac{[XC]}{X.C} = \frac{k_+}{k_-} \tag{2.2b}$$

where $[XC]$, X, and C are the equilibrium concentrations of the complex XC, the activator X and the complexing agent C respectively and k_+, and k_- are respectively the forward and reverse rate constants of the complexing equilibrium reaction (2.2a) such that $k_+ >> k_-$. If we use the complexing agent in large excess such that the initial concentration (C_0) of the complexing agent is almost equal to its concentration (C) at chemical equilibrium, (2.2a) becomes a rapid complex forming equilibrium reaction for which one can define a new constant K' such that

$$K' = KC_0 \tag{2.2c}$$

where K is the equilibrium constant for complex formation reaction (2.2a), and K' is a measure of degree of complex formation in presence of complexing agent C in large excess such that $C \approx C_0$, its initial concentration. It is essential that K' remains time-independent because here it's a measure of the degree of complex formation with the activator species X. To make K' time-independent, K and C_0 in Eq. (2.2c) must also be time-independent.

Because of complex formation reaction with the activator species X, we get a new D–R system described by

$$\partial X/\partial t = f(X,Y) - k_+c_0X + k_-[XC] + D_X\left(\partial^2 X/\partial l^2\right) \tag{2.3a}$$

$$\partial Y/\partial t = g(X,Y) + D_Y\left(\partial^2 Y/\partial l^2\right) \tag{2.3b}$$

$$\partial[XC]/\partial t = k_+c_0X - k_-[XC] \tag{2.3c}$$

(2.3a) plus (2.3c) $\Rightarrow$

$$\partial(X + [XC])/\partial t = (1 + K')\partial X/\partial t = f(X, Y) + D_X\left(\partial^2 X/\partial l^2\right) \tag{2.4a}$$

and,

$$\partial Y/\partial t = g(X, Y) + D_Y\left(\partial^2 Y/\partial l^2\right) \tag{2.4b}$$

Using the following scaling of time and space,

$$\tau = k_4 t/(1 + K') = t/(1 + K'),\ \text{and}\ l = (D_X/k_4)^{1/2}\rho = (D_X)^{1/2}\rho \tag{2.5}$$

one obtains Eq. (2.4) in dimensionless form as given below if the concentrations are assumed to have numerical values.

$$\partial X/\partial\tau = f(X, Y) + \partial^2 X/\partial\rho^2 \tag{2.6a}$$

$$\partial Y/\partial\tau = (1 + K')\left[g(X, Y) + (1/\eta^2)\partial^2 Y/\partial\rho^2\right] \tag{2.6b}$$

where

$$\eta = (D_X/D_Y)^{1/2} \tag{2.7}$$

The nature of complexing agent is left open as a subject of different possible reactions/interactions including weak H-bonding interactions with the activator species X, which may take place in a real physiological D–R system.

3 Linear stability analysis

The steady state concentration of the dimensionless D–R equations (2.6) is given by

$$X_S = a,\ Y_S = b/a \tag{3.0}$$

The Jacobian matrix elements are given below

$$\begin{aligned} f_{11} &= f'_X = -(b + 1) + 2X_S Y_S = (b - 1) \\ f_{12} &= f'_Y = X_S^2 = a^2 \\ f_{21} &= (1 + K')g'_X = (1 + K')(b - 2X_S Y_S) = -(1 + K')b \\ f_{22} &= (1 + K')g'_Y = -(1 + K')(X_S^2) = -(1 + K')a^2 \end{aligned} \tag{3.1}$$

Applying the linear operator,

$$f_{ij}(\vec{k}) = f_{ij} - D_i\vec{k}^2\delta_{ij} \tag{3.2}$$

one obtains the characteristic equation for the dimensionless partial differential equations (2.6) in the form

$$\det\begin{bmatrix} f_{11}-\vec{k}^2-\omega_k & f_{12} \\ f_{21} & f_{22}-(1+K').\frac{1}{\eta^2}\vec{k}^2-\omega_k \end{bmatrix}=0$$
$$or,\ (f_{11}-\vec{k}^2-\omega_k)\left[f_{22}-(1+K').\frac{1}{\eta^2}\vec{k}^2-\omega_{\mathrm{k}}\right]-f_{12}f_{21}=0$$
$$or,\ \omega_k{}^2+\left(\frac{1}{\eta^2}\vec{k}^2+\vec{k}^2+\frac{1}{\eta^2}\vec{k}^2K'-f_{11}-f_{22}\right)\omega_{\mathrm{k}}$$
$$+\det F-(f_{22}+f_{11}/\eta^2+f_{11}K'/\eta^2)\vec{k}^2$$
$$+(1/\eta^2)(1+K')\vec{k}^4=0$$
$$or,\ \omega(\vec{k})^2-Tr(\vec{k})\omega(\vec{k})+\Delta(\vec{k})=0 \tag{3.3}$$

where $Tr(\vec{k})$ and $\Delta(\vec{k})$ indicate the trace and determinant of the linear operator $f_{ij}(\vec{k})$) respectively given by

$$Tr(\vec{k})=f_{11}+f_{22}-\frac{1}{\eta^2}\vec{k}^2-\vec{k}^2-\frac{1}{\eta^2}\vec{k}^2K' \tag{3.4a}$$

and

$$\Delta(\vec{k})=\det F-\left(f_{22}+f_{11}/\eta^2+f_{11}K'/\eta^2\right)\vec{k}^2+(1/\eta^2)(1+K')\vec{k}^4 \tag{3.4b}$$

and $\omega(\vec{k})$ is the eigenvalue. From the characteristic Eq. (3.3), for nonzero wavevector $(\vec{k})$ mode, the condition for Hopf-wave bifurcation is given by $Tr(\vec{k})=0$, which gives

$$\vec{k}^2_{Hopf}\le\frac{f_{11}+f_{22}}{1+(1+K')/\eta^2} \tag{3.5}$$

where $\Delta(\vec{k})$ [from equation (3.4b)]>0. Turing bifurcation occurs [35, 36] when the real part of an eigenvalue of equation (3.3) becomes positive, which implies that the constant term $\Delta\,(\vec{k})$ of the characteristic Eq. (3.3)vanishes.

$$\Delta\,(\vec{k})=0 \tag{3.6a}$$

For critical wavenumber calculation the first derivative of $\Delta(\vec{k})$ with respect to $(\vec{k})$ must be zero as given in Eq. (3.6b).

$$d\Delta(\vec{k})/d\vec{k}=0 \tag{3.6b}$$

provided that

$$\left(f_{22}+f_{11}/\eta^2+f_{11}K'/\eta^2\right)>0 \tag{3.7a}$$

because of positivity of $\vec{k}^2$ from Eq. (3.6a) and

$$f_{11} + f_{22} < 0 \tag{3.7b}$$

to prevent Hopf-wave oscillations from Eq. (3.5). Substituting the values of f_{11} and f_{22} from Eqs. (3.1), Eqs. (3.7a) and (3.7b) take the forms as given in Eqs. (3.8a) and (3.8b) respectively.

$$-a^2\eta^2 + b - 1 > 0 \tag{3.8a}$$

$$b < 1 + (1 + K')a^2 \tag{3.8b}$$

The critical wavenumber (wavelength) is determined either from the degenerate root of equation (3.6a) or from equation (3.6b). The degenerate roots of equation (3.6a) give,

$$\left(f_{22} + f_{11}/\eta^2 + f_{11}K'/\eta\right)^2 \geq \frac{4}{\eta^2}(1 + K')\det F \tag{3.9}$$

This is the condition for Turing instability—the equality being the critical condition. Substituting the values of f_{11}, f_{22}, and $\det F$ from Eq. (3.1), Turing instability condition (3.9) takes the form as given in the following Eqs. (3.10a)/(3.10b) in terms of the dimensionless steady state concentrations of X_S and Y_S (see Eq. 3.0). From Eq. (3.9), we have

$$\left[-(1 + K')a^2 + \frac{1}{\eta^2}(1 + K').(b - 1)\right]^2 \geq \frac{4}{\eta^2}(1 + K')$$
$$\left[-(b - 1)\cdot(1 + K')a^2 + a^2(1 + K')b\right]$$
$$or, \left(-a^2 + \frac{b - 1}{\eta^2}\right)^2 \geq \frac{4}{\eta^2}a^2 \,[\because (1 + K') \neq 0]$$
$$or, -a^2\eta^2 + (b - 1) \geq \pm 2a\eta$$

Ignoring the negative value because of Eq. (3.8a), we have

$$-a^2\eta^2 + (b - 1) \geq 2a\eta \tag{3.10a}$$

Substituting the value of η from Eq. (2.7), one gets

$$(D_x a^2 + D_y - bD_y)^2 \geq 4a^2 D_x D_y \tag{3.10b}$$

3.1 Critical wave number: ($\vec{k}_c$)

From Eq. (3.6b) one obtains

$$-\left[f_{22} + f_{11}(1 + K')/\eta^2)2\vec{k} + (1/\eta^2)(1 + K')4\vec{k}^3 = 0\right.$$

$$\therefore \vec{k}^2 = \frac{f_{22} + (1 + K')f_{11}/\eta^2}{2(1 + K')/\eta^2}$$

Substituting the values of f_{11} and f_{22} from Eq. (3.1), we have

$$\vec{k}^2 = (-a^2\eta^2 + b - 1)/2$$

Substituting Eq. (3.10a), we get

$$\vec{k}_c \geq (a\eta)^{1/2} \tag{3.11a}$$

The critical wavenumber $\vec{k}_c$ can also be calculated from the degenerate roots of Eq. (3.6a). We have from Eqs. (3.6a) and (3.4b) after substitution of the values of f_{11} and f_{22} from Eq. (3.1),

$$\vec{k}_c^2 \geq \frac{f_{22} + (1/\eta^2)f_{11}(1 + K')}{(2/\eta^2)(1 + K')} = \frac{-(1 + K')a^2 + (1/\eta^2).(1 + K')(b - 1)}{(2/\eta^2)(1 + K')}$$

$$= \frac{-a^2 + (1/\eta^2)(b - 1)}{(2/\eta^2)} \geq a\eta \tag{3.11a}$$

$$or, 2\pi/\lambda_c \geq (a\eta)^{1/2}, \text{ which implies that } \lambda_c \leq 2\pi/(a\eta)^{1/2} \tag{3.11b}$$

where λ_c is the critical wavelength for Turing bifurcation. It is noteworthy that the value of $\vec{k}_c^2$ for the dimensionless diffusion co-ordinate ρ (see Eq. (2.5)) is different from that reported in Ref. [20] based on the geometric diffusion co-ordinate l. This difference is obviously due to the coordinate transformation from geometric diffusion coordinate l to dimensionless diffusion co-ordinate ρ used in our present manuscript. By applying the coordinate transformation Eq. (2.5), we can revert to $\vec{k}_c^{(l)}$ by multiplying $\vec{k}_c^{(\rho)}$ in Eq. (3.11a) with $D_x^{-1/2}$. This implies

$$\vec{k}_c^{(l)^2} = \left\{(a\eta)^{1/2}D_x^{-1/2}\right\}^2 = a/(D_x D_y)^{1/2}, \tag{3.11c}$$

which was really obtained in Ref. [20].

3.2 Critical b_c at the Turing bifurcation point

One can substitute the value of $\vec{k}_c^2$ from Eqs. (3.11a) in (3.6a) to get

$$\left[-a^2(b-1)+a^2b\right]-\left[-a^2+\frac{1}{\eta^2}(b-1)\right]a\eta+a^2 = 0$$
$$or, b = b_c^T = (1+a\eta)^2 \tag{3.12}$$

It is to be noted that b_c is independent of the choice of diffusion co-ordinate-the value is identical to that reported in Ref. [20], which is based on the geometric diffusion coordinate l as given in Eqs. (2.5) and (2.1). Substituting the value of b_c from Eqs. (3.12) in (3.8b), one obtains

$$(1+a\eta) < \left\{1+(1+K')a^2\right\}^{1/2} \tag{3.13}$$

Therefore, in presence of complexing reaction with the activator species X, the reference steady state $(X_s, Y_s) \equiv (a, b/a)$ may undergo a Turing bifurcation at $b_c = (1+a\eta)^2$ where $\eta = (D_x/D_y)^{1/2} < [\{1+(1+K')a^2\}^{1/2} - 1]/a$; the corresponding critical wavenumber being $\vec{k}_c \geq (a\eta)^{1/2}$.

4 Separation into linear and nonlinear parts

One obtains D–R equations (2.6) divided into linear and non-linear parts as given below.

$$\begin{aligned}\partial_\tau(x,y)^T &= [L_c+(b_c-\, b)\cdot(M+K'N)]\cdot(x,y)^T+\left(\frac{b_c}{a}\cdot x^2+2axy+x^2y\right)\\ &\cdot[+1,-(1+K')]^T-\left[(b_c-b)\frac{x^2}{a}\right]\cdot[+1,-(1+K')]^T\end{aligned} \tag{4.0}$$

where

$$L_c = \begin{bmatrix}(b_c-1)+\nabla_\rho^2 & a^2\\ -(1+K')b_c & (1+K')\cdot(-a^2+\frac{1}{\eta^2}\nabla_\rho^2)\end{bmatrix} \tag{4.1}$$
$$M = \begin{bmatrix}-1 & 0\\ +1 & 0\end{bmatrix} \tag{4.2}$$
$$N = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \tag{4.3}$$

and the symbol $\partial_\tau \Rightarrow \partial/\partial\tau$. The derivation of Eq. (4.0) is in Appendix A.

5 Nonlinear analysis

We consider that Turing structures constitute the linear superposition of three pairs of modes $(\vec{k}_i, -\vec{k}_i)_{i=1,2,3}$ which make angles of $2\pi/3$ between each pair, such that $|\vec{k}_i| = \vec{k}_c$ and $\sum_{i=1}^{3} \vec{k}_i = 0$. Let the inhomogeneous perturbations, $(x, y)^T$ unfold in terms of three amplitudes $(A_i's)$ according to the relations

$$(x, y)^T = \sum_{i=1}^{3} \left(A_i^x, A_i^y\right)^T \exp\,(i\vec{k}_i \cdot \rho) + c.c. \tag{5.0}$$

where $c.c$ indicates the complex conjugate terms. The nonlinear analysis is based on the following expansion in terms of the amplitude of the branching solutions for b (say) in the neighborhood of b_c (the critical value for Turing bifurcation) such that

$$(x, y)^T = \varepsilon(x_1, y_1)^T + \varepsilon^2(x_2, y_2)^T + \cdots \tag{5.1}$$

where,

$$b_c - b = \varepsilon b^{(1)} + \varepsilon^2 b^{(2)} + \cdots \tag{5.2}$$

$$\partial_\tau = \partial_{\tau_0} + \varepsilon\partial_{\tau_1} + \varepsilon^2\partial_{\tau_2} + \cdots \tag{5.3}$$

$$\partial_\tau A = (\partial A/\partial\tau_0) + \varepsilon(\partial A/\partial\tau_1) + \varepsilon^2(\partial A/\partial\tau_2) + \cdots, \tag{5.4}$$

where ε is the coefficient of the first order perturbation in the expansion of the inhomogeneous term (x, y) around the stable steady state $X_S = a$, $Y_S = b/a$ and A is the amplitude; $(\partial A/\partial\tau_0)$ term in Eq. (5.4) is ignored because the slow variable at zero time is time-invariant. Nonvanishing values of $b^{(1)}$ in Eq. (5.2) are possible only when triangular conditions between the wavevectors are satisfied [20]. Substituting Eqs. (5.1) to (5.3) into Eq. (4.0), one obtains the first, second and third order balance equations as given below in Eqs. (5.5), (5.6), and (5.7) respectively.

$$\varepsilon : L_c(x_1, y_1)^T = 0_{2\times1} \tag{5.5}$$

$$\varepsilon^2 : L_c(x_2, y_2)^T = \left[b^{(1)}x_1 - \frac{b_c}{a}x_1^2 - 2ax_1y_1\right].(+1, -(1+K'))^T + \partial_{\tau1}(x_1, y_1)^T$$
$$= \left(F_x, F_y\right)^T \quad \text{(say)} \tag{5.6}$$

$$\varepsilon^3 : L_c(x_3, y_3)^T = \left[b^{(2)}x_1 + (b^{(1)} - 2\frac{b_c}{a}x_1 - 2ay_1)x_2 - 2ax_1y_2 + b^{(1)}\frac{x_1^2}{a} - x_1^2y_1\right].$$
$$\left(+1, -(1+K')\right)^T + \partial_{\tau1}(x_2, y_2)^T + \partial_{\tau2}(x_1, y_1)^T = \left(G_x, G_y\right)^T \quad \text{(say)} \tag{5.7}$$

The derivation of Eqs. (5.5)–(5.7) is given in Appendix B.

