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Directional derivatives and subdifferentials for set-valued maps applied to set optimization

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Abstract
We present a general method to devise directional derivatives and subdifferentials for set-valued maps that generalize the corresponding constructions from the classical situation of real-valued functions. We show that these generalized differentiation objects enjoy some properties that, on the one hand, meaningfully extend the aforementioned case and, on the another hand, are useful to deal with the so-called ℓ-minimality in set optimization problems.

Keywords Set optimization · Generalized directional derivatives · Subdifferentials of set-valued maps · Optimality conditions · Penalization methods

Mathematics Subject Classification 54C60 · 46G05 · 90C46

1 Introduction

Set optimization has gained recently an increasing attention from many researchers in an effort to investigate various features of this special type of problems such as optimality conditions, well-posedness, stability of solutions, approximate solutions, the links with classical vector optimization approach, etc. For details and insights into development of this area of research we mention the references [9, 14]. Looking at the very recent developments, we consider that a real need of generalized differentiation objects directly issued from the analysis of the set optimization problems is constantly observable, since one can find in the literature an extended use of scalarization methods or technical assumptions that are employed, just in order to have the possibility to use derivatives or subdifferentials of single real-valued maps in this setting. Moreover, since set optimization problems are, in general, governed by set-valued maps, the use of tangential derivatives (see [1]) that depend on points in the graph

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is employed, and this is somehow not in the spirit of these problems, that in contrast to vector approach, take into consideration the whole values of the underlying set-valued map at the minimum point. We refer here to the works [2, 8, 10, 11, 13, 18] and the references therein.

In this paper we continue the effort done in [6] to introduce and to study some appropriate directional derivatives for set-valued in order to handle set-valued optimization problems under so-called set approach paradigm. In this work we significantly extend the number and the use of the generalized derivatives introduced in [6], taking advantage of their sequential characterizations. This allows us to have a more complete tableau of these constructions and to further build on it appropriate subdifferentials for set-valued maps. Moreover, we parallel our findings with the usual objects associated to the single real-valued maps and, more importantly, we show that the newly studied objects are good enough to describe optimality conditions in primal and dual forms for the minimality concept under study, in both constrained and unconstrained set optimization problems.

In the following we describe the way the present paper is organized and its main results. In the second section we make precise the notation we use and we collect some definitions and results that are useful in the sequel. A special attention is paid to the concept of ℓ-minimality for constrained problems where the objective is a set-valued map and the novelty of this section is an incompatibility result between openness and ℓ-minimality in a certain setting. The third section introduces two concepts of directional derivatives for set-valued maps and explores their basic properties and relationships with the Dini directional derivatives for real-valued functions. The fact that these objects extend in a nontrivial manner the classical Dini derivatives which are known to be very important and useful in deriving optimality conditions for optimization problems in the classical nonsmooth framework (see [5, 12]) is an impetus to consider their applications for set optimization problems. We mention that our Dini derivatives for set-valued maps are closer in spirit to set optimization, since they take into consideration a point in the domain of multifunction and not a point from the Cartesian product of domain and codomain, as, for instance, the similarly named objects defined in [16]. Therefore, Sect. 4 investigates the links of the Dini derivatives for set-valued maps defined here with the concept of ℓ-minimality. We concentrate our study on optimality conditions and penalization techniques that are devised for several classes of problems. The fifth section explores the possibility to introduce other similar objects and to extend to the case of set-valued maps the most usual of the classical subdifferentials of real-valued functions. We describe a mechanism that allows to extend basically all the subdifferentials based on generalized directional derivatives and we present the illustrating examples of Dini and Clarke subdifferentials. Moreover, we extend as well the Fréchet subdifferential. Then we show that these extensions still enjoy some basic properties of subdifferentials among which we mention a generalized Fermat rule for set optimization problems. A short section containing some concluding remarks ends the paper.

2 Preliminary notions and results

Let X, Y be normed spaces over the real field $\mathbb{R}$. The notation is fairly standard. The topological dual of X is X^*, the norm will be denoted $\|\cdot\|$, and $\mathcal{L}(X, Y)$ is the normed vector space of linear bounded operators from X to Y. We put $B(x, \varepsilon)$, $D(x, \varepsilon)$ and $S(x, \varepsilon)$ for open and closed balls and the sphere centered at $x \in X$ with the radius $\varepsilon > 0$. The distance from a point $x \in X$ to a nonempty set $A \subset X$ is

$$d(x, A) = \inf \{\|x - a\| \mid a \in A\}.$$

If A, B are nonempty subsets of X, the excess from A to B is

$$e(A, B) := \sup_{x \in A} d(x, B).$$

If at least one of the sets is empty, we impose the following conventions:

$$e(\emptyset, \cdot) := 0 \quad \text{and} \quad e(A, \emptyset) := +\infty, \ \forall A \neq \emptyset.$$

As usual, int A stands for the topological interior of A.

Consider $K \subset Y$ a closed convex pointed proper cone. We denote by K^+ the dual of K. The set approach in vector optimization is based on some order relations on sets defined by Kuroiwa: see [9] for discussion and details. We work here with one such relation. We collect some known concepts and results, mainly from [11] and [13].

Let $A, B \subset Y$ be nonempty sets. Define $\preceq_K^\ell$ by

$$A \preceq_K^\ell B \Leftrightarrow B \subset A + K.$$

If K is solid, that is int $K \neq \emptyset$, then one defines as well the strict relation $\prec_K^\ell$ by

$$A \prec_K^\ell B \Leftrightarrow B \subset A + \operatorname{int} K.$$

Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values and $M \subset X$ be a nonempty closed set. We denote by Epi $F : X \rightrightarrows Y$ the epigraphical set-valued map defined by

$$\operatorname{Epi} F(x) = F(x) + K.$$

Similarly, one can define Hypo $F : X \rightrightarrows Y$, the hypographical set-valued map, given by

$$\operatorname{Hypo} F(x) = F(x) - K.$$

Definition 2.1 An element $\overline{x} \in M$ is said to be ℓ-minimum for F on M if

$$x \in M, F(x) \preceq_K^\ell F(\overline{x}) \Rightarrow F(\overline{x}) \preceq_K^\ell F(x).$$

The local counterpart is obvious.

The above concept means that, for any $x \in M$, $F(\overline{x}) \subset F(x) + K$ implies that $F(x) \subset F(\overline{x}) + K$, or, in other words, one can have

$$F(\overline{x}) \not\subset F(x) + K \text{ or } F(\overline{x}) \subset F(x) + K \subset F(\overline{x}) + K.$$

Definition 2.2 An element $\overline{x} \in M$ is said to be ℓ-weak minimum for F on M if

$$x \in M, \ F(x) \prec_K^\ell F(\overline{x}) \Rightarrow F(\overline{x}) \prec_K^\ell F(x).$$

Remark 2.3 An element $\overline{x} \in M$ is a ℓ-yminimum for F on M if and only if $\overline{x}$ is a ℓ-minimum for Epi F on M. The same statement holds for ℓ-weak minimality.

For a set $\emptyset \neq A \subset Y$, the set of weakly minimal points (in the vector approach) is

$$\operatorname{WMin}(A, K) := \{a \in A \mid (A - a) \cap -\operatorname{int} K = \emptyset\}.$$

The results contained in the next remarks are known and easy to prove (see, for instance [11, Lemma 2.6], [10, Lemma 2.3]).

