

# Quasiclassical Theory of Spin Dynamics in Superfluid $^3\text{He}$ : Kinetic Equations in the Bulk and Spin Response of Surface Majorana States

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Received: 12 October 2017 / Accepted: 25 January 2018 / Published online: 6 February 2018  
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**Abstract** We develop a theory based on the formalism of quasiclassical Green's functions to study the spin dynamics in superfluid  $^3\text{He}$ . First, we derive kinetic equations for the spin-dependent distribution function in the bulk superfluid reproducing the results obtained earlier without quasiclassical approximation. Then, we consider spin dynamics near the surface of fully gapped  $^3\text{He}$ -B-phase taking into account spin relaxation due to the transitions in the spectrum of localized fermionic states. The lifetimes of longitudinal and transverse spin waves are calculated taking into account the Fermi-liquid corrections which lead to a crucial modification of fermionic spectrum and spin responses.

**Keywords** Superfluid  $^3\text{He}$  · Magnetic resonance · Majorana states

## 1 Introduction

The theory of spin dynamics in superfluid  $^3\text{He}$  has been developed for several decades since the pioneering works of Leggett [1–3], where the phenomenological equations were formulated explaining shifts of the transverse nuclear magnetic resonance mode and predicting longitudinal resonance in the B phase [4–6]. To study spin relaxation, that is the width of the NMR signal, Leggett and Takagi [7, 8] introduced the two-fluid model which yields qualitatively the same results as the kinetic theory [9–13].

Nowadays, the most common approach to study non-equilibrium states in different condensed matter systems including superconductors and Fermi superfluids is based on

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the quasiclassical Keldysh formalism. In this way, kinetic equations for spin-singlet superconductors were derived [14, 15] and applied to study various problems (see, e.g., the book [16] for review). Recently, this theory has been extended to study non-equilibrium spin states in spin-singlet superconducting materials [17–19].

Interestingly enough, although the general quasiclassical approach to superfluid  $^3\text{He}$  has been described [20], kinetic equations for spin dynamics in spin-triplet superconductors/superfluids have never been derived using this technique. An attempt to apply quasiclassical methods to spin dynamics has been done in Ref. [21]. However, this work does not reproduce kinetic equations obtained without the quasiclassical approximation [9–13], essentially because several important terms have been omitted during the derivation. The purpose of the present paper is partially to close this gap.

Being powerful tools to study spin dynamics in the bulk, previous kinetic theories are not capable of describing the spin response of localized quasiparticle surface states dubbed Andreev–Majorana fermions [22–25]. Recently, the frequency-dependent linear spin response of Andreev–Majorana states was found [23]. As noted in Ref. [26] to obtain the total spin susceptibility, it is necessary to take into account the self-consistent response of the spin-triplet order parameter. In the present paper, we apply quasiclassical formalism to this problem, calculating the coupled dynamics of spin waves and Andreev–Majorana surface states in the film of fully gapped superfluid  $^3\text{He-B}$ . This approach allows treating both the longitudinal and transverse magnetic resonances and taking into account Fermi-liquid corrections which are quite important in  $^3\text{He}$  [18, 27, 28]. We show that these corrections can drastically change the spin response properties shifting the threshold absorption frequency of the surface states to values several times larger than the basic Larmor frequency of magnetic precession.

## 2 Keldysh Formalism for Non-equilibrium Spin States

### 2.1 General Equations

In general, the spin density  $\mathbf{S}$  can be written in terms of the Keldysh quasiclassical Green's function (GF) as

$$\mathbf{S}(\mathbf{r}, t) = \frac{\hbar v_0}{16} \int \frac{d\Omega_p}{4\pi} \text{Tr} \left[ \hat{\tau}_3 \hat{\sigma} \hat{g}^K(t, t) \right] + \mathbf{S}_{\text{eq}}^{(n)}, \quad (1)$$

where  $v_0$  is the normal-state density of states,  $\tau_i, \sigma_i$  are Pauli matrices in Nambu and spin spaces,  $\hat{g}^K$  is the  $(2 \times 2)$  matrix Keldysh component of the matrix quasiclassical Green's function

$$\check{g} = \begin{pmatrix} \hat{g}^R & \hat{g}^K \\ 0 & \hat{g}^A \end{pmatrix}, \quad (2)$$

and  $\hat{g}^{R(A)}$  are the retarded (advanced) propagators. The additional term in (1) compensates off-shell contributions  $\mathbf{S}_{\text{eq}}^{(n)} = \chi_n \mathbf{H}_{\text{ext}}$ , where  $\chi_n$  is the normal-state susceptibility and  $\mathbf{H}_{\text{ext}}$  is an external field.

In clean superfluid, the matrix  $\check{g}$  obeys the Keldysh–Eilenberger equation [29]

$$\mathbf{v}_F \cdot \nabla \check{g} + \{ \hat{\tau}_3 \partial_t, \check{g} \}_t + i [ \check{\Sigma}, \check{g} ]_t + i [ \check{V}_Z, \check{g} ]_t = St(\check{g}), \tag{3}$$

where  $\mathbf{v}_F$  is the Fermi velocity and the commutator/anti-commutator operators are defined as

$$[X, g]_t = X(t_1)g(t_1, t_2) - g(t_1, t_2)X(t_2) \tag{4}$$

$$\{X, g\}_t = X(t_1)g(t_1, t_2) + g(t_1, t_2)X(t_2). \tag{5}$$

The l.h.s of Eq. (3) contains the spin-dependent Zeeman energy  $\check{V}_Z = \hat{I}_K \hat{V}_Z$  where  $\hat{I}_K$  is the unit matrix in Keldysh space,  $\hat{V}_Z = -\gamma \hat{\tau}_3 (\hat{\sigma} \cdot \mathbf{H}_{\text{ext}})/2$ , and  $\gamma$  is the gyromagnetic ratio of  $^3\text{He}$  nuclei. Similarly, the Keldysh component of the Hartree–Fock self-energy is absent and the spectral components are given by the superposition of three terms:  $\hat{\Sigma}^{R(A)} = \hat{\Sigma}_{FL} + \hat{\Delta} + \hat{\Sigma}_D$ . The first one is the Fermi-liquid self-energy, which we take in the form describing the correction to the Zeeman field  $\hat{\Sigma}_{FL} = \hat{\tau}_3 \gamma^2 (Z_0/8) (\hat{\sigma} \cdot \mathbf{S})/\chi_{n0}$ , where  $Z_0 \approx -3$  is the Landau parameter, describing the enhancement of spin susceptibility,  $\chi_{n0} = \gamma^2 \hbar^2 \nu_0/4$  is the normal-state susceptibility without corrections. As a result, the effective magnetic field modified by Fermi-liquid corrections is given by [8]

$$\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{ext}} - \gamma (Z_0/4) \mathbf{S} / \chi_{n0}. \tag{6}$$

