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ORIGINAL PAPER

Centrality measure in graphs

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Abstract Centrality of an edge of a graph is proposed to be viewed as a degree of global sensitivity of a graph distance function (i.e., a graph metric) on the weight of the considered edge. For different choices of distance function, contact is made with several previous ideas of centrality, whence their different characteristics are clarified, and strengths or short-comings are indicated, via selected examples. The centrality based on "resistance distance" exhibits several nice features, and might be termed "amongness" centrality.

Keywords Centrality · Betweenness · Shortest-path centrality · Resistance-distance centrality · Graph metrics · Neighborliness centrality

1 Introduction

One mathematical idea concerning centrality in a graph was presented 140 years ago, at least for the case of trees, by Jordan [1]. Also centrality has (starting before 1970) been considered for use in transportation-network theory [2–4] in communication-network theory [5–10] in psychology [11–14] in sociology [15–26] in geography [27] and in game theory [28]. Now an even more widespread interest is developing, for applications: in electrical circuits [29,30] in molecular bonding patterns [31–33] in biochemical reaction networks [34,35] in ecological or population-genetics migration patterns [36] and perhaps in food-webs [37]. The idea of centrality has become much investigated in computer science [7,9,10,38–42] and some recent articles [43–46] have been directed to a general understanding of centrality without focus to any particular application. There have been several centrality definitions advocated

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[1,2,6,13,21,29,31,38,42,43] different efficient means have been sought [8,42,43,47,48] for the computation of different centrality measures, further mathematical results have been enunciated [43,45,46,49,50] and comparisons of different definitions have been made [51–55]. Some centrality measures were developed [11,17] for directed graphs, possibly even highly directed, as occur in organizational (e.g., business or military) structures, but more often focus has been on the undirected case.

In 1979 Freeman [23] gave an informative critical discussion of centrality. He reviewed earlier work (primarily in sociology), and identified three overall views:

(1) the centrality of a graph part (site, or perhaps edge) depends on the connectedness through immediate edge connections to other sites;
(2) the centrality of a site depends on the frequency with which a graph part falls on relevant (perhaps shortest) paths between all pairs of other sites; and
(3) the centrality of a graph part depends on the degree to which it is close to the rest of the graph (measured by an average distance to the sites of the graph).

Freeman discussed a few particular centrality measures.

Generally one desires not just an identification of the most central edge (or vertex) but also a measure of the degree of centrality for each edge (or vertex). And it should apply to weighted graphs, as so often arise in practice. It might even be wished to compare centralities in different graphs.

Here for the circumstance of undirected graphs, "centrality" is viewed to depend in a global manner on a graph distance function ρ, and more particularly on global changes in ρ which occur upon modification of a considered graph part (say an edge). The "centrality" of an edge of a graph G is naturally measured by the sensitivity of such a graph metric ρ to changes in the weight of the edge. That is, centrality is naturally measured in terms of sensitivity to changes in the "communicability" or "strength" of the edge, as indicated by its influence on ρ. Indeed "betweenness" centrality has already been often framed in terms of the shortest-path distance, but as there are [56–59] different fundamental distance functions definable on a graph, this naturally indicates the possibility of different centrality definitions. Here indication is made of some characteristics, advantages, and short-comings of different consequent centrality definitions. Interrelations to earlier defined centrality measures are frequently found. Ultimately focus is directed to a centrality measure based on the so-called "resistance distance".

2 Formal framework and general definition

The discussion here is formally phrased in terms of a *weighted graph* $G = (V, E, \mathbf{A})$, where V is the *vertex* set of G, $\mathbf{A}$ is a matrix of weights with real finite elements $A_{xy} = A_{yx} \geq 0, x, y \in V$, and $E \equiv \{\{x, y\} : A_{xy} > 0\}$ is the *edge* set of G. Technically with $\mathbf{A}$ given, E is redundant, but is retained as its use is standard, especially for unweighted G. The matrix $\mathbf{A}$ might be called a weighted adjacency matrix or conductance matrix, or an *admittance* matrix. Generally it is assumed that G is *finite* and *connected*. Higher weights are interpreted to mean that there is better or closer contact or communication between the so interconnected sites. That is, this weight has an inversive relation to distances between (neighbor) sites. A general *distance function*, or metric, $\rho : V \times V \to \mathbb{R}$ satisfies

$$\begin{array}{ll} \rho(x,x) = 0, & x \in V \\ \rho(x,y) = \rho(y,x) > 0, & x \neq y \in V \\ \rho(x,y) + \rho(y,z) \geq \rho(x,z), & x,y,z \in V \end{array}$$

