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Two general higher-order derivative free iterative techniques having optimal convergence order

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Abstract

The principal objective of this study is to propose two derivative free iteration functions. Both are applicable to each earlier optimal multi-point derivative free scheme of order four and eight whose first sub step should be Steffensen's type method to develop more advanced optimal iteration techniques of order eight and sixteen, respectively. Both schemes satisfy the Kung–Traub optimality conjecture. In addition, the theoretical convergence properties of our schemes are fully explore with the help of main theorem that demonstrate the convergence order. The performance and effectiveness of our optimal iteration functions have compared with the existing competitors on some standard academic problems. Finally, on the account of results obtained, our methods are find to be more efficient as compared to some standard and robust iteration functions of same order.

Keywords Computational order of convergence · Simple zeros · Kung–Traub conjecture · Scalar equations · Steffensen's type method

1 Introduction

A remarkable recognition has given to the development of derivative free iteration functions having optimal convergence of order eight and sixteen in the last two decade. Because of the upgraded digital computer and computation programming softwares. One of the best feature of them is that they do not require any derivative evaluation of the involved function. In addition, we can simply get our needed precision in a small amount of time and number of iterations along with smaller residual errors.

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In the literature, we can easily find a good amount of optimal higher-order derivative free iteration functions which were given by various researchers in [1–11,13–18]. Among them, most of the modifications or reconstructions are based on the Steffensen's or Steffensen's like iterative procedure at the expense of more substeps to the original scheme or extra functional evaluations. However, there is no optimal scheme that useful to each derivative free iterative technique of order 4 and 8 to obtain further optimal eighth and sixteenth-order derivative free method, according to our knowledge.

In simple words, no researcher presented a general iteration function that applicable on every optimal convergent scheme of order 4 and 8 (provided whose first sub step employ Steffensen's type scheme) to produce more advance extensions of order eight and sixteen, respectively. Nowadays, such types of techniques are more fascinating and tough chore in the field of computational methods and numerical analysis instead of obtaining a particular higher-order extension of an existing iterative method.

Therefore, we are interested to develop two schemes: first one is useful for every optimal iteration function/family of iteration functions of order four to obtain further more advance reconstruction of optimal eighth-order; second one is applicable to each optimal eighth-order iterative scheme to construct further optimal sixteenth-order convergence. The construction of them based on the concept of the rational approximations. They are useful for each optimal scheme of order four and eight provided whose first substep employs Steffensen's type method. Then, it is obvious to choose any scheme from [2–9,13,15–17], etc. to further obtain eighth and sixteenth-order methods. The efficiency of them is checked on a good number of standard test problems. It is found that their performance is more useful than the existing ones of same order.

The layout of this study is given as follows: the Sect. 2 is assign to the construction of two new derivative free schemes, which are useful for each optimal scheme of order four and eight to obtain further eighth and sixteenth-order iterative functions. In addition, we examined the convergence properties of them in a main theorem that demonstrate the theoretical order of convergence. Section 3 is dedicate to illustrate the computational consequence of our schemes that were presented in the earlier Sect. 2. Therefore, several numerical experimentation are executed on some standard academic numerical problems. In addition, we also present the contrast of them with the earlier existing iterative procedure of same order. Section 4 consists of final concluding comments based on our proposed schemes.

2 Development of optimal schemes

This section is dedicated to the main objective of this study. In simple words, we propose here two optimal derivative free schemes of order eight and sixteen. First of all, we assume the following general derivative free iteration function of order four:

$$
\begin{cases}
w_k = x_k - \gamma \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \\
z_k = \phi_4(x_k, \ u_k, \ w_k),
\end{cases}
\tag{2.1}
$$

where $u_k = x_k + \gamma f(x_k)$, $\gamma \in \mathbb{R}$. We assume the rational function of the following form in order to attain the next approximation x_{k+1} to the required root,

$$x = x_k - \frac{\tau(x) - \tau(x_k) + \mu_1}{\mu_2\big(\tau(x) - \tau(x_k)\big)^2 + \mu_3\big(\tau(x) - \tau(x_k)\big) + \mu_4}, \qquad (2.2)$$

where μ_i, $i = 1, 2, 3, 4$ are free disposable parameters and can be determined by imposing the following tangency conditions

$$\tau(x_k) = f(x_k), \quad \tau(u_k) = f(u_k), \quad \tau(w_k) = f(w_k), \quad \tau(z_k) = f(z_k). \qquad (2.3)$$

By using first condition of tangency, we have

$$\mu_1 = 0. \qquad (2.4)$$

Again, by applying the last three tangency assertions, we have three linear equations in the form of μ_2, μ_3 and μ_4, which are defined as follow

$$\mu_2\big(f(u_k) - f(x_k)\big) + \mu_3 + \frac{\mu_4}{f(u_k) - f(x_k)} - \frac{1}{u_k - x_k} = 0,$$
$$\mu_2\big(f(w_k) - f(x_k)\big) + \mu_3 + \frac{\mu_4}{f(w_k) - f(x_k)} - \frac{1}{a_k} = 0, \qquad (2.5)$$
$$\mu_2\big(f(z_k) - f(x_k)\big) + \mu_3 + \frac{\mu_4}{f(z_k) - f(x_k)} - \frac{1}{b_k} = 0,$$

which further yield

$$\mu_2 = \frac{\mu_3\gamma f(x_k)(f(x_k) - f(u_k)) + f(u_k) - \mu_4\gamma f(x_k) - f(x_k)}{\gamma f(x_k)(f(u_k) - f(x_k))^2},$$
$$\mu_3 = \frac{\mu_4(f(u_k) + f(w_k) - 2f(x_k))}{(f(u_k) - f(x_k))(f(x_k) - f(w_k))} - \frac{a_k f(w_k) - a_k f(x_k) - \gamma f(u_k)f(x_k) + \gamma f(x_k)^2}{a_k\gamma f(u_k)f(x_k) - a_k\gamma f(w_k)f(x_k)},$$
$$\mu_4 = \frac{(f(u_k) - f(x_k))(f(x_k) - f(w_k))(f(x_k) - f(z_k))h_1}{a_k b_k\gamma f(x_k)(f(u_k) - f(w_k))(f(u_k) - f(z_k))(f(w_k) - f(z_k))},$$
$$(2.6)$$

where $a_k = w_k - x_k$, $b_k = z_k - x_k$, $h_1 = \gamma f(x_k)(a_k(f(u_k) - f(w_k)) + b_k(f(z_k) - f(u_k))) + a_k b_k(f(w_k) - f(z_k))$.