Remark 2.4 (i) If $A \subset Y$ is a nonempty set such that $\operatorname{WMin}(A, K) \neq \emptyset$, then

$$A \not\subset A + \operatorname{int} K.$$

(ii) Let $\overline{x} \in M$. If WMin $(F(\overline{x}), K) \neq \emptyset$, then $\overline{x}$ is ℓ-weak minimum for F on M if and only if $F(x) \not\prec_K^{\ell} F(\overline{x})$ for all $x \in M$. A similar assertion holds for local ℓ-weak minimality.
(iii) Suppose that $\overline{x}$ is a ℓ-minimum for $F$and WMin $(F(\overline{x}), K) \neq \emptyset$. Then $\overline{x}$ is a ℓ-weak minimum for F.

Probably, all the facts of the below lemma are also known, but we collect them into a single result for easy reference.

Lemma 2.5 *Let $K \subset Y$ be a closed convex pointed solid cone. Then:*

(i) *for all $y^* \in Y^* \setminus \{0\}$,*

$$y^*(B(0,1)) = \left(-\|y^*\|, \|y^*\|\right),$$

and if, additionally, Y is reflexive,

$$y^*(D(0,1)) = \left[-\|y^*\|, \|y^*\|\right];$$

(ii) *for all $y^* \in K^+ \setminus \{0\}$,*

$$y^*(\operatorname{int} K) = (0, +\infty) \text{ and } y^*(K) = [0, +\infty);$$

(iii) *for all $a, b \in Y$, one has the equivalences*

$$(a \in b + K) \Leftrightarrow \left(y^*(b) \leq y^*(a), \ \forall y^* \in K^+ \setminus \{0\}\right);$$
$$(a \in b + \operatorname{int} K) \Leftrightarrow \left(y^*(b) < y^*(a), \ \forall y^* \in K^+ \setminus \{0\}\right);$$

(iv) *for all A, B nonempty subsets of Y, one has the equivalences*

$$(A \subset B + K) \Leftrightarrow \left(y^*(A) + [0, +\infty) \subset y^*(B) + [0, +\infty), \ \forall y^* \in K^+ \setminus \{0\}\right);$$
$$(A \subset B + \operatorname{int} K) \Leftrightarrow \left(y^*(A) \subset y^*(B) + (0, +\infty), \ \forall y^* \in K^+ \setminus \{0\}\right).$$

Proof (i), (ii) These facts are well-known (and easy to prove).

(iii) If $a \in b + K$, then $a - b \in K$, and one applies the second part of (ii) to get $y^*(b) \leq y^*(a)$, for all $y^* \in K^+$. Conversely, knowing that $y^*(b) \leq y^*(a)$, for all $y^* \in K^+$, we show first that $a + \operatorname{int} K \subset b + K$. If this would not be the case it would exist $k_0 \in \operatorname{int} K$ such that $a + k_0 \notin b + K$. By a standard separation result, one gets $y^* \in Y^* \setminus \{0\}$ such that

$$y^*(a + k_0) \leq y^*(b + k), \ \forall k \in K.$$

It is also standard to get from here that $y^* \in K^+$ and

$$y^*(a) + y^*(k_0) \leq y^*(b).$$

But, $y^*(k_0) > 0$ and, by assumption, $y^*(b) \leq y^*(a)$, whence $y^*(b) < y^*(a) + y^*(k_0) \leq y^*(b)$ and this is not possible. We conclude that

$$a + \operatorname{int} K \subset b + K$$

and passing to the closure, we get $a + K \subset b + K$ which is equivalent to $a \in b + K$.

The second equivalence follows the same steps, with obvious modifications.

(iv) Using the first equivalence in (iii) we can write

$$\begin{aligned}(A \subset B + K) &\Leftrightarrow (\forall a \in A, \exists b \in B : a \in b + K) \\ &\Leftrightarrow \left(\forall a \in A, \exists b \in B : y^*(b) \leq y^*(a), \ \forall y^* \in K^+ \setminus \{0\}\right)\end{aligned}$$

$$\Leftrightarrow \left(\forall a \in A, \exists b \in B, \forall y^* \in K^+ \setminus \{0\} : y^*(a) \in y^*(b) + [0,+\infty) \subset y^*(B) + [0,+\infty)\right)$$
$$\Leftrightarrow \left(\forall a \in A, \forall y^* \in K^+ \setminus \{0\} : y^*(a) \in y^*(B) + [0,+\infty)\right)$$
$$\Leftrightarrow \left(\forall a \in A, \forall y^* \in K^+ \setminus \{0\} : y^*(a) + [0,+\infty) \subset y^*(B) + [0,+\infty)\right)$$
$$\Leftrightarrow \left(\forall y^* \in K^+ \setminus \{0\} : y^*(A) + [0,+\infty) \subset y^*(B) + [0,+\infty)\right),$$

and this is the first conclusion.

For the second conclusion, we correspondingly have

$$(A \subset B + \operatorname{int} K) \Leftrightarrow (\forall a \in A, \exists b \in B : a \in b + \operatorname{int} K)$$
$$\Leftrightarrow \left(\forall a \in A, \exists b \in B : y^*(b) < y^*(a),\ \forall y^* \in K^+ \setminus \{0\}\right)$$
$$\Leftrightarrow \left(\forall a \in A, \exists b \in B, \forall y^* \in K^+ \setminus \{0\} : y^*(a) \in y^*(b) + (0,+\infty) \subset y^*(B) + (0,+\infty)\right)$$
$$\Leftrightarrow \left(\forall a \in A, \forall y^* \in K^+ \setminus \{0\} : y^*(a) \in y^*(B) + (0,+\infty)\right)$$
$$\Leftrightarrow \left(\forall y^* \in K^+ \setminus \{0\} : y^*(A) \subset y^*(B) + (0,+\infty)\right),$$

and the proof is complete. □

Remark 2.6 One has, by Lemma 2.5 (iv),

$$\left(F(x) \preceq_K^{\ell} F(\overline{x})\right) \Leftrightarrow \left(\left(y^* \circ F\right)(x) \preceq_{[0,+\infty)}^{\ell} \left(y^* \circ F\right)(\overline{x}),\ \forall y^* \in K^+ \setminus \{0\}\right).$$

So, $\overline{x}$ is a ℓ-minimum for F (with respect to K) provided that $\overline{x}$ is a ℓ-minimum point for $y^* \circ F$ (with respect to $[0,+\infty)$) for all $y^* \in K^+ \setminus \{0\}$. Similarly, $\overline{x}$ is a ℓ-weak minimum for F (with respect to K) provided $\overline{x}$ is a ℓ-weak minimum point for $y^* \circ F$ (with respect to $[0,+\infty)$) for all $y^* \in K^+ \setminus \{0\}$.

We present now the usual concept of openness for set-valued maps.

Definition 2.7 Let $F : X \rightrightarrows Y$ and $\overline{x} \in X$ with $F(\overline{x}) \neq \emptyset$. We say that F is open at $\overline{x}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$B(F(\overline{x}), \delta) \subset F(B(\overline{x}, \varepsilon)).$$

The principle of incompatibility between openness and minimality (see, for instance, [7]) holds as well in this generalized setting. First, we recall a boundedness concept with respect to a cone.

Definition 2.8 We say that a set $A \subset Y$ is lower bounded with respect to K if there exists a bounded set $M \subset Y$ such that $A \subset M + K$.