The self-energy  $\hat{\Sigma}$  contains off-diagonal gap and dipolar interaction operators. The gap is parametrized in terms of the spin vector  $\mathbf{d}$  and is given by  $\hat{\Delta} = i \Delta_0 \hat{\tau}_1 (\hat{\sigma} \cdot \mathbf{d})$ . The general form of the dipolar interaction is not important for the present paper. Similarly, we do not specify the particle–particle collision integral on the r.h.s. of Eq. (3).

Equation (3) is complemented by the normalization condition  $\check{g} \circ \check{g} = 1$  that allows writing the Keldysh component

$$\hat{g}^K(t_1, t_2) = \hat{g}^R \circ \hat{f} - \hat{f} \circ \hat{g}^A, \tag{7}$$

where  $\hat{f} = \hat{f}(t_1, t_2)$  is the generalized distribution function and the convolution product is defined as  $(A \circ B)(t_1, t_2) = \int dt A(t_1, t) B(t, t_2)$ .

### 2.2 Spin Conservation

The spin conservation law can be obtained from the general Keldysh–Eilenberger equation (3) using the definition of spin density (1) and the self-consistency relation for the order parameter which yields  $\text{Tr} \hat{\sigma} [\hat{\Delta}, \hat{g}^K] = 0$ . Moreover, the Fermi-liquid self-energy contribution drops out as well  $\text{Tr} \hat{\sigma} [\hat{\Sigma}_{FL}, \hat{g}^K] = 0$ . Therefore, multiplying the Keldysh component of Eq. (3) with  $\hat{\sigma}$  and taking the trace we obtain the exact equation:

$$\nabla_k \mathbf{J}_k + \dot{\mathbf{S}} = \gamma \mathbf{S} \times \mathbf{H}_{\text{ext}} + \mathbf{R}_D, \tag{8}$$

where the components of the spin current  $\mathbf{J}_k$  and the dipole torque  $\mathbf{R}_D$  are defined as follows

$$\mathbf{J}_k = \frac{\hbar v_0}{16} \int \frac{d\Omega_p}{4\pi} v_{Fk} \text{Tr} \left( \hat{\sigma} \hat{g}^K \right), \quad (9)$$

$$\mathbf{R}_D = \frac{i\hbar v_0}{8} \int \frac{d\Omega_p}{4\pi} \text{Tr} \left( \sigma \left[ \hat{\Sigma}_D, \hat{g}^K \right] \right). \quad (10)$$

### 2.3 Rotating Frame: General Case

There are two possible types of driving terms in the kinetic equation (3) which generate non-equilibrium states. One of them is the time-dependent external field  $\mathbf{H}_{\text{ext}} = \mathbf{H}_{\text{ext}}(t)$  generated by external sources. The other driving term is given by the time-dependent rotating order parameter vector  $\mathbf{d}(t) = \hat{R}^{-1}(t)\mathbf{d}_0(\mathbf{k})$ , where the rotation matrix  $\hat{R}$  is determined by its angle  $\theta(t)$  and axis  $\mathbf{n}(t)$ :

$$R_{ik} = \delta_{ik} + (n_i n_k - \delta_{ik})(1 - \cos \theta) - \varepsilon_{ikl} n_l \sin \theta. \quad (11)$$

In general, the rotation matrix can be split  $\hat{R}(t) = \hat{R}_{\text{ac}}(t)\hat{R}_0$  into the time-independent part  $\hat{R}_0$ , describing the static order parameter configuration and the part  $\hat{R}_{\text{ac}}(t)$  corresponding to the time-dependent rotation (12). To simplify the kinetic equation Eq. (3), it is convenient to introduce the rotating frame by removing the time dependence of the order parameter with the help of the following SU(2) transformation

$$\check{g}(t_1, t_2) = \hat{U}^\dagger(t_1) \check{g}(t_1, t_2) \hat{U}(t_2), \quad (12)$$

where  $\hat{U}(t) = e^{i\hat{\sigma} \cdot \boldsymbol{\theta}(t)/2}$ . This transformation rotates spin vectors  $\hat{U}^\dagger \hat{\sigma} \hat{U} = \hat{R}_{\text{ac}}^{-1} \hat{\sigma}$  so that the gap function becomes time-independent  $\hat{\Delta} = i\tau_1(\hat{\sigma} \cdot \mathbf{d}_0)$  and the Zeeman energy in the rotating frame acquires a new term as follows

$$\hat{V}_Z = -\frac{\gamma}{2} \hat{\tau}_3 \left( \hat{\sigma} \cdot \hat{R}_{\text{ac}} \mathbf{H}_{\text{eff}} \right) - i\hat{\tau}_3 \hat{U}^\dagger \partial_t \hat{U}, \quad (13)$$

where the effective field in the first term is given by Eq. (6). Here the spin-dependent Fermi-liquid corrections are incorporated into the Zeeman term. For small angles  $\theta$ , one can expand the rotation matrix  $\hat{R}_{\text{ac}} \mathbf{H}_{\text{eff}} \approx \mathbf{H}_{\text{eff}} - \mathbf{H}_{\text{eff}} \times \boldsymbol{\theta}$ , and put  $\hat{U}^\dagger \partial_t \hat{U} \approx i\hat{\sigma} \cdot \partial_t \boldsymbol{\theta}/2$  which allows rewriting the kinetic equation (3) separating the time-independent term  $\hat{\Lambda}_0 = \hat{I}_K \hat{\Lambda}_0$  and the driving term  $\check{V}_{Z1} = \hat{I}_K \hat{V}_{Z1}$  as follows