Now, we want to find the final substep of our scheme. So, we assume that the rational function (2.2) cuts the $x - $ axis at a point $x = x_{k+1}$, we have

$$\tau(x_{k+1}) = 0, \qquad (2.7)$$

which further yield

$$x_{k+1} = x_k + \frac{f(x_k)}{\mu_2 f(x_k)^2 - \mu_3 f(x_k) + \mu_4}. \qquad (2.8)$$

Now, by using the expressions (2.2) and (2.8), we obtain the following new eighth-order scheme free from derivatives

$$
\begin{cases}
w_k = x_k - \gamma \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \\
z_k = \phi_4(x_k,\ u_k,\ w_k), \\
x_{k+1} = x_k + \dfrac{f(x_k)}{\mu_2 f(x_k)^2 - \mu_3 f(x_k) + \mu_4}.
\end{cases}
\tag{2.9}
$$

where $\mu_2,\ \mu_3\ \mu_4$ are defined earlier in the expression (2.6).

2.1 Sixteenth order scheme

In the similar fashion, we can easily obtain the following general family of order sixteen

$$
\begin{cases}
w_k = x_k - \gamma \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \\
z_k = \phi_4(x_k,\ u_k,\ w_k), \\
t_k = \psi_8(x_k,\ u_k,\ w_k,\ z_k), \\
x_{k+1} = x_k + \dfrac{(w_k - x_k)(z_k - x_k)(t_k - x_k)\gamma f(x_k)^2 v_1}{(z_k - x_k)f(z_k)\Big[(t_k - x_k)\gamma f(t_k)f(u_k)f(x_k)v_2 + (w_k - x_k)f(w_k)(f(w_k) - f(z_k))v_3\Big] + v_4},
\end{cases}
\tag{2.10}
$$

$v_1 = (f(t_k) - f(u_k))(f(t_k) - f(w_k))(f(t_k) - f(z_k))(f(u_k) - f(w_k))(f(u_k) - f(z_k))(f(w_k) - f(z_k)), v_2 = (f(u_k) - f(t_k))(f(t_k) - f(z_k))(f(u_k) - f(z_k))(f(w_k) - f(x_k)), v_3 = [(t_k - x_k)f(t_k)(f(t_k) - f(w_k))(f(t_k) - f(z_k))(f(u_k) - f(x_k)) - \gamma f(u_k)f(x_k)(f(t_k) - f(x_k))(f(u_k) - f(w_k))(f(u_k) - f(z_k))]$ and $v_4 = -(w_k - x_k)(t_k - x_k)\gamma f(w_k)f(t_k)f(u_k)f(x_k)(f(t_k) - f(u_k))(f(t_k) - f(w_k))(f(u_k) - f(w_k))(f(x_k) - f(z_k))$.

The construction of the scheme (2.10) is based on the following rational function

$$
x = x_k - \frac{\varphi(x) - \varphi(x_k) + \beta_1}{\beta_2\big(\varphi(x) - \varphi(x_k)\big)^3 + \beta_3\big(\varphi(x) - \varphi(x_k)\big)^2 + \beta_4\big(\varphi(x) - \varphi(x_k)\big) + \beta_5},
\tag{2.11}
$$

where $\beta_i,\ i = 1, 2, 3, 4, 5$ are free disposable parameters and can be determined by imposing the following tangency conditions

$$
\varphi(x_k) = f(x_k),\ \varphi(u_k) = f(u_k),\ \varphi(w_k) = f(w_k),\ \varphi(z_k) = f(z_k),\ \varphi(t_k) = f(t_k).
\tag{2.12}
$$

The next Theorem 2.1 demonstrates that the convergence order of our schemes (2.9) and (2.10) are eight and sixteenth-order, respectively, without using any more functional evaluations.

Theorem 2.1 *Let us assume that $f\ :\ \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ be an holomorphic function in the region $\mathbb{D}$ consists the required simple zero ξ. In addition, we consider that $\phi_4(x_k,\ u_k,\ w_k)$ and $\psi_8(x_k,\ u_k,\ w_k,\ z_k)$ are two optimal schemes of order four*

*and eight, respectively. Further, we consider that the initial guess $x = x_0$ is suffi-
ciently close to ξ for sure convergence. Then, the iteration functions (2.9) and (2.10)
have convergence order eight and sixteen, respectively. They also satisfy the following
asymptotic error constant terms, respectively*

$$e_{k+1} = \frac{\lambda_0 c_2 \left(2c_2^3 - 3c_1 c_3 c_2 + c_1^2 c_4\right)(\gamma c_1 + 1)^2}{c_1^4} e_k^8 + O\left(e_k^9\right), \qquad (2.13)$$

and

$$e_{k+1} = \lambda_0 \kappa_0 \left(5c_2^4 - 10c_2^2 c_3 + 2c_3^2 + 4c_2 c_4 - c_5\right) c_2 e_k^{16} + O\left(e_k^{17}\right). \qquad (2.14)$$

where $e_k = x_k - \xi$ and $c_j = \frac{f^{(j)}(\xi)}{j!}$ for $j = 1, 2, \ldots, 16$.