5.1 Order 1

From Eq. (5.5), $(x_1, y_1)^T$ is proportional to the right eigenvectors of L_c with zero eigenvalue at the critical mode ($b = b_c$, $|\vec{k}_i| = k_c$) is given below. For degenerate eigenvectors, the solution $(x_1, y_1)^T$ is

$$(x_1, y_1)^T = \left[\sum\nolimits_i W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right] \cdot \left(-\frac{a}{\eta(1+a\eta)}, 1\right]^T \tag{5.8}$$

The derivation of Eq. (5.8) is given in Appendix C.

5.2 Order 2

The first solvability condition requires that the inhomogeneous term $(F_x^i, F_y^i)^T$ (see $Eq.5.6$) be orthogonal to the left (row) eigenvectors of L_c^T with zero left eigenvalue. In order to get it, we calculate the eigenvectors of L_c^T with zero left eigenvalue as given in Eq. (5.9).

$$L_c^T (x_2, y_2)^T = 0_{2\times 1} \tag{5.9}$$

These eigenvecors of L_c^T with zero left eigenvalue (see $Appendix D$ for its calculation) are given in Eq. (5.10).

$$(x_2, y_2)^T = \left[\frac{(1+K')\cdot(1+a\eta)}{a\eta}, 1\right]^T \cdot \left[\sum\nolimits_i W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right] \tag{5.10}$$

The eigenvectors of L_c^T, on transposing, gives the left (row) eigenvectors of L_c^T with zero left eigenvalue are given in Eq. (5.11).

$$(x_2, y_2) = \left[\frac{(1+K')\cdot(1+a\eta)}{a\eta}, 1\right] \cdot \left[\sum\nolimits_i W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right] \tag{5.11}$$

We now calculate $(F_x^{(1)}, F_y^{(1)})^T$, the coefficient of $\exp(i\overline{k}_1 \cdot \rho)$ by incorporating Eqs. (5.8) into (5.6) as given below in Eq. (5.12) (see Appendix E for this calculation)

$$\begin{aligned}\left(F_x^{(1)}, F_y^{(1)}\right)^T &= \frac{a}{\eta(1+a\eta)}\left\{-b^{(1)}W_1 - \frac{2(1-a\eta)}{\eta}\overline{W}_2\overline{W}_3\right\}.(+1, -(1+K'))^T \\ &\quad + \frac{\partial W_1}{\partial \tau_1} \cdot \left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T\end{aligned} \tag{5.12}$$

The inhomogeneous term, relation (5.12), being orthogonal to relation (5.11), gives the first solvability condition as given in Eq. (5.13) (see Appendix F for its calculation)

$$\{\eta^2 - (1+K')\} \cdot \frac{\partial W_1}{\partial \tau_1} - \frac{(1+K')b^{(1)}W_1}{1+a\eta} - \frac{2(1+K')\cdot(1-a\eta)\overline{W}_2\overline{W}_3}{\eta(1+a\eta)} = 0 \tag{5.13}$$

We express $(x_2, y_2)^T$ in $Eq.$(5.6) as the sum of a series of Fourier terms as given below.

$$\begin{aligned}(x_2, y_2)^T &= (X_o, Y_o)^T + (X_1, Y_1)^T \exp(i\vec{k}_1\cdot\rho) + (X_2, Y_2)^T \exp(i\vec{k}_2\cdot\rho) \\ &\quad +(X_3, Y_3)^T \exp(i\vec{k}_3\cdot\rho) + (X_{11}, Y_{11})^T \exp(2i\vec{k}_1\cdot\rho) + (X_{22}, Y_{22})^T \\ &\quad \exp(2i\vec{k}_2\cdot\rho) + (X_{33}, Y_{33})^T \\ &\quad \exp(2i\vec{k}_3\cdot\rho) + (X_{12}, Y_{12})^T \exp(i(\vec{k}_1-\vec{k}_2)\cdot\rho) + (X_{23}, Y_{23})^T \\ &\quad \exp(i(\vec{k}_2-\vec{k}_3)\cdot\rho) + (X_{31}, Y_{31})^T \\ &\quad \exp(i(\vec{k}_3-\vec{k}_1)\cdot\rho) + c.c.\end{aligned} \tag{5.14}$$

The Fourier coefficient values have been derived in Appendix G.

5.3 Order 3

Substituting the values of the Fourier coefficients from Appendix G into the inhomogeneous term $\left(G_x, G_y\right)^T$, the coefficient of $\exp(i\vec{k}_1\cdot\rho)$ from Eq. (5.7) may be written as (for calculations see $Appendix\ H$).

$$\begin{aligned}\left(G_x^{(1)}, G_y^{(1)}\right)^T &= \left(+1, -(1+K')\right)^T \cdot \frac{1}{\eta(1+a\eta)} \cdot \Big[-ab^{(2)}W_1 + \{\eta(1+a\eta)X_1 \\ &\quad + \frac{2a\overline{W}_2\overline{W}_3}{\eta(1+a\eta)}\}b^{(1)} - \frac{2a(1-a\eta)}{\eta}\cdot(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) \\ &\quad -(|W_1|^2W_1).\frac{g_1}{9\eta^3(1+a\eta)} - (|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3(1+a\eta)}\Big] \\ &\quad + \left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right)\cdot\left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T\end{aligned} \tag{5.15}$$

where,

$$\begin{aligned}g_1 &= 38a\eta + 5(a\eta)^2 - 8 - 8(a\eta)^3 \\ h_1 &= 5a\eta + 7(a\eta)^2 - 3 - 3(a\eta)^3\end{aligned} \tag{5.16}$$

As the second solvability condition should satisfy that the left (row) eigenvectors (see Eq. 5.11) be orthogonal to the column matrix $\left(G_x^{(1)}, G_y^{(1)}\right)^T$ (see Eq. 5.15), we have

$$\begin{aligned}&(x_2, y_2)\cdot\left(G_x^{(1)}, G_y^{(1)}\right)^T = 0 \\ &Or,\end{aligned}$$

$$\left(\frac{(1+K')(1+a\eta)}{a\eta}G_x^{(1)}+G_y^{(1)}\right)^T=0 \tag{5.17}$$

Substituting Eq. (5.15) in Eq. (5.17), one obtains (see calculations in Appendix I),

$$\begin{aligned}&\{(1+K')-\eta^2\}\cdot(1+a\eta).\left(\frac{\partial W_1}{\partial\tau_2}+\frac{\partial Y_1}{\partial\tau_1}\right)\\&=(1+K')\cdot\Bigg\{-b^{(2)}W_1-Y_1b^{(1)}\\&\quad+\frac{2\overline{W}_2\overline{W}_3b^{(1)}}{\eta(1+a\eta)}-\frac{2(1-a\eta)}{\eta}\\&\quad\cdot\left(\overline{Y}_2\overline{W}_3+\overline{Y}_3\overline{W}_2\right)-|W_1|^2W_1\cdot\frac{g_1}{9\eta^3a(1+a\eta)}\\&\quad-(|W_2|^2+|W_3|^2)W_1\cdot\frac{h_1}{\eta^3a(1+a\eta)}\Bigg\}\end{aligned} \tag{5.18}$$

Equation (5.18) is the second solvability condition, also to be used in amplitude equation derivation in the next section.

6 Amplitude equation

Substituting the values of $(x_1, y_1)^T$ and $(x_2, y_2)^T$ from Eqs. (5.8) and (5.14) respectively into Eq. (5.1), we equate the coefficients of $\exp(i\overline{k}_i.\rho)$ from Eqs. (5.0) and (5.1).

$$\therefore\ (A_i^x, A_i^y)^T\Rightarrow\ \varepsilon\left[-\frac{a}{\eta(1+a\eta)},1\right]^T W_i+\varepsilon^2[X_i,Y_i]^T \tag{6.0}$$

Using Eq. (G10) from Appendix G, one obtains

$$\begin{aligned}&(A_i, A_i^y)^T\Rightarrow(A_i^x, A_i^y)^T\\&\quad=\varepsilon\left[-\frac{a}{\eta(1+a\eta)},1\right]^T W_i+\varepsilon^2\left[-\frac{a}{\eta(1+a\eta)},1\right]^T Y_i\end{aligned} \tag{6.1}$$

The partial differential of the amplitude $A_1$with respect to time τ from Eq. (5.4) is given below.

$$\frac{\partial A_1}{\partial\tau}=\varepsilon\frac{\partial A_1}{\partial\tau_1}+\varepsilon^2\frac{\partial A_1}{\partial\tau_2}+\cdots \tag{6.2}$$

The term $\partial A_1/\partial\tau_0$ from Eq. (5.4) has been neglected here because the slow variable at time zero is time-invariant. Substituting the value ofA_1 from Eq. (6.1) into Eq. (6.2) and neglecting terms beyond the order of ε^3, one obtains

$$\begin{aligned}\frac{\partial A_1}{\partial \tau} &= \varepsilon \frac{\partial}{\partial \tau_1}\left[\varepsilon\left(-\frac{a}{\eta(1+a\eta)}\right)W_1 \right.\\ &\left. +\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right] + \varepsilon^2\frac{\partial}{\partial \tau_2}\left[\varepsilon\left(-\frac{a}{\eta(1+a\eta)}\right)W_1\right.\\ &\left. +\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right]\\ &= \varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\left[\frac{\partial Y_1}{\partial \tau_1}+\frac{\partial W_1}{\partial \tau_2}\right] + \varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{\partial W_1}{\partial \tau_1}\end{aligned} \tag{6.3}$$

We incorporate the solvability conditions Eqs. (5.18) and (5.13) into Eq. (6.3) to get the amplitude equation as given below in Eq. (6.4) with the help of Eqs. (5.2) and (6.1)—neglecting terms involving ε of order ε^3 and higher (Derivation of Eq. (6.4) including the coefficients τ_0, μ, $\overline{\nu}$, g and h are given in Appendix J).

$$\tau_0\left(\frac{\partial A_1}{\partial \tau}\right) = \mu A_1 + \overline{\nu}\,\overline{A}_2\overline{A}_3 - g|A_1|^2A_1 - h(|A_2|^2+|A_3|^2)A_1 \tag{6.4}$$

where,

$$\begin{aligned}\tau_0 &= \frac{(1+K')-\eta^2}{(1+K')\cdot(1+a\eta)}\,;\ \mu = \frac{b-b_c}{b_c}\,;\ \overline{\nu} = \frac{2(1-a\eta)}{a(1+a\eta)}\,;\ g = \frac{g_1}{9a^3\eta(1+a\eta)}\\ &= \frac{38a\eta+5(a\eta)^2-8-8(a\eta)^3}{9a^3\eta(1+a\eta)}\,;\ h\\ &= \frac{h_1}{a^3\eta(1+a\eta)} = \frac{5a\eta+7(a\eta)^2-3-3(a\eta)^3}{a^3\eta(1+a\eta)}\end{aligned} \tag{6.5}$$

Expressions for the amplitudes A_2 and A_3 may be obtained by circular permutation of the indices from Eq. (6.4). The bifurcation scenario may be schematically presented [22, 37–41] in Fig. 1 in the form of amplitude as a function of the bifurcation parameter μ (the normalized distance from onset) for stripes (S) and hexagons (H) states respectively. For Turing pattern amplitude Eq. (6.4), the supercritical stripe branch is unstable near the critical point, but becomes stable for $\mu > \mu_S > 0$. The stable hexagonal branch is H_0 (or H_π) if $\overline{\nu} > 0$ (or $\overline{\nu} < 0$). A subcritical hexagonal branch begins at $\mu_{H1} = -\overline{\nu}^2/4(g+2h) < 0$, which remains stable until $\mu_{H2} > 0$. Both branches (stripes and hexagons) are stable in the bistable range given by

$$\mu_S < \mu < \mu_{H2} : \mu_S = \overline{\nu}^2 g/(h-g)^2,\ \mu_{H2} = (2g+h)\overline{\nu}^2/(h-g)^2 \tag{6.6}$$

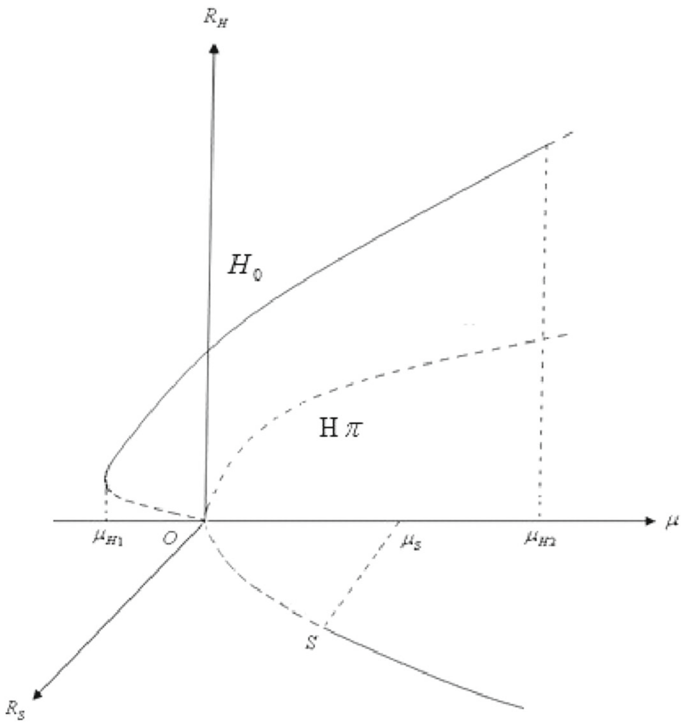

Fig. 1 Schematic bifurcation diagram of the amplitude of stripe state (R_S) and hexagon state (R_H) as a function of the bifurcation parameter μ for the amplitude Eq. (6.4) for $\overline{\nu} > 0$; H_O: hexagonal patterns with $\phi = 0$; H_π: hexagonal patterns with $\phi = \pi$; S: stripe patterns; solid lines denote stable states; dashed lines, unstable states. The limits of stability ($\mu_i^\prime S$) are given in Eq. (6.6). For $\overline{\nu} > 0$, the indices 0 and π must be interchanged

7 Results and discussion

The results from algebraic calculations are similar to those for the reversible Sel'kov model [30, 31]. Hopf-bifurcations of the homogeneous (k = 0) and inhomogeneous ($k \neq 0$) model systems strongly depend on the complex formation reaction [11, 34] with the activator species X (see Eq. 3.5). Due to the complexing reaction with the activator species X, there is decrease (increase) of Hopf-wavenumber k (wavelength λ) values—when translated to wavelengths using the relation $k = 2\pi/\lambda$, one can get the values in dimensionless unit of ρ, from which the values in the unit of real length (cm) may be obtained with the help of Eq. (2.5) for an appropriate value of $D_x(\approx 10^{-5} cm^2 s^{-1})$ in aqueous solution.

Equation (3.5) shows that Hopf wavenumber in (a, b) parameter plane decreases with the increase of K' (complex formation reaction with the activator species X for a particular value of the ratio η), whereas Eq. (3.8b) shows that, by increasing K' to an appropriate value, it is possible in principle to increase the range of b, for which the state of excitable non-oscillations may be imposed for a particular value of a. But, complex formation reaction with the activator species X neither changes the wavenumber (wavelength) of Turing patterns (see Eq. 3.11) nor does it affect the basic relation to be maintained for Turing structure formation (see Eqs. 3.10a and 3.10b).Turing wave-number $\vec{k}_c$ as a function of a is given in Eq. (3.11), which verifies that $\vec{k}_c$ indeed doesn't depend on K'—an appropriate value ofK' can only help in changing the range of b, for which the state of excitable non-oscillations may be imposed for a particular value of a (see Eq. 3.8b). This result is in agreement with the observation of De Kepper et al. [9, 10] on the chlorite–iodide–malonic acid (CIMA) reaction with varying concentrations of starch. Also, by increasing K' in Eq. (3.5) to an appropriate value it is possible, in principle, to arrest the arrival of Hopf-bifurcation in this model system—the Hopf-domain which disappears by this technique, may be utilized in generating Turing patterns by inducing inhomogeneous perturbations of nonzero k-mode.