$$i v_F \cdot \nabla \check{g} + i \left\{ \hat{\tau}_3 \partial_t, \check{g} \right\}_t - \left[ \check{\Lambda}_0, \check{g} \right] = \left[ \check{V}_{Z1}, \check{g} \right]_t + S t \left( \check{g} \right), \quad (14)$$

where  $\hat{\Lambda}_0 = i\Delta_0 \hat{\tau}_1 (\hat{\sigma} \cdot \mathbf{d}_0) - \gamma \hat{\tau}_3 (\hat{\sigma} \cdot \mathbf{H}_{\text{eff}}^{(0)})/2$  and  $\check{V}_{Z1}(t) = -\gamma \hat{\tau}_3 (\hat{\sigma} \cdot \mathbf{h})/2$ . The driving field is given by

$$\mathbf{h}(t) = \check{\mathbf{H}}_{\text{eff}} + \boldsymbol{\theta} \times \mathbf{H}_{\text{eff}}^{(0)} - \partial_t \boldsymbol{\theta} / \gamma, \quad (15)$$

where  $H_{\text{eff}}^{(0)}$  and  $\tilde{H}_{\text{eff}}$  are the constant and time-dependent parts of the effective field (6) in the rotating frame. Below we will use Eq. (14) to find the first-order non-equilibrium corrections to the Keldysh function in various situations. Namely at first we will derive kinetic equations for the spin distribution function describing the bulk NMR in superfluid  $^3\text{He}$ . Second, we will calculate the spin response in the presence of Majorana surface states in fully gapped B phase.

### 3 Spin Dynamics in the Bulk Superfluid $^3\text{He}$

In this section, we apply the general formalism described above to derive kinetic equations for the spin-dependent distribution function which were obtained before without quasiclassical approximation [9–13]. The advantage of the quasiclassical approach is that the derivation becomes much simpler and one can clearly understand the approximations made.

#### 3.1 Mixed Representation

We will use the mixed representation of Green’s functions  $\check{g}(t_1, t_2) = e^{i\varepsilon(t_2-t_1)}\check{g}(\varepsilon, t)$  where  $t = (t_1 + t_2)/2$ . Then, with the help of the gradient expansion  $[\check{X}, \check{g}]_t = [\check{X}, \check{g}] - i\{\partial_t \check{X}, \partial_\varepsilon \check{g}\}/2$  the Keldysh component of Eq. (3) can be written in the form

$$v_F \cdot \nabla \widehat{g}^K - i\varepsilon [\widehat{\tau}_3, \widehat{g}^K] + \frac{1}{2} \{ \widehat{\tau}_3, \partial_t \widehat{g}^K \} + [\widehat{\Lambda}, \widehat{g}^K] + \frac{1}{2} \{ \partial_t \widehat{\Lambda}, \partial_\varepsilon \widehat{g}^K \} = St(\check{g}), \tag{16}$$

where we denote  $\widehat{\Lambda} = \widehat{\Sigma} + \widehat{V}_Z$ . To describe non-equilibrium spin states, we use the parametrization (7) of the Keldysh function. Using the gradient expansion in the mixed representation the relation (7) can be written as follows

$$\widehat{g}^K = \widehat{g}^R \widehat{f} - \widehat{f} \widehat{g}^A - \frac{i}{2} \left( \partial_t \widehat{g}^R \partial_\varepsilon \widehat{f} + \partial_\varepsilon \widehat{f} \partial_t \widehat{g}^A \right) + \frac{i}{2} \left( \partial_\varepsilon \widehat{g}^R \partial_t \widehat{f}_1 + \partial_t \widehat{f}_1 \partial_\varepsilon \widehat{g}^A \right). \tag{17}$$

We represent the distribution function in the form  $\widehat{f}(\varepsilon, t) = f_0(\varepsilon) + \widehat{f}_1(\varepsilon, t)$ , where  $f_0 = \tanh(\varepsilon/2T)$  is the equilibrium part and  $\widehat{f}_1 = (\widehat{\sigma} \cdot \mathbf{f}_T)$  describes the spin non-equilibrium. The last two terms in Eq. (17) containing the time derivative  $\partial_t \widehat{f}_1$  can be neglected provided that the frequency  $\omega$  defined by the driving term is much smaller as compared to the typical energy scale, which is of the order of the energy gap  $\Delta$ . In this case  $\widehat{f}_1 \propto \omega$  and  $\partial_t \widehat{f}_1 \propto \omega^2$ , so that the last two terms in Eq. (17) are of the higher order in the small parameter  $\omega/\Delta$ . When we substitute the expansion  $\widehat{g}^K = \widehat{g}^R \widehat{f} - \widehat{f} \widehat{g}^A$  to (16), some terms cancel due to the Eilenberger equation (3) for spectral components  $\widehat{g}^{R,A}$ . Thus, we are left with a kinetic equation for the spin-dependent distribution function

$$v_F \cdot \nabla \left( \widehat{g}^R \widehat{f}_1 - \widehat{f}_1 \widehat{g}^A \right) + \frac{1}{2} \left\{ \widehat{\tau}_3, \left( \widehat{g}^R \partial_t \widehat{f}_1 - \partial_t \widehat{f}_1 \widehat{g}^A \right) \right\}$$

$$\begin{aligned}
 &+ i\varepsilon \left[ \hat{\tau}_3, \left( \hat{g}^R \hat{f}_1 - \hat{f}_1 \hat{g}^A \right) \right] - \frac{1}{2} \left[ \hat{\Lambda}, \partial_\varepsilon \hat{g}^R \partial_t \hat{f}_1 + \partial_t \hat{f}_1 \partial_\varepsilon \hat{g}^A \right] \\
 &+ i \left[ \hat{\Lambda}, \left( \hat{g}^R \hat{f}_1 - \hat{f}_1 \hat{g}^A \right) \right] + \frac{1}{2} \partial_\varepsilon f_0 \left\{ \partial_t \hat{\Lambda}, \hat{g}^R - \hat{g}^A \right\} = St\{f_1\}. \tag{18}
 \end{aligned}$$

Here, it is quite important to take into account modifications of spectral functions due to the Zeeman energy shift. This modification has not been taken into account in previous work on the quasiclassical theory of spin dynamics in superfluid <sup>3</sup>He [21]. As discussed in recent works, Zeeman spin splitting leads to the qualitative changes in the quasiparticle spin transport properties even in spin-singlet superconductors [17–19].