Proof We expand the function $f(x_k)$ by applying the Taylor series expansion around
the point $x = \xi$ leads to us:

$$f(x_k) = \sum_{j=1}^{16} c_j e_k^j + O\left(e_k^{17}\right). \qquad (2.15)$$

In the similar fashion, we have

$$f(u_k) = f(x_k + \gamma f(x_k))$$

$$= c_1 \eta e_k + c_2 \left(\gamma^2 c_1^2 + 3\gamma c_1 + 1\right) e_k^2 + \sum_{n=1}^{16} A_n e_k^{n+2} + O\left(e_k^{17}\right), \quad (2.16)$$

where $A_j = A_j(\gamma, c_1, c_2, \ldots, c_{16})$ are given in terms of $\gamma, c_2, c_3, \ldots, c_{16}$ for example
$A_1 = 2\gamma^2 c_1 c_2^2 + c_3 \eta^3 + 2\gamma c_2^2 + \gamma c_1 c_3$ and $A_2 = \gamma^2 c_2^3 + 2\gamma^2 c_1 c_2 c_3 + c_4 \eta^4 + 3\gamma c_2 c_3 \eta^2 + 2\gamma c_2 c_3 + \gamma c_1 c_4$, $A_3 = c_5 \eta^5 + 4\gamma c_2 c_4 \eta^3 + 3\gamma c_3 (\gamma c_2^2 + c_3 \eta)\eta + 2\gamma c_2 (\gamma c_2 c_3 + c_4 \eta) + \gamma c_1 c_5$ and $\eta = (\gamma c_1 + 1)$, etc.
By using the Eqs. (2.15) and (2.16) in the first sub step, we get

$$w_k - \xi = c_2 \left(\gamma + \frac{1}{c_1}\right) e_k^2 + \frac{\left(c_1 c_3 \left(\gamma^2 c_1^2 + 3\gamma c_1 + 2\right) - c_2^2 \left(\gamma^2 c_1^2 + 2\gamma c_1 + 2\right)\right)}{c_1^2} e_k^3$$

$$+ \sum_{n=1}^{13} \bar{A}_n e_k^{n+3} + O\left(e_k^{17}\right). \qquad (2.17)$$

which further yields

$$f(w_k) = c_2 \eta e_k^2 + \frac{c_1 c_3 \left(\gamma^2 c_1^2 + 3\gamma c_1 + 2\right) - c_2^2 \left(\gamma^2 c_1^2 + 2\gamma c_1 + 2\right)}{c_1} e_k^3$$

$$+ \sum_{n=1}^{13} \bar{\bar{A}}_n e_k^{n+3} + O\left(e_k^{17}\right). \qquad (2.18)$$

Since, $\phi_4(x_k, u_k, w_k)$ is an optimal fourth-order scheme. Therefore, the scheme will satisfy the following error equation

$$z_k - \xi = \sum_{j=0}^{12} \lambda_j e_k^{(j+4)} + O\left(e_k^{17}\right), \tag{2.19}$$

where $\lambda_j (0 \le j \le 12)$ are asymptotic error constants terms, provided $\lambda_0 \ne 0$.

Now, we can expand the function $f(z_k)$ about a point $z = \xi$ with the help of Taylor series expansion, which is given as follow

$$
\begin{aligned}
f(z_k) = {}& c_1\left(\lambda_0 e_k^4 + \lambda_1 e_k^5 + \lambda_2 e_k^6 + \lambda_3 e_k^7\right) + (c_2\lambda_0^2 + c_1\lambda_4)e_k^8 + (2c_2\lambda_0\lambda_1 + c_1\lambda_5)e_k^9 \\
&+ \left\{c_2(\lambda_1^2 + 2\lambda_0\lambda_2) + c_1\lambda_6\right\}e_k^{10} + \left\{2c_2(\lambda_1\lambda_2 + \lambda_0\lambda_3) + c_1\lambda_7\right\}e_k^{11} \\
&+ \left\{c_3\lambda_0^3 + c_2(\lambda_2^2 + 2\lambda_1\lambda_3 + 2\lambda_0\lambda_4) + c_1\lambda_8\right\}e_k^{12} + \left\{3c_3\lambda_1\lambda_0^2 \right. \\
&\left. + 2c_2(\lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_0\lambda_5) + c_1\lambda_9\right\}e_k^{13} + O\left(e_k^{14}\right).
\end{aligned}
\tag{2.20}
$$

By using the expressions (2.15)–(2.20), we further obtain

$$\frac{f(x_k)}{\mu_2 f(x_k)^2 - \mu_3 f(x_k) + \mu_4} = -e_k + \frac{\lambda_0 c_2(2c_2^3 - 3c_1c_3c_2 + c_1^2 c_4)\eta^2}{c_1^4}e_k^8 + O\left(e_k^9\right). \tag{2.21}$$

With the help of expression (2.21) and last sub step of scheme (2.9), we have

$$e_{k+1} = \frac{\lambda_0 c_2(2c_2^3 - 3c_1c_3c_2 + c_1^2 c_4)\eta^2}{c_1^4}e_k^8 + O\left(e_k^9\right). \tag{2.22}$$

Now, we want to verify the theoretical convergence order of scheme (2.10). Since, we know that $\psi_8(x_k, u_k, w_k, z_k)$ is an optimal eighth-order scheme. Therefore, it will satisfy the following error equation

$$t_k - \xi = \kappa_0 e_k^8 + \kappa_1 e_k^9 + \kappa_2 e_k^{10} + \kappa_3 e_k^{11} + \kappa_4 e_k^{12} + \kappa_5 e_k^{13} + \kappa_6 e_k^{14} + \kappa_7 e_k^{15} + \kappa_8 e_k^{16} + O\left(e_k^{17}\right), \tag{2.23}$$

where $\kappa_i (0 \le j \le 8)$ are asymptotic error constants terms, provided $\kappa_0 \ne 0$.