Proposition 2.9 *Let $F : X \rightrightarrows Y$ and suppose that there exists $y^* \in K^+ \setminus \{0\}$ such that $\overline{x}$ is a local ℓ-minimum for $y^* \circ F$. If $F(\overline{x})$ is lower bounded with respect to K, then* Epi F *is not open at $\overline{x}$.*

Proof Suppose, by contradiction, that Epi F is open at $\overline{x}$. Take $\varepsilon > 0$ such that the ℓ-minimality holds, so we can find $\delta > 0$ such that

$$\operatorname{Epi} F(\overline{x}) + B(0, \delta) \subset \operatorname{Epi} F(B(\overline{x}, \varepsilon)). \tag{2.1}$$

Without loss of generality, suppose that $\|y^*\| = 1$. Taking into account Lemma 2.5, relation (2.1) implies

$$y^*(F(\overline{x})) + (-\delta, +\infty) = y^*(\text{Epi}\, F(\overline{x})) + (-\delta, \delta) + [0, +\infty) \subset y^*(\text{Epi}\, F(B(\overline{x}, \varepsilon))).$$

Since $F(\overline{x})$ is lower bounded, $\alpha := \inf\{y^*(y) \mid y \in F(\overline{x})\} > -\infty$. For the chosen δ, we find $y_\delta \in F(\overline{x})$ such that

$$\alpha + 2^{-1}\delta > y^*(y_\delta).$$

Then $\alpha - 2^{-1}\delta \in y^*(F(\overline{x})) + (-\delta, +\infty)$, hence there exists $x_0 \in B(\overline{x}, \varepsilon)$ and $y_0 \in \text{Epi}\, F(x_0)$ such that

$$\alpha - 2^{-1}\delta \geq y^*(y_0).$$

But then,

$$y^*(y) \geq \alpha > \alpha - 2^{-1}\delta \geq y^*(y_0), \quad \forall y \in F(\overline{x}),$$
$$y^*(\text{Epi}\, F(\overline{x})) = y^*(F(\overline{x})) + [0, +\infty) \subsetneq y^*(y_0) + [0, +\infty) \subset y^*(\text{Epi}\, F(x_0)).$$

The above inclusion is strict, since $\alpha > y^*(y_0)$ implies $y_0 \notin F(\overline{x})$. But this contradicts the ℓ-minimality of $\overline{x}$. □

Corollary 2.10 *Let $F : X \rightrightarrows \mathbb{R}$ and suppose that $\overline{x}$ is a local ℓ-minimum for F. If $F(\overline{x})$ is lower bounded with respect to* $[0, +\infty)$*, then* Epi F *is not open at $\overline{x}$.*

We end the section by recalling the notion of Lipschitz set–valued map.

Definition 2.11 Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x} \in X$. One says that F is Lipschitz around $\overline{x}$ if there is $L > 0$ and $\varepsilon > 0$ such that for all $x, x' \in B(\overline{x}, \varepsilon)$, one has

$$F(x) \subset F\left(x'\right) + L\left\|x - x'\right\| D(0, 1).$$

3 Concepts of directional derivatives for set-valued mappings

Consider the following notion.

Definition 3.1 Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x}, u \in X$. One calls the upper directional derivative of F at $\overline{x}$ in direction u the set, denoted $D^+F(\overline{x})(u)$, of elements $v \in Y$ such that for all $(t_n) \downarrow 0$ and $(u_n) \to u$

$$\lim_n \frac{e(F(\overline{x}) + t_n v, F(\overline{x} + t_n u_n))}{t_n} = 0. \tag{3.1}$$

Remark 3.2 This concept was introduced in [6] under the notation $D_{H\_}F(\overline{x})(u)$. We change here this notation for the reasons that will shortly become clear. Observe that $D^+F(\overline{x})(u)$ is a closed set and the relation $v \in D^+F(\overline{x})(u)$ is sequentially characterized by: for all $(t_n) \downarrow 0$ and $(u_n) \to u$

$$\lim_n e\left(\frac{1}{t_n}F(\overline{x}) + v, \frac{1}{t_n}F(\overline{x} + t_n u_n)\right) = 0,$$

and by: for all $(t_n) \downarrow 0$ and $(u_n) \to u$, and for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

We propose now another concept inspired by the sequential characterization of $D^+F(\overline{x})(u)$.

Definition 3.3 Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x}, u \in X$. One calls the lower directional derivative of F at $\overline{x}$ in direction u the set, denoted $D^-F(\overline{x})(u)$, of elements $v \in Y$ such that for all $\varepsilon > 0$ there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

Remark 3.4 If F is Lipschitz around $\overline{x}$, then in both sequential characterizations one can take $u_n = u$ for all n.

Next we analyze the links of the above concepts with the classical Dini directional derivatives. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $\operatorname{dom} f := \{x \in X \mid f(x) \in \mathbb{R}\}$ its domain. Take $\overline{x} \in \operatorname{dom} f$ and $u \in X$. We recall that the upper Dini directional derivative of f at $\overline{x}$ in the direction u is

$$d_+ f(\overline{x}, u) = \limsup_{t \downarrow 0, u' \to u} \frac{f(\overline{x} + tu') - f(\overline{x})}{t},$$

while the lower Dini directional derivative of f at $\overline{x} \in \operatorname{dom} f$ in the direction $u \in X$ is

$$d_- f(\overline{x}, u) = \liminf_{t \downarrow 0, u' \to u} \frac{f(\overline{x} + tu') - f(\overline{x})}{t}.$$

Proposition 3.5 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$, $\overline{x} \in \operatorname{dom} f$, $u \in X$. *One has*

(i) $D^-\operatorname{Epi} f(\overline{x})(u) = [d_- f(\overline{x}, u), +\infty)$;
(ii) $D^+\operatorname{Epi} f(\overline{x})(u) = [d_+ f(\overline{x}, u), +\infty)$;
(iii) $D^- f(\overline{x})(u) \subset [d_- f(\overline{x}, u), d_+ f(\overline{x}, u)]$, *while* $D^+ f(\overline{x})(u) \neq \emptyset$ *forces* $d_- f(\overline{x}, u) = d_+ f(\overline{x}, u)$, *in which case* $D^+ f(\overline{x})(u) = \{d_- f(\overline{x}, u)\}$.

Proof We give the proof for the first item. Take a real number $v \geq d_- f(\overline{x}, u)$. Then for all $\varepsilon > 0$ there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$\frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} < v + 2^{-1}\varepsilon,$$

that is

$$f(\overline{x} + t_n u_n) < f(\overline{x}) + t_n \left(v + 2^{-1}\varepsilon\right),$$

whence

$$f(\overline{x}) + t_n \left(v + 2^{-1}\varepsilon\right) + [0, +\infty) \subset f(\overline{x} + t_n u_n) + [0, +\infty).$$

We get, for $n \geq n_\varepsilon$,

$$\begin{aligned} f(\overline{x}) + t_n v + [0, +\infty) &\subset f(\overline{x} + t_n u_n) + [0, +\infty) - t_n \cdot 2^{-1}\varepsilon \\ &\subset f(\overline{x} + t_n u_n) + [0, +\infty) + t_n(-\varepsilon, \varepsilon). \end{aligned}$$

This shows that $v \in D^-\operatorname{Epi} f(\overline{x})(u)$. Conversely, if $v \in D^-\operatorname{Epi} f(\overline{x})(u)$, then for all $\varepsilon > 0$, there are $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$f(\overline{x}) + t_n v + [0, +\infty) \subset f(\overline{x} + t_n u_n) + [0, +\infty) + t_n(-\varepsilon, \varepsilon),$$

therefore

$$[f(\overline{x}), +\infty) \subset (f(\overline{x} + t_n u_n) + t_n(-\varepsilon - v), +\infty).$$

We get, for $n \geq n_\varepsilon$

$$f(\overline{x} + t_n u_n) + t_n(-\varepsilon - v) < f(\overline{x}),$$

so

$$\frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} < v + \varepsilon.$$

Taking into account the sequential characterization of lim inf, we get

$$d_- f(\overline{x}, u) < v + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $d_- f(\overline{x}, u) \leq v$, and this ends the proof of (i).