Since the Zeeman shift is quite small as compared to the gap amplitude, it is enough to use the first-order expansions in terms of the effective magnetic field

$$\hat{g}^R = \hat{g}_0^R + (\gamma/2) (\mathbf{h} \cdot \mathbf{d}_0) \hat{g}_1^R \tag{19}$$

$$\hat{g}_0^R = \hat{\tau}_3 G_0 - i \hat{\tau}_1 (\hat{\sigma} \cdot \mathbf{d}_0) F_0 \tag{20}$$

$$\hat{g}_1^R = \hat{\tau}_3 (\hat{\sigma} \cdot \mathbf{d}_0) \partial_\varepsilon G_0 - i \hat{\tau}_1 \partial_\varepsilon F_0, \tag{21}$$

where  $G_0 = \varepsilon/\sqrt{\varepsilon^2 - \Delta^2}$  and  $F_0 = \Delta/\sqrt{\varepsilon^2 - \Delta^2}$ . The advanced function is given by the usual relation  $\hat{g}^A = -\hat{\tau}_3 \hat{g}^{R\dagger} \hat{\tau}_3$ . Then, after some algebra we transform the kinetic equation to the following form

$$\begin{aligned}
 &G_0 \partial_t \mathbf{f}_T - \Delta \partial_\varepsilon F_0 \partial_t \mathbf{f}_T^\perp + \gamma G_0 (\mathbf{h} \times \mathbf{f}_T) - \gamma \Delta \partial_\varepsilon F_0 (\mathbf{d}_0 \cdot \mathbf{h}) (\mathbf{d}_0 \times \mathbf{f}_T) \\
 &= \frac{\gamma}{2} G_0 \partial_\varepsilon f_0 \mathbf{h} + St\{\mathbf{f}_T\}, \tag{22}
 \end{aligned}$$

where we separate the transverse component of the distribution functions with respect to  $\mathbf{d}_0$  such that  $\mathbf{f}_T^\perp = \mathbf{f}_T - \mathbf{d}_0 (\mathbf{d}_0 \cdot \mathbf{f}_T)$ . For the reasons discussed above, this quasiclassical equation (22) is different from that obtained in Ref. [21]. However, it coincides with the one derived without using the quasiclassical approximation. To demonstrate that, let us introduce the distribution function

$$\mathbf{v} = \mathbf{d}_0 f_T^\parallel + G_0 \mathbf{f}_T^\perp \tag{23}$$

Then, with the help of (22) one obtains the Boltzmann-like kinetic equation

$$\partial_t \mathbf{v} - 2\delta \mathbf{E} \times \mathbf{v} = -\partial_\varepsilon f_0 \partial_t (\delta \mathbf{E}) + St\{\mathbf{v}_T\} \tag{24}$$

where the effective shift of energy levels under the action of the Zeeman field is given by

$$\delta \mathbf{E} = -\frac{\gamma}{2} \left( h_\parallel \mathbf{d}_0 + \frac{\mathbf{h}_\perp}{G_0} \right). \tag{25}$$

Equation (24) is identical to previous results obtained without quasiclassical approximation [9–13].

## 4 Spin Relaxation Due to Surface Majorana States in Superfluid <sup>3</sup>He-B

The main simplification used in the derivation of the bulk kinetic equation is based on the truncation of the gradient expansion to the first term in Eq. (17). This approximation is not valid to describe resonant transitions between the energy levels corresponding to the surface bound states. These transitions happen at the fixed momentum projection to the surface plane, so that the spectrum consists of the discrete energy levels  $\varepsilon_n$ . In this case, the spectral functions have isolated poles  $\widehat{g}^{R,A}(\varepsilon) \propto \delta(\varepsilon - \varepsilon_n)$ , so that, for example, the last term in the expansion (17) is much larger than the first one. However, the gradient expansion is still applicable in some cases with the discrete spectrum. For example, it can be used for the description of localized fermionic states in the vortex cores [16,30,31]. The interlevel distance corresponding to the localized vortex core states is so small that the quasiclassical approximation yields continuous spectral branches. In case of the surface Andreev–Majorana states [22,24], the situation is different since their spectrum at fixed momentum projection is discrete even within the quasiclassics. Thus, the gradient expansion is not applicable and it is not possible to derive the Boltzmann-like kinetic equation. Below we will treat this problem by finding the Keldysh function, which is a solution of the full Eilenberger equation (14), using the exact form of the spectrum and wave functions near the surface.

### 4.1 Andreev–Majorana Bound States on the Surface of <sup>3</sup>He-B

In this section we derive the spectrum of localized states at the surface of <sup>3</sup>He-B taking into account the Fermi-liquid corrections. We assume that the surface normal is oriented along  $z$ , the equilibrium order parameter is defined by the rotation matrix  $\widehat{R}_0$  and  $\mathbf{d}_0 = (q_x \Delta_\perp, q_y \Delta_\perp, q_z \Delta_\parallel) / \Delta$  where the components depend on the distance to the surface  $\Delta_{\perp,\parallel} = \Delta_{\perp,\parallel}(z)$ .

