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ORIGINAL PAPER

A variational principle for a thin film equation

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Abstract
Thin film arises in various applications from electrochemistry to nano devices, many mathematical tools were adopted to study the problem, e.g. Lie symmetries and conservation laws, however, the variational approach is rare. This paper shows that the semi-inverse method is an effective approach to establishment of a variational formulation for the thin film equation. A detailed derivation process is given, a special skill for construction of a heuristic trial-functional is elucidated.

Keywords Variational theory · Special function · Trial-functional · Nanoscale adhesion · Coating · Wetting

1 Introduction

Variational principle plays a key role in both numerical and analytical analyses of a practical problem, it suggests an energy conservation for the whole solution domain, and a variational-based numerical algorithm guarantees the energy conservation at each point, while a variational-based analytical solution is an optimal one for a given trial-solution and valid for the whole solution domain.

Recently Recio et al. [1] studied the following equation

$$u_t + (f(u)u_{xxx} + g(u)u_x)_x + h(u) = 0 \tag{1}$$

where f, g and h are functions of u.

Equation (1) can describe a thin film problem, which can be found widely applications in electrochemistry [2], cell culture [3], fiber fabrication [4], nanoscale adhesion [5], coating [6], wetting [7] and micro/nano devices [8]. Many analytical methods and

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numerical methods were applied to study such problems [9–15]. This paper aims at establishing a variational formulation for Eq. (1) by the semi-inverse method [16–27].

2 The semi-inverse method and the variational formulation

The semi-inverse method [16–19] is to establish a variational principle directly from governing equations. In order to effectively use the method, we write Eq. (1) in a conservation form

$$u_t + [(F(u))_{xxx} + (G(u))_x + H(u)]_x = 0 \tag{2}$$

where F, G and H satisfy the following relationships

$$(F(u))_{xxx} = f(u)u_{xxx} \tag{3}$$

$$(G(u))_x = g(u)u_x \tag{4}$$

$$(H(u))_u = h(u) \tag{5}$$

According to Eq. (2), we can introduce a special function φ, satisfying the following relations

$$\frac{\partial \varphi}{\partial x} = u \tag{6}$$

$$\frac{\partial \varphi}{\partial t} = -[(F(u))_{xxx} + (G(u))_x + H(u)] \tag{7}$$

The defined special function (φ) is potential-like function. By the semi-inverse method [16–19], we can establish a trial functional in the form

$$J(u, \varphi) = \iint \{u\varphi_t + [(F(u))_{xxx} + (G(u))_x + H(u)]\varphi_x + \sigma\}dxdt \tag{8}$$

where σ is an unknown function of u and its derivatives. The advantage of the above trial-functional is the stationary condition with respect to φ is Eq. (2), but we can not identify σ, so we modify Eq. (8) in the form

$$J(u, \varphi) = \iint \{au\varphi_t + b\varphi_x\varphi_t + [(F(u))_{xxx} + (G(u))_x + H(u)]\varphi_x + \sigma\}dxdt \tag{9}$$

where a and b are constants to be further determined.

The Euler–Lagrange equations of Eq. (9) are

$$-au_t - 2b\varphi_{xt} - [(F(u))_{xxx} + (G(u))_x + H(u)]_x = 0 \tag{10}$$

$$a\varphi_t - \frac{\partial F}{\partial u}\varphi_{xxxx} - \frac{\partial G}{\partial u}\varphi_{xx} + \frac{\partial H}{\partial u}\varphi_x + \frac{\delta \sigma}{\delta u} = 0 \tag{11}$$

where $\delta\sigma/\delta u$ is the variational derivative. In this paper it can be written in the form

$$\frac{\delta\sigma}{\delta u} = \frac{\partial\sigma}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial\sigma}{\partial u_x}) - \frac{\partial^3}{\partial x^3}(\frac{\partial\sigma}{\partial u_{xxx}}) \tag{12}$$

In view of Eq. (6), we can write Eq. (10) in the form

$$(a+2b)u_t + [(F(u))_{xxx} + (G(u))_x + H(u)]_x = 0 \tag{13}$$

which should be Eq. (2), so we have

$$\mathrm{a} + 2\mathrm{b} = 1 \tag{14}$$

In view of Eqs. (6) and (7), we can write Eq. (11) in the form

$$-a[(F(u))_{xxx} + (G(u))_x + H(u)] - fu_{xxx} - gu_x + hu + \frac{\delta\sigma}{\delta u} = 0 \tag{15}$$

or

$$-a[f(u)u_{xxx} + g(u)u_x + H(u)] - fu_{xxx} - gu_x + hu + \frac{\delta\sigma}{\delta u} = 0 \tag{16}$$