It further is recognized that plausibly such a ρ should be sensitive to the edge weights (or more directly to the inverses, of the non-zero weights), and should vary with these weights such that distances are nonincreasing with increasing weight. The [3] respective conditions above mean that each singleton, doublet, or triplet of points can be faithfully (or isometrically) embedded in 0-, 1-, or 2-dimensional Euclidean space. If the third condition is not met then ρ "degenerates" to a *pseudometric*. A distance function varying with all non-zero weights is termed **A**-*sensitive*. A general way to take an overall measure of such a metric is by way of the *Wiener-Bavelas ρ-sum*

$$W_\rho(G) \equiv \frac{1}{2} \sum_{x,y}^{\in V} \rho(x,y)$$

in rememberance of Harry Wiener [60,61] and of Alex Bavelas [15,16], each of whom early on used such a sum with ρ the "shortest-path" metric.

Now the proposed *ρ-centrality* measure for an edge $\{u,v\} \in E$ is

$$c_\rho(u,v) \equiv \frac{A_{uv}^{-1}}{n} \frac{\partial W_\rho(G)}{\partial A_{uv}^{-1}}$$

where $n \equiv |V|$ divides out some effects of the size of a graph. From a conventional graph-theoretic perspective for unweighted graphs (i.e., those with all $A_{xy} \in \{0, 1\}$), it may be noted that this centrality is an "intrinsic" measure for ordinary (unit-weighted) G, when the A_{uv}^{-1}-derivative is taken for a given $\{u,v\} \in E$ and evaluated at $A_{uv} = 1$—at least this result is "intrinsic" so long as ρ itself is "intrinsic". For molecular graphs, one could naturally be interested in the Euclidean metric for different embeddings of the molecule in 3-dimensional space—though this is not "intrinsic" to the (unembedded) graph. Still in many (perhaps most) applications, weights will be desired and depend on "extrinsic" features of what is being modelled. But even if weights reflect extrinsic features, the definition is still "intrinsic" relative to the weighted graph. Regardless of such considerations, the *centrality* of a vertex $u \in V$ is naturally expressed in terms of the sensitivities of the edges to which it is incident, thusly

$$c_\rho(u) \equiv \sum_{v}^{\{u,v\} \in E} c_\rho(u,v)$$

Further a *compactness* invariant for the (weighted) graph might also be defined as

$$c_\rho(G) \equiv \sum_{\{u,v\}}^{\in E} c_\rho(u, v)$$

Presumably the edge centrality, vertex centrality, and compactness functions can all be distinguished by their arguments—thereby avoiding some proliferation of symbols.

With these formal definitions in hand, the next matter at hand is to explore these ideas in the context of different choices for the distance function ρ, thereby looking at different possible centrality measures.

3 Neighborliness or degree centrality

A first "degenerate" case is based on a very simple choice for ρ, as

$$\rho_{neighbor}(x, y) \equiv \left\{ \begin{matrix} 0, & x = y \\ 1/A_{xy}, & x \neq y \end{matrix} \right\}, \quad x, y \in V$$

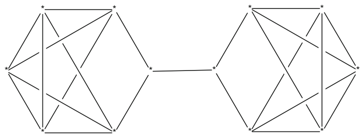
This is "degenerate" in that when $x \neq y$ & $\{x, y\} \notin E$, the result $\rho_{neighbor}(x, y)$ is infinite, $+\infty$. Then this function does not generally satisfy the third (triangle-inequality) for a metric, and is only a pseudometric. Moreover, the Wiener $\rho_{neighbor}$-sum is usually $+\infty$. But if it is assumed that the formally infinite part of this sum is constant, one still obtains a formula for the related centrality of an edge $\{u, v\} \in E$, as