The proofs for (ii) and (iii) are similar. □

Remark 3.6 The first two conclusions of the above result suggest that D^- and D^+ are natural extensions to set-valued case of d_- and d_+, respectively.

We list below some immediate properties.

Proposition 3.7 *In the above notation, if $F, G : X \rightrightarrows Y$ are set-valued maps, $\overline{x}, u \in X$, then the following facts hold:*

(i) $D^+F(\overline{x})(u) \subset D^-F(\overline{x})(u)$;
(ii) $y \in D^+F(\overline{x})(u)$ *if and only if* $-y \in D^+(-F)(\overline{x})(u)$;
(iii) $y \in D^-F(\overline{x})(u)$ *if and only if* $-y \in D^-(-F)(\overline{x})(u)$;
(iv) $D^+F(\overline{x})(u) + K \subset D^+(\text{Epi } F)(\overline{x})(u)$;
(v) $D^-F(\overline{x})(u) + K \subset D^-(\text{Epi } F)(\overline{x})(u)$;
(vi) $D^+F(\overline{x})(u) + D^+G(\overline{x})(u) \subset D^+(F+G)(\overline{x})(u)$;
(vii) $D^+F(\overline{x})(u) + D^-G(\overline{x})(u) \subset D^-(F+G)(\overline{x})(u)$;
(viii) *if F is Lipschitz around $\overline{x}$, then* $0 \in D^+F(\overline{x})(0)$;
(ix) *if* $y \in D^+F(\overline{x})(u)$ *and* $\alpha > 0$, *then* $\alpha y \in D^+F(\overline{x})(\alpha u)$;
(x) *if* $y \in D^-F(\overline{x})(u)$ *and* $\alpha > 0$, *then* $\alpha y \in D^-F(\overline{x})(\alpha u)$.

Proof Some of these assertions are proved in [6, Proposition 3.4], while the others are similar or straightforward. □

Proposition 3.8 *Let $F : X \rightrightarrows Y$ be a set-valued map with convex graph. Consider $\overline{x}, u, v \in X$ and $\lambda \in (0, 1)$. Then*

(i) $\lambda D^+F(\overline{x})(u) + (1-\lambda) D^-F(\overline{x})(v) \subset D^-F(\overline{x})(\lambda u + (1-\lambda) v)$;
(ii) $\lambda D^+F(\overline{x})(u) + (1-\lambda) D^+F(\overline{x})(v) \subset D^+F(\overline{x})(\lambda u + (1-\lambda) v)$, *that is, the set-valued map $u \rightrightarrows D^+F(\overline{x})(u)$ has convex graph.*

Proof (i) Let $y \in D^+F(\overline{x})(u)$ and $z \in D^-F(\overline{x})(v)$. Take $\varepsilon > 0$. Then there are $(t_n) \downarrow 0$, $(v_n) \to v$ and $n_\varepsilon \in \mathbb{N}$ such that for all $(u_n) \to u$ and $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n y \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon),$$
$$F(\overline{x}) + t_n z \subset F(\overline{x} + t_n v_n) + t_n B(0, \varepsilon).$$

Since, in particular, F has convex values, $\lambda F(\overline{x}) + (1-\lambda) F(\overline{x}) = F(\overline{x})$, so

$$\begin{aligned} F(\overline{x}) + t_n(\lambda y + (1-\lambda) z) &\subset \lambda F(\overline{x} + t_n u_n) + (1-\lambda) F(\overline{x} + t_n v_n) + t_n B(0, 2\varepsilon) \\ &\subset F(\overline{x} + t_n(\lambda u_n + (1-\lambda) v_n)) + t_n B(0, 2\varepsilon), \end{aligned}$$

where the last inclusion uses the convexity of the graph of F. This shows that $\lambda y + (1-\lambda) z \in D^- F(\overline{x})(\lambda u + (1-\lambda) v)$.

(ii) Let $y \in D^+ F(\overline{x})(u)$ and $z \in D^+ F(\overline{x})(v)$. Take $\varepsilon > 0$, $(t_n) \downarrow 0$ and $(w_n) \to \lambda u + (1-\lambda) v$. Then, clearly,

$$\begin{aligned} (v_n) &= \left(\frac{w_n - \lambda u}{1-\lambda} \right) \to v \\ (u_n) &= \left(\frac{w_n - (1-\lambda) v_n}{\lambda} \right) \to u \end{aligned}$$

and

$$(\lambda u_n + (1-\lambda) v_n) = (w_n).$$

Therefore, for n large enough,

$$\begin{aligned} F(\overline{x}) + t_n y &\subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon), \\ F(\overline{x}) + t_n z &\subset F(\overline{x} + t_n v_n) + t_n B(0, \varepsilon), \end{aligned}$$

and we conclude as above. □

4 Links with ℓ-minimality

In this section we present several applications to ℓ-minimality of the generalized directional derivatives for set-valued mappings introduced before.

First of all, observe that on the basis of Lemma 2.5, we get the following result.

Proposition 4.1 *Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x}, u \in X$. If $v \in D^- \operatorname{Epi} F(\overline{x})(u)$, then*

$$y^*(v) \in D^- \operatorname{Epi}\left(y^* \circ F\right)(\overline{x})(u), \ \forall y^* \in K^+ \setminus \{0\}.$$

Proof Recall that $v \in D^- \operatorname{Epi} F(\overline{x})(u)$ iff for all $\varepsilon > 0$, there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + K + t_n v \subset F(\overline{x} + t_n u_n) + K + t_n B(0, \varepsilon).$$

By applying $y^* \in K^+ \setminus \{0\}$, and using Lemma 2.5 (i), (ii) and (iv) we get

$$\left(y^* \circ F\right)(\overline{x}) + [0, +\infty) + t_n y^*(v) \subset \left(y^* \circ F\right)(\overline{x} + t_n u_n) + [0, +\infty) + t_n \left(-\varepsilon \left\| y^* \right\|, \varepsilon \left\| y^* \right\|\right),$$

showing that $y^*(v) \in D^- \operatorname{Epi}(y^* \circ F)(\overline{x})(u)$. □

We give now a necessary optimality condition for local ℓ-weak minimality. As usual, for a nonempty set $M \subset X$ and for $x \in M$, the Bouligand tangent cone to M at x is

$$T_B(M, x) = \{u \in X \mid \exists (t_n) \downarrow 0, \ \exists (u_n) \to u : x + t_n u_n \in M, \ \forall n\},$$

while the Ursescu tangent cone to M at x is

$$T_U(M,x)=\{u\in X \mid \forall (t_n)\downarrow 0,\ \exists (u_n)\to u : x+t_nu_n\in M,\ \forall n\}.$$

We start with a known result (see [6, Proposition 4.1]).

