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ORIGINAL PAPER

A fast-converging iterative scheme for solving a system of Lane–Emden equations arising in catalytic diffusion reactions

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Abstract

In this paper, we present a fast-converging iterative scheme to approximate the solution of a system of Lane–Emden equations arising in catalytic diffusion reactions. In this method, the original system of differential equations subject to a Neumann boundary condition at $X = 0$ and a Dirichlet boundary condition at $X = 1$, is transformed into an equivalent system of Fredholm integral equations. The resulting system of integral equations is then efficiently treated by the optimized homotopy analysis method. A numerical example is provided to verify the effectiveness and accuracy of the method. Results have been compared with those obtained by other existing iterative schemes to show the advantage of the proposed method. It is shown that the residual error in the present method solution is seven orders of magnitude smaller than in Adomian decomposition method and five orders of magnitude smaller than in the modified Adomian decomposition method. The proposed method is very simple and accurate and it converges quickly to the solution of a given problem.

Keywords System of Lane–Emden boundary value problems · Optimized homotopy analysis method · Iterative scheme · Fast-converging scheme · Modified Adomian decomposition method

Mathematics Subject Classification 65L10 · 65L60 · 34B16

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1 Introduction

We consider a system of Lane–Emden singular equations of the form [1]:

$$y''(x) + \frac{2}{x}y'(x) = -f(y(x), z(x)), \tag{1}$$

$$z''(x) + \frac{2}{x}z'(x) = -g(y(x), z(x)), \tag{2}$$

subject to the following boundary conditions:

$$y'(0) = 0, \quad y(1) = \alpha, \quad z'(0) = 0, \quad z(1) = \beta, \tag{3}$$

where α and β are finite real constants. This problem appears in the modelling of several physical phenomena in applied science, such as chemical reactions, population evolution, pattern formation, and so on [2–4]. Equations (1) and (2) have singularity at the origin. The existence of a solution to the general form of the coupled Lane–Emden Eqs. (1)–(2) subjected to the homogeneous Dirichlet boundary condition has been established in [4].

It is worth noting that much work has been done to obtain numerical solution of scalar Lane–Emden singular equation [5–9]. We emphasize that, to our knowledge, very little work has been done on a system of Lane–Emden singular equations. Rach et al. [1] used Adomian decomposition method (ADM) and its modified version, called the modified Adomian decomposition method (MADM), for solving the problem under consideration. These methods produce approximate solutions to the given problem in the form of series, but need a large number iterations to obtain results with a high degree of accuracy.

The main aim of this study is to present a fast-converging and straight forward iterative algorithm to obtain numerical solution of the coupled Lane–Emden equations (CLEE) described by equations (1)–(3). In this method, we first convert the system of differential Eqs. (1)–(2) subject to the boundary conditions (3) into an equivalent system of Fredholm integral equations. Then the resulting system is efficiently tackled by the OHAM. The OHAM is an improved version of the standard homotopy analysis method [10,11] which contains a convergence control parameter h. In the standard HAM, this parameter is computed by means of plotting so-called the h-curve. But, it may be very difficult or impossible to identify the value from the h-curve which ensures the fastest convergence series solution. In the OHAM, the parameter h is computed by minimizing the squared residual error of the governing equation. Unlike the homotopy perturbation method (HPM) [12–15], ADM [1], MADM [1] and variational iteration method (VIM) [16], the OHAM is always guaranteed to converge to the solution of a given problem. A numerical example is presented in order to illustrate the efficiency and accuracy of the proposed technique. The results obtained by the present method are compared with those obtained by other existing methods applied to the CLEE (1)–(3). Comparison shows that the proposed technique has certain advantages over the existing methods.

This paper is organized as follows. In Sect.2, we construct an iterative technique based on an OHAM for approximating the solutions of the coupled Lane–Emden equations (1)–(3). In Sect. 3, we present and analyze numerical results of the problem considered and compare our results with other existing methods. Finally, we summarize the main conclusions of this work in Sect. 4.