Then, upon specular reflection from the surface the quasiparticle momentum projection  $q_z$  and therefore the  $z$ -component of the order parameter vector  $\mathbf{d}_0$  change the signs which leads to the formation of surface bound states. Their spectrum is determined by the Andreev equation for the quasiparticle wave function  $\widehat{\psi} = \widehat{\psi}(z)$ , which is a four-component vector in spin-Nambu space

$$\widehat{H}(z, \partial_z) \widehat{\psi} = \varepsilon \widehat{\psi}. \tag{26}$$

To find the discrete spectrum of (26), we employ the usual procedure of splitting the Hamiltonian in two parts  $\widehat{H} = \widehat{H}_0 + \widehat{H}_1$ , so that

$$\widehat{H}_0 = -i \widehat{\tau}_3 v_F q_z \partial_z + \widehat{\tau}_2 \Delta_\perp \widehat{\sigma}_z q_z \tag{27}$$

$$\widehat{H}_1 = \Delta_\parallel \widehat{\tau}_2 (\widehat{\sigma} \cdot \mathbf{q}_\perp) - \frac{\gamma}{2} (\widehat{\sigma} \cdot \widehat{R}_0 \mathbf{H}_{\text{eff}}^{(0)}), \tag{28}$$

where  $\mathbf{q}_\perp = (q_x, q_y, 0)$  and  $\widehat{R}_0$  is the equilibrium order parameter rotation matrix. The Hamiltonian (27) has zero-energy eigenvalues corresponding to the degenerate surface bound states. The correction from perturbation  $\widehat{H}_1$  results in the spectrum

$$\varepsilon_{1,2} = \pm \sqrt{C^2 p_{\perp}^2 + E_g^2/4}, \quad (29)$$

where  $p_{\perp} = p_F q_{\perp}$ . The corresponding wave functions were found in Refs. [22, 23]. The velocity and mass of the spectrum (29) are given by

$$C = \frac{1}{p_F L_{\xi}} \int_0^{\infty} \Delta_{\parallel} \exp[-2K(z)] dz \quad (30)$$

$$E_g = \frac{\gamma}{L_{\xi}} \int_0^{\infty} (\mathbf{n}_s \cdot \mathbf{H}_{\text{eff}}^{(0)}) \exp[-2K(z)] dz \quad (31)$$

where  $\mathbf{n}_s = \hat{R}_0 \hat{\mathbf{z}}$  is the spin quantization axis,  $K(z) = \frac{1}{\hbar v_F} \int_0^z \Delta_{\perp}(z) dz$ ,  $L_{\xi} = 4 \int_0^{\infty} e^{-2K(z)} dz$  is a normalization length which is of the order of the coherence length  $\xi = \hbar v_F / \Delta$ .

The minigap  $E_g$  is induced by the external magnetic field. Here, we point out that Fermi-liquid corrections lead to a strong re-normalization of  $E_g$  as compared to its 'bare' value given by  $E_g^{(0)} = \hbar \omega_L$ , where  $\omega_L = \gamma H_{\text{ext}}^{(0)}$  is the Larmor frequency. The effective field in (31) is given by (6) which can be written in terms of the local spin susceptibility  $\chi = \chi(z)$  so that  $\mathbf{H}_{\text{eff}}^{(0)}(z) = \mathbf{H}_{\text{ext}}(1 - (Z_0/4)\chi/\chi_{n0})$ . To calculate an exact value of  $E_g$ , one has to determine  $\chi(z)$  self-consistently [28]. For the estimation, we can assume that  $\chi \approx \chi_n$ , where  $\chi_n = \chi_{n0}/(1 + Z_0/4)$  is the renormalized bulk normal-state susceptibility. Then, for  $Z_0 \approx -3$  the spectral minigap is given by  $E_g \approx 4\hbar \omega_L$ . As we will see below at smaller frequencies  $\omega < E_g/\hbar$  surface bound states give no absorption signal. In particular, this situation is realized for the experimentally interesting domain of frequencies in the close vicinity of the main NMR peak located at the Larmor frequency  $\omega \approx \omega_L < E_g/\hbar$ .

The localized states of the Andreev equation (26) with discrete spectrum  $\varepsilon_n$  provide singular contributions to the Green's functions:

$$\widehat{G}^{R,A}(z_1, z_2, \mathbf{q}_{\perp}, \varepsilon) = \sum_n \frac{|\hat{\psi}_n(z_1)\rangle \langle \hat{\psi}_n(z_2)|}{\varepsilon_n - \varepsilon \mp i0}. \quad (32)$$

This expression will be used below to calculate the singular part of quasiclassical propagators determined by the contribution of surface Andreev–Majorana states.

## 4.2 Quasiclassical Propagators

To proceed, we need to calculate spectral functions  $g^{R,A}$  near the surface of  $^3\text{He-B}$ . We will use a general relation [16] for the singular part of quasiclassical propagators in terms of the Green's functions (32) of Andreev equations

$$(\widehat{g}^R - \widehat{g}^A)(z) = -2i v_z \hat{\tau}_3 (\widehat{G}^R - \widehat{G}^A) \quad (33)$$



where  $v_z = (\mathbf{v}_F \cdot \hat{\mathbf{z}})$  and  $\widehat{G}^{R,A} = \widehat{G}^{R,A}(z_1 = z_2 = z, \mathbf{q}_\perp, \varepsilon)$  are the Green’s functions of Andreev equations (26) which depend on the coordinate  $z$  perpendicular to the surface, momentum projection on  $xy$  plane  $\hat{\mathbf{p}}_\perp$  and energy  $\varepsilon$ . We are interested in the contribution of bound states and therefore can use the expression  $(\widehat{G}^R - \widehat{G}^A)(z_1 = z_2) = 2\pi i \sum_n \delta(\varepsilon - \varepsilon_n) |\hat{\psi}_n\rangle \langle \hat{\psi}_n|$ , so that

$$\widehat{g}^R - \widehat{g}^A = 4\pi v_z \sum_n \delta(\varepsilon - \varepsilon_n) |\hat{\tau}_3 \hat{\psi}_n\rangle \langle \hat{\psi}_n|. \tag{34}$$

The equilibrium Keldysh function is given by  $\widehat{g}^K(\varepsilon) = f_0(\varepsilon)(\widehat{g}^R - \widehat{g}^A)$ , where  $f_0(\varepsilon) = \tanh(\varepsilon/2T)$  is the equilibrium distribution function. Hence, in the time domain we get  $\widehat{g}^K(t, t') = \int \widehat{g}^K(\varepsilon) \exp(i\varepsilon(t' - t)) d\varepsilon$  so that

$$\widehat{g}_{\text{eq}}^K(t, t') = 4\pi v_z \sum_n e^{i\varepsilon_n(t'-t)} f_0(\varepsilon_n) |\hat{\tau}_3 \hat{\psi}_n\rangle \langle \hat{\psi}_n|. \tag{35}$$