Proposition 4.2 *Let $\overline{x}$ be a local ℓ-weak minimum for F on M and* WMin $(F(\overline{x}),K)\neq\emptyset$. *Then*

$$D^+F(\overline{x})(u)\cap -\operatorname{int}K=\emptyset,\ \forall u\in T_B(M,\overline{x}).$$

The next result devise a sufficient condition for ℓ-minimality.

Proposition 4.3 *Let X be finite dimensional. Let $\overline{x}\in M$ and suppose that* WMin $(F(\overline{x}),K)$ $\neq\emptyset$. *If*

$$0\notin D^-\operatorname{Epi}F(\overline{x})(T_B(M,\overline{x})\setminus\{0\}), \tag{4.1}$$

then for all $e\in Y$ there is $\mu>0$ such that $\overline{x}$ is a local ℓ-weak minimum for $x\rightrightarrows F(x)-\mu\|x-\overline{x}\|e$ on M.

Proof Suppose, by way of contradiction, that $\overline{x}$ does not enjoy the property in the conclusion for a fixed $e\in Y$. For all $\mu>0$, the set-valued map $x\rightrightarrows F(x)-\mu\|x-\overline{x}\|e$ satisfies the property in Remark 2.4 (ii). Then there exists a sequence (x_n) of elements in M such that

$$F(\overline{x})\subset F(x_n)-\frac{1}{n}\|x_n-\overline{x}\|e+\operatorname{int}K.$$

Clearly, $x_n\neq\overline{x}$ for all n since otherwise $F(\overline{x})\subset F(\overline{x})+\operatorname{int}K$, which is not possible. Then we denote $(\|x_n-\overline{x}\|)=(t_n)\downarrow 0$ and we have

$$F(\overline{x})\subset F\left(\overline{x}+t_n\frac{x_n-\overline{x}}{t_n}\right)-\frac{1}{n}t_ne+\operatorname{int}K.$$

Since the sequence $(u_n)=\left(\frac{x_n-\overline{x}}{t_n}\right)$ is bounded and X is finite dimensional, one can suppose, without loss of generality, that $(u_n)\to u$. It is easy to see that $u\in T_B(M,\overline{x})\setminus\{0\}$. Now, for all $\varepsilon>0$, for n large enough,

$$\begin{aligned}F(\overline{x})+t_n\cdot 0+\operatorname{int}K&\subset F(\overline{x}+t_nu_n)-\frac{1}{n}t_ne+\operatorname{int}K\\&\subset F(\overline{x}+t_nu_n)+\operatorname{int}K+t_nB(0,\varepsilon),\end{aligned}$$

whence $0\in D^-\operatorname{Epi}F(\overline{x})(u)$, and this is a contradiction. □

Remark 4.4 Taking into account Proposition 3.5, we get a well-known result: if X is finite dimensional, $f:X\to\mathbb{R}$ is a function and $d_-f(\overline{x},u)>0$ for all $u\in T_B(M,\overline{x})\setminus\{0\}$, then $\overline{x}$ is a local minimum point for f on M.

Let us present now an illustrative example for Proposition 4.3.

Example 4.5 Consider $F:\mathbb{R}\to\mathbb{R}^2$ given by

$$F(x):=\{\cos x\}\times[\sin x,\sin x+1),\quad \forall x\in\mathbb{R},$$

$M:=[0,1]$, $K:=\mathbb{R}^2_+$ and $\overline{x}:=0$. Then, obviously, $T_B(M,\overline{x})=[0,+\infty)$, $(1,0)\in$ WMin $(F(\overline{x}),K)$ and

$$\operatorname{Epi}F(x):=[\cos x,+\infty)\times[\sin x,+\infty),\quad\forall x\in\mathbb{R}.$$

Consider $(v_1, v_2) \in D^{-}\mathrm{Epi}\, F(0)(u)$ for $u > 0$. Then, for any $\varepsilon > 0$, there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that, for any $n \geq n_\varepsilon$, we have

$$\mathrm{Epi}\, F(0) + t_n(v_1, v_2) \subset \mathrm{Epi}\, F(t_n u_n) + t_n(-\varepsilon, \varepsilon)^2, \text{ or}$$
$$[1, +\infty) \times [0, +\infty) + t_n(v_1, v_2) \subset [\cos(t_n u_n), +\infty) \times [\sin(t_n u_n), +\infty) + t_n(-\varepsilon, \varepsilon)^2.$$

But this means

$$\begin{cases} 1 + t_n v_1 \geq \cos(t_n u_n) - t_n \varepsilon \\ t_n v_2 \geq \sin(t_n u_n) - t_n \varepsilon, \end{cases}$$

and, furthermore,

$$\begin{cases} v_1 \geq \dfrac{\cos(t_n u_n) - 1}{t_n} - \varepsilon \\ v_2 \geq \dfrac{\sin(t_n u_n)}{t_n} - \varepsilon. \end{cases}$$

Passing to the limit for $n \to +\infty$ and taking into account that $\varepsilon > 0$ was arbitrarily chosen, we obtain $v_1 \geq 0$ and $v_2 \geq u$, hence $(v_1, v_2) \neq (0, 0)$ and condition (4.1) is satisfied.

Consider next arbitrary $e = (e_1, e_2) \in \mathbb{R}^2$. In order to show that there exists $\mu > 0$ such that $\overline{x}$ is a local ℓ-weak minimum for $x \rightrightarrows G(x) := F(x) - \mu \|x - \overline{x}\| e$ on M, remark that $(1, 0) \in \mathrm{WMin}(G(\overline{x}), K)$, so is enough to prove that

$$G(0) \not\subset G(x) + \mathrm{int}\, K, \quad \forall x \in M \setminus \{0\}.$$

For this, it is enough to prove that there exists $\mu > 0$ such that

$$\sin x - \mu x e_2 > 0, \quad \forall x \in (0, 1].$$

If $e_2 \leq 0$, this is obvious for any $\mu > 0$, and if $e_2 > 0$, one can choose $\mu \in \left(0, \dfrac{2}{\pi e_2}\right)$.

Several penalization results can be as well obtained on the basis given by our concepts. Let $M \subset X$ be a closed set. Consider a function $\varphi : X \to [0, +\infty)$ with the property that $\varphi(x) = 0$ iff $x \in M$ and a function $\psi : X \to \mathbb{R}$ with the property that $\psi(x) \leq 0$ iff $x \in M$. On $\mathbb{R}$ one considers the usual ordering cone $\mathbb{R}_+ = [0, +\infty)$.