Again, we can expand the function $f(t_k)$ by using Taylor series expansion about a point $z = \xi$. Then, we have

$$
\begin{aligned}
f(t_k) = {}& \left(\kappa_0 e_k^8 + \kappa_1 e_k^9 + \kappa_2 e_k^{10} + \kappa_3 e_k^{11} + \kappa_4 e_k^{12} + \kappa_5 e_k^{13} + \kappa_6 e_k^{14} + \kappa_7 e_k^{15}\right)c_1 \\
&+ (c_2\kappa_0^2 + c_1\kappa_8)e_k^{16} + O\left(e_k^{17}\right).
\end{aligned}
\tag{2.24}
$$

By using the expressions (2.15)–(2.20), (2.23) and (2.24), we further obtain

$$
\frac{(w_k - x_k)(z_k - x_k)(t_k - x_k)\gamma f(x_k)^2 v_1}{(z_k - x_k)f(z_k)\Big[(t_k - x_k)\gamma f(t_k)f(u_k)f(x_k)v_2 + (w_k - x_k)f(w_k)(f(w_k) - f(z_k))v_3\Big] + v_4}
$$
$$
= e_k - \lambda_0 \kappa_0 \Big(5c_2^4 - 10c_2^2 c_3 + 2c_3^2 + 4c_2 c_4 - c_5\Big)c_2 e_k^{16} + O\left(e_k^{17}\right).
$$
(2.25)

Finally, by inserting the expression (2.25), in the last sub step of scheme (2.10), we get

$$
e_{k+1} = \lambda_0 \kappa_0 \Big(5c_2^4 - 10c_2^2 c_3 + 2c_3^2 + 4c_2 c_4 - c_5\Big)c_2 e_k^{16} + O\left(e_k^{17}\right). \qquad (2.26)
$$

The expressions (2.22) and (2.26) reveal that the schemes (2.9) and (2.10) reach maximum eighth and sixteenth-order convergence, respectively. In addition, both schemes have optimal convergence according to Kung–Traub conjecture. This completes the proof.   □

3 Numerical experimentation

This section is dedicated to illustrate the computational behavior of our theoretical results which are present in the earlier Sect. 2. Therefore, we assume several standard academic nonlinear scalar problems which are depicted in the Table 1. Since, both of the schemes are in general way. That's mean, we can consider any exiting iterative scheme of order four and eight in our iteration functions (2.9) and (2.10) respectively, to obtain further eighth and sixteenth-order convergence. Firstly, we propose some new optimal eighth-order methods, which are consider as below:

(i) We assume an expression (11) from Li et al. [6] and use in our general scheme (2.9) with $\gamma = 1$. Then, we have

$$
\begin{cases}
w_k = x_k - \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \quad u_k = x_k + f(x_k), \\[2mm]
z_k = w_k - \dfrac{f[u_k, x_k] + f[w_k, x_k] - f[u_k, w_k]}{f[w_k, x_k]^2} f(w_k), \\[2mm]
x_{k+1} = x_k + \dfrac{f(x_k)}{\mu_2 f(x_k)^2 - \mu_3 f(x_k) + \mu_4}.
\end{cases} \qquad (3.1)
$$

where $f[u_k, x_k] = \frac{f(x_k) - f(u_k)}{x_k - u_k}$ is finite difference of first-order. We denote the scheme (3.1) by $(M1_8)$, for computational point of view.

(ii) Now, we choose another expression (12) from Ren et al. in [5] and use in scheme (2.9) with $\gamma = 1$. Then, we further yield another new optimal scheme of order eight, which is defined as follows:

Table 1 Test problems

$f(x)$	x_0	$Root(r)$
$f_1(x) = 10xe^{-x^2} - 1$; [5]	1.5	1.6796306104284499406749203388379704
$f_2(x) = e^{-x} - \tan^{-1} x - 1$; [2]	0.3	0
$f_3(x) = \sin^{-1}(x^2 - 1) - \frac{x}{2} + 1$; [3]	1.5	0.59481096839836917752265623515213618
$f_4(x) = \sin^2 x - x^2 + 1$; [4]		1.4044916482153412260350868177868681
$f_5(x) = \log\left(x^2 - x + 1\right) - 4\sin(x - 1)$; [7]	0.6	1

$$\begin{cases} w_k = x_k - \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \quad u_k = x_k + f(x_k), \\[3mm] z_k = w_k - \dfrac{f(w_k)}{a(w_k - u_k)(w_k - x_k) + f[u_k, w_k] - f[u_k, x_k] + f[w_k, x_k]}, \\[3mm] x_{k+1} = x_k + \dfrac{f(x_k)}{\mu_2 f(x_k)^2 - \mu_3 f(x_k) + \mu_4}. \end{cases}$$
(3.2)

We recall the above scheme by $(M2_8)$ with $a = 1$.

There are two ways to obtain optimal derivative free sixteenth-order methods/families of iterative methods. First one is corresponding to choose an optimal fourth-order derivative free iteration function and insert in scheme (2.9). Then, again reuse the newly obtain eighth-order method in (2.10) to further obtain sixteenth-order convergence. Second one is corresponding to consider an existing eighth-order derivative free method and use that one in scheme (2.10) to further obtain optimal sixteenth-order convergence. We define the new sixteenth-order methods in both ways, which are given as follow:

(i) Let us insert the expression (3.1) in scheme (2.10). In this way, we have the following new optimal scheme of order sixteen

$$\begin{cases} w_k = x_k - \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \quad u_k = x_k + f(x_k), \\[3mm] z_k = w_k - \dfrac{f[u_k, x_k] + f[w_k, x_k] - f[u_k, w_k]}{f[w_k, x_k]^2} f(w_k), \\[3mm] t_n = x_k + \dfrac{f(x_k)}{\alpha_2 f(x_k)^2 - \alpha_3 f(x_k) + \alpha_4}, \\[3mm] x_{k+1} = x_k + \dfrac{(w_k - x_k)(z_k - x_k)(t_k - x_k)\gamma f(x_k)^2 v_1}{(z_k - x_k)f(z_k)\Big[(t_k - x_k)\gamma f(t_k)f(u_k)f(x_k)v_2 + (w_k - x_k)f(w_k)(f(w_k) - f(z_k))v_3\Big] + v_4}. \end{cases}$$
(3.3)

We denote the expression (3.3) by $(M3_{16})$, for computational work.