$$c_{neighbor}(u, v) = A_{uv}^{-1}/n$$

Indeed were all the non-edges $\{x, y\}$ to be given a small weight ε, the value of $\rho_{neighbor}(x, y)$ becomes $= 1/\varepsilon$, whence if the limit $\varepsilon \to 0$ is taken after the derivative $\partial/\partial A_{uv}^{-1}$ for $\{u, v\} \in E$, the centrality is just $1/A_{uv}n$, while for $\{x, y\} \notin E$ the result is 0. The (now likely undesirable) factor of $1/n$ occurs because of the extreme "degeneracy" of this pseudometric, with only a single value of $\rho_{neighbor}(x, y)$ depending on A_{uv}. The associated vertex centrality then is

$$c_{neighbor}(u) = n^{-1} \sum_{v}^{\{u,v\}\in E} A_{uv}^{-1}$$

such as has already been taken [11,12,23] as a centrality measure, and sometimes (without the n^{-1} factor) referred to as "degree" centrality–and sometimes this is referred to as the "eccentricity" of the vertex. These centralities of course do not reflect (directly) on how the non-neighbor pairs are influenced by the weight for a given edge (or site). As an illustration of this point consider the (unweighted) graph:



Here one sees a clear intuitive choice for a most "central" edge (in the center of the graph), though this is the least "central" according to the current neighborliness centrality measure (with or without the $1/n$ factor).

4 Shortest-path centrality

The next centrality index is based on the most common distance function, which for an unweighted graph is simply the length of a shortest path between the 2 considered vertices. In fact, in Buckley and Harary's seminal *Distance in Graphs* [62] this is the only metric considered. For the weighted case it is given as

$$\rho_{sp}(x, y) \equiv \min_{\pi : x \leftrightarrow y} \sum_{\{u,v\}}^{\in E(\pi)} A_{uv}^{-1}$$

where the minimum is taken over all paths π between x & y. The Wiener *sp*-sum is defined straightforwardly for the weighted case. But to illustrate a particular behavior of the centrality measure, we temporarily specialize to the unweighted case, whence it is noted that its derivative is in general ill-defined, as it can be different for a left derivative or a right derivative, when there are alternative $x, y - geodesics$ which are minimum weight paths between x & y. That is, this devolves to two possibilities, c_{sp+} & c_{sp-} as the derivative with respect to A_{uv}^{-1} is evaluated as A_{uv}^{-1} approaches 1 from above (+) or below (−). That is,

$$c_{sp\pm}(u, v) \equiv \frac{A_{uv}^{-1}}{n} \left. \frac{\partial W_{sp}(G)}{\partial A_{uv}^{-1}} \right]_{A_{uv}^{-1} \to 1\pm}$$

To further develop this, it is useful to introduce the set G_{xy} of all x, y-geodesics, and let $\delta(uv\forall G_{xy}) = 1$ if $\{u, v\} \in E$ is in all geodesics in G_{xy}, while otherwise $\delta(uv\forall G_{xy}) = 0$. That is, this δ-function characterizes just which geodesics contribute to c_{sp-}. Yet further let $\delta(uv\exists G_{xy}) = 1$ if $\{u, v\} \in E$ is in some geodesic in G_{xy}, and otherwise $\delta(uv\exists G_{xy}) = 0$. Thence $\delta(uv\exists G_{xy})$ characterizes the geodesics which contribute to c_{sp+}. As a consequence:

Proposition 1 *For G an unweighted graph with $c_{sp\pm}$, $\delta(uv\forall G_{xy})$, & $\delta(uv\exists G_{xy})$ defined as above for an edge $\{u, v\} \in E$,*

$$c_{sp-}(u, v) = \frac{1}{2n} \sum_{x,y}^{\in V} \delta(uv\forall G_{xy}) \quad \& \quad c_{sp+}(u, v) = \frac{1}{2n} \sum_{x,y}^{\in V} \delta(uv\exists G_{xy})$$

In as much as these two centrality indices are obtained from looking at deviations of weights in one direction or another (from 1), one also might define an average shortest-path centrality, as

$$c_{sp0}(u, v) \equiv \{c_{sp+}(u, v) + c_{sp-}(u, v)\}/2$$

All this may be compared to the betweenness centrality index of Freeman [15,23](and Anthonoisse, as quoted by Freeman [23]). This entails the fraction $g_{xy}(u, v)$ of geodesics in G_{xy} which contain $\{u, v\}$, whence their *betweenness* centrality is given by

$$c_{betweenness}(u, v) = \frac{1}{2n} \sum_{x,y}^{\in V} g_{xy}(u, v)$$

The intimacy of relation of the various centrality indices of this section is indicated in a table indicating the values of the summands in the different cases:

