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Exact solutions of the Schrödinger equation with position-dependent effective mass via general point canonical transformation

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Exact solutions of the Schrödinger equation are obtained for the Rosen–Morse and Scarf potentials with the position-dependent effective mass by appliying a general point canonical transformation. The general form of the point canonical transformation is introduced by using a free parameter. Two different forms of mass distributions are used. A set of the energy eigenvalues of the bound states and corresponding wave functions for target potentials are obtained as a function of the free parameter.

KEY WORDS: position-dependent mass, point canonical transformation, effective mass Schrödinger equation, Rosen–Morse potential, Scarf potential

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1. Introduction

Exact solutions of the effective mass Schrödinger equations for some physical potentials have much attention. They have found important applications in the fields of material science and condensed matter physics such as semiconductors [1], quantum well and quantum dots [2], ^3H clusters [3], quantum liquids [4], graded alloys and semiconductor heterostructures [5,6]. Recently, number of exact solutions on these topics increased [6–23]. Various methods are used in the calculations. The point canonical transformations (PCT) is one of these methods providing exact solution of energy eigenvalues and corresponding eigenfunctions [24–27]. It is also used for solving the Schrödinger equation with position-dependent effective mass for some potentials [8–13]. In the present work, we

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solve two different potentials with the two mass distributions. The point canonical transformation is taken in the more general form introducing a free parameter. This general form of the transformation will provide us a set of solutions for different values of the free parameter.

The contents of the paper is as follows. In section 2, we present briefly the solution of the Schrödinger by using point canonical transformation. In section 3, we introduce some applications for specific potentials. Results are discussed in section 4.

2. Method

To introduce a general form of PCT with a free parameter, we start from a time independent Schrödinger equation for a potential $V(y)$

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+V(y)\right)\phi(y)=E\phi(y) \tag{1}$$

where the atomic unit $\hbar=1$ and the constant mass $M=1$ are taken. We define a transformation $y\to x$ for a mapping $y=f(x)$, we rewrite the wave functions in the form of

$$\phi(y)=m^{\alpha}(x)\psi(x) \tag{2}$$

Here we assume that $m(x)$ is the position dependent mass. The transformed Schrödinger equation takes

$$\left\{-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\left(\alpha\frac{m'}{m}-\frac{f''}{2f'}\right)\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\alpha}{2}\left[\frac{m''}{m}+(\alpha-1)\left(\frac{m'}{m}\right)^2-\left(\frac{m'}{m}\right)\frac{f''}{f'}\right]\right.$$
$$\left.+\left(f'\right)^2\ V(f(x))\right\}\psi(x)=(f')^2E\ \psi(x), \tag{3}$$

where the prime denotes differentiation with respect to x. On the other hand the one dimensional Schrödinger equation with position dependent mass can be written as

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{M(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right]+\tilde{V}(x)\psi(x)=\tilde{E}\psi(x), \tag{4}$$

where $M(x)=m_0\ m(x)$, and the dimensionless mass distribution $m(x)$ is real function. For simplicity, we take $m_0=1$. Thus, equation (4) takes the form

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{m'}{2m}\frac{\mathrm{d}}{\mathrm{d}x}+m\tilde{V}(x)\right)\psi(x)=m\tilde{E}\psi(x) \tag{5}$$

Comparing equations (3) and (5), we get the following identities

$$\frac{f''}{2f'} - \alpha\frac{m'}{m} = \frac{m'}{2m} \tag{6}$$

and

$$\tilde{V}(x) - (\tilde{E}) = \frac{f'^2}{m}[V(f(x)) - E] - \frac{\alpha}{2m}\left[(\alpha-1)\left(\frac{m'}{m}\right)^2 - \left(\frac{m'}{m}\right)\left(\frac{f''}{f'}\right) + \frac{m''}{m}\right] \tag{7}$$

From equation (6), we find

$$f' = m^{2\alpha+1} \tag{8}$$

Substituting f' into equation (7), we obtain the new potential as

$$\tilde{V}(x) = \frac{f'^2}{m}V(f(x)) + (1 - m^{4\alpha+1})E - \frac{\alpha}{2m}\left[\frac{m''}{m} - (\alpha+2)\left(\frac{m'}{m}\right)^2\right] \tag{9}$$

Therefore, the energy eigenvalues and corresponding wave functions for the potential $V(y)$ as E_n and $\phi_n(y)$ become

$$\begin{aligned}\tilde{E}_n &= E_n \\ \psi_n(x) &= \tfrac{1}{m^{\alpha}(x)}\phi_n(y).\end{aligned} \tag{10}$$

For $\alpha = -1/4$ equations (9) and (10) reduce to the same form given in [13].

3. Some applications

In this section, we use two different position-dependent mass distributions. The reference potentials are taken as the Rosen–Morse [28,29] and Scarf [30] potentials to get some target potentials providing us the exact solutions.