### 4.3 Non-equilibrium Spin Surface States

Having in hand the equilibrium Keldysh function (35) we can proceed to study non-equilibrium spin polarization of surface bound states given by the general kinetic equation (14). We search for the first-order correction to the Keldysh function  $\widehat{g}_1^K = \widehat{g}_1^K(t, t')$  substituting the equilibrium function in the form (35) to the r.h.s. of Eq. (14). Assuming that the driving field is  $\mathbf{h}(t) = \mathbf{h}_\omega e^{i\omega t}$  and neglecting the collision integral we can find the analytical solution of Eq. (14) in the following form

$$\begin{aligned} \widehat{g}_1^K(t, t') &= 2\pi\gamma v_z \sum_{n \neq m} \frac{\langle \hat{\psi}_n | \mathbf{h}_\omega \cdot \hat{\boldsymbol{\sigma}} | \hat{\psi}_m \rangle}{\omega + \varepsilon_n - \varepsilon_m} \\ &\times \left( f_0(\varepsilon_m) e^{i\varepsilon_m t' + i(\omega - \varepsilon_m)t} - f_0(\varepsilon_n) e^{i(\varepsilon_n + \omega)t' - i\varepsilon_n t} \right) |\hat{\tau}_3 \hat{\psi}_n\rangle \langle \hat{\psi}_m|, \end{aligned} \tag{36}$$

where  $\mathbf{h}_\omega$  denotes the gauge-invariant effective field obtained from Eq. (15)

$$\mathbf{h}_\omega = \tilde{\mathbf{H}}_{\text{eff}} + \boldsymbol{\theta} \times \mathbf{H}_{\text{eff}}^{(0)} - i\omega\boldsymbol{\theta}/\gamma. \tag{37}$$

Calculating the Fourier component  $\widehat{g}_1^K(\omega) = \int e^{-i\omega t} \widehat{g}_1^K(t, t') dt$ , using the definition (1) and the matrix element  $\langle \hat{\psi}_1 | \hat{\boldsymbol{\sigma}} | \hat{\psi}_2 \rangle = (Cp/\varepsilon_1) \mathbf{n}_s$  we obtain the frequency-dependent spin polarization of Andreev–Majorana bound states per unit surface area

$$\mathbf{S}_{bs}(\omega) = \frac{\chi_{bs}(\omega)}{\gamma} (\mathbf{n}_s \cdot \bar{\mathbf{h}}_\omega) \mathbf{n}_s, \tag{38}$$

where  $\mathbf{n}_s$  is the spin quantization axis of surface states and

$$\bar{\mathbf{h}}_\omega = L_\xi^{-1} \int_0^\infty \mathbf{h}_\omega(z) \exp[-2K(z)] dz \tag{39}$$

is the driving field (37) averaged over the surface bound state localization scale. In the absence of Fermi-liquid corrections  $\bar{\mathbf{h}}_\omega = \mathbf{h}_\omega$ . The longitudinal susceptibility  $\chi_{bs}(\omega)$  coincides with the expression found in Refs. [23,26]

$$\chi_{bs} = \frac{\gamma^2 C^2}{8\pi \hbar^2} \int_0^{p_F} p^3 dp \frac{(\varepsilon_2 - \varepsilon_1) [f_0(\varepsilon_1) - f_0(\varepsilon_2)]}{\varepsilon_1^2 [(\hbar\omega)^2 - (\varepsilon_2 - \varepsilon_1)^2]}. \quad (40)$$

The dissipation rate is determined by the imaginary part of  $\chi_{bs}$ , which is nonzero at the frequencies larger than the minigap  $\omega > E_g/\hbar$

$$\text{Im}\chi_{bs} = - \left( \frac{\gamma}{4\hbar C} \right)^2 \frac{\hbar\omega}{2} f_0 \left( \frac{\hbar\omega}{4T} \right) \left( 1 - \frac{E_g^2}{\hbar^2 \omega^2} \right). \quad (41)$$

In contrast to the previous considerations [23,26] which neglected Fermi-liquid corrections to the spectrum, the result (41) demonstrates that the absorption threshold is given by  $E_g/\hbar$  which is significantly larger than the Larmor frequency as discussed above.

#### 4.4 Gauge-Invariant Theory of Surface Relaxation

Let us consider the influence of the transitions in the spectrum of surface states on the dissipation rate of spin waves in the fully gapped superfluid  $^3\text{He-B}$ . In general, the dynamics of the total spin in the laboratory frame is governed by Eq. (8). Let us consider a monochromatic signal  $\mathbf{S} \propto e^{i\omega t}$  to obtain from (8)

$$\nabla_k \mathbf{J}_k + i\omega \mathbf{S} = \mathbf{S} \times \boldsymbol{\omega}_L + \mathbf{R}_D, \quad (42)$$

where  $\boldsymbol{\omega}_L = \gamma \mathbf{H}_{\text{ext}}$  and the dipole interaction  $\mathbf{R}_D$  is a function of the rotation vector  $\boldsymbol{\theta}$  which parametrizes deviations of the order parameter from equilibrium determined by the rotation matrix (11). In the bulk B-phase, the dipole torque is given by [32]

$$\mathbf{R}_D = \frac{\chi_B \Omega_B^2}{\gamma} \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\theta}), \quad (43)$$

where  $\Omega_B$  is the B phase longitudinal resonance frequency [3,32] and  $\chi_B$  is the B phase bulk susceptibility. The total spin density  $\mathbf{S} = \mathbf{S}(\omega, z)$  is given by the superposition of the bulk contribution and that of the Andreev–Majorana bound states, localized at the distance of the coherence length  $\xi$  near the surface. Since all length-scales which determine the spin dynamics in the bulk are much larger than  $\xi$ , we can write  $\mathbf{S}$  in the rotating frame as follows

$$\mathbf{S}(\omega, z) = \frac{\chi_{B0}}{\gamma} \mathbf{h}_\omega + \mathbf{S}_{bs}(\omega) \delta(z), \quad (44)$$

where  $\chi_{B0}$  is the bulk susceptibility without Fermi-liquid corrections, while the amplitude of the second term is given by Eq. (38).