Different numerical values of the ratio $\eta << 1$ (see Eq. 2.7) can be realized in experiments by adopting a regulatory mechanism based on immobilization of the activator X in a real D–R system. This can also be realized in BZ-reaction experiments in a water-in-oil aerosol OT (AOT) micro-emulsion [42–44], because small molecular species such as Br_2, Br^-, or BrO_2. may display very different diffusion coefficient (D) values in oil and dispersed phases—D of Br_2, Br^- species is of the order of the diffusion coefficient of oil molecules (~10^{-5} cm^2 s^{-1}). For the $BrO_2\cdot$ molecules responsible for the formation of activator species $HBrO_2$ (X), D is decided by the diffusion of droplets in oil phase [45] and is of the order of 10^{-7}–10^{-8} cm^2 s^{-1}. D–R systems with such unequal diffusion coefficients may have values of the ratio η very much less than 1 (see Eq. 2.7) and possess unusually complex properties [46–48] including Turing structure formation. Incorporation of complexing reaction with the activator species X in the Brusselator model [16] enhances these interesting features further, as given in the last paragraph in presence of an AOT micro-emulsion, which generates very much unequal diffusion coefficients (see Eq. 2.7) leading to Turing structure formation.

However, the mathematical treatment undertaken in this manuscript remains valid also for equal diffusion coefficients for this two component system for which $\eta = (D_X/D_Y)^{1/2} \sim 1$. Vastano et al. [49] reported that discrete Turing type patterns remain hidden in a D–R system having equal diffusion coefficients. These stable patterns can't organize spontaneously, but require finite perturbation of the homogeneous steady state. This is because primary Turing bifurcation is unstable, which becomes stabilized by a secondary supercritical Hopf-bifurcation.

8 Conclusion

The manuscript reports an amplitude equation Eq. (6.4) derivation for a D–R Brusselator model [16] in the framework of a weakly nonlinear theory in presence of complexing reactions [11, 34] with the activator species X. The amplitude equation derivation is interesting enough from mathematical point of view because the kinetic steady state is simple looking in algebraic form [12, 21–24].

The kinetic stability analysis of the amplitude Eq. (6.4) interprets the structural transitions and stability [23, 37–41] of stripes and hexagonal patterns (see Fig. 1).

Complexing reaction with the activator species X strongly influences the wavenumber (wavelength) of Hopf-wave bifurcations (see Eq. 3.5), whereas Turing pattern wavenumber (wavelength) is independent of complexing reaction with the activator species X (see Eq. 3.11). We may obtain the same information immediately just by looking at the amplitude equation itself-time dependent amplitudes in Eq. (6.4) such as in Hopf-wave bifurcations, strongly depend on K' (degree of complexing reaction with the activator species X), whereas time-independent amplitudes, such as in Turing structures, are independent of complexing reaction with the activator species X; K' included in the expression for τ_0 (Eqs. 6.5) only.

Complexing reactions arrest the arrival of Hopf-bifurcations and the additional domain of excitable non-oscillations such created may be utilized for Turing structure generation by inducing inhomogeneous perturbations of nonzero k-mode (see Eq. 3.11), for which very low values of the positive ratio $\eta << 1$ is useful [35, 36]. Complexing reaction with the activator species seems to have a major role in biological oscillatory networks including glycolytic oscillations [30, 31], because any kind of interaction with the activator species X is bound to alter the domain of Hopf/Turing bifurcations (see Eq. 6.4) enforcing additional changes in the existing order in self-organizing physiological systems.

9 Note added in proof

Reference note 50

$$\begin{aligned}&\left\{\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)+c.c.\right\}^2\\&=\left(\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)\right)^2+\left(\textstyle\sum_{i=1}^{i=3} \overline{W}_i \exp(-i\vec{k}_i\cdot\rho)\right)^2\\&+2\left(\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)\right)\cdot\left(\textstyle\sum_{i=1}^{i=3} \overline{W}_i \exp(-i\vec{k}_i\cdot\rho)\right)=0+2\overline{W}_2\overline{W}_3;\end{aligned}$$

$$\begin{aligned}&\left\{\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)+c.c.\right\}^3\\&=\left(\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)\right)^3+\left(\textstyle\sum_{i=1}^{i=3} \overline{W}_i \exp(-i\vec{k}_i\cdot\rho)\right)^3\\&+3\left(\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)\right)^2\cdot\left(\textstyle\sum_{i=1}^{i=3} \overline{W}_i \exp(-i\vec{k}_i\cdot\rho)\right)+3\left(\textstyle\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)\right)\\&\cdot\left(\textstyle\sum_{i=1}^{i=3} W_i \exp(-i\vec{k}_i\cdot\rho)\right)^2=0+0+3(W_1^2\overline{W}_1+2W_1W_2\overline{W}_2+2W_3\overline{W}_3W_1)+3.0\end{aligned}$$

Appendix A

Let the inhomogeneous perturbation (x, y) around the stable steady state $(X_S = a, Y_S = b/a)$ be represented by the following relations.

$$x = X - a\,;\; y = Y - b/a \tag{A1}$$

As (X_S, Y_S) is the dimensionless steady state solution of the D–R equations (2.6), we have

$$a-(b+1)X_S+X_S^2Y_S=0\,;\; bX_S-X_S^2Y_S=0 \tag{A2}$$

Incorporating the values of (X, Y) from Eq. (A1) into the D–R equations (2.6), one obtains

$$\partial_\tau (x + a) = a - (b + 1).(x + a) + (x + a)^2 \cdot (y + b/a) + \nabla_\rho^2 (x + a) \quad \text{(A3)}$$

$$\partial_\tau (y + b/a) = (1 + K') \cdot [b(x + a) - (x + a)^2 (y + b/a) + \frac{1}{\eta^2} \nabla_\rho^2 (y + b/a)] \quad \text{(A4)}$$

Therefore,

$$\partial_\tau x = a - (b + 1)x - (b + 1)a + (x^2 + 2xa + a^2) \cdot \left(y + \frac{b}{a} \right) + \nabla_\rho^2 x \quad \text{(A5)}$$

$$\partial_\tau y = (1 + K') \cdot \left[bx + ab - \left(x^2 y + x^2 \frac{b}{a} + 2xay + 2xb + a^2 y + ab \right) + \frac{1}{\eta^2} \nabla_\rho^2 y \right] \quad \text{(A6)}$$

Incorporating the steady state conditions, Eqs. (A2) into (A5) and (A6) one obtains,

$$\begin{aligned} \partial_\tau x &= [a - (b + 1)a + ab] - (b + 1)x + a^2 y + \left(x^2 y + x^2 \frac{b}{a} + 2axy \right) + 2xb + \nabla_\rho^2 x \\ &= [(b - 1) + \nabla_\rho^2]x + a^2 y + \left[x \left(\frac{b}{a} x + 2ay \right) + x^2 y \right] \quad \text{(A7)} \\ \partial_\tau y &= (1 + K') \cdot \left[(ab - ab) + (bx - 2bx) + \left(-a^2 y + \frac{1}{\eta^2} \nabla_\rho^2 y \right) \right. \\ &\quad \left. - \left(x^2 y + x^2 \frac{b}{a} + 2axy \right) \right] \quad \text{(A8)} \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_\tau (x, y)^T &= \begin{bmatrix} (b - 1) + \nabla_\rho^2 & a^2 \\ -b(1 + K') & (-a^2 + \frac{1}{\eta^2} \nabla_\rho^2) \cdot (1 + K') \end{bmatrix} \boldsymbol{.} (x, y)^T \\ &\quad + \left[x \left(\frac{b}{a} x + 2ay \right) + x^2 y \right] \cdot [+1, -(1 + K')]^T \quad \text{(A9)} \\ &= \left\{ \begin{bmatrix} (b_c - 1) + \nabla_\rho^2 & a^2 \\ -b_c(1 + K') & (-a^2 + \frac{1}{\eta^2} \nabla_\rho^2) \cdot (1 + K') \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} -b_c + b & 0 \\ (1 + K') \cdot (b_c - b) & 0 \end{bmatrix} \right\} (x, y)^{\mathbf{T}} \\ &\quad + \left[\left(\frac{b_c}{a} x^2 + 2axy + x^2 y \right) - (b_c - b) \frac{x^2}{a} \right] \cdot [+1, -(1 + K')]^T \quad \text{(A10)} \\ &= \left[L_c + (b_c - b) \begin{bmatrix} -1 & 0 \\ (1 + K') & 0 \end{bmatrix} \right] \boldsymbol{.} (x, y)^T \end{aligned}$$

$$+\left(\frac{b_c}{a}\cdot x^2+2axy\ +x^2y\right)\cdot[+1,-(1+K^{'})]^T$$
$$-\left[(b_c-b)\frac{x^2}{a}\right]\cdot\ [+1,-(1+K^{'})]^T \tag{A11}$$

$$\begin{aligned}&=[L_c+(b_c-\ b)\cdot(M+K'N)]\cdot(x,y)^T\\&+\left(\frac{b_c}{a}x^2+2axy+x^2y\right)\cdot[+1,-(1+K')]^T\\&-\left[(b_c-b)\frac{x^2}{a}\right]\cdot[+1,-(1+K')]^T\end{aligned} \tag{4.0}$$

where

$$L_c=\begin{bmatrix}(b_c-1)+\nabla_\rho^2 & a^2\\ -(1+K')b_c & (1+K').(-a^2+\frac{1}{\eta^2}\nabla_\rho^2)\end{bmatrix} \tag{4.1}$$

$$M=\begin{bmatrix}-1 & 0\\ +1 & 0\end{bmatrix} \tag{4.2}$$

$$N=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \tag{4.3}$$

and the symbol $\partial_\tau\ \Rightarrow\ \partial/\partial\tau$

Appendix B

Substituting Eqs. (5.1)–(5.3) into equation Eq. (4.0), one obtains

$$\begin{aligned}&(\partial_{\tau_0}\ +\varepsilon\partial_{\tau_1}+\varepsilon^2\partial_{\tau 2}\ +\cdots)\cdot[\varepsilon(x_1,y_1)^T+\varepsilon^2(x_2,y_2)^T+\varepsilon^3(x_3,y_3)^T+\ldots]\\&=[L_c+(M+K'N)\cdot\ (\varepsilon b^{(1)}+\varepsilon^2 b^{(2)}+\cdots)].[\varepsilon(x_1,y_1)^T+\varepsilon^2(x_2,y_2)^T+\varepsilon^3(x_3,y_3)^T\\&\quad+\cdots]-\left[\left(\varepsilon b^{(1)}\ +\varepsilon^2 b^{(2)}+\cdots\right)\frac{x^2}{a}\right].(+1,-(1+K^{'}))^T\\&\quad+\left(b_c\frac{x^2}{a}+2axy+x^2y\right)\cdot(+1,-(1+K^{'}))^T\end{aligned} \tag{B1}$$

Equating the coefficients of order ε, ε^2, and ε^3 from both sides, one obtains

$$\varepsilon\ :\ \varepsilon L_c(x_1,y_1)^T\ =0_{2\times 1} \tag{B2/5.5}$$

$$\varepsilon^2 : \varepsilon^2 L_c(x_2, y_2)^T = \varepsilon^2 \partial_{\tau_0}(x_2, y_2)^T + \varepsilon^2 \partial_{\tau 1}(x_1, y_1)^T - \varepsilon^2 (M + K'N) b^{(1)} (x_1, y_1)^T - \varepsilon^2 \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right) \cdot (+1, -(1 + K'))^T \tag{B3}$$

Therefore,

$$\begin{aligned} L_c(x_2, y_2)^T &= -(M + K'N) b^{(1)} \cdot (x_1, y_1)^T - \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right) \\ &\quad \cdot (+1, -(1 + K'))^T + \partial_{\tau 1}(x_1, y_1)^T \\ &= -b^{(1)} \left(\begin{bmatrix} -1 & 0 \\ +1 & 0 \end{bmatrix} + K' \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) .(x_1, y_1)^T \\ &\quad - \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right) .(+1, -(1 + K^{'}))^T + \partial_{\tau 1}(x_1, y_1)^T \\ &= -b^{(1)} \left(\begin{bmatrix} -1 & 0 \\ (1 + K)' & 0 \end{bmatrix}\right) \cdot (x_1, y_1)^T \\ &\quad - \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right) .(+1, -(1 + K'))^T + \partial_{\tau 1}(x_1, y_1)^T \\ &= -b^{(1)} x_1 (-1, (1 + K'))^T - \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right) \\ &\quad \cdot (+1, -(1 + K^{'}))^T + \partial_{\tau 1}(x_1, y_1)^T \\ &= b^{(1)} x_1 (1, -(1 + K'))^T - \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right) \\ &\quad \cdot (+1, -(1 + K^{'}))^T + \partial_{\tau 1}(x_1, y_1)^T \\ &= \left[b^{(1)} x_1 - \left(\frac{b_c}{a} x_1^2 + 2a x_1 y_1\right)\right] \cdot (+1, -(1 + K^{'}))^T + \partial_{\tau 1}(x_1, y_1)^T \end{aligned}$$

Therefore,

$$\begin{aligned} L_c(x_2, y_2)^T &= \left(b^{(1)} x_1 - \frac{b_c}{a} x_1^2 - 2a x_1 y_1\right) \cdot (+1, -(1 + K'))^T + \partial_{\tau 1}(x_1, y_1)^T \\ &= \left(F_x, F_y\right)^T \text{ (say)} \end{aligned} \tag{B4/5.6}$$

$$\begin{aligned} \varepsilon^3 : \varepsilon^3 L_c(x_3, y_3)^T &= \varepsilon^3 \left[\partial_{\tau 0}(x_3, y_3)^T + \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}(x_1, y_1)^T\right] \\ &\quad - \varepsilon^3 \left[b^{(1)}(M + K'N) \cdot (x_2, y_2)^T + b^{(2)}(M + K'N) \cdot (x_1, y_1)^T\right] \\ &\quad - \varepsilon^3 \left[2\frac{b_c}{a} x_1 x_2 + 2a(x_1 y_2 + x_2 y_1) + x_1^2 y_1 - b(1)\frac{x_1^2}{a}\right] \\ &\quad \cdot (+1, -(1 + K'))^T \end{aligned} \tag{B5}$$

$$= \varepsilon^3\left[\partial_{\tau 0}(x_3, y_3)^T + \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}\,(x_1, y_1)^T\right] - \varepsilon^3(M + K'N).[b^{(1)}(x_2, y_2)^T + b^{(2)}\,(x_1, y_1)^T]$$
$$-\varepsilon^3\left[2\frac{b_c}{a}x_1x_2 + 2a(x_1y_2 + x_2y_1) + x_1^2y_1 - b^{(1)}\frac{x_1^2}{a}\right]\cdot(+1, -(1 + K'))^T \tag{B6}$$

$$\therefore L_c(x_3, y_3)^T = \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}\,(x_1, y_1)^T - \left(\begin{bmatrix}-1 & 0\\ +1 & 0\end{bmatrix} + K'\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)\cdot\left[b^{(1)}(x_2, y_2)^T + b^{(2)}\,(x_1, y_1)^T\right] - \left[2\frac{b_c}{a}x_1x_2 + 2a(x_1y_2 + x_2y_1) + x_1^2y_1 - b^{(1)}\frac{x_1^2}{a}\right]\cdot(+1, -(1 + K'))^T \tag{B7}$$

$$= \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}\,(x_1, y_1)^T - \left(\begin{bmatrix}-1 & 0\\ (1 + K') & 0\end{bmatrix}\cdot\left[b^{(1)}(x_2, y_2)^T + b^{(2)}\,(x_1, y_1)^T\right]\right. \tag{B8/5.7}$$

$$- [2\frac{b_c}{a}x_1x_2 + 2a(x_1y_2 + x_2y_1) + x_1^2y_1 - b^{(1)}\frac{x_1^2}{a}]\cdot(+1, -(1 + K'))^T$$
$$= \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}\,(x_1, y_1)^T - b^{(1)}(-x_2,\ (1 + K')x_2)^T - b^{(2)}(-x_1, (1 + K')x_1)^T - \left[2\frac{b_c}{a}x_1x_2 + 2a(x_1y_2 + x_2y_1) + x_1^2y_1 - b^{(1)}\frac{x_1^2}{a}\right].(+1, -(1 + K'))^T$$
$$= \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}\,(x_1\ , y_1)^T + b^{(1)}x_2(1, -(1 + K'))^T + b^{(2)}x_1\left(1, -(1 + K')\right)^T - \left[2\frac{b_c}{a}x_1x_2 + 2a(x_1y_2 + x_2y_1) + x_1^2y_1 - b^{(1)}\frac{x_1^2}{a}\right]\cdot(+1, -(1 + K'))^T$$
$$\therefore L_c(x_3, y_3)^T = \left[b^{(2)}x_1 + \left(b^{(1)} - 2\frac{b_c}{a}x_1 - 2ay_1\right)x_2 - 2ax_1y_2 + b^{(1)}\frac{x_1^2}{a} - x_1^2y_1\right].\left(1, -(1 + K')\right)^T + \partial_{\tau 1}(x_2, y_2)^T + \partial_{\tau 2}\,(x_1, y_1)^T = \left(G_x, G_y\right)^T \text{ (say)}$$