2 Derivation of the proposed iterative method

The aim of this section is to construct an iterative algorithm based on an OHAM for solving the system of Eqs. (1)–(2) with boundary conditions (3). To construct the method, we first convert the given system of singular differential equations into an equivalent system of integral equations. To do so, we first set $y(x)$ and $z(x)$ as follows (see [17]):

$$y(x) = y(0) - \int_0^x t\left(1 - \frac{t}{x}\right) f(y(t), z(t))dt, \tag{4}$$

$$z(x) = z(0) - \int_0^x t\left(1 - \frac{t}{x}\right) g(y(t), z(t))dt, \tag{5}$$

where $y(0)$ and $z(0)$ are the unknown constants to be determined. By using the boundary conditions at $x = 1$ from (3) in Eqs. (4)–(5), we obtain

$$\alpha = y(0) - \int_0^1 t\,(1-t)\, f(y(t), z(t))dt, \tag{6}$$

$$\beta = z(0) - \int_0^1 t\,(1-t)\, g(y(t), z(t))dt. \tag{7}$$

The above equations imply that:

$$y(0) = \alpha + \int_0^1 t\,(1-t)\, f(y(t), z(t))dt, \tag{8}$$

$$z(0) = \beta + \int_0^1 t\,(1-t)\, g(y(t), z(t))dt. \tag{9}$$

Substituting the undetermined constants obtained in (8)–(9) into Eqs. (4)–(5), we get the following system of Fredholm–Volterra integral equations:

$$y(x) = \alpha + \int_0^1 t\,(1-t)\, f(y(t),\ z(t))dt - \int_0^x t\left(1 - \frac{t}{x}\right) f(y(t), z(t))dt, \tag{10}$$

$$z(x) = \beta + \int_0^1 t\,(1-t)\, g(y(t),\ z(t))dt - \int_0^x t\left(1 - \frac{t}{x}\right) g(y(t), z(t))dt. \tag{11}$$

Next, the above coupled integral equations are solved by the optimized homotopy analysis method [18,19]. To apply the method, we first rewrite the system (10)–(11) in the following operator forms:

$$\begin{aligned} M[y,z] &= y(x) - \alpha - \int_0^1 t\,(1-t)\, f(y(t), z(t))dt \\ &\quad + \int_0^x t\left(1 - \frac{t}{x}\right) f(y(t), z(t))dt, \end{aligned} \tag{12}$$

$$\begin{aligned} N[y,z] &= z(x) - \beta - \int_0^1 t\,(1-t)\, g(y(t), z(t))dt \\ &\quad + \int_0^x t\left(1 - \frac{t}{x}\right) g(y(t), z(t))dt. \end{aligned} \tag{13}$$

According to the OHAM, we represent $y(x)$ and $z(x)$ in infinite series form as

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x), \quad z(x) = z_0(x) + \sum_{m=1}^{\infty} z_m(x). \tag{14}$$

The components $y_m(x)$ and $z_m(x)$, $m \geq 1$ of the series solutions (14) are obtained by solving the following system of m-th order deformation equations, respectively

$$G\left(y_m(x) - \chi_m y_{m-1}(x)\right) = h_1 L_1(x) R_{1,m}\left(\overrightarrow{y}_{m-1}(x), \overrightarrow{z}_{m-1}(x)\right), \tag{15}$$

$$H\left(z_m(x) - \chi_m z_{m-1}(x)\right) = h_2 L_2(x) R_{2,m}\left(\overrightarrow{y}_{m-1}(x), \overrightarrow{z}_{m-1}(x)\right), \tag{16}$$

where

$$\begin{aligned} &\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \\ &\overrightarrow{y}_n(x) = \{y_0(x), y_1(x), \ldots, y_n(x)\}, \\ &\overrightarrow{z}_n(x) = \{z_0(x), z_1(x), \ldots, z_n(x)\}, \end{aligned} \tag{17}$$

$$\begin{aligned} R_{1,m}(\overrightarrow{y}_{m-1}(x), \overrightarrow{z}_{m-1}(x)) &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} M\left[\sum_{m=1}^{\infty} y_m(x)p^m, \sum_{m=1}^{\infty} z_m(x)p^m\right]}{\partial p^{m-1}} \right|_{p=0}, \\ R_{2,m}(\overrightarrow{y}_{m-1}(x), \overrightarrow{z}_{m-1}(x)) &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N\left[\sum_{m=1}^{\infty} y_m(x)p^m, \sum_{m=1}^{\infty} z_m(x)p^m\right]}{\partial p^{m-1}} \right|_{p=0}, \end{aligned} \tag{18}$$