We denote by (F, f) the set-valued map acting between X and $Y \times \mathbb{R}$ given by

$$(F, f)(x) = \{(y, f(x)) \mid y \in F(x)\},$$

where $f \in \{\varphi, \psi\}$. The space $Y \times \mathbb{R}$ is partially ordered by the cone $K \times [0, +\infty)$, so the associated epigraphical set-valued map is $\mathrm{Epi}\,(F, f) : X \rightrightarrows Y \times \mathbb{R}$ given by

$$\mathrm{Epi}\,(F, f)(x) = \{(y, v) \mid y \in F(x) + K, v \in [f(x), +\infty)\}.$$

Proposition 4.6 *Let $\overline{x} \in M$ be a ℓ-minimum for F on M. Then $\overline{x}$ is a ℓ-minimum for* $\mathrm{Epi}\,(F, \varphi)$ *on X.*

Proof Suppose that there is $x \in X$ such that

$$\mathrm{Epi}\,(F, \varphi)(\overline{x}) \subset \mathrm{Epi}\,(F, \varphi)(x) + K \times [0, +\infty).$$

This implies that

$$F(\overline{x}) \subset F(x) + K$$

and

$$\varphi(\overline{x}) \in [\varphi(x), +\infty).$$

The latter inclusion means that $\varphi(x) \le \varphi(\overline{x}) = 0$, so $x \in M$ and $\varphi(x) = 0$. Now, the former inclusion and the ℓ-minimum for F on M imply

$$F(x) \subset F(\overline{x}) + K,$$

so we have

$$\mathrm{Epi}(F, \varphi)(x) \subset \mathrm{Epi}(F, \varphi)(\overline{x}) + K \times [0, +\infty),$$

which proves our assertion. □

Proposition 4.7 *Let $\overline{x} \in M$ be a ℓ-weak minimum for F on M. Suppose that* WMin $(F(\overline{x}), K) \neq \emptyset$. *Then $\overline{x}$ is a ℓ-weak minimum for* Epi (F, ψ) *on X.*

Proof We use Remark 2.4 (ii). Since WMin $(F(\overline{x}), K) \neq \emptyset$, for all $x \in M$

$$F(\overline{x}) \not\subset F(x) + \mathrm{int}\, K.$$

On the other hand, clearly, WMin $((F, \psi)(\overline{x}), K \times [0, +\infty)) \neq \emptyset$, so we have to show that (according to the Remark 2.3, as well)

$$(F, \psi)(\overline{x}) \not\subset (F, \psi)(x) + \mathrm{int}\, K \times (0, +\infty), \ \forall x \in X.$$

Suppose, by way of contradiction, that there is $u \in X$ such that

$$F(\overline{x}) \subset F(u) + \mathrm{int}\, K$$
$$\psi(\overline{x}) \in (\psi(u), +\infty).$$

Then, $\psi(u) < \psi(\overline{x}) \le 0$, whence $u \in M$, and now the inclusion $F(\overline{x}) \subset F(u) + \mathrm{int}\, K$ is in contradiction with the ℓ-weak minimality of $\overline{x}$. □

Remark 4.8 Similar assertions to Propositions 4.6, 4.7 hold for the concepts of local minimality.

Proposition 4.9 *Let $\overline{x} \in M$ be a ℓ-weak minimum for F on M. Suppose that* WMin $(F(\overline{x}), K) \neq \emptyset$. *Then*

$$\left(D^{+}\mathrm{Epi}\, F(\overline{x}), D^{-}\mathrm{Epi}\psi(\overline{x})\right)(u) \cap (-\mathrm{int}\, K \times (-\infty, 0)) = \emptyset,\ \forall u \in X.$$

Proof According to Proposition 4.7 and the subsequent remark, $\overline{x}$ is a ℓ-weak minimum for Epi (F, ψ) on X and, based on Remark 2.4, Epi $(F, \psi)(x) \not\prec_K^{\ell}$ Epi $(F, \psi)(\overline{x})$ for all $x \in X$.

Now suppose, by way of contradiction, that there is $u \in X$, $v \in D^{+}\mathrm{Epi}\, F(\overline{x})(u) \cap -\mathrm{int}\, K$ and $z \in D^{-}\mathrm{Epi}\, \psi(\overline{x})(u)$ such that $z < 0$. Take $\varepsilon \in (0, -z)$ such that $v + B(0, \varepsilon) \subset -\mathrm{int}\, K$. The relation $z \in D^{-}\mathrm{Epi}\psi(\overline{x})(u)$ implies that there are $(t_n) \downarrow 0$ and $(u_n) \to u$ such that for all n large enough,

$$\psi(\overline{x}) + [0, +\infty) + t_n z \subset \psi(\overline{x} + t_n u_n) + [0, +\infty) + t_n(-\varepsilon, \varepsilon),$$

that is

$$\begin{aligned}\mathrm{Epi}\psi(\overline{x}) &\subset \mathrm{Epi}\psi(\overline{x} + t_n u_n) + t_n(-z - \varepsilon, -z + \varepsilon) + [0, +\infty)\\ &\subset \mathrm{Epi}\psi(\overline{x} + t_n u_n) + (0, +\infty).\end{aligned}$$

Now, according to the definition of D^+, for ε chosen above there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$\operatorname{Epi} F(\overline{x}) + t_n v \subset \operatorname{Epi} F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

Therefore, for all n large enough,

$$\operatorname{Epi} F(\overline{x}) \subset \operatorname{Epi} F(\overline{x} + t_n u_n) + t_n(-v + B(0, \varepsilon)) \subset \operatorname{Epi} F(\overline{x} + t_n u_n) + \operatorname{int} K.$$

We conclude that for large n,

$$\operatorname{Epi}(F, \psi)(\overline{x} + t_n u_n) \prec^{\ell}_{K \times \mathbb{R}_+} \operatorname{Epi}(F, \psi)(\overline{x}),$$

and this is a contradiction. □

One considers now another constrained problem by introducing a new normed vector space Z partially ordered by a closed convex pointed cone Q and a set-valued map $G : X \rightrightarrows Z$. The problem is

$$\preceq^{\ell}_{K} - \min F(x) \text{ subject to } G(x) \preceq^{\ell}_{Q} \{0\},$$

that is

$$\preceq^{\ell}_{K} - \min F(x) \text{ subject to } 0 \in G(x) + Q.$$

Proposition 4.10 *If $\overline{x}$ is a solution of this problem and* WMin $(F(\overline{x}), K) \neq \emptyset$*, then for all $u \in X$*

$$\left(D^+ \operatorname{Epi} F(\overline{x}), D^- \operatorname{Epi} G(\overline{x})\right)(u) \cap (-\operatorname{int} K \times -\operatorname{int} Q) = \emptyset,$$

and

$$\left(D^- \operatorname{Epi} F(\overline{x}), D^+ \operatorname{Epi} G(\overline{x})\right)(u) \cap (-\operatorname{int} K \times -\operatorname{int} Q) = \emptyset.$$

Proof We prove only the first assertion, the second one being similar. Suppose, by way of contradiction, that there are $u \in X$ and $(v, z) \in Y \times Z$ in the intersection

$$\left(D^+ \operatorname{Epi} F(\overline{x}), D^- \operatorname{Epi} G(\overline{x})\right)(u) \cap (-\operatorname{int} K \times -\operatorname{int} Q).$$

Let $\varepsilon > 0$ such that $B(v, \varepsilon) \subset -\operatorname{int} K$ and $B(z, \varepsilon) \subset -\operatorname{int} Q$. Then there are $(t_n) \downarrow 0$ and $(u_n) \to u$ such that for all n large enough,

$$F(\overline{x}) + K + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon) + K,$$
$$G(\overline{x}) + Q + t_n z \subset G(\overline{x} + t_n u_n) + t_n B(0, \varepsilon) + Q.$$

From the latter inclusion, and the choice of ε one gets $0 \in G(\overline{x}) + Q \subset G(\overline{x} + t_n u_n) + Q$, whence $\overline{x} + t_n u_n$ is a feasible point of the problem. Now, the former relation ensures that

$$F(\overline{x}) \subset F(\overline{x} + t_n u_n) + \operatorname{int} K,$$

and we get a contradiction. □

5 Generalized differentiation for set-valued mappings

In this section we extend our ideas for getting generalization (sub)differentiation objects for set-valued maps. In fact, we see, according to Proposition 3.5, that D^+ and D^- are suited for generalize superdifferentials of functions. However, the mechanism employed in Sect. 3 allows us to define as well

$$D_-F(\overline{x})(u) = \{v \in Y \mid \forall \varepsilon > 0, \forall (t_n) \downarrow 0, \forall (u_n) \to u, \forall n \in \mathbb{N},\\ F(\overline{x} + t_n u_n) \subset F(\overline{x}) + t_n v + t_n B(0, \varepsilon)\}.$$