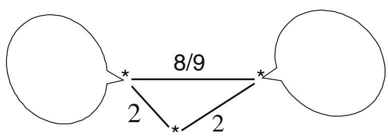
	xy-summand for		
$g_{xy}(u, v) \equiv p$	*sp*+	*sp*0	*sp*
0	0	0	0
0< p <1	1	1/2	0
1	1	1	1

Though clearly closely related, Freeman's betweenness centrality is evidently more discriminating, and one evidently also has:

Proposition 2 *For G an unweighted graph, the shortest-path centrality indices for any edge* $\{u, v\} \in E$ *satisfy*

$$c_{sp-}(u, v) \le c_{sp0}(u, v) \le c_{sp+}(u, v) \,\&\, c_{sp-}(u, v) \le c_{betweenness}(u, v) \le c_{sp+}(u, v)$$

Further it should be noted that the shortest-path metric in the weighted case is not always **A**-sensitive. A case in point is provided by the weighted graph:



where the graph is rather arbitrary within each of the "balloons". Recall that the geodesics are given in terms of the inverses of these indicated weights, whence one sees that no geodesic for any pair of vertices $x, y \in V$ contain the 8/9-weighted edge. Thence any of the shortest-path centralities for this 8/9-weighted edge is 0–though this edge carries almost as much traffic as the two 2-weighted edges. For many choices of the rest of the graph in the two "balloon" parts, the above 2-weighted edges turn out to be the central edges, though in using the shortest-path (or betweenness) centralities, with the 8/9-weighted edge gradually increased in weight (by up to a little over 11%), it would abruptly change to become of maximum centrality.

5 Resistive centrality

The resistive centrality is based on a more recently recognized metric [56], which may be defined in several different ways. One such way is to view $G = (V, E, \mathbf{A})$ as an electrical network with resistors of values A_{uv}^{-1} on each edge, and take the *resistance distance* $\rho_\Omega(x, y)$ to be the effective resistance between nodes $x \in V$ & $y \in V$ (as when the two poles of a battery are connected to x & y). Alternatively ρ_Ω can be given in a formula in terms of the *Laplacian* matrix $\mathbf{L} \equiv \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix whose xth diagonal element is $D_{xx} \equiv \sum_y^{\in V} A_{xy}$. This matrix $\mathbf{L}$ has a 0-eigenvalue for the (eigen-) vector $\vec{e}$ of all 1s, and this eigenvector is unique (up to scalar multiples) so long as G is connected (as we have assumed). [This and some related things are reviewed in [56], though $\mathbf{L}\vec{e} = 0$ has long been known—with many results appearing in various standard electrical-circuits texts, back to Maxwell [63], and some results go back to Kirchoff [64].] In the connected case there then is a matrix Γ which is the inverse to $\mathbf{L}$ on the subspace orthogonal to $\vec{e}$ and otherwise is 0, whence the product of Γ & $\mathbf{L}$ is the idempotent projector $\mathbf{O}_{(\vec{e})}$ onto this subspace orthogonal to $\vec{e}$. Now in terms of this matrix Γ, Kirchoff's laws lead to the resistance distance being given as

$$\rho_\Omega(x, y) = \Gamma_{xx} - \Gamma_{xy} - \Gamma_{yx} + \Gamma_{yy}, \quad x, y \in V$$

A further physical interpretation [57] of ρ_Ω again starts with the Laplacian matrix $\mathbf{L}$, noting that its action on a vector is essentially just a discretized form of the continuum Laplacian ∇^2 of mathematical physics. As a consequence the eigenvectors $\vec{\psi}_\varepsilon$ of $\mathbf{L}$ should be viewable as discretized versions of standing waves, with eigenvalues ε corresponding to wave energies, which are generally expected to be lower for the longer wavelength such waves. Then granted that these eigenvectors $\vec{\psi}_\varepsilon$ (with $\Gamma\vec{\psi}_\varepsilon = \varepsilon\vec{\psi}_\varepsilon$) are normalized, as

$$\vec{\psi}_\varepsilon^\dagger \cdot \vec{\psi}_\varepsilon = \sum_u^{\in V} |\psi_{\varepsilon u}|^2 = 1$$

the resistance distance may be rewritten as

$$\rho_\Omega(x, y) = \sum_\varepsilon^{>0} (\psi_{\varepsilon x} - \psi_{\varepsilon y})^2/\varepsilon, \quad x, y \in V$$

and interpreted as a wave-amplitude "correlation function" with more major contributions from the higher weighted longer wave-length (lower energy) standing waves. There is also a combinatorial interpretation [65], as well as a probablistic (random-walk-based) interpretation [66]. All of these interpretations point [57] to a fundamentality for this metric.