The main difficulty for the further analytical calculations is the influence of the Fermi-liquid corrections on the spin of localized states in Eq. (38). To obtain qualitative results, we neglect these corrections in the surface term and keep them for the bulk contribution. In this case, the gauge-invariant driving field is given by its ‘bare’ form  $\mathbf{h}_\omega^{(b)} \approx \tilde{\mathbf{H}}_{\text{ext}} + \boldsymbol{\theta} \times \mathbf{H}_{\text{ext}}^{(0)} - i\omega\boldsymbol{\theta}/\gamma$ . Hence we will use the approximate expression for the spin signal

$$\mathbf{S}(\omega, z) = \frac{\chi_B}{\gamma} \mathbf{h}_\omega^{(b)} + \mathbf{S}_{bs}(\omega)\delta(z), \tag{45}$$

where  $\chi_B$  is the total B-phase bulk susceptibility and the second term is given by Eq. (38) with  $\mathbf{h}_\omega^{(b)}$  instead of  $\mathbf{h}_\omega$ .

#### 4.4.1 Longitudinal Resonance in Thin Film of $^3\text{He-B}$

First we consider the influence of Andreev–Majorana states on the decay of longitudinal modes [32] when  $\boldsymbol{\theta} = \theta_z \mathbf{z}$ . For simplicity, we assume that the superfluid is homogeneous without any underlying texture so that the rotation axis of the matrix  $\hat{R}$  is directed along the surface normal  $\mathbf{n} \parallel \mathbf{z}$ . In this case, the spin density can be written as  $\mathbf{S}(\omega, z) = \tilde{\chi}(z)\mathbf{h}_\omega^{(b)}$ , where

$$\tilde{\chi}(\omega, z) = \chi_B + \chi_{bs}(\omega)\delta(z). \tag{46}$$

Note that  $\tilde{\chi}$  is not the spin susceptibility, since the driving field  $\mathbf{h}_\omega^{(b)}$  contains the dynamical variable  $\boldsymbol{\theta}$  in addition to the external field. In order to find the true susceptibility  $\hat{\chi}$  which relates the total spin density and external magnetic field  $\mathbf{S} = \hat{\chi}\mathbf{H}_{\text{ext}}$  the angle  $\boldsymbol{\theta}$  has to be determined from the general equation for the total spin dynamics (42). Then, writing for the spin current  $\nabla_z \mathbf{J}_z = c_z^2 \nabla_z^2 \theta_z$ , where  $c_z$  is the spin wave velocity, and combining Eqs. (42, 43, 44) we get

$$\left( c_z^2 \nabla_z^2 + \Omega_B^2 - \omega^2 \frac{\tilde{\chi}}{\chi_B} \right) \theta_z = i\omega\gamma \tilde{\mathbf{H}}_{\text{ext}} \frac{\tilde{\chi}}{\chi_B}. \tag{47}$$

The mode with the smallest frequency  $\omega_0 \approx \Omega_B$  is given by the space-homogeneous state  $\theta_z(z) = \text{const}$ . Its life time is given by the imaginary part of the frequency which can be found by averaging over the coordinate  $z$  the homogeneous Eq. (47) with  $\tilde{\mathbf{H}}_{\text{ext}} = 0$

$$\tau^{-1} = -\frac{\Omega_B \text{Im} \chi_{bs}}{2L}, \tag{48}$$

where  $L$  is the film thickness. In case if there are surface bound states on both surfaces of the film, the relaxation rate (48) is doubled.

The longitudinal magnetic susceptibility  $\chi_{zz}$  which relates the total spin density and external magnetic field  $S = \chi_{zz} H_{\text{ext}}$  can be found using Eqs. (48) and (42):

$$\chi_{zz} = \bar{\chi} \frac{\chi_B \Omega_B^2}{\chi_B \Omega_B^2 - \omega^2 \bar{\chi}}, \tag{49}$$

where  $\bar{\chi}$  is the function (46) averaged over the film thickness. In the absence of Fermi-liquid corrections, this result coincides with the one obtained before [26] from the effective action approach. The most significant difference is, however, in the behavior of the bound states spin susceptibility  $\text{Im}\chi_{bs}$  given by (41) which becomes finite at the frequencies larger than  $E_g \approx 4\hbar\omega_L$  rather than at the bare Larmor frequency.

#### 4.4.2 Transverse Resonance in a Texture

The basic measurement tools in superfluid  $^3\text{He}$  experiments are frequency shifts and dissipation rates of the transverse magnetic precession modes in the presence of a large static magnetic field component. Recently, the technique based on the relaxation of a magnon condensate has been developed [33–38]. In principle, it can be used for the identification of surface Andreev–Majorana states although this approach has several difficulties discussed below.

Let us assume that the constant magnetic field is directed along the surface normal  $\mathbf{H}_{\text{ext}}^{(0)} \parallel \mathbf{z}$  and the non-equilibrium spin state is driven by the time-dependent perpendicular component  $\tilde{\mathbf{H}}_{\text{ext}} \perp \mathbf{H}_{\text{ext}}^{(0)}$ . If the rotation axis is parallel to the constant field  $\mathbf{n} \parallel \mathbf{z}$  and the spin quantization axis is  $\mathbf{n}_s \parallel \mathbf{z}$ , then according to Eq. (38) the oscillating transverse field component cannot induce transitions of the surface bound states. Therefore, the presence of Andreev–Majorana states shows up in the transverse resonance only if  $\mathbf{n}$  is deflected with a finite angle  $\beta_n$  from the  $\mathbf{z}$ -axis so that the effective driving field  $\mathbf{h}$  has a component parallel to the spin quantization axis  $\mathbf{n}_s$ . Physically, the texture of  $\mathbf{n}$  can appear via the interplay between gradient and dipole energies. However, in typical experimental setups [37] the angle  $\beta_n$  is rather small near the surface which leads to a strong suppression of the spin response from the surface states.