Similarly,

$$D_+F(\overline{x})(u) = \{v \in Y \mid \forall \varepsilon > 0, \exists (t_n) \downarrow 0, \exists (u_n) \to u, \forall n \in \mathbb{N},\\ F(\overline{x} + t_n u_n) \subset F(\overline{x}) + t_n v + t_n B(0, \varepsilon)\}.$$

Remark 5.1 Observe that, if $f : X \to Y$ is a function, then for any $\overline{x}, u \in X$, one has

$$v \in D_-\text{Epi}\, f(\overline{x})(u) \Leftrightarrow v \in D^+\text{Hypo}\, f(\overline{x})(u),\\ v \in D_+\text{Epi}\, f(\overline{x})(u) \Leftrightarrow v \in D^-\text{Hypo}\, f(\overline{x})(u).$$

Indeed, consider arbitrary $\varepsilon > 0$, $(t_n) \downarrow 0$, and $(u_n) \to u$. Then

$$\begin{aligned} v \in D_-\text{Epi}\, f(\overline{x})(u) &\Leftrightarrow f(\overline{x} + t_n u_n) + K \subset f(\overline{x}) + t_n v + t_n B(0,\varepsilon) + K \\ &\Leftrightarrow (-f)(\overline{x}) + t_n(-v) + K \subset (-f)(\overline{x} + t_n u_n) + t_n B(0,\varepsilon) + K \\ &\Leftrightarrow \text{Hypo}\, f(\overline{x}) + t_n v \subset \text{Hypo}\, f(\overline{x} + t_n u_n) + t_n B(0,\varepsilon) \\ &\Leftrightarrow v \in D^+\text{Hypo}\, f(\overline{x})(u). \end{aligned}$$

The other case is similar.

Remark that, for $F = f$ being a real-valued function, for all u

$$D_-\text{Epi}\, f(\overline{x})(u) = (-\infty, d_- f(\overline{x}, u)],\\ D_+\text{Epi}\, f(\overline{x})(u) = (-\infty, d_+ f(\overline{x}, u)].$$

Therefore, for a multifunction F which takes values in $\mathcal{P}(\mathbb{R})$, the construction

$$\partial_- F(\overline{x}) = \left\{x^* \in X^* \mid x^*(u) \in D_-\text{Epi}\, F(\overline{x})(u),\ \forall u \in X\right\} \tag{5.1}$$

generalizes the Dini subdifferential (see, e.g., [17, Chapter 8]).

For a real-valued function, we use the notation $\widehat{\partial}$ for the Fréchet subdifferential. It is known that

$$\widehat{\partial} f(\overline{x}) := \left\{x^* \in X^* \mid \liminf_{x \to \overline{x}} \frac{f(x) - f(\overline{x}) - x^*(x - \overline{x})}{\|x - \overline{x}\|} \geq 0\right\} \subset \partial_- f(\overline{x}), \tag{5.2}$$

and the equality holds provided X is finite dimensional (see [17, Exercise 8.4]). Relations (5.1) and (5.2) inspire us to define subdifferential objects for general set-valued mappings, $F : X \rightrightarrows Y$.

Definition 5.2 Let $F : X \rightrightarrows Y$, and $\overline{x} \in X$. The Fréchet subdifferential of F at $\overline{x}$ is

$$\widehat{\partial} F(\overline{x}) = \left\{T \in \mathcal{L}(X, Y) \mid \lim_{x \to \overline{x}} \frac{e(\text{Epi}\, F(x), \text{Epi}\, F(\overline{x}) + T(x - \overline{x}))}{\|x - \overline{x}\|} = 0\right\}. \tag{5.3}$$

Equivalently, $T \in \widehat{\partial} F(\overline{x})$ iff $T \in \mathcal{L}(X, Y)$ and

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\overline{x}, \delta) : \text{Epi}\, F(x) \subset \text{Epi}\, F(\overline{x}) + T(x - \overline{x}) + \varepsilon \|x - \overline{x}\| B(0, 1). \tag{5.4}$$

Similarly, we can define the upper subdifferential of F at $\overline{x}$ as follows

$$\widehat{\partial}^{+} F(\overline{x}) = \left\{ T \in \mathcal{L}(X, Y) \mid \lim_{x \to \overline{x}} \frac{e(\text{Epi}\, F(\overline{x}) + T(x - \overline{x}), \text{Epi}\, F(x))}{\|x - \overline{x}\|} = 0 \right\}. \tag{5.5}$$

The Dini lower subdifferential of F at $\overline{x}$ is

$$\partial_{-} F(\overline{x}) = \{T \in \mathcal{L}(X, Y) \mid T(u) \in D_{-}\text{Epi}\, F(\overline{x})(u), \ \forall u \in X\}. \tag{5.6}$$

Remark that, if $f : X \to \mathbb{R}$ is a function, then relation (5.4) can be equivalently written as

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\overline{x}, \delta) : f(x) - f(\overline{x}) - x^{*}(x - \overline{x}) > -\varepsilon \|x - \overline{x}\|,$$

i.e., $x^{*} \in \widehat{\partial} f(\overline{x})$. Similarly for $\widehat{\partial}^{+} f(\overline{x})$. Moreover, $\partial_{-} F(\overline{x})$ reduces to the Dini subdifferential defined in (5.1) for the case $Y = \mathbb{R}$.

Remark 5.3 Observe that many subdifferential objects for set-valued mappings exist in literature. We mention here only the regular/Frechet and the basic/Mordukhovich subdifferentials developed in [3, 4], [15, Chapter 9], which are successully used for deriving optimality conditions in set-valued optimization, in the vector approach.

As mentioned in the Introduction, the main difference between the subdifferentials from Definition 5.2 and other objects from literature, including the aforementioned ones, is that we want to take into consideration a point $\overline{x}$ in the domain of the multifunction F and the whole set $F(\overline{x})$, and not a pair $(\overline{x}, \overline{y})$ from the graph of the multifunction F, in order to be closer in spirit to the set approach in set-valued optimization and appropriately formulate optimality conditions by means of these constructions. Of course, our intention is to use the limiting procedure, which generates the basic/Mordukhovich subdifferential starting from the regular one, in order to develop a Mordukhovich type subdifferential in our framework. This will be subject to future work.

We explore the situation of differentiable functions.

Proposition 5.4 *Let $f : X \to Y$ be a Fréchet differentiable function, and $\overline{x} \in X$.*

(i) *For every $u \in X$, one has that*

$$\nabla f(\overline{x})(u) - K \subset D_{-}\,\text{Epi}\, f(\overline{x})(u). \tag{5.7}$$

If, in addition, $K + D(0, 1)$ is closed, then

$$D_{-}\text{Epi}\, f(\overline{x})(u) = D_{+}\text{Epi}\, f(\overline{x})(u) = \nabla f(\overline{x})(u) - K \tag{5.8}$$

and, moreover,

$$\partial_{-} f(\overline{x}) = \{\nabla f(\overline{x})\}.$$

(ii) *One has*

$$\widehat{\partial} f(\overline{x}) = \widehat{\partial}^{+} f(\overline{x}) = \{\nabla f(\overline{x})\}.$$

Proof (i) Taking into account Remark 5.1, and using Proposition 3.7 (ii), we have that

$$\nabla f(\overline{x})(u) - K \subset D_{-}\text{Epi}\, f(\overline{x})(u) \Leftrightarrow \nabla f(\overline{x})(u) - K \subset D^{+}\text{Hypo}\, f(\overline{x})(u)$$
$$\Leftrightarrow \nabla(-f)(\overline{x})(u) + K \subset D^{+}(-\text{Hypo}\, f)(\overline{x})(u) = D^{+}\text{Epi}\,(-f)(\overline{x})(u),$$

and the last relation is known from [6, Proposition 3.5], with equality if $K + D(0,1)$ is closed.

