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ORIGINAL PAPER

Homotopy perturbation method for Fangzhu oscillator

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Abstract
An accurate frequency-amplitude relationship is very needed to elucidate the properties of the oldest device of Fangzhu for collecting water from the air. The Fangzhu oscillator was derived and solved approximately (He et al. in Math Methods Appl Sci, 2020, https://doi.org/10.1002/mma.6384), here we show that the singular Duffing-like oscillator can be more effectively solved by the homotopy perturbation method and a criterion is obtained for the existence of a periodic solution for the singular differential equation. The results obtained in this paper are helpful for the optimal design of the Fangzhu device.

Keywords Homotopy perturbation method · Frequency expansion method · Periodic solution · Fangzhu oscillator

Mathematics Subject Classification 34-K13 · 34-K27 · 34-L30 · 37-J25 · 37-K45 · 41-A10 · 42-A15 · 42-B20

1 Introduction

The Fangzhu device was considered as the oldest nanotechnology used in ancient China for collecting water from the air, its nano-scale surface morphology plays an important role in water collection efficiency, the super-hydrophobic surface is designed to attract water molecules from the air, and its super-hydrophilic partner is used to deliver the

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attracted water molecules to the water collector. Its mechanism was fully elucidated by a singular differential equation by He et al. [1]:

$$\ddot{y} + \omega_0^2 y + Q y^{-\alpha} = g(t). \tag{1}$$

where y is the distance of the attracted molecule from its equilibrium position. The understanding of each parameter is given in Ref. [1]. A low frequency is beneficial for the attracted molecule to be transmitted from the super-hydrophobic surface of the super-hydrophilic surface, so an accurate estimation of its solution is very needed.

The nano-scale surface has a high surface energy or geometric potential, different geometric patterns result in different wetting property of the surface [2–11].

Equation (1) is a Duffing-like oscillator, and we call it as Fangzhu oscillator, the periodic property or the instability property of the absorbed water on the Fangzhu's surface plays an important role [1]. There are many analytical methods to solve such an equation [12–22], and this paper adopts the homotopy perturbation method [23–28] to reveal the periodic property of Eq. (1).

The approximate analytical solutions derived by HPM for the enzyme kinetic model of the double nucleic acid strand synthesis during the PCR cycle are presented by Fedorov et al. [29]. The analytic approximate solution using the Homotopy Perturbation Method is used by Bayón et al. [30] to solve the nonlinear differential equations that appear in an irreversible linear pathway with the enzyme kinetics. Recently, a modification to the Homotopy perturbation method has been demonstrated by Shen and El-Dib [31] and El-Dib and Elgazery [32]. This scheme allows us to convert the original equation, to an easy alternative one. The approach is concerned to replace the original auxiliary linear operator with a new linear auxiliary one.

2 Homotopy construction to expand $(\sec x)^{\alpha}$ in the periodic form

Since there is no exact expansion when the function has a negative or fractional power, therefore, we need to derive an approximate periodic expansion of the function $(\sec x)^{\alpha}$ where α it may be a positive integer or perhaps a fraction.

Here, we try to use the concept of the homotopy perturbation method [33] to obtain an approximate expansion. To accomplish this aim, it is convenient to rewrite the function $(\sec x)^{\alpha}$ in the form

$$(\sec x)^{\alpha} = (\cos x)^{-\alpha} = \lim_{\rho \to 1} [1 + \rho(\cos x - 1)]^{-\alpha}, \quad \rho \in [0, 1] \tag{2}$$

where ρ any parameter and may be taken as a small parameter defined by $\rho \in [0, 1]$. Clearly, as $\rho \to 1$, the original function is found. The following binomial expansion will be used:

$$(1 + \rho u)^{-\alpha} = 1 - \alpha \rho u + \frac{\alpha(\alpha + 1)}{2!} \rho^2 u^2 - \frac{\alpha(\alpha + 1)(\alpha + 2)}{3!} \rho^3 u^3 + \cdots. \tag{3}$$