(ii) Now, we use an optimal eighth-order family of derivative free iteration functions proposed by Khattri and Steihaug [4] in scheme (2.10). Then, we further yield

$$
\begin{cases}
w_k = x_k - \gamma \dfrac{f(x_k)^2}{f(u_k) - f(x_k)}, \quad u_k = x_k + \gamma f(x_k), \\[2mm]
z_k = w_k - \dfrac{f(w_k)}{-\frac{f(u_k)(x_k - w_k)}{\gamma f(x_k)(\gamma f(x_k) - w_k + x_k)} + \frac{\gamma f(x_k) - w_k + x_k}{\gamma (x_k - w_k)} - \frac{f(w_k)(\gamma f(x_k) - 2w_k + 2x_k)}{(x_k - w_k)(\gamma f(x_k) - w_k + x_k)}}, \\[3mm]
t_k = z_k - \dfrac{f(z_k)(u_k - w_k)(u_k - x_k)(u_k - z_k)(x_k - w_k)(w_k - z_k)(x_k - z_k)}{(w_k - z_k)^2 \left(f(u_k)(x_k - w_k)(x_k - z_k)^2 - f(x_k)(u_k - w_k)(u_k - z_k)^2 \right) + a_1}, \\[3mm]
x_{k+1} = x_k + \dfrac{(w_k - x_k)(z_k - x_k)(t_k - x_k)\gamma f(x_k)^2 v_1}{(z_k - x_k) f(z_k)\left[(t_k - x_k)\gamma f(t_k) f(u_k) f(x_k) v_2 + (w_k - x_k) f(w_k)(f(w_k) - f(z_k)) v_3 \right] + v_4},
\end{cases}
$$
$$\tag{3.4}$$

where $a_1 = f(w_k)(u_k - x_k)(u_k - z_k)^2 (x_k - z_k)^2 - f(z_k)(u_k - w_k)(u_k - x_k)(x_k - w_k)\left[u_k(w_k + x_k - 2z_k) + x_k(w_k - 2z_k) + z_k(3z_k - 2w_k) \right]$. This is another new optimal sixteenth-order family of derivative free methods. Let us consider $\gamma = \frac{1}{2}$ in scheme (3.4), called as $(M4_{16})$.

First of all, we demonstrate the comparison of our eighth-order iteration functions with existing optimal derivative free methods of same order. In this regards, we compare them with an optimal scheme proposed by Zheng et al. [2], among this family, we consider the method (8) (for $\gamma = 1$), called by (ZM_8). In addition, we contrast them with another Steffensen-type iteration functions having optimal convergence of eighth order, which was constructed by Soleymani and Vanani [3], out of them, we consider the method (21), recalled by (SM_8). Finally, we compare them with one more derivative-free iterative technique given by Khattri and Steihaug [4], among them, we choose method (17) (for $\gamma = 1$), denoted by (KM_8).

Then, we also illustrate the convergence behavior of our sixteenth-order methods. Therefore, we compare them with a family of fast 16th-order derivative free iteration functions, proposed by Geum and Kim [11], among them we choose the method (5), called by GM_{16}. In addition, we show the contrast with an optimal sixteenth-order method given by Kung and Traub [13], denoted by (KM_{16}). In the last, we contrast them with another derivative-free optimal methods of sixteenth-order which is recently proposed by Sharma and Gupta [16], out of them, we assume the expression (28) (for $\gamma = 1$), known as (SM_{16}).

In order to demonstrate a better comparison of our methods with the existing schemes, we depicted the number of iteration indexes (k), estimated roots (x_k), modulus value of residual errors ($|f(x_k)|$), errors in the consecutive iterations $|x_{k+1} - x_k|$, $\left| \frac{x_{k+1} - x_k}{(x_k - x_{k-1})^p} \right|$, the asymptotic error constant $\eta = \lim\limits_{k \to \infty} \left| \frac{x_{kk+1} - x_k}{(x_k - x_{k-1})^p} \right|$, (where p is either 8 or 16) and ρ, corresponding to each considered problem, in the Tables 2, 3, 4 5 and 6. In addition, we verify the computational convergence order (ρ) of our proposed methods, by using the following formula

$$
\rho = \left| \frac{(x_{k+1} - x_k)/\eta}{(x_k - x_{k-1})} \right|, \quad k = 1, 2, 3.
$$

All the values like residual errors, asymptotic error constants, computational order of convergence, etc. have been calculated minimum 3000 significant digits in order to minimize the round off error. But due to the restricted paper space, we mention the value of estimated zeros x_k and computational convergence order ρ in 20 and 5 digits,

Table 2 Convergence behavior of different methods for $f_1(x)$

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\frac{e_{n+1}}{e_n^p}\right|$	η	ρ
ZM_8	1	1.6796205386647263701	2.8e(−005)	1.0e(−005)			
	2	1.6796306104284499407	5.0e(−040)	1.8e(−040)	1.703282805	1.703121240	7.999
	3	1.6796306104284499407	3.3e(−318)	1.9e(−318)	1.703121240		8.000
SM_8	1	1.6794934867385129054	3.8e(−004)	1.4e(−004)			
	2	1.6796306104284499407	1.1e(−029)	3.9e(−030)	30.94287780	30.91187283	7.999
	3	1.6796306104284499407	4.3e(−234)	1.5e(−234)	30.91187283		8.000
KM_8	1	1.6796205386647263701	2.8e(−005)	1.0e(−005)			
	2	1.6796306104284499407	5.0e(−040)	1.8e(−040)	1.703282805	1.703121240	7.999
	3	1.6796306104284499407	5.3e(−318)	1.9e(−318)	1.703121240		8.000
$M1_8$	1	1.6796285830566095169	5.6e(−006)	2.0e(−006)			
	2	1.6796306104284499407	1.1e(−045)	3.9e(−046)	1.358020268	1.358001549	8.000
	3	1.6796306104284499407	1.9e(−363)	6.9e(−364)	1.358001549		8.000
$M2_8$	1	1.6796260127931330853	1.3e(−005)	4.6e(−006)			
	2	1.6796306104284499407	1.5e(−042)	5.3e(−043)	2.647641435	2.647572768	8.000
	3	1.6796306104284499407	4.5e(−338)	1.6e(−338)	2.647572768		8.000
GM_{16}	1	1.6796306102345856422	5.4e(−010)	1.9e(−010)			
	2	1.6796306104284499407	5.1e(−151)	1.9e(−151)	46498.51670	46498.51715	16.000
	3	1.6796306104284499407	2.4e(−2407)	8.8e(−2408)	46498.51715		16.000
KM_{16}	1	1.6796303122042883160	8.2e(−007)	3.0e(−007)			
	2	1.6796306104284499407	6.7e(−101)	2.4e(−101)	6232.011315	6231.977614	16.000
	3	1.6796306104284499407	2.7e(−1606)	9.8e(−1607)	6231.977614		16.000