Granted this resistance-distance metric, the associated centrality index may be addressed. Evidently

$$c_\Omega(u,v) = \frac{A_{uv}^{-1}}{2n}\frac{\partial}{\partial A_{uv}^{-1}}\sum_{x,y}^{\in V}(\Gamma_{xx} - \Gamma_{xy} - \Gamma_{yx} + \Gamma_{yy})$$

But since $\vec{e}$ is a 0-eigenvector to Γ, $\sum_y \Gamma_{xy}{=}0{=}\sum_x \Gamma_{yx}$, and also $\sum_{x,y} \Gamma_{xx}{=}n \cdot \mathrm{tr}\Gamma$, where tr denotes the trace operation. Thence the centrality expression simplifies to

$$c_\Omega(u,v) = A_{uv}^{-1}\frac{\partial}{\partial A_{uv}^{-1}}\mathrm{tr}\{\Gamma\}$$

But further noting that $\Gamma = \mathrm{O}_{(\vec{e})}\Gamma = \Gamma\mathbf{L}\Gamma$, one has

$$\frac{\partial\Gamma}{\partial\xi} = \frac{\partial}{\partial\xi}\{\Gamma\mathbf{L}\Gamma\} = \frac{\partial\Gamma}{\partial\xi}\cdot\mathbf{L}\Gamma + \Gamma\cdot\frac{\partial\mathbf{L}}{\partial\xi}\cdot\Gamma + \Gamma\mathbf{L}\cdot\frac{\partial\Gamma}{\partial\xi} = 2\cdot\frac{\partial\Gamma}{\partial\xi} + \Gamma\cdot\frac{\partial\mathbf{L}}{\partial\xi}\cdot\Gamma$$

with ξ being an arbitrary variable (such as A_{uv}^{-1}). Thence $\frac{\partial\Gamma}{\partial\xi} = -\Gamma\cdot\frac{\partial\mathbf{L}}{\partial\xi}\cdot\Gamma$, and for $\xi \equiv A_{uv}^{-1}$, one has $\frac{\partial\mathbf{L}}{\partial A_{uv}^{-1}} = -A_{uv}^2\cdot\frac{\partial\mathbf{L}}{\partial A_{uv}} = -A_{uv}^2\cdot(\mathbf{e}_{uu} - \mathbf{e}_{uv} - \mathbf{e}_{vu} + \mathbf{e}_{vv})$ where $\mathbf{e}_{xy}$ is the matrix of all 0s except in the $(x, y)th$ position where there is a 1. Then

$$c_\Omega(u,v) = -A_{uv}^{-1}\cdot\mathrm{tr}\left\{\Gamma\cdot\frac{\partial\mathbf{L}}{\partial A_{uv}^{-1}}\cdot\Gamma\right\} = A_{uv}\cdot\mathrm{tr}\{\Gamma\cdot(\mathbf{e}_{uu} - \mathbf{e}_{uv} - \mathbf{e}_{vu} + \mathbf{e}_{vv})\cdot\Gamma\}$$

Now $tr\{\Gamma\mathbf{e}_{yz}\Gamma\} = \sum_x \Gamma_{xy}\Gamma_{zx} = (\Gamma^2)_{zy}$, so that one obtains:

Proposition 3 *Let $G = (V, E, A)$ be a weighted graph, with ρ_Ω the resistance distance, and associated edge centrality as in the introduction. Then for $u, v \in V$,*

$$c_\Omega(u,v){=}A_{uv}\cdot\{(\Gamma^2)_{uu} - (\Gamma^2)_{uv} - (\Gamma^2)_{vu} + (\Gamma^2)_{vv}\} = A_{uv}\cdot\sum_{\varepsilon}^{>0}(\psi_{\varepsilon u} - \psi_{\varepsilon v})^2/\varepsilon^2$$