Expanding the formula (2) about the parameter ρ, and using (3), we obtain

$$
\begin{aligned}
(\cos x)^{-\alpha} &= \lim_{\rho\to 1}[1+\rho(\cos x-1)]^{-\alpha} = \lim_{\rho\to 1}[(1-\rho)+\rho\cos x]^{-\alpha} = \lim_{\rho\to 1}(1-\rho)^{-\alpha}\left[1+\left(\frac{\rho\cos x}{1-\rho}\right)\right]^{-\alpha} \\
&= \lim_{\rho\to 1}(1-\rho)^{-\alpha}\left[1-\alpha\rho(1-\rho)^{-1}\cos x+\frac{\alpha(\alpha+1)}{2}\rho^2(1-\rho)^{-2}\cos^2 x-\frac{\alpha(\alpha+1)(\alpha+2)}{6}\rho^3(1-\rho)^{-3}\cos^3 x-\cdots\right]
\end{aligned}
\tag{4}
$$

We, again, apply the binomial expansion to expand $(1-\rho)^{-n-\alpha}$; $n = 1, 2, 3, \ldots$. Then (4) becomes

$$
\begin{aligned}
(\cos x)^{-\alpha} &= \lim_{\rho\to 1}\left[1+\alpha\rho+\tfrac{1}{2}\alpha(\alpha+1)\rho^2+\tfrac{1}{6}\left(\alpha^3+3\alpha^2+2\alpha\right)\rho^3+\cdots\right] \\
&\quad-\lim_{\rho\to 1}\alpha\rho\left[1+(1+\alpha)\rho+\tfrac{1}{2}(2+3\alpha+\alpha^2)\rho^2+\left(1+\frac{11}{6}\alpha+\alpha^2+\frac{1}{6}\alpha^3\right)\rho^3+\cdots\right]\cos x \\
&\quad+\lim_{\rho\to 1}\frac{\alpha(\alpha+1)}{2}\rho^2\left[1+(2+\alpha)\rho+\tfrac{1}{2}(6+5\alpha+\alpha^2)\rho^2+\cdots\right]\cos^2 x \\
&\quad-\lim_{\rho\to 1}\frac{\alpha(\alpha+1)(\alpha+2)}{6}\rho^3(1+(3+\alpha)\rho+\cdots)\cos^3 x+\cdots
\end{aligned}
\tag{5}
$$

If we approximate the above series to the cubic order for the parameter ρ and take the limit as $\rho \to 1$, Eq. (5) becomes

$$
\begin{aligned}
(\cos x)^{-\alpha} &= \left(1+\tfrac{11}{6}\alpha+\alpha^2+\tfrac{1}{6}\alpha^3\right)-\tfrac{1}{2}\alpha\left(9+5\alpha+\alpha^2\right)\cos x \\
&\quad+\tfrac{1}{2}\alpha(\alpha+1)(3+\alpha)\cos^2 x-\tfrac{1}{6}\alpha(\alpha+1)(\alpha+2)(4+\alpha)\cos^3 x+\cdots
\end{aligned}
\tag{6}
$$

The cubic approximate expansion of $(\cos x)^{-\alpha}$ can be rewritten in the form

$$
\begin{aligned}
(\cos x)^{-\alpha} &= \left(1+\tfrac{31}{12}\alpha+2\alpha^2+\tfrac{5}{12}\alpha^3\right)-\tfrac{1}{2}\alpha\left(9+4\alpha-\tfrac{1}{8}\alpha^2(\alpha+1)(\alpha+6)\right)\cos x \\
&\quad+\tfrac{1}{4}\alpha(\alpha+1)(3+\alpha)\cos 2x-\frac{1}{24}\alpha(\alpha+1)(\alpha+2)(4+\alpha)\cos 3x+\cdots
\end{aligned}
\tag{7}
$$

Accordingly, we can write the periodic form of the function $\sec x$ as

$$
\sec x \cong 6-\frac{45}{8}\cos x+2\cos 2x-\frac{5}{4}\cos 3x+\cdots \tag{8}
$$