and G and H are auxiliary linear operators satisfying the properties $G(0) = 0$ and $H(0) = 0$. By using Eqs. (14) and Eqs. (12)–(13), from (18) we obtain

$$R_{1,m}(\overrightarrow{y}_{m-1}(x), \overrightarrow{z}_{m-1}(x)) = y_{m-1}(x) - (1-\chi_m)F(x) - \int_0^1 t\,(1-t)\,U_{1,m-1}(t)dt + \int_0^x t\left(1-\frac{t}{x}\right)U_{1,m-1}(t)dt, \tag{19}$$

$$R_{2,m}(\overrightarrow{y}_{m-1}(x), \overrightarrow{z}_{m-1}(x)) = z_{m-1}(x) - (1-\chi_m)P(x) - \int_0^1 t\,(1-t)\,U_{2,m-1}(t)dt + \int_0^x t\left(1-\frac{t}{x}\right)U_{2,m-1}(t)dt. \tag{20}$$

where

$$F(x) = \alpha, \quad P(x) = \beta, \tag{21}$$

$$U_{1,m-1}(x) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1} f\left(\sum_{m=1}^{\infty} y_m(x)p^m, \sum_{m=1}^{\infty} z_m(x)p^m\right)}{\partial p^{m-1}}\right)\Bigg|_{p=0}, \tag{22}$$

and

$$U_{2,m-1}(x) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1} g\left(\sum_{m=1}^{\infty} y_m(x)p^m, \sum_{m=1}^{\infty} z_m(x)p^m\right)}{\partial p^{m-1}}\right)\Bigg|_{p=0}. \tag{23}$$

Substituting Eqs. (19) and (20) into the m-th order deformation Eqs. (15) and (16) respectively, we get

$$G\,(y_m(x) - \chi_m y_{m-1}(x)) = h_1 L_1(x)\Bigg(y_{m-1}(x) - (1-\chi_m)F(x) - \int_0^1 t\,(1-t)\,U_{1,m-1}(t)dt + \int_0^x t\left(1-\frac{t}{x}\right)U_{1,m-1}(t)dt\Bigg), \tag{24}$$

$$H\,(z_m(x) - \chi_m z_{m-1}(x)) = h_2 L_2(x)\Bigg(z_{m-1}(x) - (1-\chi_m)P(x) - \int_0^1 t\,(1-t)\,U_{2,m-1}(t)dt + \int_0^x t\left(1-\frac{t}{x}\right)U_{2,m-1}(t)dt\Bigg). \tag{25}$$

We assume that $y_0(x) = F(x)$, $L_1(x) = 1$ and $G(u) = u$. Taking into account these conditions, from Eq. (24), we can obtain the following terms for OHAM approximation to the solution $y(x)$ of (10):

$$
\begin{aligned}
y_0(x) &= F(x),\\
y_1(x) &= h_1\left[-\int_0^1 t\,(1-t)\,U_{1,0}(t)dt + \int_0^x t\left(1-\frac{t}{x}\right)U_{1,0}(t)dt\right],\\
y_k(x) &= (1+h_1)y_{k-1}(x) + h_1\left[-\int_0^1 t\,(1-t)\,U_{1,k-1}(t)dt\right.\\
&\quad\left.+\int_0^x t\left(1-\frac{t}{x}\right)U_{1,k-1}(t)dt\right],\ k\geq 2.
\end{aligned} \tag{26}
$$

Further, we assume that $z_0(x) = P(x)$, $L_2(x) = 1$ and $H(u) = u$. Taking into account these conditions, from Eq. (25), we can obtain the following terms for OHAM approximation to the solution $z(x)$ of (11):

$$
\begin{aligned}
z_0(x) &= P(x),\\
z_1(x) &= h_2\left[-\int_0^1 t\,(1-t)\,U_{2,0}(t)dt + \int_0^x t\left(1-\frac{t}{x}\right)U_{2,0}(t)dt\right],\\
z_k(x) &= (1+h_2)z_{k-1}(x) + h_2\left[-\int_0^1 t\,(1-t)\,U_{2,k-1}(t)dt\right.\\
&\quad\left.+\int_0^x t\left(1-\frac{t}{x}\right)U_{2,k-1}(t)dt\right],\ k\geq 2.
\end{aligned} \tag{27}
$$