Table 2 continued

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\dfrac{e_{n+1}}{e_n^p}\right|$	η	ρ
SM_{16}	1	1.6796306104687432700	1.1e(−010)	4.0e(−011)		0.4907775715	16.000
	2	1.6796306104284499407	6.5e(−167)	2.4e(−167)	0.4907775711		16.000
	3	1.6796306104284499407	1.3e(−2666)	4.8e(−2667)	0.4907775715		
$M3_{16}$	1	1.6796306101700027706	7.1e(−010)	2.6e(−010)		14.06173919	16.000
	2	1.6796306104284499407	1.5e(−152)	5.6e(−153)	14.06173926		16.000
	3	1.6796306104284499407	3.4e(−2435)	1.2e(−2435)	14.06173919		
$M4_{16}$	1	1.6796306104284459974	1.1e(−014)	3.9e(−015)		0.0001078113772	16.000
	2	1.6796306104284499407	1.0e(−234)	3.7e(−235)	0.0001078113772		16.000
	3	1.6796306104284499407	3.4e(−3755)	1.2e(−3755)	0.0001078113772		

Table 3 Convergence behavior of different methods for $f_2(x)$

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\frac{e_{n+1}}{e_n^p}\right|$	η	ρ
ZM_8	1	$-4.7115231588098e(-0007)$	$9.4e(-007)$	$4.7e(-007)$			
	2	$-1.26800021474312e(-054)$	$2.5e(-054)$	$1.3e(-054)$	0.0005221901622	0.0005221896701	8.000
	3	$-3.48965821455630e(-0435)$	$7.4e(-435)$	$3.5e(-435)$	0.0005221896701		8.000
SM_8	1	$-2.57499475484165e(-0006)$	$5.1e(-006)$	$2.6e(-006)$			
	2	$-3.80144307772904e(-0048)$	$7.6e(-048)$	$3.8e(-048)$	$0.00196669 8117$	$0.00196668 8368$	8.000
	3	$-8.57676446442166e(-0382)$	$1.7e(-382)$	$8.6e(-383)$	$0.00196668 8368$		8.000
KM_8	1	$-4.71152315888097e(-0007)$	$9.4e(-007)$	$4.7e(-007)$			
	2	$-1.26800021474312e(-054)$	$2.5e(-054)$	$1.3e(-054)$	0.0005221901622	0.000522189670	8.000
	3	$-3.48965821455630e(-0435)$	$7.0e(-435)$	$3.5e(-435)$	0.000522189670		8.000
$M1_8$	1	$-1.94761168980544e(-0007)$	$3.9e(-007)$	$1.9e(-007)$			
	2	$-1.06701448099953e(-0057)$	$2.1e(-057)$	$1.1e(-057)$	0.0005154082109	0.0005154079861	8.000
	3	$-8.65991544330789e(-0460)$	$1.7e(-459)$	$8.7e(-460)$	0.0005154079861		8.000
$M2_8$	1	$-5.73202108870614e(-0007)$	$1.1e(-006)$	$5.7e(-007)$			
	2	$-2.68707451996246e(-0054)$	$5.4e(-054)$	$2.7e(-054)$	0.000230577791	0.0002305772569	8.000
	3	$-6.26692005442966e(-0433)$	$1.3e(-432)$	$6.3e(-433)$	0.0002305772569		8.000
GM_{16}	1	$-3.74128946374869470031e(-0010)$	$7.5e(-010)$	$3.7e(-010)$			
	2	$2.99544167889859926722e(-0154)$	$6.0e(-154)$	$3.0e(-154))$	0.002032901262	0.002032901261	16.000
	3	$8.53949993167126227700e(-2460)$	$1.7e(-2459)$	$8.5e(-2460)$	0.002032901261		16.000
KM_{16}	1	$-7.73216947357456184444e(-0012)$	$1.5e(-011)$	$7.7e(-012)$			
	2	$-6.29506180853735611 35e(-0184)$	$1.3e(-183)$	$6.3e(-184)$	$3.856365328e(-6)$	$3.856365327e(-6)$	16.000
	3	$-2.34519061393879270 79e(-2937)$	$4.7e(-2937)$	$2.3e(-2937)$	$3.856365327e(-6)$		16.000

Table 3 continued

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\frac{e_{n+1}}{e_n^p}\right|$	η	ρ
SM_{16}	1	$1.3290440490813870046e(-0012)$	$2.7e(-012)$	$1.3e(-012)$		$8.910802411e(-7)$	16.000
	2	$8.4439726267172439385e(-0197)$	$1.7e(-196)$	$8.4e(-197)$	$8.910802411e(-7)$		16.000
	3	$5.9520405581576804091e(-3144)$	$1.2e(-3143)$	$6.0e(-3144)$	$8.910802411e(-7)$		
$M3_{16}$	1	$6.5970057177738583357e(-0015)$	$1.3e(-014)$	$6.6e(-015)$		$6.745381610e(-9)$	16.000
	2	$-8.6807308751711217004e(-0236)$	$1.7e(-235)$	$8.7e(-236)$	$6.745381610e(-9)$		16.000
	3	$-7.0130542650643310955e(-3770)$	$1.4e(-3769)$	$7.0e(-3770)$	$6.745381610e(-9)$		
$M4_{16}$	1	$2.5571565668791280433 6e(-0019)$	$5.1e(-019)$	$2.6e(-019)$		$1.168314155e(-3685)$	16.000
	2	$-9.6327073869051604255e(-0456)$	$1.9e(-456)$	$9.6e(-460)$	$2.881583793(-162)$		16.000
	3	$-6.4199777832040922528e(-11029)$	$1.3e(-11029)$	$6.4e(-11030)$	$1.168314155e(-3685)$		