3 The solution of Eq. (1) by the homotopy perturbation method

As usual, we need to replace Eq. (1) by its corresponding homotopy equation and solve it. Thus we have

$$
\ddot{y}+\omega_0^2 y+\rho(Qy^{-\alpha}-g(t)) = 0;\ \rho\in[0,1] \tag{9}
$$

Supposing that the solution of Eq. (9) and ω_0^2 can be expressed in the form

$$y(t) = y_0 + \rho y_1 + \rho^2 y_2 + \cdots \tag{10}$$

$$\omega^2 = \omega_0^2 + \rho\omega_1 + \rho^2\omega_2 + \cdots \tag{11}$$

If we focus on the case of $g(t) = 0$, then one can substitute (10) and (11) into Eq. (9) and equating coefficients of like powers ρ, to become zero yields the following equations:

$$\ddot{y}_0 + \omega^2 y_0 = 0 \tag{12}$$

$$\ddot{y}_1 + \omega^2 y_1 = \omega_1 y_0 - Q y_0^{-\alpha}. \tag{13}$$

Solving Eq. (12) results in

$$y_0(t) = A\cos(\omega t + \theta), \tag{14}$$

where A and θ are constants determined by initial conditions. Employing (14) in Eq. (13), using (7) and avoiding the presence of a secular term needs:

$$\omega_1 = -\tfrac{1}{2}\alpha\Big(9 + 4\alpha - \tfrac{1}{8}\alpha^2(\alpha+1)(\alpha+6)\Big)QA^{-1-\alpha}. \tag{15}$$

The approximate frequency-amplitude relation is

$$\omega^2 = \omega_0^2 - \tfrac{1}{2}\alpha\Big(9 + 4\alpha - \tfrac{1}{8}\alpha^2(\alpha+1)(\alpha+6)\Big)QA^{-1-\alpha}. \tag{16}$$

Without a secular term, solution of Eq. (13) is

$$\begin{aligned} y_1 = -QA^{-\alpha}\Bigg[&\frac{1}{\omega^2}\Big(1 + \tfrac{31}{12}\alpha + 2\alpha^2 + \tfrac{5}{12}\alpha^3\Big) - \frac{\alpha(\alpha+1)(3+\alpha)}{12\omega^2}\cos 2(\omega t + \theta) \\ &+ \frac{\alpha(\alpha+1)(\alpha+2)(4+\alpha)}{192\omega^2}\cos 3(\omega t + \theta)\Bigg] \end{aligned} \tag{17}$$

Its first-order approximation is sufficient, and then we have:

$$\begin{aligned} y(t) = &-\frac{Q}{A^\alpha\omega^2}\Big(1 + \tfrac{31}{12}\alpha + 2\alpha^2 + \tfrac{5}{12}\alpha^3\Big) + A\cos(\omega t + \theta) + \frac{\alpha(\alpha+1)(3+\alpha)Q}{12\omega^2 A^\alpha}\cos(2\omega t + 2\theta) \\ &- \frac{\alpha(\alpha+1)(\alpha+2)(4+\alpha)Q}{192\omega^2 A^\alpha}\cos(3\omega t + 3\theta) + \cdots \end{aligned} \tag{18}$$

It is worthwhile to note the oscillatory solution is available when

$$\omega_0^2 > \tfrac{1}{2}\alpha\Big(9 + 4\alpha - \tfrac{1}{8}\alpha^2(\alpha+1)(\alpha+6)\Big)QA^{-1-\alpha}. \tag{19}$$

4 To estimate the most suitable α for the periodic solution

In this section, one can derive an alternative form of Eq. (1) free of $y^{-\alpha}$, By applying the modified homotopy perturbation method [31, 32]. This scheme allows us to convert the original equation, to an easy alternative one. The approach is concerned to replace the original auxiliary linear operator with a new linear auxiliary one. This can be accomplished by operating to Eq. (1) by D^2 then adding a new auxiliary part $\omega^2(D^2+\omega_0^2)y$ to both sides, yields

$$\left(D^2+\omega^2\right)\left(D^2+\omega_0^2\right)y=\omega^2\left(D^2+\omega_0^2\right)y-\alpha Q\left((\alpha+1)\frac{\dot{y}^2}{y^2}-\frac{\ddot{y}}{y}\right)y^{-\alpha}. \tag{20}$$