Appendix C

We have

$$L_c(x_1, y_1)^T = 0 \tag{5.5}$$

Substituting the value of L_c from Eq. (4.1), Eq. (5.5) in component form may be written for $b = b_c$; $|\vec{k}_i| = \vec{k}_c$ as

$$[(1 + a\eta)^2 - 1 + \nabla_\rho^2 \,]x_1 + a^2 y_1 = 0 \tag{C1}$$

$$-(1 + K').(1 + a\eta)^2 x_1 + (1 + K').\left(-a^2 + \frac{1}{\eta^2}\nabla_\rho^2\right) y_1 = 0 \tag{C2}$$

Let. $(\overline{x}_1, \overline{y}_1)^T$ be the solution for non-degenerate eigenvectors such that the degenerate solution $(x_{1,} y_1)^T$ may be written as a linear combination of such non-degenerate eigenvectors as given below

$$x_1 = \overline{x}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] \tag{C3a}$$

$$y_1 = \overline{y}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] \tag{C3b}$$

We, therefore, obtain from the first component Eq. (C1),

$$[(1 + a\eta)^2 - 1]\overline{x}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] + (i\vec{k}_i)^2\overline{x}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c]$$
$$+ a^2\overline{y}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] = 0 \tag{C4}$$

As $|\vec{k}_i| = \vec{k}_c$, we have

$$[(1 + a\eta)^2 - 1]\overline{x}_1 - \vec{k}_c^2\,\overline{x}_1 + a^2\overline{y}_1 = 0$$
$$Or, [(1 + a\eta)^2 - 1 - \vec{k}_c^2\,]\overline{x}_1 + a^2\overline{y}_1 = 0 \tag{C5}$$

Substituting $\vec{k}_c^2$ from Eq. (3.11), one obtains

$$[(1 + a\eta)^2 - 1 - a\eta]\ \overline{x}_1 + a^2\overline{y}_1 = 0$$
$$Or,\ (a\eta^2 + \eta)\ \overline{x}_1 + a\overline{y}_1 = 0 \tag{C6}$$

which gives the non-degenerate eigenvectors solution

$$(\overline{x}_{1,}\overline{y}_1)^T = \left(-\frac{a}{\eta(1 + a\eta)}, 1\right)^T \tag{C7}$$

For degenerate eigenvectors solution, we obtain from Eq. (C3), the solution of $(x_1, y_1)^T$ as shown in Eq. (5.8).

$$(x_1, y_1)^T = \sum\nolimits_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c.)] \cdot \left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T \quad \text{(C8/5.8)}$$

We could also get the same solution starting from the second equation Eq. (C2), and substitution of the values of b_c and $\vec{k}_c^2$ from Eqs. (3.12) and (3.11) respectively. For calculations, see below:

$$-(1+a\eta)^2 x_1 + \left(-a^2 + \frac{1}{\eta^2}\nabla_\rho^2\right) y_1 = 0 \quad \text{(C2}'\text{)}$$

Therefore,

$$-(1+a\eta)^2\,\overline{x}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] + \left(-a^2 + \frac{1}{\eta^2}\nabla_\rho^2\right)\overline{y}_1$$
$$\sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] = 0$$
$$\text{Or, } -(1+a\eta)^2\,\overline{x}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c] + (-a^2)\overline{y}_1 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c]$$
$$+\frac{1}{\eta^2}(i\vec{k}_i)^2 \sum_{i=1}^{3}[W_i \exp(i\vec{k}_i \cdot \rho) + c.c]\,\overline{y}_1 = 0$$
$$\text{Or, } -(1+a\eta)^2\,\overline{x}_1 + \left[-a^2 + \frac{1}{\eta^2}(-\vec{k}_c^2)\right]\overline{y}_1 = 0$$
$$\text{Or, } -(1+a\eta)^2\,\overline{x}_1 + \left[-a^2 + \frac{1}{\eta^2}(-a\eta)\right]\overline{y}_1 = 0$$
$$\text{Or,} (1+a\eta)^2\,\overline{x}_1 + \left(\frac{a^2\eta + a}{\eta}\right)\overline{y}_1 = 0$$

The non-degenerate eigenvectors solution is given by
$(\overline{x}_1, \overline{y}_1)^T = (-\frac{a}{\eta(1+a\eta)}, 1)^T$, which is the same as that in Eq. (C7).

Appendix D

$$L_c^T (x_2, y_2)^T = 0_{2\times 1} \quad (5.9)$$

Substituting the values of L_c and b_c from Eqs. (4.1) and (3.12) respectively into Eq. (5.9) and considering that $(\overline{x}_2, \overline{y}_2)^T$ be the non-degenerate eigenvectors solution such that

$$x_2 = \overline{x}_2 \sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] \tag{D1}$$

$$y_2 = \overline{y}_2 \sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] \tag{D2}$$

One obtains Eq. (5.9) divided into component forms given in Eqs. (D3) and (D4).

$$[(1 + a\eta)^2 - 1 + \nabla_\rho^2 \,]x_2 - (1 + K').(1 + a\eta)^2 y_2 = 0 \tag{D3}$$

and

$$a^2 x_2 + \left(-a^2 + \frac{1}{\eta^2}\nabla_\rho^2\right) \cdot (1 + K')y_2 \;= 0 \tag{D4}$$

$$\text{Or, } [(1 + a\eta)^2 - 1 + (i\vec{k}_c)^2]\,\overline{x}_2 \sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] - (1 + K') \cdot (1 + a\eta)^2$$

$$\overline{y}_2 \sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] = 0$$

$$\text{Or, } [(1 + a\eta)^2 - 1 - \vec{k}_c^2]\,\overline{x}_2 - (1 + K') \cdot (1 + a\eta)^2 \overline{y}_2 = 0 \tag{D5}$$

and,

$$a^2 \overline{x}_2 \sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] + \left(-a^2 + \frac{1}{\eta^2}(ik_c)^2\right) \cdot (1 + K')\overline{y}_2$$

$$.\sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] = 0$$

$$\text{Or, } a^2 \overline{x}_2 + (1 + K')\left(-a^2 - \frac{1}{\eta^2}\vec{k}_c^2\right)\overline{y}_2 \;= 0 \tag{D6}$$

Substituting the value of $\vec{k}_c^2$ from Eq. (3.11) into Eq. (D5) one obtains,

$$[(1 + a\eta)^2 - 1 - a\eta]\,\overline{x}_2 - (1 + K') \cdot (1 + a\eta)^2 \overline{y}_2 = 0, \tag{D7}$$

which gives the non-degenerate eigenvectors solution

$$(\overline{x}_2, \overline{y}_2)^T = \left[(1 + K').\left(\frac{1 + a\eta}{a\eta}\right), 1\right]^T$$

and the degenerate solution

$$(x_2, y_2)^T = \left[(1 + K').\left(\frac{1 + a\eta}{a\eta}\right), 1\right]^T \sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c] \tag{D8/5.10}$$

Equation (D6) can also give identical non-degenerate eigenvectors solution as given below

$$(\overline{x}_2, \overline{y}_2)^T = \left[(1+K').\left(\frac{1+a\eta}{a\eta}\right), 1\right]^T$$

Appendix E

Let us calculate $(F_x^{(1)}, F_y^{(1)})^T$, the coefficient of $\exp(i\overline{k}_1 \cdot \rho)$ by incorporating Eq. (5.8) into Eq. (5.6).

$$\begin{aligned}(F_x^{(1)}, F_y^{(1)})^T \Rightarrow & \Bigg[b^{(1)}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}\right\} \\ & -\frac{b_c}{a}\left\{\sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c]\right\}^2 \\ & \cdot\left(-\frac{a}{\eta(1+a\eta)}\right)^2 - 2a\left\{\sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c]\right\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}\right\} \\ & \cdot\left\{\sum_i [W_i \exp(i\vec{k}_i \cdot \rho) + c.c]\right\}\Bigg] \cdot (+1, -(1+K^{'}))^T + \partial_{\tau 1}\Big[\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \\ & \cdot\left\{-\frac{a}{\eta(1+a\eta)}, 1\right\}^T\Big] \end{aligned} \tag{E1}$$

$$\begin{aligned} = & \Bigg[b^{(1)}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}\right\} + \left\{-\frac{(1+a\eta)^2}{a} \cdot \frac{a^2}{\eta^2(1+a\eta)^2} + \frac{2a^2}{\eta(1+a\eta)}\right\} \\ & \cdot\left\{\sum_{i=1}^{3} W_i \exp(i\overline{k}_i \cdot \rho) + c.c.\right\}^2\Bigg] \cdot (+1, -(1+K^{'}))^T + \partial_{\tau 1}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \\ & \cdot\left\{-\frac{a}{\eta(1+a\eta)}, 1\right\}^T \end{aligned} \tag{E2}$$

$$\begin{aligned} = & \Bigg[b^{(1)}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}\right\} - \frac{a}{\eta^2}\left(1 - \frac{2a\eta}{1+a\eta}\right) \cdot \{W_1 \exp(i\overline{k}_1 \cdot \rho) \\ & + W_1^* \exp(-i\overline{k}_1 \cdot \rho) \\ & + W_2 \exp(i\overline{k}_2 \cdot \rho) + W_2^* \exp(-i\overline{k}_2 \cdot \rho) \\ & + W_3 \exp(i\overline{k}_3 \cdot \rho) + W_3^* \exp(-i\overline{k}_3 \cdot \rho)\}^2\Bigg] \cdot (+1, -(1+K^{'}))^T \\ & + \partial_{\tau 1}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}, 1\right\}^T \end{aligned} \tag{E3}$$

$$\begin{aligned} = & \left[b^{(1)}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}\right\} - \frac{a}{\eta^2}\frac{1-a\eta}{1+a\eta} 2W_2^* W_3^* \exp(i\overline{k}_1 \cdot \rho)\right]. \\ & (+1, -(1+K^{'}))^T + \partial_{\tau 1}\{W_1 \exp(i\overline{k}_1 \cdot \rho)\} \cdot \left\{-\frac{a}{\eta(1+a\eta)}, 1\right\}^T \end{aligned} \tag{E4}$$

$$\therefore (F_x^{(1)}, F_y^{(1)})^T = \frac{a}{\eta(1+a\eta)}\left\{-b^{(1)}W_1 - \frac{2(1-a\eta)}{\eta}\overline{W}_2\overline{W}_3\right\}\cdot(+1, -(1+K^{'}))^T + \frac{\partial W_1}{\partial \tau_1}.\left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T \tag{E5/5.12}$$

Appendix F

As the left (row) eigenvector of L_c^T with zero eigenvalue, given by Eq. (5.11) is orthogonal to $(F_x^{(1)}, F_y^{(1)})^T$ as given by Eq. (5.12), one obtains

$$(x_2, y_2)\cdot(F_x^{(1)}, F_y^{(1)})^T = 0 \tag{F1}$$

$$\text{Therefore, } x_2 F_x^{(1)} + y_2 F_y^{(1)} = 0 \tag{F2}$$

Substituting the values in Eqs. (5.11) and (5.12), one obtains

$$\frac{(1+K')\cdot(1+a\eta)}{a\eta}F_x^{(1)} + F_y^{(1)} = 0 \qquad \therefore \tag{F3}$$

$$Or, \frac{(1+K')\cdot(1+a\eta)}{a\eta}\left[\frac{a}{\eta(1+a\eta)}\left\{-b^{(1)}W_1 - \frac{2(1-a\eta)}{\eta}\overline{W}_2\overline{W}_3\right\} - \frac{a}{\eta(1+a\eta)}\frac{\partial W_1}{\partial \tau_1}\right] + \frac{a}{\eta(1+a\eta)}\cdot\left\{-b^{(1)}W_1 - \frac{2(1-a\eta)}{\eta}\overline{W}_2\overline{W}_3\right\}\cdot(-1)\cdot(1+K^{'}) + \frac{\partial W_1}{\partial \tau_1} = 0 \tag{F4}$$

$$Or, (1+K^{'})\frac{1}{\eta^2}\cdot\left\{-b^{(1)}W_1 - \frac{2(1-a\eta)}{\eta}\overline{W}_2\overline{W}_3\right\} - (1+K^{'})\frac{1}{\eta^2}\frac{\partial W_1}{\partial \tau_1} + \left\{-\frac{b^{(1)}aW_1}{\eta(1+a\eta)} - \frac{2a(1-a\eta)}{\eta^2(1+a\eta)}\overline{W}_2\overline{W}_3\right\}\cdot(-1).(1+K^{'}) + \frac{\partial W_1}{\partial \tau_1} = 0$$

Therefore,

$$\left\{1-(1+K^{'})\frac{1}{\eta^2}\right\}\frac{\partial W_1}{\partial \tau_1} + b^{(1)}W_1(1+K^{'})\cdot\left\{-\frac{1}{\eta^2} + \frac{a}{\eta(1+a\eta)}\right\} +2(1-a\eta)\overline{W}_2\overline{W}_3\cdot(1+K^{'})\cdot\left\{-\frac{1}{\eta^3} + \frac{a}{\eta^2(1+a\eta)}\right\} = 0 \tag{F5}$$

$$Or, \{\eta^2 - (1+K^{'})\}\frac{\partial W_1}{\partial \tau_1} + b^{(1)}W_1(1+K^{'})\cdot\left\{-1 + \frac{a\eta}{1+a\eta}\right\} +2(1-a\eta)\overline{W}_2\overline{W}_3\cdot(1+K^{'}).\left\{-\frac{1}{\eta} + \frac{a}{1+a\eta}\right\} = 0$$

Or,

$$\{\eta^2-(1+K')\}\cdot\frac{\partial W_1}{\partial\tau_1}-\frac{(1+K')b^{(1)}W_1}{1+a\eta}-\frac{2(1+K')\cdot(1-a\eta)\overline{W}_2\overline{W}_3}{\eta(1+a\eta)}=0 \tag{F6/5.13}$$

Appendix G

Calculation of the coefficient $(X_o, Y_o)^T\exp(0)$:

$$\text{Let } (x_2, y_2)^T=(X_o, Y_o)^T\exp(0)+\cdots\cdots\cdots\cdots \tag{G1}$$

Substituting Eq. (G1) into Eq. (5.6) and equating the coefficient of exp(0) (i.e. constant term) from both sides,

$$\begin{aligned}
LHS(\text{Left hand side}) \Rightarrow L_c(x_2, y_2)^T &= \begin{bmatrix}(b_c-1)+\nabla_\rho^2 & a^2\\ -b_c(1+K') & \left(-a^2+\frac{1}{\eta^2}\nabla_\rho^2\right).(1+K')\end{bmatrix}.\\
&\quad (X_o, Y_o)^T\\
&=\begin{bmatrix}(b_c-1) & a^2\\ -b_c(1+K') & -a^2.(1+K')\end{bmatrix}.(X_o, Y_o)^T\\
&=\begin{bmatrix}(1+a\eta)^2-1 & a^2\\ -(1+a\eta)^2(1+K') & -a^2.(1+K')\end{bmatrix}.(X_o, Y_o)^T
\end{aligned} \tag{G2}$$

$$\begin{aligned}
RHS\ (\text{Right hand side}) &\Rightarrow (+1, -(1+K'))^T\cdot\left[b^{(1)}x_1-\frac{b_c}{a}x_1^2-2ax_1y_1\right]+\partial_{\tau 1}(x_1, y_1)^T\\
&=(+1, -(1+K'))^T\cdot\left[-\frac{(1+a\eta)^2}{a}\left(-\frac{a}{\eta(1+a\eta)}\right)^2\cdot\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i.\rho)+c.c.\right)^2\right.\\
&\quad\left.-2a\cdot\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho)+c.c.\right)^2\right]+0\\
&=(+1, -(1+K'))^T\cdot\left[-\frac{a}{\eta^2}+\frac{2a^2}{\eta(1+a\eta)}\right]\cdot\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho)+c.c.\right)^2\\
&=(+1, -(1+K'))^T\cdot\frac{a}{\eta}\left(-\frac{1}{\eta}+\frac{2a}{(1+a\eta)}\right)\cdot 2(W_1W_1^*+W_2W_2^*+W_3W_3^*)\\
&=(+1, -(1+K'))^T\cdot\frac{a}{\eta}\left(\frac{-1+a\eta}{\eta(1+a\eta)}\right)\cdot 2(|W_1|^2+|W_2|^2+|W_3|^2)
\end{aligned} \tag{G3}$$

$$\begin{aligned}
\therefore \begin{bmatrix}(1+a\eta)^2-1 & a^2\\ -(1+a\eta)^2(1+K') & -a^2\cdot(1+K')\end{bmatrix}.(X_o, Y_o)^T\\
=(+1, -(1+K'))^T\cdot\left(-\frac{a}{\eta^2}\right)\cdot\frac{1-a\eta}{1+a\eta}
\end{aligned}$$

$$\cdot 2(|W_1|^2 + |W_2|^2 + |W_3|^2)$$

Or,

$$((1 + a\eta)^2 - 1\,)X_o + a^2 Y_o = -\frac{a}{\eta^2} \cdot \frac{1 - a\eta}{1 + a\eta} \cdot 2(|W_1|^2 + |W_2|^2 + |W_3|^2) \tag{G4a}$$

and

$$-\,(1 + a\eta)^2 X_o\; - a^2 Y_o = \frac{a}{\eta^2} \cdot \frac{1 - a\eta}{1 + a\eta} \cdot 2(|W_1|^2 + |W_2|^2 + |W_3|^2) \tag{G4b}$$

$$\begin{aligned} &(G4a) \text{ plus } (G4b) \Rightarrow X_o\; = 0 \\ &(G4a) \text{ minus } (G4b)\; \Rightarrow Y_o\; = -\frac{2(1 - a\eta)}{a\eta^2(1 + a\eta)} \cdot (|W_1|^2 + |W_2|^2 + |W_3|^2) \end{aligned} \tag{G5}$$