The n-th order OHAM approximations for the system of Eqs. (1)–(3) are given by

$$
\Phi_n(x) = y_0(x) + \sum_{m=1}^{n} y_m(x), \quad \Psi_n(x) = z_0(x) + \sum_{m=1}^{n} z_m(x). \tag{28}
$$

The approximation solutions Φ_n and Ψ_n, as defined in (28), involve the convergence control parameters h_1, h_2. We next find the optimal values of h_1 and h_2 by minimizing the squared residual errors of governing Eqs. (10)–(11). The squared residual errors of the coupled integral Eqs. (12) and (13) are given by

$$
S_{1,n}(h_1,h_2) = \int_0^1 (M[\Phi_n(x),\Psi_n(x)]^2)dx, \tag{29}
$$

$$
S_{2,n}(h_1,h_2) = \int_0^1 (N[\Phi_n(x),\Psi_n(x)]^2)dx. \tag{30}
$$

At the minimum, we have

$$
\frac{\partial}{\partial h_1}\int_0^1 (M[\Phi_n(x),\Psi_n(x)]^2)dx = 0, \tag{31}
$$

$$
\frac{\partial}{\partial h_2}\int_0^1 (N[\Phi_n(x),\Psi_n(x)]^2)dx = 0. \tag{32}
$$

We note that in some cases it may be difficult or too time-consuming for evaluating the above integrals. To avoid this drawback, the minimization can be done via the discrete version for (29)–(30) using a sufficient number of equally spaced points in the domain associated with the problem. Once the optimal values of the parameters h_1, h_2 have been computed, one can obtain the optimal solution of the system of Eqs. (1)–(3) by using Eqs. (26) and (27).

3 Numerical illustration

In this section, in order to illustrate the accuracy and efficiency of our technique described in the previous section, we apply it on the following example. We compare our findings with those obtained by ADM [1] and its modified version given in [1].

Example Consider the following coupled Lane–Emden singular boundary value problem arising in the study of catalyst diffusion reactions [2]:

$$\begin{aligned} y'' + \frac{2}{x}y' &= ay^2 + byz, \\ z'' + \frac{2}{x}z' &= cy^2 + dyz, \end{aligned} \tag{33}$$

subject to the boundary conditions:

$$y'(0) = 0, \quad y(1) = \alpha, \quad z'(0) = 0, \quad z(1) = \beta. \tag{34}$$

The problem (33)–(34) corresponds to (1)–(3) with $f(y, z) = ay^2+byz$ and $g(y, z) = cyz + dz^2$. The parameters $a, b, c, d, \alpha, \beta$ can be described for the real chemical reactions. The authors of [2] have established the existence of solution to the problem (33)–(34).

We solve the above problem with $a = 1, b = 2/5, c = 1/2, d = 1, \alpha = 1$ and $\beta = 2$ using present method (28) for different values of n. Making use of (28) with $n = 4$, we get the following fourth order approximate solutions:

$$\begin{aligned} \Phi_4(x, h_1, h_2) = \frac{1}{1134000000}&(1134000000 + 144h_1^4(5675139 - 7318255x^2 \\ &+ 1791111x^4 - 152265x^6 + 4270x^8) \\ &+ 625h_1(-1 + x^2)(-2177280 + 30240h_2(-7 + 3x^2) \\ &- 480h_2^2(325 - 144x^2 + 3x^4) \\ &+ h_2^3(-43101 + 19679x^2 - 775x^4 + 5x^6)) \\ &+ 50h_1^2(-1 + x^2)(272160(-199 + 21x^2) \\ &- 480h_2(6452 - 3411x^2 + 411x^4) \\ &+ h_2^2(-877237 + 478963x^2 - 68255x^4 + 1985x^6)) \\ &+ 80h_1^3(-1 + x^2)(-540(55973 \end{aligned}$$

$$
\begin{aligned}
&- 11304x^2 + 519x^4) + h_2(-819273 \\
&+ 492682x^2 - 96830x^4 + 4945x^6))), \qquad (35) \\
\Psi_4(x, h_1, h_2) = &\frac{1}{453600000}(907200000 + 625h_2^4(419781 \\
&- 471860x^2 + 53214x^4 - 1140x^6 \\
&+ 5x^8) + 864h_2(-1 + x^2)(-875000 \\
&+ 47250h1(-7 + 3x^2) - 150h_1^2(1961 \\
&- 930x^2 + 57x^4) + h_1^3(-99153 + 51227x^2 \\
&- 5830x^4 + 200x^6)) + 50h_2^3(-1 \\
&+ x^2)(-6000(3139 - 270x^2 + 3x^4) \\
&+ h_1(-1519117 + 800343x^2 - 97575x^4 \\
&+ 1885x^6)) + 120h_2^2(-1 + x^2)(157500(-67 + 3x^2) \\
&- 600h_1(3365 - 1626x^2 + 117x^4) \\
&+ h_1^2(-692289 + 369041x^2 - 49195x^4 + 2315x^6))). \qquad (36)
\end{aligned}
$$