Replacing the term $Qy^{-\alpha}$ by bits of help of the original Eq. (1), then applying the operator $\left(D^2+\omega_0^2\right)^{-1}$ to both sides of Eq. (20), final, write its corresponding homotopy equation which becomes

$$\left(D^2+\omega^2\right)y=\rho\left[\omega^2 y-\frac{\alpha}{\left(D^2+\omega_0^2\right)}\left((\alpha+1)\frac{\dot{y}^2}{y^2}-\frac{\ddot{y}}{y}\right)\left(\ddot{y}+\omega_0^2 y\right)\right]. \tag{21}$$

Employing the expansion (10) in Eq. (21) and equating the identical power of ρ to zero, we have the zero-order problem as given above in (12), while the first-order problem is derived as

$$\left(D^2+\omega^2\right)y_1=\omega^2 y_0-\frac{\alpha}{\left(D^2+\omega_0^2\right)}\left((\alpha+1)\frac{\dot{y}_0^2}{y_0^2}-\frac{\ddot{y}_0}{y_0}\right)\left(\ddot{y}_0+\omega_0^2 y_0\right). \tag{22}$$

Inserting the zero-order solution (14) into Eq. (22), converted to

$$\left(D^2+\omega^2\right)y_1=\left(1-\alpha^2\right)\omega^2 A\cos(\omega t+\theta)+\frac{\alpha(\alpha+1)A\omega^2\left(\omega^2-\omega_0^2\right)}{\left(D^2+\omega_0^2\right)}\sec(\omega t+\theta). \tag{23}$$

Due to the difficulty in estimating the particular integral of $\sec(\omega t+\theta)$, us may use the approximate expiration of $\sec(\omega t+\theta)$ as given in (8), yields

$$\begin{aligned}\left(D^2+\omega^2\right)y_1=&-\frac{1}{8}(\alpha+1)\left(\alpha-\frac{8}{53}\right)\omega^2 A\cos(\omega t+\theta)+\alpha(\alpha+1)A\omega^2\left(\omega^2-\omega_0^2\right)\\&\times\left(\frac{6}{\omega_0^2}-\frac{2}{\left(4\omega^2-\omega_0^2\right)}\cos(2\omega t+2\theta)+\frac{5}{4\left(9\omega^2-\omega_0^2\right)}\cos(3\omega t+3\theta)+\cdots\right).\end{aligned} \tag{24}$$

Avoid the secular term requires that $\alpha=-1$ or $\alpha=\frac{8}{53}$. It is noted that the first is the trivial value refers to the linear form of Eq. (1), therefore we consider the second value only, to evaluate the nonlinear analysis of Eq. (1). Accordingly, the solution of Eq. (24) has the form

$$y_1 = \frac{488}{2809}A\left(\omega^2 - \omega_0^2\right)\left(\frac{6}{\omega_0^2} + \frac{2}{3\left(4\omega^2 - \omega_0^2\right)}\cos(2\omega t + 2\theta) - \frac{5}{32\left(9\omega^2 - \omega_0^2\right)}\cos(3\omega t + 3\theta) + \cdots\right). \tag{25}$$

Substitute the zero-order solution and the first-order one into expansion (10) to obtain the first-order approximate solution in the form

$$y(t) = A\cos(\omega t + \theta) + \frac{488}{2809}A\left(\omega^2 - \omega_0^2\right) \left(\frac{6}{\omega_0^2} + \frac{2}{3\left(4\omega^2 - \omega_0^2\right)}\cos(2\omega t + 2\theta) - \frac{5}{32\left(9\omega^2 - \omega_0^2\right)}\cos(3\omega t + 3\theta) + \cdots\right). \tag{26}$$

5 The Fangzhu oscillator with an external periodic force

In this section, we investigate the influence of the external periodic force, therefore, assume that the inhomogeneous part in Eq. (1) become a nonzero. For example, let $g(t) = f\cos(\Omega t + \theta)$; f represents the amplitude of the external force and Ω denotes its frequency.