Calculation of the coefficient $(X_1,Y_1)^T$:

$$\text{Let } (x_2, y_2)^T\; = (X_1, Y_1)^T \exp(i\vec{k}_1 \cdot \rho) + \cdots\cdots \tag{G6}$$

Substituting Eq. (G6) into Eq. (5.6) and equating the coefficient of $\exp(i\vec{k}_1 \cdot \rho)$ from both sides,

$$\begin{aligned} &LHS(\text{Left hand side})\; \Rightarrow L_c(x_2, y_2)^T \\ &= \begin{bmatrix} (1 + a\eta)^2 - 1 + \nabla_\rho^2 & a^2 \\ -(1 + a\eta)^2(1 + K') & \left(-a^2 + \frac{1}{\eta^2}\nabla_\rho^2\right) \cdot (1 + K') \end{bmatrix} .(X_1, Y_1)^T \exp(i\vec{k}_1 \cdot \rho) \\ &= \begin{bmatrix} (1 + a\eta)^2 - 1 - \vec{k}_c^2 & a^2 \\ -(1 + K') \cdot (1 + a\eta)^2 & (1 + K')(-a^2 + \frac{1}{\eta^2}(-\vec{k}_c^2)) \end{bmatrix} \cdot (X_1, Y_1)^T \exp(i\vec{k}_1 \cdot \rho) \\ &= \begin{bmatrix} (1 + a\eta)^2 - 1 - a\eta & a^2 \\ -(1 + K') \cdot (1 + a\eta)^2 & (1 + K') \cdot \left(-a^2 - \frac{a}{\eta}\right) \end{bmatrix} \cdot (X_1, Y_1)^T \exp(i\vec{k}_1 \cdot \rho) \end{aligned} \tag{G7}$$

$$\begin{aligned} &RHS(\text{Right hand side}) \Rightarrow (+1, -(1 + K^{'}))^T . \Bigg[b^{(1)} \left(-\frac{a}{\eta(1 + a\eta)}\right) W_1 \exp(i\vec{k}_1.\rho) - \\ &\frac{(1 + a\eta)^2}{a} . \left(-\frac{a}{\eta(1 + a\eta)}\right)^2 . \left(\sum\nolimits_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right)^2 \\ &-2a\left(-\frac{a}{\eta(1 + a\eta)}\right) \cdot \left(\sum\nolimits_{i=1}^{i=3} W_i \exp(i\vec{k}_i.\rho) + c.c.\right)^2 \Bigg] + \partial_{\tau 1} (x_1, y_1)^T \\ &= (+1, -(1 + K^{'}))^T \cdot \Bigg[-\frac{ab^{(1)} W_1}{\eta(1 + a\eta)} \cdot \exp(i\vec{k}_i \cdot \rho) + \left(\frac{2a^2}{\eta(1 + a\eta)} - \frac{a}{\eta^2}\right) \end{aligned}$$

$$\cdot\left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho)+c.c.\right)^2\Bigg]$$

$$= (+1, -(1+K^{'}))^T\cdot\left[-\frac{ab^{(1)}W_1}{\eta(1+a\eta)}-\frac{a}{\eta}\left(\frac{1-a\eta}{\eta(1+a\eta)}\right)\cdot 2\overline{W}_2\overline{W}_3\right]\exp(i\vec{k}_1\cdot\rho)$$

$$= (+1, -(1+K^{'}))^T\cdot\frac{a}{\eta(1+a\eta)}\cdot\left[-b^{(1)}W_1-\left(\frac{1-a\eta}{\eta}\right)\right.$$
$$\left.\cdot 2\overline{W}_2\overline{W}_3\right]\exp(i\vec{k}_1\cdot\rho) \tag{G8}$$

From Eqs. (G7) and (G8), one gets

$$\begin{bmatrix}(1+a\eta)^2-1-a\eta & a^2\\ -(1+K')\cdot(1+a\eta)^2 & (1+K')\cdot\left(-a^2-\frac{a}{\eta}\right)\end{bmatrix}\cdot(X_1, Y_1)^T$$
$$= (+1, -(1+K^{'}))^T\cdot\frac{a}{\eta(1+a\eta)}$$
$$\cdot\left[-b^{(1)}W_1-\left(\frac{1-a\eta}{\eta}\right)\cdot 2\overline{W}_2\overline{W}_3\right]$$
$$\therefore [(1+a\eta)^2-1-a\eta]X_1+a^2Y_1=\frac{a}{\eta(1+a\eta)}$$
$$\cdot\left[-b^{(1)}W_1-\left(\frac{1-a\eta}{\eta}\right)\cdot 2\overline{W}_2\overline{W}_3\right] \tag{G9a}$$

and,

$$-(1+a\eta)^2X_1+(-a^2-\frac{a}{\eta})Y_1=-\frac{a}{\eta(1+a\eta)}\cdot\left[-b^{(1)}W_1-\frac{1-a\eta}{\eta}.2\overline{W}_2\overline{W}_3\right] \tag{G9b}$$

$$(G9a)\text{ plus }(G9b)\ \Rightarrow X_1=-\frac{aY_1}{\eta(1+a\eta)} \tag{G10}$$

Calculation of the coefficient $(X_{11}, Y_{11})^T$:

$$\text{Let } (x_2, y_2)^T = (X_{11}, Y_{11})^T\exp(2i\vec{k}_1\cdot\rho)+\cdots\cdots \tag{G11}$$

Substituting Eq. (G11) into Eq. (5.6) and equating the coefficient of $\exp(2i\vec{k}_1\cdot\rho)$ from both sides,

$$LHS(\text{Left hand side}) \Rightarrow L_c(x_2, y_2)^T$$
$$=\begin{bmatrix}(1+a\eta)^2-1+\nabla_\rho^2 & a^2\\ -(1+a\eta)^2(1+K') & \left(-a^2+\frac{1}{\eta^2}\nabla_\rho^2\right)\cdot(1+K')\end{bmatrix}\cdot(X_{11}, Y_{11})^T\exp(2i\vec{k}_1\cdot\rho)$$
$$=\begin{bmatrix}(1+a\eta)^2-1+(2ik_c)^2 & a^2\\ -(1+a\eta)^2(1+K') & \left(-a^2+\frac{1}{\eta^2}(2ik_c)^2\right)\cdot(1+K')\end{bmatrix}\cdot(X_{11}, Y_{11})^T\exp(2i\vec{k}_1\cdot\rho)$$

$$= \begin{bmatrix} (1+a\eta)^2 - 1 - 4a\eta & a^2 \\ -(1+a\eta)^2(1+K') & \left(-a^2 + \frac{1}{\eta^2}(-4a\eta)\right)\cdot(1+K') \end{bmatrix} \cdot (X_{11}, Y_{11})^T \exp(2i\vec{k}_1 \cdot \rho)$$

$$= \begin{bmatrix} a^2\eta^2 - 2a\eta & a^2 \\ -(1+a\eta)^2(1+K') & \left(-a^2 - \frac{4a}{\eta}\right)\cdot(1+K') \end{bmatrix} \cdot (X_{11}, Y_{11})^T \exp(2i\vec{k}_1 \cdot \rho) \quad \text{(G12)}$$

$$RHS(\text{Right hand side}) \Rightarrow$$

$$(+1, -(1+K^{'}))^T \cdot \left[b^{(1)}x_1 - \frac{b_c}{a}x_1^2 - 2ax_1y_1\right] + \partial_{\tau 1}(x_1, y_1)^T$$

$$= (+1, -(1+K^{'}))^T \cdot \left[0 - \frac{(1+a\eta)^2}{a} \cdot \left(-\frac{a}{\eta(1+a\eta)}\right)^2 \cdot \left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right)^2 - 2a\left(-\frac{a}{\eta(1+a\eta)}\right) \cdot \left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right)^2\right] + 0$$

$$= (+1, -(1+K^{'}))^T \cdot \left[-\frac{a}{\eta^2} + \frac{2a^2}{\eta(1+a\eta)}\right] \cdot \left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right)^2$$

$$= (+1, -(1+K^{'}))^T \cdot \frac{a}{\eta} \cdot \left[-\frac{1}{\eta} + \frac{2a}{1+a\eta}\right] \cdot W_1^2 \exp(2ik_1 \cdot \rho)$$

$$= (+1, -(1+K^{'}))^T \cdot \frac{aW_1^2(a\eta - 1)}{\eta^2(1+a\eta)} \cdot \exp(2ik_1 \cdot \rho) \quad \text{(G13)}$$

From Eq. (G12) and (G13) one obtains,

$$(a^2\eta^2 - 2a\eta)X_{11} + a^2Y_{11} = -\frac{aW_1^2(1-a\eta)}{\eta^2(1+a\eta)} \quad \text{(G14a)}$$

$$-(1 + a^2\eta^2 + 2a\eta)X_{11} + \left(-a^2 - \frac{4a}{\eta}\right)Y_{11} = \frac{aW_1^2(1-a\eta)}{\eta^2(1+a\eta)} \quad \text{(G14b)}$$

$$Eq.(\text{G14a}) \text{ plus } (\text{G14b}) \Rightarrow Y_{11} = \frac{\eta}{4a}(-4a\eta - 1)X_{11} \quad \text{(G14c)}$$

Substituting Eq. (G14c) in Eq. (G14a) one obtains,

$$X_{11} = \frac{4W_1^2(1-a\eta)}{9\eta^3(1+a\eta)} \quad \text{(G15a)}$$

Substituting X_{11} from (G15a) in (G14c) one obtains,

$$Y_{11} = -\frac{W_1^2(1+4a\eta)(1-a\eta)}{9a\eta^2(1+a\eta)} \quad \text{(G15b)}$$

Calculation of the coefficient $(X_{12}, Y_{12})^T$:

$$\text{Let } (x_2, y_2)^T = (X_{12}, Y_{12})^T \exp(i(\vec{k}_1 - \vec{k}_2) \cdot \rho) \tag{G16}$$

Substituting Eq. (G16) into (5.6) and equating the coefficient of $\exp(i(\vec{k}_1 - \vec{k}_2) \cdot \rho)$from both sides, one obtains

$$\begin{aligned}
&LHS(\text{Left hand side}) \Rightarrow L_c(x_2, y_2)^T \\
&= \begin{bmatrix} (1+a\eta)^2 - 1 + \nabla_\rho^2 & a^2 \\ -(1+a\eta)^2(1+K') & \left(-a^2 + \frac{1}{\eta^2}\nabla_\rho^2\right)\cdot(1+K') \end{bmatrix} \\
&\quad .(X_{12}, Y_{12})^T \exp(i(\vec{k}_1 - \vec{k}_2)\cdot\rho) \\
&= \begin{bmatrix} (1+a\eta)^2 - 1 + i^2(\vec{k}_1 - \vec{k}_2)^2 & a^2 \\ -(1+a\eta)^2(1+K') & \left(-a^2 + \frac{1}{\eta^2} i^2 \left(\vec{k}_1 - \vec{k}_2\right)^2\right)\cdot(1+K') \end{bmatrix} \\
&\quad .(X_{12}, Y_{12})^T . \exp(i(\vec{k}_1 - \vec{k}_2)\cdot\rho) \\
&= \begin{bmatrix} (1+a\eta)^2 - 1 - 3\vec{k}_c^2 & a^2 \\ -(1+a\eta)^2.(1+K') & \left(-a^2 + \frac{1}{\eta^2}\left(-3k_c^2\right)\right)\cdot(1+K') \end{bmatrix} \\
&\quad \cdot(X_{12}, Y_{12})^T . \exp(i(\vec{k}_1 - \vec{k}_2)\cdot\rho) \\
&= \begin{bmatrix} (1+a\eta)^2 - 1 - 3a\eta & a^2 \\ -(1+a\eta)^2.(1+K') & \left(-a^2 - \frac{3}{\eta^2}a\eta\right)\cdot(1+K') \end{bmatrix} \\
&\quad \cdot(X_{12}, Y_{12})^T \cdot \exp(i(\vec{k}_1 - \vec{k}_2)\cdot\rho)
\end{aligned} \tag{G17}$$

$$\begin{aligned}
&RHS(\text{Right hand side}) \Rightarrow \\
&= (+1, -(1+K^{'}))^T \cdot \left[0 - \frac{(1+a\eta)^2}{a}\cdot\left(-\frac{a}{\eta(1+a\eta)}\right)^2\right. \\
&\quad \cdot\left(\sum\nolimits_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho) + c.c\right)^2 \\
&\quad \left. -2a\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\left(\sum\nolimits_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho) + c.c.\right)^2\right] + 0 \\
&= (+1, -(1+K^{'}))^T \cdot \left[\frac{2a^2}{\eta(1+a\eta)} - \frac{a}{\eta^2}\right]\cdot\left(\sum\nolimits_{i=1}^{i=3} W_i \exp(i\vec{k}_i\cdot\rho) + c.c\right)^2 \\
&= (+1, -(1+K^{'}))^T \cdot \frac{a}{\eta}\left(\frac{2a}{1+a\eta} - \frac{1}{\eta}\right)\cdot 2W_1\overline{W}_2 . \exp(i(\vec{k}_1 - \vec{k}_2)\cdot\rho) \\
&= (+1, -(1+K^{'}))^T \cdot \frac{2aW_1\overline{W}_2(a\eta - 1)}{\eta^2(1+a\eta)}\cdot\exp(i(\vec{k}_1 - \vec{k}_2).\rho)
\end{aligned} \tag{G18}$$

From Eq. (G17) and (G18) one obtains,

$$\begin{bmatrix} a^2\eta^2 - a\eta & a^2 \\ -(1+a\eta)^2 \cdot (1+K') & \left(-a^2 - \frac{3a}{\eta}\right) \cdot (1+K') \end{bmatrix} \cdot (X_{12}, Y_{12})^T$$

$$= (+1, -(1+K^{'}))^T \cdot \frac{2aW_1\overline{W}_2(a\eta - 1)}{\eta^2(1+a\eta)}$$

$$\therefore (a^2\eta^2 - a\eta)X_{12} + a^2 Y_{12} = -\frac{2aW_1\overline{W}_2(1-a\eta)}{\eta^2(1+a\eta)} \tag{G19a}$$

and

$$-(1+a\eta)^2 X_{12} + \left(-a^2 - \frac{3a}{\eta}\right) Y_{12} = \frac{2aW_1\overline{W}_2(1-a\eta)}{\eta^2(1+a\eta)} \tag{G19b}$$

$$Eq.(\text{G19a}) \text{ plus } (\text{G19b}) \Rightarrow Y_{12} = \frac{\eta}{3a}(-3a\eta - 1)X_{12} \tag{G20}$$

Substituting Eq. (G20) in Eq. (G19a), one gets

$$X_{12} = \frac{3W_1\overline{W}_2(1-a\eta)}{2\eta^3(1+a\eta)} \tag{G21a}$$

Substituting X_{12} from Eq. (G21a) in Eq. (G20), one gets

$$Y_{12} = -\frac{W_1\overline{W}_2(1+3a\eta)\cdot(1-a\eta)}{2a\eta^2(1+a\eta)} \tag{G21b}$$