This result for c_Ω may be compared with the two formulas for ρ_Ω as given in the preceding paragraph. Evidently c_Ω is essentially as easy to compute as ρ_Ω, for which the requisite matrix inversion is generally understood to be an $O(n^3)$ process—see e.g. [67]. If one is interested only in the centrality of one or a few edges, then the process of computing $\Gamma(\vec{e}_u - \vec{e}_v)$ & $\Gamma^2(\vec{e}_u - \vec{e}_v)$ (as needed for the associated ρ_Ω & c_Ω values—$\vec{e}_x$ being the vector of 0s except for a single 1 in the xth position) is just an $O(n^2)$ process. And if the graph is suitably sparse, say with a degree limited to 4 (as for organic molecular graphs), then these process for such a few centralities can be reduced to $O(n)$—by treating $\mathbf{L}$ as an array of non-zero matrix-element values and their positions in $\mathbf{L}$.

Further from these expressions it is seen that the resistance distance ρ_Ω is $\mathbf{A}$-sensitive, whence also is c_Ω. That is, c_Ω gives non-zero centralities for the cases in the figures of the preceding sections, where previous centrality measures encountered difficulties.

6 Inter-relations to resistive centrality

There is an earlier definition by Newman [42] for centrality based on random walks, though particularly the formulas are based on electrical network theory, which is in fact intimately related [66] to random walks on networks. Newman's definition proposes viewing the graph as corresponding to an electric network with resistors on each edge, considering the connection of a power source to each different pair of vertices, and taking the average magnitude of current flowing through an edge as its centrality. The first two steps are just as in one of the interpretations in the preceding section. The current magnitude is used, rather than just the current, as the current can be positive or negative, with different contributions corresponding to different battery attachments, and so cancelling. But beyond Newman's considerations [42], voltage difference across an edge might also be relevant for centrality–as an edge far on the periphery will typically develop little potential across it for most connections of the battery, while one centrally located should typically develop a more substantial potential difference. Of course, the potential difference and the current through an edge are proportional, via Ohm's law. But it might be imagined that what is more fundamental is the net activity in the edge, as measured by the electric power dissipated in the considered edge–power being equal to the product of the potential difference and current. In another language, not only the net traffic flow, but also its speed matters–the speed depending on the pressure (or potential) difference between the two ends of a traffic channel. That is, of two channels carrying the same amount of traffic, one might be narrower carrying traffic at higher speeds and thereby cause more problems if interrupted. Granted potentials φ_w^{xy} at a vertex w along with currents $I_{u\to v}^{xy}$ through edge $\{u, v\} \in E$ when the battery is connected to $x \in V$ & $y \in V$, the *power* dissipated in the edge $\{u, v\}$ is

$$I_{u\to v}^{xy} \cdot (\varphi_u^{xy} - \varphi_v^{xy}) \equiv P_{uv}^{xy}$$

Newman's earlier proposed [42] current-based centrality then is $n^{-1} \sum_{x,y}^{\in V} |I_{u\to v}^{xy}|$.

But if cognizance is taken of the preceding remarks, a quite natural *power* centrality might be defined as

$$c_{power}(u, v) \equiv \frac{1}{2n} \sum_{x,y}^{\in V} P_{uv}^{xy}$$

Notably there is no problem about taking absolute values (of currents or voltages) as the signs are correlated, so that the resultant power (dissipated) is positive. Then

$$c_{power}(u, v) = \frac{1}{2n} \sum_{x,y}^{\in V} I_{u\to v}^{xy} \cdot (\varphi_u^{xy} - \varphi_v^{xy}) = \frac{1}{2n} \sum_{x,y}^{\in V} (\varphi_u^{xy} - \varphi_v^{xy})^2 \cdot A_{uv}$$

But as this potential difference is [56] given by $\varphi_u^{xy} - \varphi_v^{xy} = \Gamma_{ux} - \Gamma_{uy} - \Gamma_{vx} + \Gamma_{vy}$, one may substitute this, then use the $\mathbf{L}\vec{e} = 0$ relation along with the symmetry of Γ, to obtain

$$\begin{aligned} c_{power}(u, v) &= \frac{A_{uv}}{n} \cdot \sum_{x,y}^{\in V} (\Gamma_{ux} - \Gamma_{uy} - \Gamma_{vx} + \Gamma_{vy})^2 \\ &= \frac{A_{uv}}{n} \cdot \sum_{x,y}^{\in V} \{(\Gamma_{ux} - \Gamma_{vx})^2 + (\Gamma_{vy} - \Gamma_{uy})^2\} \\ &= A_{uv} \cdot \{(\Gamma^2)_{uu} - (\Gamma^2)_{uv} - (\Gamma^2)_{vu} + (\Gamma^2)_{vv}\} \end{aligned}$$