Table 4 Convergence behavior of different methods for $f_3(x)$

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\dfrac{e_{n+1}}{e_n^p}\right|$	η	ρ
ZM_8	1	0.59481086658917665593	1.1e(−007)	1.0e(−007)			
	2	0.59481096839836917752	3.2e(−058)	3.0e(−058)	0.02620066308	0.02620068712	8.000
	3	0.59481096839836917752	1.9e(−462)	1.8e(−462)	0.02620068712		8.000
SM_8	1	0.59481102727180669464	6.2e(−008)	5.9e(−008)			
	2	0.59481096839836917752	1.1e(−060)	1.0e(−060)	0.00703242626280	0.00703242423951	8.000
	3	0.59481096839836917752	8.4e(−483)	7.9e(−483)	0.00703242423951		8.000
KM_8	1	0.59481086658917665593	1.1e(−007)	1.0e(−007)			
	2	0.59481096839836917752	3.2e(−58)	3.0e(−58)	0.02620066308	0.02620068712	8.000
	3	0.59481096839836917752	1.9e(−462)	1.8e(−462)	0.02620068712		8.000
$M1_8$	1	0.59481097926567975830	1.2e(−008)	1.1e(−008)			
	2	0.59481096839836917752	5.8e(−067)	5.4e(−067)	0.002801124722	0.002801124357	8.000
	3	0.59481096839836917752	2.3e(−533)	2.2e(−533)	0.002801124357		8.000
$M2_8$	1	0.59481368494016044661	2.9e(−006)	2.7e(−006)			
	2	0.59481096839836917752	7.9e(−046)	7.5e(−046)	0.2523219191	0.2523199578	8.000
	3	0.59481096839836917752	2.6e(−362)	2.5e(−362)	0.2523199578		8.000
GM_{16}	1	0.59481096839836901617	1.7e(−016)	1.6e(−016)			
	2	0.59481096839836917752	6.1e(−258)	5.8e(−258)	0.00002744572336	0.00002744572336	16.000
	3	0.59481096839836917752	7.0e(−4121)	4.4e(−4121)	0.00002744572336		16.000
KM_{16}	1	0.59481096839836721379	2.1e(−015)	2.0e(−015)			
	2	0.59481096839836917752	8.8e(−242)	8.3e(−242)	1.703671135e(−006)	1.703671135e(-006)	16.000
	3	0.59481096839836917752	9.7e(−3864)	9.2e(−3864)	1.703671135e(−006)		16.000

Table 4 continued

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\dfrac{e_{n+1}}{e_n^p}\right|$	η	ρ
SM_{16}	1	0.59481096839836836640	8.6e(−016)	8.1e(−016)			16.000
	2	0.59481096839836917752	2.0e(−246)	1.9e(−246)	0.0000539889893115	0.0000539889893115	16.000
	3	0.59481096839836917752	1.6e(−3936)	1.5e(−3936)	0.0000539889893115		16.000
$M3_{16}$	1	0.59481096839836917446	3.2e(−018)	3.1e(−018)			16.000
	2	0.59481096839836917752	4.4e(−287)	4.2e(−0287)	7.054095279e(−0007)	7.054095279e(−0007)	16.000
	3	0.59481096839836917752	7.0e(−4589)	6.6e(−4589)	7.054095279e(−0007)		16.000
$M4_{16}$	1	0.59481096839836917555	2.1e(−018)	2.0e(−018)			16.000
	2	0.59481096839836917752	1.1e(−289)	9.9e(−290)	1.817748633e(−006)	1.817748633e(−006)	16.000
	3	0.59481096839836917752	1.7e(−4630)	1.6e(−4630)	1.817748633e(−006)		16.000

Table 5 Convergence behavior of different methods for $f_4(x)$

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\dfrac{e_{n+1}}{e_n^p}\right|$	η	ρ
ZM_8	1	1.4044916540409303481	1.4e(−008)	5.8e(−009)			8.000
	2	1.4044916482153412260	1.5e(−066)	5.9e(−067)	0.4455616048	0.4455615881	8.000
	3	1.4044916482153412260	1.6e(−530)	6.6e(−531)	0.4455615881		8.000
SM_8	1	1.4044916971026842645	1.2e(−007)	4.9e(−008)			8.000
	2	1.4044916482153412260	3.1e(−058)	1.3e(−058)	3.879227039	3.879225932	8.000
	3	1.4044916482153412260	6.3e(−463)	2.6e(−463)	3.879225932		8.000
KM_8	1	1.4044916540409303481	1.4e(−008)	5.8e(−009)			8.000
	2	1.4044916482153412260	1.55e(−066)	5.9e(−067)	0.4455616048	0.4455615881	8.000
	3	1.4044916482153412260	1.6e(−530)	6.6e(−531)	0.4455615881		8.000
$M1_8$	1	1.4044916496757786194	3.6e(−009)	1.5e(−009)			8.000
	2	1.4044916482153412260	5.2e(−072)	2.1e(−072)	0.1013068688	0.1013068674	8.000
	3	1.4044916482153412260	9.4e(−575)	3.8e(−575)	0.1013068674		8.000
$M2_8$	1	1.4044916493812343257	2.9e(−009)	1.2e(−009)			8.000
	2	1.4044916482153412260	9.7e(−073)	3.9e(−073)	0.1148023458	0.1148023452	8.000
	3	1.4044916482153412260	1.6e(−580)	6.4e(−581)	0.1148023452		8.000
GM_{16}	1	1.4044916482153382325	7.4e(−015)	3.0e(−015)			16.000
	2	1.4044916482153412260	1.8e(−229)	7.1e(−230)	2.560922533	2.560922533	16.000
	3	1.4044916482153412260	2.0e(−3663)	7.9e(−3664)	2.560922533		16.000
KM_{16}	1	1.4044916482153572537	4.0e(−014)	1.6e(−014)			16.000
	2	1.4044916482153412260	3.7e(−219)	1.5e(−219)	79.45850676	79.45850676	16.000
	3	1.4044916482153412260	1.4e(−3499)	5.6e(−3500)	79.45850676		16.000