In order to identify σ in Eq. (16), we set

$$\mathrm{a} = -1 \tag{17}$$

Equation (16) becomes

$$\frac{\delta\sigma}{\delta u} = -H(u) - h(u)u \tag{18}$$

From Eq. (18) we can determine σ easily, which satisfies the following relation

$$\frac{\partial\sigma}{\partial u} = -H(u) - h(u)u \tag{19}$$

We, therefore, obtain the following variational formulation

$$J(u,\varphi) = \iint \{-u\varphi_t + \varphi_x\varphi_t + [(F(u))_{xxx} + (G(u))_x + H(u)]\varphi_x + \sigma\}dxdt \tag{20}$$

where σ is defined in Eq. (19).

Remark if a = 1 as suggested in Eq. (8), we have difficulty in identifying σ from Eq. (16).

3 An example

We consider a special case of Eq. (1)

$$u_t + (u_{xxx} + uu_x)_x + u^2 = 0 \tag{21}$$

Hereby $f(u) = 1$, $g(u) = u$,$h(u) = u^2$, $F(u) = u$,$G(u) = \frac{1}{2}u^2$, $H(u) = \frac{1}{3}u^3$, and

$$\frac{\partial \varphi}{\partial x} = u \tag{22}$$

$$\frac{\partial \varphi}{\partial t} = -(u_{xxx} + uu_x + \frac{1}{3}u^3) \tag{23}$$

$$\frac{\partial \sigma}{\partial u} = -H(u) - h(u)u = -\frac{1}{3}u^3 - u^3 = -\frac{4}{3}u^3 \tag{24}$$

From Eq. (24) σ can be identified as

$$\sigma = -\frac{1}{3}u^4 \tag{25}$$

We, therefore, obtain the following variational principle for Eq. (21):

$$J(u, \varphi) = \iint \left\{ -u\varphi_t + \varphi_x\varphi_t + \left[u_{xxx} + \frac{1}{2}(u^2)_x + \frac{1}{3}u^3 \right] \varphi_x - \frac{1}{3}u^4 \right\} dxdt \tag{26}$$

which is subject to the constraint of Eq. (22).

Proof The stationary conditions of Eq. (26) with respect to φ and u are

$$u_t - 2\varphi_{xt} - \left[u_{xxx} + \frac{1}{2}(u^2)_x + \frac{1}{3}u^3 \right]_x = 0 \tag{27}$$

$$-\varphi_t - \varphi_{xxxx} - u\varphi_{xx} + u^2\varphi_x - \frac{4}{3}u^3 = 0 \tag{28}$$

In view of Eq. (22), we find Eqs. (27) and (28) are equivalent to Eq. (21) and Eq. (23), respectively.

4 Discussion and conclusion

A suitable construction of a trial functional is of great importance for the establishment of a variational principle. Equation (8) does not work, because if we set a = 1, Eq. (16) becomes

$$-[f(u)u_{xxx} + g(u)u_x + H(u)] - fu_{xxx} - gu_x + hu + \frac{\delta \sigma}{\delta u} = 0 \tag{29}$$

or

$$-2f(u)u_{xxx} - 2g(u)u_x - H(u) + hu + \frac{\delta\sigma}{\delta u} = 0 \tag{30}$$

In Eq. (30), we have difficulty in identification of σ due to the terms involving u_{xxx} and u_x.

We can easily determine σ for all even-order derivatives from the following equation

$$\frac{\delta\sigma}{\delta u} = m(u) + \alpha u_{xx} + \beta u_{xxxx} + \delta u_{xxxxxx} \tag{31}$$

where α, β, and δ are constants, m is a function of u. From Eq. (31) σ can be determined as

$$\sigma = M(u) - \frac{1}{2}\alpha(u_x)^2 + \frac{1}{2}\beta(u_{xx})^2 - \frac{1}{2}\delta(u_{xxx})^2 \tag{32}$$

where M is defined as

$$\frac{\partial M}{\partial u} = m \tag{33}$$

In this paper, we find that the semi-inverse method provides an effective tool to finding a needed variational principle for a practical problem, the derivation process is explained step by step, so that it can be easily followed. Recently Wang et al. [27] successfully applied the semi-inverse method to fractal calculus [28, 29], and obtained a variational principle for wave traveling in a fractal space.

The variational principle is a foundation of the variational iteration method [30, 31], which is now widely applied in fractional calculus, and the present paper might give a hint for an effective identification of Lagrange multiplier in the fractional variational iteration method [30–36].

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