Thus:

Proposition 4 *Let G be a weighted graph with resistance distance ρ_Ω, associated Ω-centrality c_Ω, and power centrality c_{power}. Then $c_{power} = c_\Omega$.*

That is, c_{power} reproduces the result of the preceding section—now providing a perhaps not unsurprising alternative interpretation of this (resistive) centrality measure: $c_{power} = c_\Omega$ is the power dissipated in the considered edge averaged over all patterns of connection of the battery.

A further comment may be made in view of Borgatti & Everett's note [55] that "all of the [centrality] measures evaluate a node's involvement in the walk structure of a network". Here the "walk structure" can refer to paths, geodesics, and weighted walks. In reviewing the field, Borgatti & Everett evidently did not entertain the idea of random walks, such as may be perceived to be involved in the resistive centrality introduced here. Now since $\varphi_u^{xy} - \varphi_v^{xy} = I_{uv}^{xy} A_{uv}^{-1}$, one may rewrite the expression half a dozen lines before Proposition 4 as

$$c_{power}(u, v) = \frac{1}{n} \sum_{x,y}^{\in V} (I_{u \to v}^{xy})^2 \cdot A_{uv}^{-1}$$

But as the current $I_{u \to v}^{xy}$ turns out to equal *[66]* the expected net number $\#_{uv}^{x \to y}$ of times that a random walker, starting at x and waking till reaching y will move along the edge from u to v, we then have:

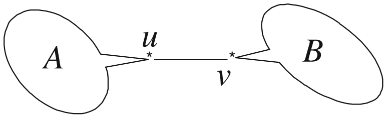
Proposition 5 *Let G be a weighted graph with resistance distance ρ_Ω. Then*

$$c_\Omega(u, v) = \frac{A_{uv}^{-1}}{n} \sum_{x,y}^{\in V} (\#_{uv}^{x \to y})^2$$

That is, a "walk structure" interpretation, of the general type of Borgatti & Evertt has been achieved.

7 Disconnectional ideas, etc

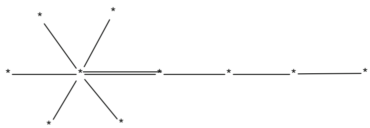
A few words might be said about graphs G which have *cut edges* (i.e., an edge which if deleted from G leaves a disconnected result). A cut edge $\{u, v\}$ appears as:



In this case, either all or none of the geodesics between a pair of points pass through the cut edge. If $x \in V$ is in the A-part of G, while y is in the B-part, then all geodesics pass through $\{u, v\}$; whereas, for $x, y \in V$ with either both in A or both in B, none of the geodesics pass through $\{u, v\}$. As a consequence, all the shortest-path centralities for $\{u, v\}$ are the same, and moreover (following Wiener [60,61]) the number of contributing geodesics is just the product of the numbers of sites in the A- & B-parts. If $n_{u(v)}$ denotes the number of sites in G closer to u than v, then this is just the number of sites in the A-part, and the number of sites in the B-part is similarly just $n_{v(u)}$, whence $c_{sp}(u, v) = A_{uv} \cdot n_{u(v)} n_{v(u)}/n$. Also the resistance distance manifests similar characteristics. Again if $x \in V$ & $y \in V$ are either both in A or both in B, then $\rho_\Omega(x, y)$ is independent of A_{uv}. And if x & y are in different parts, $\rho_\Omega(x, y)$ turns out to be a sum $\rho_\Omega(x, u) + A_{uv}^{-1} + \rho_\Omega(v, y)$, with $\rho_\Omega(x, u)$ & $\rho_\Omega(v, y)$ independent of A_{uv}. That is, though the resistance distance between $x \in A$ & $y \in B$ are generally different than the corresponding shortest-path distance, the rates of change of each of these distances with respect to A_{uv}^{-1} are the same. Thus we have:

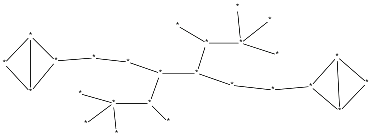
Proposition 6 *Let G be a weighted graph with cut edge $\{u, v\} \in E$. Then the shortest-path and resistive centralities for $\{u, v\}$ are the same* $= A_{uv}^{-1} \cdot n_{u(v)} n_{v(u)}/n$.