Table 5 continued

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\frac{e_{n+1}}{e_n^p}\right|$	η	ρ
SM_{16}	1	1.4044916482153412372	$2.8e(-017)$	$1.1e(-017)$			
	2	1.4044916482153412260	$7.9e(-273)$	$3.2e(-273)$	0.05676542802	0.05676542802	16.000
	3	1.4044916482153412260	$1.6e(-4361)$	$6.4e(-4362)$	0.05676542802		16.000
$M3_{16}$	1	1.4044916482153412456	$4.7e(-017)$	$1.9e(-017)$			
	2	1.4044916482153412260	$4.5e(-269)$	$1.8e(-269)$	0.06830038366	0.06830038366	16.000
	3	1.4044916482153412260	$2.0e(-4301)$	$8.0e(-4302)$	0.06830038366		16.000
$M4_{16}$	1	1.4044916482153412260	$4.3e(-022)$	$1.7e(-022)$			
	2	1.4044916482153412260	$4.8e(-355)$	$1.9e(-355)$	$3.356166697e(-7)$	$3.356166697e(-7)$	16.000
	3	1.4044916482153412260	$3.2e(-5682)$	$1.3e(-5682)$	$3.356166697e(-7)$		16.000

Table 6 Convergence behavior of different methods for $f_5(x)$

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\dfrac{e_{n+1}}{e_n^p}\right|$	η	ρ
ZM_8	1	0.99999117240896172740	2.6e(−5)	8.8e(−6)			
	2	1.00000000000000000000	1.2e(−43)	4.0e(−44)	0.00108601329	0.001085962506	8.0000
	3	1.00000000000000000000	2.2e(−350)	7.2e(−351)	0.00108596250 6		8.0000
SM_8	1	0.99997694309 1378	6.9e(−5)	2.3e(−5)			
	2	1.00000000000000000000	5.8e(−40)	1.9e(−40)	0.00242936084 2	0.00242926658	7.9999
	3	1.00000000000000000000	1.5e(−320)	4.9e(−321)	0.002429126658		8.0000
KM_8	1	0.99999117240896172740	2.6e(−5)	8.0e(−6)			
	2	1.00000000000000000000	1.2e(−43)	4.0e(−44)	0.00108601329	0.001085962506	8.0000
	3	1.00000000000000000000	2.2e(−350)	7.2e(−351)	0.00108596250 6		8.0000
$M1_8$	1	0.99999187657614305623	2.4e(−5)	8.6e(−6)			
	2	1.00000000000000000000	3.2e(−44)	1.1e(−44)	0.0005573094681	0.0005572702332	7.9999
	3	1.00000000000000000000	2.6e(−355)	8.7e(−356)	0.0005572702332		8.0000
$M2_8$	1	1.00011427470096153	3.4e(−5)	1.1e(−5)			
	2	1.00000000000000000000	4.4e(−2)	1.5e(−42)	0.005029512834	0.005029721079	8.0000
	3	1.00000000000000000000	3.2e(−337)	1.1e(−337)	0.005029721079		8.0000
GM_{16}	1	0.97075500462797713557	8.8e(−2)	2.9e(−2)			
	2	1.00000000000000000000	6.4e(−27)	2.1e(−27)	0.0074801681 0	0.007965130040	16.018
	3	1.00000000000000000000	4.7e(−429)	1.6e(−429)	0.007965130040		16.000
KM_{16}	1	0.99999999899865540584	3.0e(−9)	1.0e(−9)			
	2	1.00000000000000000000	3.0e(−149)	9.9e(−150)	9.717639193e(−6)	9.717639121e(−6)	16.000
	3	1.00000000000000000000	2.6e(−2389)	8.7e(−2390)	9.717639121e(−6)		16.000

Table 6 continued

Cases	n	x_n	$|f(x_n)|$	$|e_n|$	$\left|\frac{e_{n+1}}{e_n^p}\right|$	η	ρ
SM_{16}	1	0.9999999990659235882	2.8e(−10)	9.3e(−11)			16.000
	2	1.000000000000000000	1.1e(−166)	3.8e(−167)	1.127590242e(−6)	1.127590241e(−6)	16.000
	3	1.000000000000000000	6.0e(−2669)	2.0e(−2669)	1.127590241e(−6)		16.000
$M3_{16}$	1	0.9999999999850223986	2.7e(−10)	9.1e(−11)			16.000
	2	1.000000000000000000	2.9e(−167)	9.8e(−168)	4.056275860e(−7)	4.056275854e(−7)	16.000
	3	1.000000000000000000	8.6e(−2679)	2.9e(−2679)	4.056275854e(−7)		16.000
$M4_{16}$	1	0.9999999999999996128	1.2e(−16)	3.9e(−17)			16.000
	2	1.000000000000000000	1.8e(−273)	6.0e(−274)	2.351966590e(−11)	2.351966590e(−11)	16.000
	3	1.000000000000000000	2.0e(−4382)	6.7e(−4383)	2.351966590e(−11)		16.000

respectively. Further, we also depict the values of $\left|\frac{x_{k+1}-x_k}{(x_k-x_{k-1})^p}\right|$ and asymptotic error constant term η in 10 digits. Furthermore, modulus value of residual errors $|f(x_k)|$ and $|x_{k+1}-x_k|$ are display in 2 digits with exponent power in the Tables 2, 3, 4 5 and 6. Finally, the estimated zeros up to 35 digits and initial approximations are also mentioned in the Table 1.

For the computer programming, we use the programming package *Mathematica* 9. Further, $x(\pm y)$ stands for $x \times 10^{(\pm y)}$ in the Tables 2, 3, 4 5 and 6.

4 Conclusions

In this paper, we present two schemes which are applicable to each earlier optimal multi-point derivative free scheme of order four and eight whose first sub step should be Steffensen's type method to construct further new optimal iteration functions of order eight and sixteen, respectively. In addition, they do not use any kind of derivative of the considered function and also satisfy the classical Kung–Traub optimality conjecture. We can easily obtain many new optimal iteration functions of order eight and sixteen by applying our schemes (2.9) and (2.10) on the earlier existing methods of order four and eight. In the similar fashion, we can obtain optimal derivative schemes of order 32, 64, etc. by consider high degree of rational functions. Finally, the efficiency of them is checked on a good number of standard test problems. It is find that their performance is more useful than the existing ones of same order.

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