To quantify the effect of surface bound states on the transverse resonance, we can use the general theory described in Sect. (4.4). Here, we need to take into account that the expression for the non-equilibrium spin density Eq. (45) is obtained in the rotating frame, while Eq. (42) is written in the laboratory frame. Hence, the non-equilibrium spin density has to be rotated back to the laboratory frame  $\hat{R}_{\text{ac}}^{-1}\mathbf{S} \approx \mathbf{S} + \mathbf{S} \times \boldsymbol{\theta}$  so that Eq. (45) changes as follows

$$\mathbf{S}(\omega, z) = \frac{\chi_B}{\gamma} (\tilde{\mathbf{H}}_{\text{ext}} - i\omega\boldsymbol{\theta}/\gamma) + \mathbf{S}_{\text{bs}}(\omega)\delta(z), \quad (50)$$

where the surface contribution  $\mathbf{S}_{\text{bs}}$  remains unchanged given by the Eq. (44) with the driving field  $\mathbf{h}_\omega^{(b)}$ . To describe the transverse resonance which occurs at frequencies close to the Larmor frequency  $\omega \approx \omega_L$ , we project the above general expressions (42, 43, 50) on the spin state corresponding to the optical magnons [34, 38, 39]  $\boldsymbol{\theta} = \Psi \mathbf{s}_{\text{opt}}$ , where  $\Psi$  is the complex amplitude which can be considered as a wave function and the polarization vector is  $\mathbf{s}_{\text{opt}} = (1, i, 0)/\sqrt{2}$ .

To find the relaxation time of transverse optical magnons due to the excitation of surface bound states, we collect Eqs. (42, 43, 50), put  $\tilde{\mathbf{H}}_{\text{ext}} = 0$  to obtain the following equation for the wave function

$$\omega\Psi = \left(\hat{L} + U\right)\Psi + \frac{\chi_{bs}(\omega - \omega_L)^2}{2\chi_B\omega_L} \sin^2\beta_s\Psi\delta(z). \tag{51}$$

Here  $\sin\beta_s = \sqrt{1 - (\mathbf{n}_sz)^2}$  and  $\hat{L}$  denotes the gradient terms coming from the spin current divergence,  $U$  is the effective potential energy

$$\hat{L} = -\frac{\hbar\nabla_\perp^2}{2m_\perp} - \frac{\hbar\nabla_z^2}{2m_z} \tag{52}$$

$$U = \frac{\Omega_B^2}{2\omega_L} \sin^2\beta_n + \omega_L, \tag{53}$$

where  $\sin\beta_n = \sqrt{1 - (\mathbf{n}z)^2}$ . Effective masses  $m_z = \hbar\omega_L/2c_z^2$  and  $m_\perp = \hbar\omega_L/2c_\perp^2$  are related to the longitudinal  $c_z$  and transverse  $c_\perp$  spin wave velocities. Apart from the surface-related last term on the r.h.s., Eq. (51) coincides with the equations considered before to describe spin waves on the optical branch modified by the  $\mathbf{n}$ -vector texture [34, 39]. The surface term in (51) yields a finite lifetime of the spin waves at the frequencies where  $\text{Im}\chi_{bs} \neq 0$ . To quantify the relaxation effect, we calculate the inverse lifetime of the magnon with frequency  $\omega_n$  with the spatial distribution of spin described by the wave function  $\Psi_n$

$$\tau^{-1} = \frac{\text{Im}\chi_{bs}}{\chi_B} \frac{(\omega_n - \omega_L)^2}{\omega_L} \frac{\int_S d^2r \sin^2\beta_s |\Psi_n|^2}{\int_V d^3r |\Psi_n|^2} \tag{54}$$

where  $S$  and  $V$  denote the surface and the volume of  $^3\text{He-B}$ . Bearing in mind that NMR occurs at frequencies that are rather close to the Larmor frequency  $\omega_n \approx \omega_L$ , one can see that minigap re-normalization by Fermi-liquid corrections (31) makes it impossible to excite Andreev–Majorana states by the transverse magnetic resonance. Indeed in this case the absorption threshold is shifted to the frequencies  $E_g/\hbar \sim 4\omega_L$  much larger than the driving frequency of the magnetic precession  $\omega_n \approx \omega_L$ .

However, Majorana states can be excited by transverse spin waves which have frequencies higher than  $E_g$  and correspondingly wavelengths of the order of  $v_F/E_g$ . Provided that  $E_g \ll \Delta$  the required wavelengths are much larger than the coherence length  $\xi$ , which determines the localization scale of the surface bound states. Therefore, to describe the interactions of Majorana states with these spin waves one can consider the magnetization precession which is locally homogeneous in space and use the same equations for the fermionic spin response as considered above.

### 5 Conclusions

To conclude, we have presented the quasiclassical theory of spin dynamics in superfluid  $^3\text{He}$ . Starting from the most general quasiclassical Keldysh–Eilenberger equation, we derived the kinetic equation for the spin-dependent distribution function in the bulk phase. The result coincides with that obtained some time ago without quasiclassical approximation. Applying this technique, we have obtained the frequency-dependent

lifetimes of longitudinal and transverse spin waves due to their interaction with Andreev–Majorana states localized at the surface of  $^3\text{He}$ -B phase. With the help of the quasiclassical approach, the crucial role of Fermi-liquid corrections in the magnetic response is demonstrated.

An important qualitative conclusion is that such relaxation mechanism can be effective only for the longitudinal spin resonance. At the same time, the spatially homogeneous transverse NMR mode does not excite Andreev–Majorana surface because of two reasons. First, due to the Fermi-liquid corrections the minigap in the surface state spectrum is much larger than NMR frequency which is close to the Larmor frequency. Second, the matrix element of transitions in the Andreev–Majorana spectrum is proportional to the deflection angle  $\beta_s$  of the spin quantization axis from the constant magnetic field component. Usually, the texture of the vector  $\mathbf{n}$  is flat near the surface so that the deflection is rather small,  $|\beta_s| \ll 1$ . The finite-momentum transverse spin waves with frequencies larger than  $E_g$  can excite Majorana states. The interaction of such spin waves with the surface states can be described within the same theoretical framework as considered in this paper.

**Acknowledgements** We thank V. El'tsov and G. Volovik for stimulating discussions. This work was supported by the Academy of Finland.

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