3.1. *Mass distribution $m(x) = a^2/(q + x^2)$*

The deformed Rosen–Morse and Scarf potentials are

$$V_{\mathrm{RMT}}(y) = -V_1\,\mathrm{sech}_q^2(\beta y) - V_2\,\tanh_q(\beta y) \tag{11}$$

and

$$V_{\mathrm{ST}}(y) = -V_1\,\mathrm{sech}_q^2(\beta y) - V_2\,\mathrm{sech}_q(\beta y)\tanh_q(\beta y), \tag{12}$$

where the parameters V_1 and V_2 are real. The deformed hyperbolic functions [31] are

$$\sinh_q y = \frac{e^y - qe^{-y}}{2}, \quad \cosh_q y = \frac{e^y + qe^{-y}}{2}, \quad \tanh_q y = \frac{\sinh_q y}{\cosh_q y} \tag{13}$$

and

$$\mathrm{cosech}_q\, y = \frac{1}{\sinh_q y}, \quad \mathrm{sech}_q\, y = \frac{1}{\cosh_q y}, \quad \coth_q y = \frac{1}{\tanh_q y}, \tag{14}$$

where q is a real, positive parameter. Recently, it is found that the deformed hyperbolic potentials can be reduced to the non-deformed hyperbolic potentials by using a coordinate translational transformation [32].

The Schrödinger equation with a constant mass for the Rosen–Morse and Scarf potentials are solved by using Nikiforov–Uvarov and the function analysis methods [28,29]. The exact bound state solutions of the Klein–Gordon and Dirac equations with equal scalar and vector potentials for these potentials are also obtained [33,34].

We define a parameter η to use as a combination of some parameters

$$\eta = n + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{V_1}{q\beta^2}}. \tag{15}$$

The energy eigenvalues and corresponding wave functions for the reference potential $V_{\mathrm{RMT}}(y)$ are respectively

$$E_{\mathrm{RMT}}(n) = -\frac{V_2^2}{4\beta^2}\frac{1}{\eta^2} - \beta^2\eta^2 \tag{16}$$

and

$$\phi_n(y) = \left(\cosh_q(\beta y)\right)^{\eta} \exp\left[-\frac{V_2}{2\beta\eta} y\right] P_n^{-2P_+,-2P_-}(-\tanh_q(\beta y)), \tag{17}$$

where the quantum number is defined as

$$n = 0, 1, 2, \ldots < \sqrt{\frac{1}{4} + \frac{V_1}{q\beta^2}} - \frac{1}{2} \tag{18}$$

The parameters P_+ and P_- are given by

$$P_\pm = \frac{1}{2}\left[\eta \pm \frac{V_2}{2\beta^2}\frac{1}{\eta}\right] \tag{19}$$

Similarly, the energy eigenvalues and the corresponding wave functions for the reference potential $V_{\mathrm{ST}}(y)$ are respectively

$$E_{\mathrm{ST}}(n) = -\beta^2\left[n+\frac{1}{2}-\frac{1}{2}\left(\sigma\sqrt{\frac{1}{4}+\frac{V_1}{q\beta^2}+\frac{V_2}{i\beta^2 q^{1/2}}}\right.\right. \\ \left.\left. +\tau\sqrt{\frac{1}{4}+\frac{V_1}{q\beta^2}-\frac{V_2}{i\beta^2 q^{1/2}}}\right)\right]^2 \tag{20}$$

and

$$\tilde{\phi}_n(y) = \frac{1}{[\cosh_q(\beta y)]^{\omega_+ + \omega_-}}\ \exp\left[(\omega_+ - \omega_-)\ \tanh^{-1}(iq^{-1/2}\sinh_q(\beta y))\right] \\ \times P_n^{-2\omega_+ - \frac{1}{2},\ -2\omega_- - \frac{1}{2}}(iq^{-1/2}\sinh_q(\beta y)), \tag{21}$$

where, the quantum number is

$$n = 0, 1, 2, \ldots < Re\left[\frac{1}{2}\left(\sigma\sqrt{\frac{1}{4}+\frac{V_1}{q\beta^2}+\frac{V_2}{i\beta^2 q^{1/2}}}\right.\right. \\ \left.\left. +\tau\sqrt{\frac{1}{4}+\frac{V_1}{q\beta^2}-\frac{V_2}{i\beta^2 q^{1/2}}}\right)\right]-\frac{1}{2} \tag{22}$$

The parameters ω_+ and ω_- are given by

$$\omega_+ = -\frac{1}{4}+\frac{\sigma}{2}\sqrt{\frac{1}{4}+\frac{V_1}{q\beta^2}+\frac{V_2}{i\beta^2 q^{1/2}}}, \qquad \omega_- = -\frac{1}{4}+\frac{\tau}{2}\sqrt{\frac{1}{4}+\frac{V_1}{q\beta^2}-\frac{V_2}{i\beta^2 q^{1/2}}}, \tag{23}$$

where

$$\sigma = \pm 1 \quad \text{and} \quad \tau = \pm 1. \tag{24}$$

Now, we consider the mass distributions

$$m(x) = \frac{a^2}{q+x^2}. \tag{25}$$

The mapping function becomes

$$y = f(x) = \int m(x)^{2\alpha+1}\,\mathrm{d}x \\ = a^{4\alpha+2}\int \frac{\mathrm{d}x}{(q+x^2)^{2\alpha+1}} \tag{26}$$