Equations (35) and (36) involve the convergence control parameters h_1 and h_2. These parameters are computed by minimizing the squared residual error of the governing equations. Using Eqs. (31)–(32), we obtain $h_1 = -0.7993$ and $h_2 = -0.8281$ for $n = 4$. Substituting the values $h_1 = -0.7993$ and $h_2 = -0.8281$ into Eqs. (35)–(36), we obtain the following fourth-order OHAM approximations:

$$
\begin{aligned}
\Phi_4(x) &= 0.781778548 + 0.18906843x^2 + 0.026091404x^4 \\
&\quad + 0.00265316833x^6 + 0.000408448870x^8, \qquad (37) \\
\Psi_4(x) &= 1.69120102 + 0.26999473x^2 + 0.034897236x^4 \\
&\quad + 0.0033800509x^6 + 0.000526960667x^8. \qquad (38)
\end{aligned}
$$

The exact solutions of (33)–(34) are not known. To measure the efficiency and accuracy of the proposed method, we define the maximum residual error functions as follows:

$$
\begin{aligned}
E_{1,R}(x) &= \max_{0\le x\le 1} \left| x\Phi_n''(x) + 2\Phi_n'(x) - x\left(\Phi_n^2(x) + \frac{2}{5}\Phi_n(x)\Psi_n(x)\right)\right|, \\
E_{2,R}(x) &= \max_{0\le x\le 1} \left| x\Psi_n''(x) + 2\Psi_n'(x) - x\left(\frac{1}{2}\Phi_n^2(x) + \Phi_n(x)\Psi_n(x)\right)\right|,
\end{aligned} \qquad (39)
$$

where $\Phi_n(x)$, $\Psi_n(x)$ represent the n-th order approximations of $y(x)$ and $z(x)$ respectively.

The optimal values of h_1, h_2 for different values of n are tabulated in Table 1. For the case of first singular Lane–Emden equation given in (1), Table 2 presents the maximum residual errors, for different values of n, obtained using the proposed method, ADM [1] and modified ADM [1]. One can observe from this table that the maximum residual error in the 15-th order proposed OHAM approximation is 6.3116×10^{-10}, whereas

Table 1 Numerical values of the convergence control parameters h_1, h_2

n	h_1	h_2
3	− 0.80822	− 0.82852
6	− 0.78670	− 0.80950
9	− 0.77641	− 0.79010
12	− 0.76985	− 0.78382
15	− 0.76576	− 0.77563

Table 2 Numerical results of maximum residual error of $y(x)$

n	Present method	ADM in [1]	Modified ADM in [1]
3	1.2697×10^{-2}	0.4428	0.2363
6	1.6953×10^{-4}	7.1558×10^{-2}	2.0937×10^{-2}
9	2.5134×10^{-6}	1.4907×10^{-2}	3.3520×10^{-3}
12	3.9764×10^{-8}	3.5455×10^{-3}	5.1124×10^{-4}
15	6.3116×10^{-10}	9.1389×10^{-4}	7.5440×10^{-5}

Table 3 Numerical results of maximum residual error of $z(x)$

n	Present method	ADM in [1]	Modified ADM in [1]
3	1.2606×10^{-2}	0.5706	0.3099
6	1.1949×10^{-4}	8.9823×10^{-2}	2.5111×10^{-2}
9	1.9772×10^{-6}	1.8550×10^{-2}	3.2833×10^{-3}
12	2.6049×10^{-8}	4.3938×10^{-3}	5.8961×10^{-4}
15	4.5961×10^{-10}	1.1298×10^{-3}	8.6060×10^{-5}