Appendix H

$$\left(G_x^{(1)}, G_y^{(1)}\right)^T \Rightarrow \left(+1, -(1+K')\right)^T \cdot \left[b^{(2)}\left(-\frac{a}{\eta(1+a\eta)}\right)W_1 + b^{(1)}X_1\right.$$
$$-\frac{2(1+a\eta)^2}{a} \cdot \left(-\frac{a}{\eta(1+a\eta)}\right) \cdot \left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right) \cdot \{X_o + \overline{X}_2 \exp(-i\overline{k}_2 \cdot \rho)$$
$$+\overline{X}_3 \exp(-i\overline{k}_3 \cdot \rho) + X_{11} \exp(+2i\overline{k}_1 \cdot \rho) +$$
$$X_{12} \exp(i(\overline{k}_1 - \overline{k}_2) \cdot \rho) + \overline{X}_{31} \exp(-i(\overline{k}_3 - \overline{k}_1) \cdot \rho\} - 2a(X_o W_1 + \overline{X}_2 \overline{W}_3 + \overline{X}_3 \overline{W}_2 +$$
$$X_{11}\overline{W}_1 + X_{12}W_2 + \overline{X}_{31}W_3) - 2a\left(-\frac{a}{\eta(1+a\eta)}\right) \cdot (Y_o W_1 + \overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2 + Y_{11}\overline{W}_1$$
$$+Y_{12}W_2 + \overline{Y}_{31}W_3) + \frac{b^{(1)}}{a}\left(-\frac{a}{\eta(1+a\eta)}\right)^2 \cdot \left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right)^2$$
$$\left.-\left(-\frac{a}{\eta(1+a\eta)}\right)^2 \cdot \left(\sum_{i=1}^{i=3} W_i \exp(i\vec{k}_i \cdot \rho) + c.c.\right)^3\right]$$

$$+\partial_{\tau 1}(X_1, Y_1)^T + \partial_{\tau 2}\left(-\frac{aW_1}{\eta(1+a\eta)}, W_1\right)^T \tag{H1}$$

Or,

$$\left(G_x^{(1)}, G_y^{(1)}\right)^T \Rightarrow \left(+1, -(1+K')\right)^T \cdot \left[b^{(2)}\left(-\frac{a}{\eta(1+a\eta)}\right)W_1 + b^{(1)}X_1 \right.$$
$$-\frac{2(1+a\eta)^2}{a}\cdot\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot(X_oW_1 + \overline{X}_2\overline{W}_3 + \overline{X}_3\overline{W}_2 + X_{11}\overline{W}_1 + X_{12}W_2$$
$$+\overline{X}_{31}W_3) - 2a(X_oW_1 + \overline{X}_2\overline{W}_3 + \overline{X}_3\overline{W}_2 +$$
$$X_{11}\overline{W}_1 + X_{12}W_2 + \overline{X}_{31}W_3) - 2a\left(-\frac{a}{\eta(1+a\eta)}\right).(Y_oW_1 + \overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2 + Y_{11}\overline{W}_1$$
$$+Y_{12}W_2 + \overline{Y}_{31}W_3) + \frac{b^{(1)}}{a}\left(-\frac{a}{\eta(1+a\eta)}\right)^2\cdot\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho) + c.c.\right)^2$$
$$\left.-\left(-\frac{a}{\eta(1+a\eta)}\right)^2\cdot\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho) + c.c.\right)^3\right]$$
$$+\partial_{\tau 1}\left(-\frac{aY_1}{\eta(1+a\eta)}, Y_1\right)^T + \partial_{\tau 2}\left(-\frac{aW_1}{\eta(1+a\eta)}, W_1\right)^T \tag{H2}$$

$$= \left(+1, -(1+K')\right)^T \cdot \left[-\frac{ab^{(2)}W_1}{\eta(1+a\eta)} + b^{(1)}X_1 + (X_oW_1 + \overline{X}_2\overline{W}_3 + \overline{X}_3\overline{W}_2\right.$$
$$+X_{11}\overline{W}_1 + X_{12}W_2 +$$
$$\overline{X}_{31}W_3)\cdot\left(-2a + \frac{2(1+a\eta)}{\eta}\right) + \frac{2a^2}{\eta(1+a\eta)}\cdot(Y_oW_1 + \overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2 + Y_{11}\overline{W}_1$$
$$+Y_{12}W_2 + \overline{Y}_{31}W_3) + \frac{b^{(1)}}{a}\left(-\frac{a}{\eta(1+a\eta)}\right)^2\cdot\left\{\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho)\right)^2\right.$$
$$+\left(\sum_{i=1}^{i=3}\overline{W}_i\exp(-i\vec{k}_i\cdot\rho)\right)^2 + 2\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho)\right)$$
$$\left.\left(\sum_{i=1}^{i=3}\overline{W}_i\exp(-i\vec{k}_i\cdot\rho)\right)\right\} - \frac{a^2}{\eta^2(1+a\eta)^2}\cdot\left\{\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho) + c.c.\right)^3\right.$$
$$+\left(\sum_{i=1}^{i=3}\overline{W}_i\exp(-i\vec{k}_i\cdot\rho)\right)^3 + 3\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho)\right)^2\cdot\left(\sum_{i=1}^{i=3}\overline{W}_i\exp(-i\vec{k}_i\cdot\rho)\right)$$
$$\left.\left.+3\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho)\right).\left(\sum_{i=1}^{i=3}\overline{W}_i\exp(-i\vec{k}_i\cdot\rho)\right)^2\right\}\right]$$
$$+\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right)\cdot\left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T \tag{H3}$$

Substituting the values of $\left(\sum_{i=1}^{i=3}W_i\exp(i\vec{k}_i\cdot\rho) + c.c.\right)^n$ for $n = 2, 3$ in Eq. (H3) from reference note 50 (see Sect. 9), one gets

$$\left(G_x^{(1)}, G_y^{(1)}\right)^T$$

$$= \left(+1, -(1+K')\right)^T \cdot \left[-\frac{ab^{(2)}W_1}{\eta(1+a\eta)} + b^{(1)}X_1 + 2\left(\frac{1+a\eta}{\eta} - a\right) \cdot \left(X_o W_1 + \overline{X}_2\overline{W}_3\right.\right.$$

$$+\overline{X}_3\overline{W}_2 + X_{11}\overline{W}_1 +$$

$$\left.+X_{12}W_2 + \overline{X}_{31}W_3\right) + \frac{2a^2}{\eta(1+a\eta)} \cdot \left(Y_o W_1 + \overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2 + Y_{11}\overline{W}_1 + Y_{12}W_2 + \overline{Y}_{31}W_3\right)$$

$$+\frac{ab^{(1)}}{\eta^2(1+a\eta)^2}.(0+0+2\overline{W}_2\overline{W}_3) - \frac{a^2}{\eta^2(1+a\eta)^2} \cdot \{0+0+3(W_1^2\overline{W}_1 + 2W_1W_2\overline{W}_2 +$$

$$\left.2W_3\overline{W}_3W_1) + 3.(0)\}\right] + \left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right) \cdot \left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T \tag{H4}$$

$$= \left(+1, -(1+K')\right)^T . \left[-\frac{ab^{(2)}W_1}{\eta(1+a\eta)} + b^{(1)}X_1 + \frac{2}{\eta}(X_o W_1 + \overline{X}_2\overline{W}_3 + \overline{X}_3\overline{W}_2 + X_{11}\overline{W}_1 + \right.$$

$$+X_{12}W_2 + \overline{X}_{31}W_3) + \frac{2a^2}{\eta(1+a\eta)}.(Y_o W_1 + \overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2 + Y_{11}\overline{W}_1 + Y_{12}W_2 + \overline{Y}_{31}W_3)$$

$$\left.+\frac{2ab^{(1)}}{\eta^2(1+a\eta)^2}\overline{W}_2\overline{W}_3 - \frac{3a^2}{\eta^2(1+a\eta)^2} \cdot (W_1^2\overline{W}_1 + 2W_1W_2\overline{W}_2 + 2W_3\overline{W}_3W_1)\right]$$

$$+\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right).\left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T \tag{H5}$$

$$= \left(+1, -(1+K')\right)^T \cdot \left[-\frac{ab^{(2)}W_1}{\eta(1+a\eta)} + b^{(1)}X_1 + \frac{2}{\eta}(\overline{X}_2\overline{W}_3 + \overline{X}_3\overline{W}_2)\right.$$

$$+\frac{2}{\eta} \cdot \frac{4W_1^2(1-a\eta)\overline{W}_1}{9\eta^3(1+a\eta)}$$

$$+\frac{2}{\eta} \cdot \frac{3W_1\overline{W}_2(1-a\eta)W_2}{2\eta^3(1+a\eta)} + \frac{2}{\eta} \cdot \frac{3\overline{W}_3W_1(1-a\eta)W_3}{2\eta^3(1+a\eta)} + \frac{2a^2}{\eta(1+a\eta)}$$

$$\cdot\left\{\frac{-2(1-a\eta)\cdot(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1}{a\eta^2(1+a\eta)}\right\} + \frac{2a^2}{\eta(1+a\eta)} \cdot (\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2)$$

$$+\frac{2a^2}{\eta(1+a\eta)}\left\{\frac{-(1+4a\eta)W_1^2(1-a\eta)}{9a\eta^2(1+a\eta)} \cdot \overline{W}_1 + \frac{-(1+3a\eta)W_1\overline{W}_2(1-a\eta)}{2a\eta^2(1+a\eta)} \cdot W_2\right.$$

$$\left.+\frac{-(1+3a\eta)\overline{W}_3W_1(1-a\eta)}{2a\eta^2(1+a\eta)} \cdot W_3\right\}$$

$$\left.+\frac{2ab^{(1)}}{\eta^2(1+a\eta)^2}\overline{W}_2\overline{W}_3 - \frac{3a^2}{\eta^2(1+a\eta)^2}\{|W_1|^2W_1 + 2(|W_2|^2 + |W_3|^2)W_1\}\right]$$

$$+\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right) \cdot \left(-\frac{a}{\eta(1+a\eta)}, 1\right)^T \tag{H6}$$

$$= \left(+1, -(1+K')\right)^T \cdot \left[-\frac{ab^{(2)}W_1}{\eta(1+a\eta)} + b^{(1)}X_1 + \frac{2}{\eta}(\overline{X}_2\overline{W}_3\right.$$

$$+\overline{X}_3\overline{W}_2) + \frac{2a^2}{\eta(1+a\eta)} \cdot (\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2)$$
$$+(|W_1|^2 W_1) \cdot \frac{1}{\eta^2(1+a\eta)} \left\{ \frac{8(1-a\eta)}{9\eta^2} - \frac{4a(1-a\eta)}{\eta(1+a\eta)} - \frac{2a(1+4a\eta)(1-a\eta)}{9\eta(1+a\eta)} - \frac{3a^2}{1+a\eta} \right\}$$
$$+(|W_2|^2 + |W_3|^2) W_1 \cdot \frac{1}{\eta^2(1+a\eta)} \left\{ \frac{3(1-a\eta)}{\eta^2} - \frac{4a(1-a\eta)}{\eta(1+a\eta)} - \frac{a(1+3a\eta)(1-a\eta)}{\eta(1+a\eta)} \right.$$
$$\left. - \frac{6a^2}{1+a\eta} \right\}$$
$$\left. + \frac{2ab^{(1)}}{\eta^2(1+a\eta)^2} \overline{W}_2\overline{W}_3 \right] + \left(\frac{\partial W_1}{\partial \tau_2} + \frac{\partial Y_1}{\partial \tau_1} \right) \cdot \left(-\frac{a}{\eta(1+a\eta)},\ 1 \right)^T \tag{H7}$$

$$= \left(+1, -(1+K')\right)^T \cdot \left[-\frac{ab^{(2)} W_1}{\eta(1+a\eta)} + b^{(1)} X_1 + \frac{2ab^{(1)}}{\eta^2(1+a\eta)^2} \overline{W}_2\overline{W}_3 \right.$$
$$+ \frac{2}{\eta} (\overline{X}_2\overline{W}_3 + \overline{X}_3\overline{W}_2) + \frac{2a^2}{\eta(1+a\eta)} \cdot (\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2)$$
$$+(|W_1|^2 W_1). \frac{1}{\eta^2(1+a\eta)} \cdot \frac{1}{9\eta^2(1+a\eta)} \left\{ 8(1-a^2\eta^2) - 4a(1-a\eta).9\eta \right.$$
$$\left. -2a\eta(1+4a\eta)(1-a\eta) - 3a^2 \cdot 9\eta^2 \right\} + (|W_2|^2 + |W_3|^2) W_1 \cdot \frac{1}{\eta^2(1+a\eta)}$$
$$\left. \cdot \frac{1}{\eta^2(1+a\eta)} \{3(1-a^2\eta^2) - 4a\eta(1-a\eta) - a\eta(1+3a\eta)(1-a\eta) - 6a^2\eta^2\} \right]$$
$$+ \left(\frac{\partial W_1}{\partial \tau_2} + \frac{\partial Y_1}{\partial \tau_1} \right) \cdot \left(-\frac{a}{\eta(1+a\eta)},\ 1 \right)^T \tag{H8}$$

$$= \left(+1, -(1+K')\right)^T \cdot \frac{1}{\eta(1+a\eta)} \left[-ab^{(2)} W_1 + \eta(1+a\eta) \cdot b^{(1)} X_1 + \frac{2ab^{(1)}}{\eta(1+a\eta)} \overline{W}_2\overline{W}_3 \right.$$
$$+ \frac{2}{\eta} \cdot \eta(1+a\eta)$$
$$\cdot \left\{ -\frac{a\overline{Y}_2}{\eta(1+a\eta)} \overline{W}_3 - \frac{a\overline{Y}_3}{\eta(1+a\eta)} \cdot \overline{W}_2 \right\} + 2a^2(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) + (|W_1|^2 W_1)$$
$$\cdot \frac{1}{9\eta^3(1+a\eta)} \left\{ -38a\eta - 5a^2\eta^2 + 8 + 8a^3\eta^3 \right\} + (|W_2|^2 + |W_3|^2) W_1 \cdot \frac{1}{\eta}$$
$$\left. \cdot \frac{1}{\eta^2(1+a\eta)} \left\{ -5a\eta - 7a^2\eta^2 + 3 + 3a^3\eta^3 \right\} \right]$$
$$+ \left(\frac{\partial W_1}{\partial \tau_2} + \frac{\partial Y_1}{\partial \tau_1} \right) \cdot \left(-\frac{a}{\eta(1+a\eta)},\ 1 \right)^T \tag{H9}$$

$$= \left(+1, -(1+K')\right)^T . \frac{1}{\eta(1+a\eta)} \left[-ab^{(2)} W_1 + \{\eta(1+a\eta) X_1 + \frac{2a}{\eta(1+a\eta)} \overline{W}_2\overline{W}_3\} b^{(1)} \right.$$
$$- \frac{2a}{\eta} \cdot (\overline{Y}_2\overline{W}_3 + \overline{Y}_3 \cdot \overline{W}_2)$$
$$+2a^2(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_3) - (|W_1|^2 W_1) \cdot \frac{1}{9\eta^3(1+a\eta)} \{38a\eta + 5a^2\eta^2 - 8 - 8a^3\eta^3\}$$

$$-(|W_2|^2+|W_3|^2)W_1.\frac{1}{\eta^3(1+a\eta)}\{5a\eta+7a^2\eta^2-3-3a^3\eta^3\}\Big]$$
$$+\left(\frac{\partial W_1}{\partial \tau_2}+\frac{\partial Y_1}{\partial \tau_1}\right)\cdot\left(-\frac{a}{\eta(1+a\eta)},1\right)^T \tag{H10}$$