To derive the solution at the harmonic resonance case, we proceeded to operate on Eq. (1) by $D^2\left(D^2 + \omega_0^2\right)^{-1}$, and replacing the function $y^{-\alpha}$ by the help of the original equation resulting

$$D^2 y = \frac{\alpha}{\left(D^2 + \omega_0^2\right)}\left((\alpha + 1)\frac{\dot{y}^2}{y^2} - \frac{\ddot{y}}{y}\right)\left(\ddot{y} + \omega_0^2 y - f\cos\Omega t\right) + \frac{\Omega^2}{\left(\Omega^2 - \omega_0^2\right)}f\cos(\Omega t + \theta). \tag{27}$$

Introducing a new auxiliary parameter $\Omega^2 y$, and construct the corresponding homotopy equation, which has the following configuration:

$$\left(D^2 + \Omega^2\right)y = \rho\left\{\Omega^2 y + \frac{\alpha}{\left(D^2 + \omega_0^2\right)}\left((\alpha + 1)\frac{\dot{y}^2}{y^2} - \frac{\ddot{y}}{y}\right)\left(\ddot{y} + \omega_0^2 y - f\cos(\Omega t + \theta)\right) + \frac{\Omega^2 f}{\left(\Omega^2 - \omega_0^2\right)}\cos(\Omega t + \theta)\right\}. \tag{28}$$

Using the expansion (10), we obtain the unknown function $y_1(t)$ of the harmonic resonance case is governed by

$$\left(D^2 + \Omega^2\right)y_1 = \left(1 - \alpha^2\right)\Omega^2\left(A + \frac{f}{\left(\Omega^2 - \omega_0^2\right)}\right) - \frac{\alpha(\alpha + 1)\Omega^2\left(A\left(\Omega^2 - \omega_0^2\right) + f\right)}{\left(D^2 + \omega_0^2\right)}\sec(\Omega t + \theta), \tag{29}$$

where the solution of the zero-order problem is still given similarly Eq. (14). Using the approximate periodic form of the function $\sec(\Omega t + \theta)$ as defined by (8), then removing the secular terms yields the following condition:

$$(\alpha+1)\left(\alpha-\frac{8}{53}\right)\left(A+\frac{f}{\left(\Omega^2-\omega_0^2\right)}\right)=0. \tag{30}$$

This condition can be read as mentioned before, the most suitable $\alpha=\frac{8}{53}$, and the frequency Ω is related to the linear frequency ω_0 by the relationship

$$\Omega^2=\omega_0^2-\frac{f}{A}. \tag{31}$$

It is worthwhile to note that the detuning parameter σ [34], in this case, is $\sigma=-\frac{f}{A}$. At this end, the solution of Eq. (29) is inserted in the expansion (10), becomes

$$y(t)=A\cos(\Omega t+\theta)-\frac{488}{2809}\left(A\left(\Omega^2-\omega_0^2\right)+f\right)\left(\frac{6}{\omega_0^2}+\frac{2}{3\left(4\Omega^2-\omega_0^2\right)}\cos(2\Omega t+2\theta)\right.$$
$$\left.-\frac{5}{32\left(9\Omega^2-\omega_0^2\right)}\cos(3\Omega t+3\theta)+\cdots\right). \tag{32}$$

6 Conclusion

This short note shows a singular second-order equation might admit a periodic solution, the frequency-amplitude relationship given in Eq. (16) shows the main parameters affecting the frequency property and can be used to design the concave hydrophilic and convex hydrophobic morphologies on Fangzhu's surface. The criterion of the periodic solution given in Eq. (19) is of great importance for the optimal design of the Fangzhu's surface. A lotus-like surface can not meet the criterion of Eq. (19), as a result, the water absorbed on the surface can not be continuously transmitted for collection. This paper shows that the homotopy perturbation is an effective tool for the Fangzhu oscillator, and the results are given in this paper to help rebuild the Fangzhu device, which has been lost for thousands of years.

Compliance with ethical standards

Conflict of interest This work does not have any conflicts of interest.

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