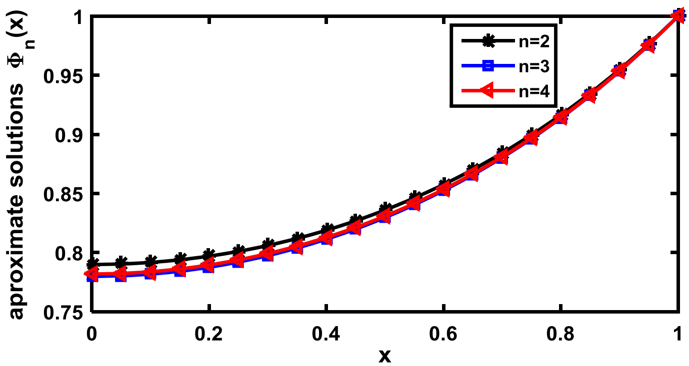

Fig. 1 Approximate solutions Φ_n (x) for n = 2, 3, 4

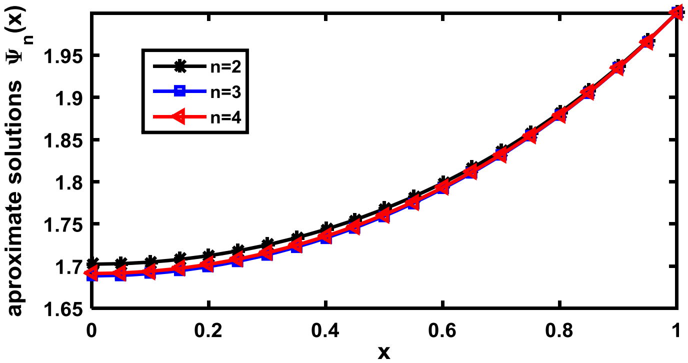

Fig. 2 Approximate solutions of Ψ_n (x) for n = 2, 3, 4

the maximum residual errors in ADM [1] and modified ADM [1] are 9.1389×10^{-4} and 7.5440×10^{-5}, respectively. This indicates that the error in OHAM solution is 6 and 5 orders of magnitude smaller than in ADM and its modified version solution, respectively with the same number of terms in the solution series. On the other hand, for the second singular Lane–Emden equation of (2), Table 3 presents the maximum residual errors for different values of n obtained using the proposed method, ADM [1] and modified ADM [1]. One can see from this table that the maximum residual error in the 15th order OHAM approximation is 4.5961×10^{-10}, whereas the maximum residual error in ADM [1] and modified ADM [1] are 1.1298×10^{-3} and 8.6060×10^{-5}, repectively, this suggest that the error in OHAM solution is 7 and 5 orders of magnitude smaller than in the ADM and its modified version, respectively. Tables 2 and 3 indicate that our method converges faster to the solution of the problem considered than the methods in [1]. Moreover, from these tables, it can be observed that our method with a few terms approximates the solution of the problem very well and the error decreases sharply with the increase in n. The approximate solutions $\Phi_n(x)$ and $\Psi_n(x)$ for $n = 2, 3, 4$ are plotted in Figs. 1 and 2, respectively. Figures 3 and 4 depict the approximate solutions $\Phi_n(x)$ and $\Psi_n(x)$ for $n = 4, 5, 6, 7$. One can observe from these figures that, the approximate solution converges to a certain function as the number of terms in the solution series increases. The logarithmic plots of residual errors in the approximate solutions for different values of n are presented in Figs. 5 and 6, which shows that as the number of solution components increases, the residual error decreases.

4 Conclusion

An iterative method based on an OHAM has been proposed to obtain an accurate approximate solution of the system of Lane–Emden singular boundary value problems arising in catalytic diffusion reactions. The formulation of this scheme is divided into two steps. In the first step, the original system of boundary value problems is converted into an equivalent system of Fredholm integral equations. In the second