Therefore,

$$(G_x{}^{(1)},G_y{}^{(1)})^T=(+1,-(1+K'))^T\cdot\frac{1}{\eta(1+a\eta)}\cdot\Big[-ab^{(2)}W_1+\{\eta(1+a\eta)X_1$$
$$+\frac{2a\overline{W}_2\overline{W}_3}{\eta(1+a\eta)}\}b^{(1)}-\frac{2a(1-a\eta)}{\eta}\cdot(\overline{Y}_2\overline{W}_3+\overline{Y}_3\overline{W}_2)$$
$$-(|W_1|^2W_1)\cdot\frac{g_1}{9\eta^3(1+a\eta)}-(|W_2|^2+|W_3|^2)W_1\cdot\frac{h_1}{\eta^3(1+a\eta)}\Big]$$
$$+\left(\frac{\partial W_1}{\partial \tau_2}+\frac{\partial Y_1}{\partial \tau_1}\right)\cdot\left(-\frac{a}{\eta(1+a\eta)},1\right)^T \tag{H11/5.15}$$

where, $g_1=38a\eta+5(a\eta)^2-8-8(a\eta)^3$

$$h_1=5a\eta+7(a\eta)^2-3-3(a\eta)^3 \tag{H12/5.16}$$

Appendix I

Substituting Eq. (5.15) in Eq. (5.17) one obtains,

$$\frac{(1+K')(1+a\eta)}{a\eta}\cdot\left[\frac{1}{\eta(1+a\eta)}\left\{-ab^{(2)}W_1+\eta(1+a\eta)X_1b^{(1)}+\frac{2a\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}b^{(1)}\right.\right.$$
$$-\frac{2a(1-a\eta)}{\eta}$$
$$\left.\cdot(\overline{Y}_2\overline{W}_3+\overline{Y}_3\overline{W}_2)-(|W_1|^2W_1)\cdot\frac{g_1}{9\eta^3(1+a\eta)}-(|W_2|^2+|W_3|^2)W_1\cdot\frac{h_1}{\eta^3(1+a\eta)}\right\}$$
$$-\frac{a}{\eta(1+a\eta)}$$
$$\left.\cdot\left(\frac{\partial W_1}{\partial \tau_2}+\frac{\partial Y_1}{\partial \tau_1}\right)\right]+\left[-\frac{(1+K')}{\eta(1+a\eta)}\left\{-ab^{(2)}W_1+\eta(1+a\eta)X_1b^{(1)}+\frac{2a\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}b^{(1)}\right.\right.$$
$$-\frac{2a(1-a\eta)}{\eta}$$
$$\left.\cdot(\overline{Y}_2\overline{W}_3+\overline{Y}_3\overline{W}_2)-(|W_1|^2W_1).\frac{g_1}{9\eta^3(1+a\eta)}-(|W_2|^2+|W_3|^2)W_1\cdot\frac{h_1}{\eta^3(1+a\eta)}\right\}+$$
$$\left.\left(\frac{\partial W_1}{\partial \tau_2}+\frac{\partial Y_1}{\partial \tau_1}\right)\right]=0 \tag{I1}$$

$$Or, \quad (1+K').\frac{1}{a\eta^2}\left\{-ab^{(2)}W_1 - aY_1b^{(1)} + \frac{2a\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}b^{(1)} - \frac{2a(1-a\eta)}{\eta}\right.$$
$$\left..(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) - |W_1|^2W_1.\frac{g_1}{9\eta^3(1+a\eta)} - (|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3(1+a\eta)}\right\} - \frac{(1+K')}{\eta^2}.$$
$$\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right) - \frac{(1+K')}{\eta(1+a\eta)}.\left\{-ab^{(2)}W_1 - aY_1b^{(1)} + \frac{2a\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}b^{(1)}\right.$$
$$\left.-\frac{2a(1-a\eta)}{\eta}.(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) - |W_1|^2W_1.\frac{g_1}{9\eta^3(1+a\eta)} - (|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3(1+a\eta)}\right\}$$
$$+\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right) = 0 \tag{I2}$$

$$Or, \quad \left\{-ab^{(2)}W_1 - aY_1b^{(1)} + \frac{2a\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}b^{(1)} - \frac{2a(1-a\eta)}{\eta}.(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) -\right.$$
$$\left.|W_1|^2W_1.\frac{g_1}{9\eta^3(1+a\eta)} - (|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3(1+a\eta)}\right\}.\left\{\frac{1+K'}{a\eta^2} - \frac{1+K'}{\eta(1+a\eta)}\right\}$$
$$+\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right).\left(1 - \frac{1+K'}{\eta^2}\right) = 0 \tag{I3}$$

$$\therefore \left(\frac{1+K'}{\eta^2} - 1\right)\eta^2(1+a\eta).\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right) = (1+K').\left\{\frac{1}{a\eta^2} - \frac{1}{\eta(1+a\eta)}\right\}\eta^2(1+a\eta).$$
$$\left\{-ab^{(2)}W_1 - aY_1b^{(1)} + \frac{2a\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}b^{(1)} - \frac{2a(1-a\eta)}{\eta}.(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2)\right.$$
$$\left.-|W_1|^2W_1.\frac{g_1}{9\eta^3(1+a\eta)} - (|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3(1+a\eta)}\right\} \tag{I4}$$

$$Or, \quad \{(1+K') - \eta^2\}.(1+a\eta).\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right)$$
$$= (1+K').\left\{\frac{1+a\eta}{a} - \eta\right\}.\left\{-ab^{(2)}W_1 - aY_1b^{(1)} + \frac{2a\overline{W_2}\,\overline{W_3}b^{(1)}}{\eta(1+a\eta)}\right.$$
$$-\frac{2a(1-a\eta)}{\eta}.(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) - |W_1|^2W_1.\frac{g_1}{9\eta^3(1+a\eta)} -$$
$$\left.(|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3(1+a\eta)}\right\} \tag{I5}$$

$$Or, \quad \{(1+K') - \eta^2\}.(1+a\eta).\left(\frac{\partial W_1}{\partial\tau_2} + \frac{\partial Y_1}{\partial\tau_1}\right) = (1+K').\left\{-b^{(2)}W_1 - Y_1b^{(1)} +\right.$$
$$\frac{2\overline{W}_2\overline{W}_3b^{(1)}}{\eta(1+a\eta)} - \frac{2(1-a\eta)}{\eta}.(\overline{Y}_2\overline{W}_3 + \overline{Y}_3\overline{W}_2) - |W_1|^2W_1.\frac{g_1}{9\eta^3a(1+a\eta)} - (|W_2|^2$$

$$+|W_3|^2)W_1.\frac{h_1}{\eta^3 a(1+a\eta)}\Bigg\} \tag{I6/5.18}$$

Appendix J

Incorporating the solvability conditions Eq. (5.18) and (5.13) into Eq. (6.3), one obtains using Eq. (5.2) and (6.1),

$$\begin{aligned}\frac{\partial A_1}{\partial \tau} &= \varepsilon^3(-\frac{a}{\eta(1+a\eta)}).\frac{(1+k')}{\{(1+k')-\eta^2\}.(1+a\eta)}.\Big\{-b^{(2)}W_1 - b^{(1)}Y_1\\ &+\frac{2b^{(1)}\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)} - \frac{2(1-a\eta)}{\eta}\\ &\cdot(\overline{Y}_2\overline{W}_3+\overline{Y}_3\overline{W}_2) - |W_1|^2W_1.\frac{g_1}{9\eta^3 a(1+a\eta)} - (|W_2|^2+|W_3|^2)W_1.\frac{h_1}{\eta^3 a(1+a\eta)}\Big\}\\ &+\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right).\frac{(1+K')}{\{\eta^2-(1+K')\}}\cdot\left\{\frac{b^{(1)W_1}}{1+a\eta}+\frac{2(1-a\eta)\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)}\right\}\end{aligned} \tag{J1}$$

$$\begin{aligned}&= -\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}}\cdot\frac{b^{(1)}W_1}{1+a\eta} - \varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\\ &\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}(1+a\eta)}\\ &\cdot b^{(2)}W_1 - \varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}}\cdot\frac{2(1-a\eta)\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)} + \varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\\ &\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}(1+a\eta)}\left\{-b^{(1)}Y_1 + \frac{2b^{(1)}\overline{W_2}\,\overline{W_3}}{\eta(1+a\eta)} - \frac{2(1-a\eta)}{\eta}.(\overline{Y}_2\overline{W}_3+\overline{Y}_3\overline{W}_2)\right.\\ &\left.-\frac{|W_1|^2W_1g_1}{9\eta^3 a(1+a\eta)} - \frac{(|W_2|^2+|W_3|^2)W_1h_1}{\eta^3 a(1+a\eta)}\right\}\end{aligned} \tag{J2}$$

$$\begin{aligned}&= \frac{(1+K')}{b_c}.(\varepsilon b^{(1)}+\varepsilon^2 b^{(2)})W_1\cdot\left\{\varepsilon\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\left(\frac{1}{(1+K')-\eta^2}\right)\cdot\frac{1}{1+a\eta}\right\}\\ &\cdot(1+a\eta)^2 -\\ &\frac{(1+K')}{\{(1+K')-\eta^2\}}\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{2(1-a\eta)}{\eta(1+a\eta)}\left\{\overline{A_2}-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y_2}\right\}\\ &\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)\\ &\cdot\left\{\overline{A_3}-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y_3}\right\}\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)+\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\\ &\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}}\cdot\frac{1}{1+a\eta}(-b^{(1)}Y_1)+\\ &\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{(1+K')2b^{(1)}}{\{(1+K')-\eta^2\}.(1+a\eta)\cdot\eta(1+a\eta)}\Big\{\overline{A_2}\end{aligned}$$

$$
\begin{aligned}
&-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y_2}\Big\}.\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)\\
&\cdot\left\{\overline{A_3}-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y_3}\right\}\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)+\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\\
&\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}}\cdot\frac{1}{1+a\eta}\\
&\Big(-\frac{2(1-a\eta)}{\eta}\overline{Y_2}\left\{\overline{A_3}-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y_3}\right\}\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)\\
&+\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}}\cdot\\
&\cdot\frac{1}{1+a\eta}\left(-\frac{2(1-a\eta)}{\eta}\right)\overline{Y_3}\left\{\overline{A_2}-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y_2}\right\}\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)\\
&-\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\\
&\cdot\frac{(1+K')}{\{(1+K')-\eta^2\}}\cdot\frac{1}{1+a\eta}\left\{\frac{|W_1|^2W_1g_1}{9\eta^3a(1+a\eta)}-\frac{(|W_2|^2+|W_3|^2)W_1h_1}{\eta^3a(1+a\eta)}\right\}\\
&=\left(1+K'\right)\cdot\frac{b-b_c}{b_c}\cdot W_1\left\{-\frac{\varepsilon a}{\eta}\cdot\frac{1}{(1+K')-\eta^2}\right\}-\frac{\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}}\\
&\cdot\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{2(1-a\eta)}{\eta(1+a\eta)}\\
&\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)^2\cdot\left\{\overline{A}_2-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\right\}\cdot\left\{\overline{A}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_3\right\}
\end{aligned}
\tag{J3}
$$

$$
\begin{aligned}
&-\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right).\\
&\frac{\left(1+K'\right)b^{(1)}Y_1}{\left\{(1+K')-\eta^2\right\}}\frac{1}{1+a\eta}+\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{\left(1+K'\right)2b^{(1)}}{\left\{(1+K')-\eta^2\right\}(1+a\eta)n(1+a\eta)}\\
&\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)^2\cdot\{\\
&\overline{A}_2-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\Big\}\cdot\left\{\overline{A}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_3\right\}\\
&+\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}}\\
&\cdot\frac{1}{1+a\eta}\left(\frac{-2(1-a\eta)}{\eta}\right)\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)\overline{Y}_2\left\{\overline{A}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_3\right\}\\
&+\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right).\\
&\frac{\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}}\frac{1}{1+a\eta}\left(\frac{-2(1-a\eta)}{\eta}\right)\cdot\left(-\frac{\eta(1+a\eta)}{a\varepsilon}\right)
\end{aligned}
$$

$$\overline{Y}_3\left\{\overline{A}_2-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\right\}-\varepsilon^3$$

$$\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{\left(1+K'\right)}{\left\{\left(1+K'\right)-\eta^2\right\}}\frac{1}{1+a\eta}\left\{\frac{|W_1|^2\,W_1g_1}{9\eta^3a(1+a\eta)}+\frac{\left(|W_2|^2+|W_3|^2\right)W_1h_1}{\eta^3a(1+a\eta)}\right\} \tag{J4}$$

$$=\left(1+K'\right)\mu\left\{A_1-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right\}\cdot\left(\frac{\eta(1+a\eta)}{-a\varepsilon}\right)\cdot\left\{-\frac{\varepsilon a}{\eta\left(\left(1+K'\right)-\eta^2\right)}\right\}-$$

$$\frac{\left(1+K'\right)}{\left\{\left(1+K'\right)-\eta^2\right\}}\cdot\frac{-2a(1-a\eta)}{a^2}\left\{\overline{A}_2\overline{A}_3\right.$$

$$-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{A}_2\overline{Y}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\overline{A}_3+\varepsilon^4$$

$$\left.\left(-\frac{a}{\eta(1+a\eta)}\right)^2\overline{Y}_2\overline{Y}_3\right\}-\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{\left(1+K'\right)b^{(1)}Y_1}{\left\{\left(1+K'\right)-\eta^2\right\}}\frac{1}{1+a\eta}$$

$$+\varepsilon\frac{-a}{a^2}\cdot\frac{\left(1+K'\right)2b^{(1)}}{\left\{\left(1+K'\right)-\eta^2\right\}}$$

$$\cdot\frac{1}{1+a\eta}\left\{\overline{A}_2\overline{A}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{A}_2\overline{Y}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\overline{A}_3\right.$$

$$\left.+\varepsilon^4\left(-\frac{a}{\eta(1+a\eta)}\right)^2\overline{Y}_2\overline{Y}_3\right\}+\varepsilon^2$$

$$\frac{\left(1+K'\right)}{\left\{\left(1+K'\right)-\eta^2\right\}}\cdot\frac{1}{1+a\eta}\left(\frac{-2(1-a\eta)}{\eta}\right)\overline{Y}_2\left\{\overline{A}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_3\right\}$$

$$+\varepsilon^2\frac{\left(1+K'\right)}{\left\{\left(1+K'\right)-\eta^2\right\}}\cdot\frac{1}{1+a\eta}$$

$$\cdot\left(\frac{-2(1-a\eta)}{\eta}\right)\overline{Y}_3\left\{\overline{A}_2-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\right\}-\varepsilon^3\left(-\frac{a}{\eta(1+a\eta)}\right)$$

$$\cdot\frac{\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}}\frac{1}{1+a\eta}$$

$$\left\{\overline{A}_1-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_1\right\}\cdot\left\{A_1-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right\}^2\cdot\left(\frac{\eta(1+a\eta)}{-a\varepsilon}\right)^3$$

$$\cdot\frac{g_1}{9\eta^3a(1+a\eta)}-\varepsilon^3$$

$$\left(-\frac{a}{\eta(1+a\eta)}\right)\cdot\frac{\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}}\cdot\frac{1}{1+a\eta}\frac{h_1}{\eta^3a(1+a\eta)}\left[\left\{A_2-\varepsilon^2(-\frac{a}{\eta(1+a\eta)})Y_2\right\}\cdot\left\{\overline{A}_2\right.\right.$$

$$\left.-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_2\right\}\cdot\left\{A_1-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right\}\cdot\left(\frac{\eta(1+a\eta)}{-a\varepsilon}\right)^3$$

$$+\left\{A_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_3\right\}\cdot$$

$$\left.\left\{\overline{A}_3-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)\overline{Y}_3\right\}\cdot\left\{A_1-\varepsilon^2\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right\}\cdot\left(\frac{\eta(1+a\eta)}{-a\varepsilon}\right)^3\right] \tag{J5}$$