This last proposition of course applies for every edge of a tree. It does not however, generally give the same most central edge as Jordan's [1] original definition, e.g., as witnessed with the tree:



Here (when the tree is unweighted) the "doubled" edge has the maximum Ω-centrality (of $6 \times 4 = 24$), whereas Jordan's [1] central edge is obtained by successive prunings of leaves (i.e., deletions of degree-1 vertices), to leave (after two stages of such prunings) the edge just to the right of the "doubled" one.

Jordan [1] defined this center just for trees. But one might seek to extend the idea to general graphs, say by first undergoing successive deletions of degree-1 vertices, till this can be done no longer, then deleting degree-2 vertices, iteratively (while deleting any degree-1 vertices which might arise along the way) till nothing more is possible, then continuing with degree-3 vertices, etc. For a single cycle, this extended scheme deletes all the vertices at once, leaving the empty graph, and thereby indicating that all edges (or vertices) have the same degree of centrality. But on the other hand, such an extension does some anomalous things, as for the graph:



The indicated extension of Jordan's deconstruction yields the 6 edges (& sites) of the two triangles comprised of degree-3 sites as the most central, although these are on what is plausibly the "periphery" of the graph. The neighborliness centrality gives the two horizontal edges connected between degree-3 & -4 sites to be of maximum centrality. On the other hand, via the shortest-path or resistive choices, the "center" edge here has the highest centrality. Generally both the "extension" of Jordan's idea as well as the neighborliness centralities depend overly strongly on local features—without accounting for the global graph structure.

Yet another sort of centrality index which has been suggested [18] and much used [7,9,20,25] is that of "eigenvector centrality", which for a vertex u is essentially just the uth component of the maximum-eigenvalue eigenvector to the adjacency matrix **A**. This has many desirable features—in not being over-sensitive solely to the neighbors or changing abruptly as geodesics change. This behavior is similar to the resistive centrality, though formal connections do not seem apparent.

As to the general difficulty and multiplicity of centrality definitions, this might but signal that in fact centrality is a partially ordered property. The different centrality measures might just be different order-consistent (ordered) scalar representations of the underlying partial order.

8 Conclusion

A general metric-based definition of edge centrality has been presented, as the sensitivity of a distance-sum to changes in the weight assigned to the considered edge. This idea is noted to specialize under different choices of graph metric to give different centrality notions, which have different characteristics, advantages, & short-comings. Those centralities based on the neighborliness pseudometric or on the shortest-path metric relate most closely to earlier studied & advocated centrality measures, while the approach taken here appears in a unified way looking at the effect of a local structure on a metric as a whole. The shortest-path metric (and even more-so the pseudometric) have simplistic "all-or-none" features, with results sometimes changing abruptly upon infinitesimal changes in weightings, and some aspects of this all-or-nothing feature carries over to the associated centralities, evidently as recognized by Borgatti & Everett [55] when they comment "the reliance on geodesic paths alone may be undesirable". The new Ω-centrality measure, based on the resistance distance, seems especially attractive, in terms of: motivation, global sensitivity, and algorithmic felicity. This resistive centrality measure looks at more averaged features, e.g., as detected by waves, (weighted) random walks, or current flows in the network—with the currents being interpretable in an electrical or hydrological fluid vein. Indeed, the flows & random walks should be interpretable in terms of the movement of goods (in a transportation network), or of information transmission (in a communication network), or of migration of members of a population (in an eco-network), or of fluxes of biomolecules (in a biomolecular reaction network). Often it is said [25,39,40,42,43,49,50] that the shortest-path centrality measures "betweenness", and so in some sense does the Ω- centrality, though with a more global detection of the whole network and its manner of interconnection. One might better say that the resistive centrality measures "amongness", and alternatively call it an *amongness centrality*.

More generally it is proposed that various interpretational concepts beyond the centrality explored here can be profitably developed in terms of the different metrics available. A nontrivial intrinsic geometry of graphs might [57,68–70] be possible, with diverse applications.

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