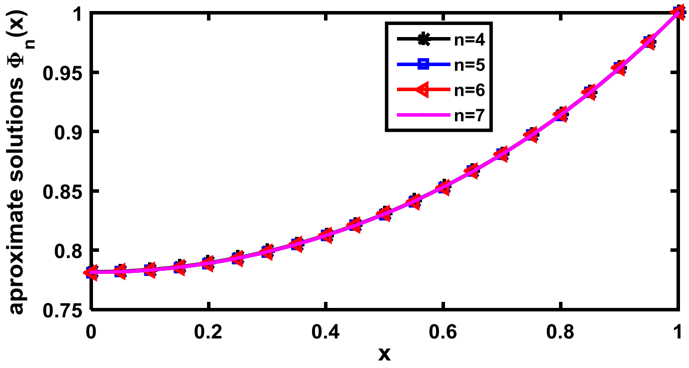

Fig. 3 Approximate solutions Φ_n (x) for n = 4, 5, 6, 7

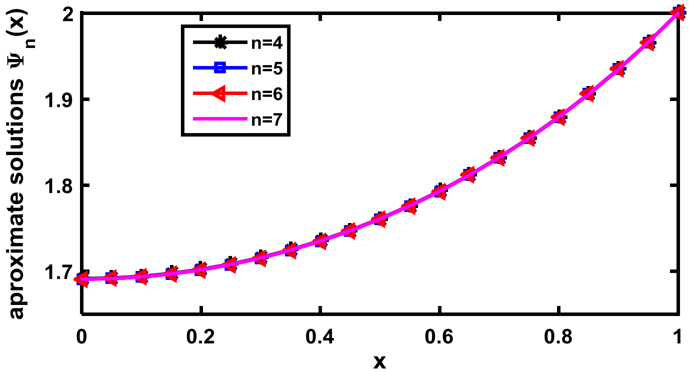

Fig. 4 Approximate solutions Ψ_n (x) for n = 4, 5, 6, 7

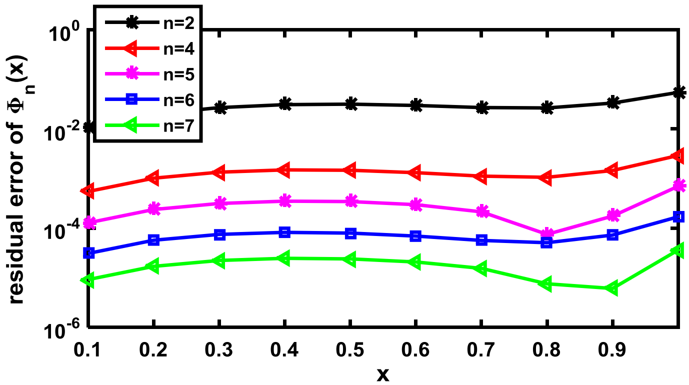

Fig. 5 Residual errors in Φ_n (x)

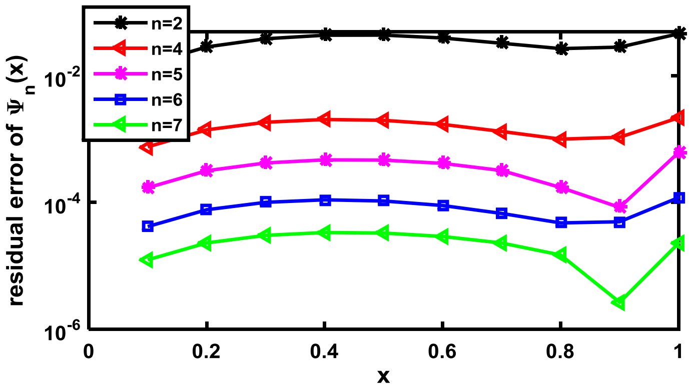

Fig. 6 Residual errors in Ψ_n (x)

step, the resulting problem is efficiently tackled by the optimized homotopy analysis method. A numerical example was considered to evidence the efficiency of the proposed method. In order to show the advantage of the proposed method, the obtained results have been compared with those obtained by existing iterative schemes such as ADM or MADM. It has been observed that the residual error in the proposed OHAM solution is seven orders of magnitude smaller than in ADM, whereas it is five orders of magnitude smaller than in MADM with the same number of terms in the series solution. This suggest that our method converges quickly to the solution of the the coupled Lane–Emden singular equations than the ADM and MADM. Unlike the ADM and its modified version, the presented method is guaranteed to converge to the solution of the problem. To conclude, the proposed iterative algorithm works well for the system of differential equations subjected to Neumann and Dirichlet boundary conditions. Moreover, it is a straight forward, highly accurate and very fast convergent. In this study, the optimized homotopy analysis method has been developed to tackle a system of singular equations which models reaction diffusion process in a catalyst. Without doubt, this method can be used to solve other similar problems arising in Chemistry and other branches of science.

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