$$
\begin{aligned}
&= \mu \left\{ A_1 + \frac{\varepsilon^2 a Y_1}{\eta(1+a\eta)} \right\} \cdot \frac{\left(1+K'\right)(1+a\eta)}{\left\{(1+K')-\eta^2\right\}} + \frac{2(1-a\eta)}{a} \cdot \frac{\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}} \\
&\quad \left\{ \overline{A}_2\overline{A}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{A}_2\overline{Y}_3 \right.\\
&\left. + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_2\overline{A}_3 + \frac{\varepsilon^4 a^2}{\eta^2(1+a\eta)^2} \overline{Y}_2\overline{Y}_3 \right\} - \varepsilon^3 \left(\frac{-a}{\eta(1+a\eta)^2} \right) \cdot \frac{\left(1+K'\right) b^{(1)} Y_1}{\left\{(1+K')-\eta^2\right\}} \\
&\quad - \frac{\varepsilon}{a} \cdot \frac{\left(1+K'\right) 2b^{(1)}}{\left\{(1+K')-\eta^2\right\}} \\
&\cdot \frac{1}{1+a\eta} \left\{ \overline{A}_2\overline{A}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{A}_2\overline{Y}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_2\overline{A}_3 + \frac{\varepsilon^4 a^2}{\eta^2(1+a\eta)^2} \overline{Y}_2\overline{Y}_3 \right\} - \\
&\frac{2\varepsilon^2 \left(1+K'\right)(1-a\eta)}{\eta\left\{(1+K')-\eta^2\right\}(1+a\eta)} \overline{Y}_2 \left\{ \overline{A}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_3 \right\} - \frac{2\varepsilon^2(1-a\eta)\left(1+K'\right)}{\eta\left\{(1+K')-\eta^2\right\}(1+a\eta)} \overline{Y}_3 \left\{ \overline{A}_2 \right.\\
&\left. + \frac{a\varepsilon^2}{\eta(1+a\eta)} \overline{Y}_2 \right\} + \frac{\varepsilon^3 a \left(1+K'\right)}{\eta(1+a\eta)^2} \cdot \frac{1}{\left\{(1+K')-\eta^2\right\}} \cdot \frac{\eta^3(1+a\eta)^3}{-a^3\varepsilon^3} \cdot \frac{g_1}{9\eta^3 a(1+a\eta)} \left\{ \overline{A}_1 \right.\\
&\left. + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_1 \right\} \cdot \left\{ A_1^2 + \varepsilon^4 \left(-\frac{a}{\eta(1+a\eta)} \right)^2 Y_1^2 - 2A_1\varepsilon^2 \left(-\frac{a}{\eta(1+a\eta)} \right) Y_1 \right\} \\
&\quad + \frac{a\varepsilon^3\left(1+K'\right)}{\eta(1+a\eta)^2}. \\
&\frac{1}{\left\{(1+K')-\eta^2\right\}} \cdot \frac{\eta^3(1+a\eta)^3}{-a^3\varepsilon^3} \cdot \frac{h_1}{\eta^3 a(1+a\eta)} \left[\{A_2 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} Y_2\} \cdot \{\overline{A}_2 \right.\\
&\quad + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_2\} \cdot \left\{ A_1 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} Y_1 \right\} \\
&\left. + \left\{ A_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} Y_3 \right\} \cdot \left\{ \overline{A}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_3 \right\} \cdot \left\{ A_1 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} Y_1 \right\} \right] \qquad \text{(J6)} \\
&= \frac{\mu A_1(1+a\eta)\left(1+K'\right)}{\left\{(1+K')-\eta^2\right\}} + \frac{\mu\varepsilon^2 a Y_1\left(1+K'\right)}{\eta\left\{(1+K')-\eta^2\right\}} + \frac{2(1-a\eta)\left(1+K'\right)}{a\left\{(1+K')-\eta^2\right\}} \\
&\quad \left\{ \overline{A}_2\overline{A}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{A}_2\overline{Y}_3 \right.\\
&\left. + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_2\overline{A}_3 + \frac{\varepsilon^4 a^2}{\eta^2(1+a\eta)^2} \overline{Y}_2\overline{Y}_3 \right\} + \frac{\varepsilon^3 a\left(1+K'\right) b^{(1)} Y_1}{\eta(1+a\eta)^2\left\{(1+K')-\eta^2\right\}} \\
&\quad - \frac{2\varepsilon b^{(1)}\left(1+K'\right)}{a(1+a\eta)\left\{(1+K')-\eta^2\right\}}. \\
&\left\{ \overline{A}_2\overline{A}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{A}_2\overline{Y}_3 + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_2\overline{A}_3 \right\} - \frac{2\varepsilon^2\left(1+K'\right)(1-a\eta)}{\eta\left\{(1+K')-\eta^2\right\}(1+a\eta)} \overline{Y}_2 \left\{ \overline{A}_3 \right.\\
&\left. + \frac{\varepsilon^2 a}{\eta(1+a\eta)} \overline{Y}_3 \right\} - \frac{2\varepsilon^2(1-a\eta)\left(1+K'\right)}{\eta\left\{(1+K')-\eta^2\right\}(1+a\eta)} \overline{Y}_3 \left\{ \overline{A}_2 + \frac{a\varepsilon^2}{\eta(1+a\eta)} \overline{Y}_2 \right\} \\
&\qquad\qquad - \frac{g_1\left(1+K'\right)}{9\eta a^3\left\{(1+K')-\eta^2\right\}}.
\end{aligned}
$$

$$\left\{\overline{A}_1 A_1^2+\varepsilon^4\overline{A}_1\left(-\frac{a}{\eta(1+a\eta)}\right)^2 Y_1^2-2\varepsilon^2\overline{A}_1 A_1\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1+\frac{\varepsilon^2 a\overline{Y}_1 A_1^2}{\eta(1+a\eta)}\right.$$

$$+\varepsilon^6\frac{a}{\eta(1+a\eta)}\overline{Y}_1.$$

$$\left.\left(-\frac{a}{\eta(1+a\eta)}\right)^2 Y_1^2-\frac{\varepsilon^4 a\overline{Y}_1}{\eta(1+a\eta)}2A_1\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1\right\}-\frac{h_1\left(1+K'\right)}{\eta a^3\left\{(1+K')-\eta^2\right\}}\left[\left\{A_2\overline{A}_2\right.\right.$$

$$\left.+A_2\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_2+\frac{\varepsilon^2 a}{\eta(1+a\eta)}Y_2\overline{A}_2+\frac{\varepsilon^4 a^2 Y_2\overline{Y}_2}{\eta^2(1+a\eta)^2}\right\}\left(A_1+\frac{\varepsilon^2 a}{\eta(1+a\eta)}Y_1\right)+\left\{A_3\overline{A}_3\right.$$

$$\left.\left.+A_3\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_3+\frac{\varepsilon^2 a}{\eta(1+a\eta)}Y_3\overline{A}_3+\frac{\varepsilon^4 a^2}{\eta^2(1+a\eta)^2}Y_3\overline{Y}_3\right\}\left(A_1.+\frac{\varepsilon^2 a}{\eta(1+a\eta)}Y_1\right)\right] \tag{J7}$$

Therefore,

$$\left\{\frac{\left(1+K'\right)-\eta^2}{(1+a\eta)\left(1+K'\right)}\right\}\frac{\partial A_1}{\partial\tau}=\mu A_1+\frac{\mu\varepsilon^2 a Y_1}{\eta(1+a\eta)}+\frac{2(1-a\eta)}{a(1+a\eta)}\left\{\overline{A}_2\overline{A}_3+\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{A}_2\overline{Y}_3\right.$$

$$\left.+\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_2\overline{A}_3\right\}+\frac{\varepsilon^3 ab^{(1)}Y_1}{\eta(1+a\eta)^3}-\frac{2\varepsilon b^{(1)}}{a(1+a\eta)^2}\left\{\overline{A}_2\overline{A}_3\right.$$

$$\left.+\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{A}_2\overline{Y}_3+\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_2\overline{A}_3\right\}-$$

$$\frac{2\varepsilon^2(1-a\eta)\overline{Y}_2}{\eta(1+a\eta)^2}\left\{\overline{A}_3+\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_3\right\}-\frac{2\varepsilon^2(1-a\eta)\overline{Y}_3}{\eta(1+a\eta)^2}.$$

$$\left\{\overline{A}_2+\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_2\right\}-\frac{g_1}{9\eta a^3(1+a\eta)}$$

$$\left\{\overline{A}_1 A_1^2-2\varepsilon^2\overline{A}_1 A_1\left(-\frac{a}{\eta(1+a\eta)}\right)Y_1+\frac{\varepsilon^2 a\overline{Y}_1 A_1^2}{\eta(1+a\eta)}\right\}-\frac{h_1}{\eta a^3(1+a\eta)}.$$

$$\left[\left\{A_1 A_2\overline{A}_2+A_1 A_2\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_2+\right.\right.$$

$$\left.\frac{A_1\varepsilon^2 a Y_2\overline{A}_2}{\eta(1+a\eta)}+\frac{\varepsilon^2 a Y_1 A_2\overline{A}_2}{\eta(1+a\eta)}+\frac{\varepsilon^2 a Y_1}{\eta(1+a\eta)}\cdot\frac{A_2\varepsilon^2 a\overline{Y}_2}{\eta(1+a\eta)}+\frac{\varepsilon^2 a Y_1}{\eta(1+a\eta)}\cdot\frac{\varepsilon^2 a Y_2\overline{A}_2}{\eta(1+a\eta)}\right\}+$$

$$\left\{A_1 A_3\overline{A}_3+A_1 A_3\frac{\varepsilon^2 a}{\eta(1+a\eta)}\overline{Y}_3+\frac{A_1\varepsilon^2 a Y_3\overline{A}_3}{\eta(1+a\eta)}\right.$$

$$\left.\left.+\frac{\varepsilon^2 a Y_1 A_3\overline{A}_3}{\eta(1+a\eta)}+\frac{\varepsilon^2 a Y_1}{\eta(1+a\eta)}\cdot\frac{A_3\varepsilon^2 a\overline{Y}_3}{\eta(1+a\eta)}+\frac{\varepsilon^2 a Y_1}{\eta(1+a\eta)}.\frac{\varepsilon^2 a Y_3\overline{A}_3}{\eta(1+a\eta)}\right\}\right] \tag{J8}$$

$$=\left[\mu A_1+\frac{2(1-a\eta)}{a(1+a\eta)}\overline{A}_2\overline{A}_3-\frac{g_1}{9\eta a^3(1+a\eta)}\left|A_1\right|^2 A_1\right.$$

$$\left.-\frac{h_1}{\eta a^3(1+a\eta)}\left\{\left|A_2\right|^2 A_1+\left|A_3\right|^2 A_1\right\}\right]+$$

$$
\begin{aligned}
&+\frac{\varepsilon^2 a Y_1}{b_c\eta(1+a\eta)}\left\{-\left(\varepsilon b^{(1)}+\varepsilon^2 b^{(2)}\right)\right\}+\frac{2\varepsilon^2(1-a\eta)\overline{A}_2\overline{Y}_3}{\eta(1+a\eta)^2}+\frac{2\varepsilon^2(1-a\eta)\overline{Y}_2\overline{A}_3}{\eta(1+a\eta)^2}\\
&+\frac{\varepsilon^3 a b^{(1)} Y_1}{b_c\eta(1+a\eta)}\\
&-\frac{2\varepsilon b^{(1)}\overline{A}_2\overline{A}_3}{a(1+a\eta)^2}-\frac{2\varepsilon^3 b^{(1)}\overline{A}_2\overline{Y}_3}{\eta(1+a\eta)^3}-\frac{2\varepsilon^3 b^{(1)}\overline{Y}_2\overline{A}_3}{\eta(1+a\eta)^3}-\frac{2\varepsilon^2(1-a\eta)\overline{Y}_2\overline{A}_3}{\eta(1+a\eta)^2}-\frac{2\varepsilon^4 a(1-a\eta)\overline{Y}_2\overline{Y}_3}{\eta^2(1+a\eta)^3}\\
&-\frac{2\varepsilon^2(1-a\eta)\overline{Y}_3\overline{A}_2}{\eta(1+a\eta)^2}-\frac{2a\varepsilon^4(1-a\eta)\overline{Y}_2\overline{Y}_3}{\eta^2(1+a\eta)^3}-\frac{g_1}{9\eta a^3(1+a\eta)}\left\{\frac{2\varepsilon^2 a\overline{A}_1 A_1 Y_1}{\eta(1+a\eta)}+\frac{\varepsilon^2 a\overline{Y}_1 A_1{}^2}{\eta(1+a\eta)}\right\}\\
&-\frac{h_1}{a^3\eta(1+a\eta)}\left[\left\{\frac{\varepsilon^2 a}{\eta(1+a\eta)}A_1 A_2\overline{Y}_2+\frac{\varepsilon^2 a A_1 Y_2\overline{A}_2}{\eta(1+a\eta)}+\frac{\varepsilon^2 a Y_1 A_2\overline{A}_2}{\eta(1+a\eta)}\right\}\right.\\
&+\left\{\frac{\varepsilon^2 a A_1 A_3\overline{Y}_3}{\eta(1+a\eta)}+\frac{\varepsilon^2 a A_1 Y_3\overline{A}_3}{\eta(1+a\eta)}\right.\\
&\left.\left.+\frac{\varepsilon^2 a Y_1 A_3\overline{A}_3}{\eta(1+a\eta)}\right\}\right]
\end{aligned}
\tag{J9}
$$

Therefore,

$$
\begin{aligned}
&\tau_0\left(\frac{\partial A_1}{\partial\tau}\right)=\left\{\mu A_1+\overline{v}\overline{A}_2\overline{A}_3-g\left|A_1\right|^2 A_1-h\left(\left|A_2\right|^2+\left|A_3\right|^2\right)A_1\right\}-\frac{2\varepsilon b^{(1)}\overline{A}_2\overline{A}_3}{a(1+a\eta)^2}\\
&-\frac{2\varepsilon^3 b^{(1)}\overline{Y}_3}{\eta(1+a\eta)^3}\varepsilon\left(-\frac{a}{\eta(1+a\eta)}\right)\left(\overline{W}_2+\varepsilon\overline{Y}_2\right)-\frac{2\varepsilon^3 b^{(1)}\overline{Y}_2}{\eta(1+a\eta)^3}\varepsilon\left(-\frac{a}{\eta(1+a\eta)}\right)\left(\overline{W}_3+\varepsilon\overline{Y}_3\right)\\
&-\frac{g_1\varepsilon^2}{9\eta^2 a^2(1+a\eta)^2}\left(2\overline{A}_1 A_1 Y_1+\overline{Y}_1 A_1{}^2\right)-\frac{h_1\varepsilon^2}{\eta^2 a^2(1+a\eta)^2}\\
&\cdot\left[\left\{A_1 A_2\overline{Y}_2+A_1 Y_2\overline{A}_2+Y_1 A_2\overline{A}_2\right\}\right.\\
&\left.+\left\{A_1 A_3\overline{Y}_3+A_1 Y_3\overline{A}_3+Y_1 A_3\overline{A}_3\right\}\right]
\end{aligned}
\tag{J10}
$$

where,

$$
\begin{aligned}
&\tau_0=\frac{\left(1+K'\right)-\eta^2}{(1+K')\cdot(1+a\eta)};\ \mu=\frac{b-b_c}{b_c};\ \overline{v}=\frac{2(1-a\eta)}{a(1+a\eta)};\ g=\frac{g_1}{9a^3\eta(1+a\eta)}\\
&=\frac{38a\eta+5(a\eta)^2-8-8(a\eta)^3}{9a^3\eta(1+a\eta)};\\
&h=\frac{h_1}{a^3\eta(1+a\eta)}=\frac{5a\eta+7(a\eta)^2-3-3(a\eta)^3}{a^3\eta(1+a\eta)}
\end{aligned}
\tag{J11/6.5}
$$

$$
\begin{aligned}
&\therefore\tau_0\left(\frac{\partial A_1}{\partial\tau}\right)-\left\{\mu A_1+\overline{v}\overline{A}_2\overline{A}_3-g\left|A_1\right|^2 A_1-h\left(\left|A_2\right|^2+\left|A_3\right|^2\right)A_1\right\}\\
&=-\frac{2\varepsilon b^{(1)}\varepsilon^2}{a(1+a\eta)^2}\left(-\frac{a}{\eta(1+a\eta)}\right)^2\cdot\left(\overline{W}_2+\varepsilon\overline{Y}_2\right)\cdot\left(\overline{W}_3+\varepsilon\overline{Y}_3\right)\\
&\quad-\frac{g_1\varepsilon^2\varepsilon}{9\eta^2 a^2(1+a\eta)^2}\left(-\frac{a}{\eta(1+a\eta)}\right)
\end{aligned}
$$

$$\cdot (W_1 + \varepsilon Y_1)\left(2\overline{A}_1 Y_1 + \overline{Y}_1 A_1\right) - \frac{h_1 \varepsilon^2 \varepsilon^2}{\eta^2 a^2 (1 + a\eta)^2}$$
$$\left[\left(-\frac{a}{\eta(1 + a\eta)}\right)^2 \left\{(W_1 + \varepsilon Y_1)(W_2 + \varepsilon Y_2)\overline{Y}_2\right.\right.$$
$$+ (W_1 + \varepsilon Y_1) Y_2 \left(\overline{W}_2 + \varepsilon \overline{Y}_2\right) + Y_1 (W_2 + \varepsilon Y_2)\left(\overline{W}_2 + \varepsilon \overline{Y}_2\right)\Big\}$$
$$+ \Big\{(W_1 + \varepsilon Y_1) \cdot (W_3 + \varepsilon Y_3)\overline{Y}_3 + (W_1 + \varepsilon Y_1)$$
$$\left.\left. Y_3\left(\overline{W}_3 + \varepsilon \overline{Y}_3\right) + Y_1 (W_3 + \varepsilon Y_3) \cdot \left(\overline{W}_3 + \varepsilon \overline{Y}_3\right)\right\}\right] = 0 \qquad \text{(J12)}$$

Therefore,

$$\tau_0 \left(\frac{\partial A_1}{\partial \tau}\right) = \mu A_1 + \overline{v}\overline{A}_2\overline{A}_3 - g\left|A_1\right|^2 A_1 - h\left(\left|A_2\right|^2 + \left|A_3\right|^2\right) A_1 \qquad \text{(J13/6.4)}$$

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