For $\alpha = -1/4$, equation (26) reduces to [13]. Therefore, we calculate the target potentials for the Rosen–Morse and Scarf potentials

$$\tilde{V}_1(x) = m^{4\alpha+1}\left[-V_1 \ \mathrm{sech}_q^2\left(\frac{y}{a}\right) - V_2 \ \tanh_q\left(\frac{y}{a}\right)\right] + (1 - m^{4\alpha+1})E_{\mathrm{RMT}}(n) + \frac{\alpha}{a^2}\left(1 + 2\alpha\frac{x^2}{q + x^2}\right) \tag{27}$$

and

$$\tilde{V}_2(x) = m^{4\alpha+1}\left[-V_1 \ \mathrm{sech}_q^2\left(\frac{y}{a}\right) - V_2 \ \mathrm{sech}_q\left(\frac{y}{a}\right) \ \tanh_q\left(\frac{y}{a}\right)\right] + (1 - m^{4\alpha+1})E_{\mathrm{ST}}(n) + \frac{\alpha}{a^2}\left(1 + 2\alpha\frac{x^2}{q + x^2}\right). \tag{28}$$

Again for $\alpha = -1/4$, the target potentials reduce to the ones in [13]. In order to express the target potentials $\tilde{V}_1$ and $\tilde{V}_2$ in terms of x, we consider the following special cases:

(i) For $\alpha = 0$

$$y = \frac{a^2}{\sqrt{q}}\tan^{-1}\left(\frac{x}{\sqrt{q}}\right). \tag{29}$$

(ii) For $\alpha = -1/4$

$$y = a \ \ln\left[x + \sqrt{q + x^2}\right]. \tag{30}$$

(iii) For $\alpha = 1$

$$y = q^{-\frac{5}{2}} \ a^6\left[\frac{3}{8}\theta + \frac{3}{8}\sin\theta \ \cos\theta + \frac{1}{4}\sin\theta \ \cos^2\theta\right], \tag{31}$$

where $\theta = \tan^{-1}\left(\frac{x}{\sqrt{q}}\right)$.

(iv) For any α.

Substituting $x = \sqrt{a} \ \tan\theta$ into the equation (26), we get

$$y = \frac{a^{4\alpha+1}}{q^{2\alpha+1/2}}\int \cos^{4\alpha}\theta \ \mathrm{d}\theta \tag{32}$$

3.2. Mass distribution $m(x) = a^2/(b+x^2)^2$

We list below the special cases as

(i) For $\alpha = 0$

$$y = \frac{a^2}{b^{3/2}}\left(\frac{1}{2}\sin\theta \ \cos\theta + \frac{1}{2}\theta\right), \tag{33}$$

where $\theta = \tan^{-1}\left(\frac{x}{\sqrt{b}}\right)$.

(ii) For $\alpha = -1/4$

$$y = a \ \tan^{-1}\left(\frac{x}{\sqrt{b}}\right) \tag{34}$$

(iii) For $\alpha = 1/2$

$$y = \frac{a^4}{b^{7/2}}\left[\frac{5\theta}{2} + \frac{5}{16}\sin\theta \ \cos\theta + \frac{5}{24}\sin\theta \ \cos^2\theta + \frac{1}{6}\sin\theta \ \cos^5\theta\right], \tag{35}$$

where $\theta = \tan^{-1}\left(\frac{x}{\sqrt{b}}\right)$.

(iv) For any α

$$y = a^{4\alpha+2}\int \frac{\mathrm{d}x}{(b+x^2)^{4\alpha+2}}. \tag{36}$$

Substituting $x = \sqrt{b}\tan\theta$ into above equation, we get

$$y = \frac{a^{4\alpha+2}}{b^{3/2}}\int \cos^{8\alpha+2}\theta \,\mathrm{d}\theta. \tag{37}$$

3.3. Mass distribution $m(x) = e^{(-\alpha x)}$

From equation (26), we get

$$\begin{aligned} y = f(x) &= \int e^{-(2\alpha+1)ax}\,\mathrm{d}x \\ &= -\frac{1}{(2\alpha+1)a}\ e^{-(2\alpha+1)ax} \end{aligned} \tag{38}$$

and

$$x = -\frac{1}{(2\alpha+1)a}\ \ln[-(2\alpha+1)ay]. \tag{39}$$

4. Conclusions

We have applied the point canonical transformation in a general form by introducing a free parameter to solve the Schrödinger equation for the Rosen–Morse and Scarf potentials with spatially dependent mass. We have obtained a set of exactly solvable target potentials by using two position-dependent mass distributions. Energy eigenvalues and corresponding wave functions for the target potentials are written in the compact form.

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