To end the proof of the first item, observe that inclusion (5.7) shows that

$$\nabla f(\overline{x})(u) \in D_{-}\text{Epi}\, f(\overline{x})(u), \ \forall u \in X,$$

hence $\nabla f(\overline{x}) \in \partial_{-} f(\overline{x})$. Furthermore, take $T \in \partial_{-} f(\overline{x})$. Using (5.8), we have that for any $u \in X$,

$$T(u) \in \nabla f(\overline{x})(u) - K \Leftrightarrow (T - \nabla f(\overline{x}))(u) \in (-K).$$

Since u is arbitrary, it follows that for any $u \in X$, $(T - \nabla f(\overline{x}))(u) \in K \cap (-K) = \{0\}$, hence $T = \nabla f(\overline{x})$.

(ii) One obviously has that $\nabla f(\overline{x}) \in \widehat{\partial} f(\overline{x})$ and $\nabla f(\overline{x}) \in \widehat{\partial}^{+} f(\overline{x})$. Take arbitrary $T \in \widehat{\partial} f(\overline{x})$. Then

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\overline{x}, \delta) : f(x) + K \subset f(\overline{x}) + T(x - \overline{x}) + \varepsilon \|x - \overline{x}\| B(0,1) + K.$$

On the other hand, using the Fréchet differentiability, we have that

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\overline{x}, \delta) : -f(x) + f(\overline{x}) + \nabla f(\overline{x})(x - \overline{x}) \in \varepsilon \|x - \overline{x}\| B(0,1).$$

By combining the previous two relations, we obtain that

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\overline{x}, \delta) : (\nabla f(\overline{x}) - T)(x - \overline{x}) + K \subset 2\varepsilon \|x - \overline{x}\| B(0,1) + K$$

and, taking into account that K is a cone, that

$$\forall \varepsilon > 0 : (\nabla f(\overline{x}) - T)(S(0,1)) + K \subset 2\varepsilon B(0,1) + K.$$

Employing Lemma 2.5, this is equivalent, successively, to

$$\forall \varepsilon > 0, \forall y^{*} \in K^{+} \cap S_{Y^{*}}(0,1) : y^{*} \circ (\nabla f(\overline{x}) - T)(S(0,1)) + [0, +\infty) \subset 2\varepsilon(-1,1) + [0,+\infty),$$
$$\forall \varepsilon > 0, \forall y^{*} \in K^{+} \cap S_{Y^{*}}(0,1), \forall x \in S(0,1) : y^{*} \circ (\nabla f(\overline{x}) - T)(x) \geq -2\varepsilon$$
$$\forall y^{*} \in K^{+} \cap S_{Y^{*}}(0,1), \forall x \in S(0,1) : y^{*} \circ (\nabla f(\overline{x}) - T)(x) \geq 0$$
$$\forall x \in S(0,1), (\nabla f(\overline{x}) - T)(x) \in K^{++} = K.$$

Using that the set $S(0,1)$ is symmetric, we obtain that the previous relation is equivalent to

$$\forall x \in S(0,1), (\nabla f(\overline{x}) - T)(x) \in K \cap (-K) = \{0\},$$

hence $T = \nabla f(\overline{x})$. The case of upper subdifferential is similar. □

The next result, inspired by [6, Proposition 3.7], generalizes the inclusion (5.2).

Proposition 5.5 *Let $F : X \rightrightarrows Y$ be a multifunction, and $\overline{x} \in X$. Then $\widehat{\partial} F(\overline{x}) \subset \partial_{-} F(\overline{x})$, with equality in case X is finite dimensional.*

Proof Suppose $T \in \widehat{\partial} F(\overline{x})$ and arbitrary $u \in X$. Moreover, take arbitrary $\varepsilon > 0$, $(t_n) \downarrow 0$, and $(u_n) \to u$. By (5.4), for any n large enough, we have

$$\begin{aligned}\text{Epi}\, F(\overline{x} + t_n u_n) &\subset \text{Epi}\, F(\overline{x}) + t_n T(u_n) + t_n \frac{\|u_n\|}{\|u\| + 2} \varepsilon B(0,1) \\ &\subset \text{Epi}\, F(\overline{x}) + t_n T(u) + t_n T(u_n - u) + t_n \frac{\|u\| + 1}{\|u\| + 2} B(0, \varepsilon).\end{aligned}$$

Since $(u_n) \to u$ and T is continuous, for any large n, $T(u_n - u) \in B\left(0, (\|u\| + 2)^{-1} \varepsilon\right)$. Hence,

$$\text{Epi}\, F(\overline{x} + t_n u_n) \subset \text{Epi}\, F(\overline{x}) + t_n T(u) + t_n B(0, \varepsilon).$$

Hence, $T(u) \in D_-(\text{Epi}\, F)(\overline{x})(u)$, and since u was arbitrarily chosen, we have $\widehat{\partial} F(\overline{x}) \subset \partial_- F(\overline{x})$.

Conversely, suppose $T(u) \in D_-(\text{Epi}\, F)(\overline{x})(u)$ for all u, and, by contradiction, that $T \notin \widehat{\partial} F(\overline{x})$. Then there exist $\theta > 0$ and $x_n \to \overline{x}$ such that

$$\text{Epi}\, F(x_n) \not\subset \text{Epi}\, F(\overline{x}) + T(x_n - \overline{x}) + \theta \|x_n - \overline{x}\| B(0,1). \tag{5.9}$$

Observe that $x_n \neq \overline{x}$ for all n, and since X is finite dimensional we may suppose that the sequence

$$u_n := \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}$$

converges to some $u \in X$. Define $(t_n) = (\|x_n - \overline{x}\|)$. Then, by assumption, for large n,

$$\text{Epi}\, F(\overline{x} + t_n u_n) \subset \text{Epi}\, F(\overline{x}) + t_n T(u) + t_n B\left(0, 2^{-1}\theta\right).$$

But, since T is continuous and $u_n \to u$, one has $T(u) - T(u_n) \in B\left(0, 2^{-1}\theta\right)$ for n large enough. Therefore, for such n,

$$\text{Epi}\, F(\overline{x} + t_n u_n) \subset \text{Epi}\, F(\overline{x}) + t_n T(u_n) + t_n \theta B(0,1),$$

and this contradicts (5.9). □

The expected and useful generalized Fermat rule holds, as shown below.

Proposition 5.6 *If $\overline{x}$ is a ℓ-weak minimum for $F : X \rightrightarrows \mathbb{R}$ and $F(\overline{x})$ is lower bounded with respect to $[0, +\infty)$, then $0 \in \widehat{\partial} F(\overline{x}) \subset \partial_- F(\overline{x})$.*

Proof Clearly, one has that $\text{WMin}\,(F(\overline{x}), [0, +\infty)) \neq \emptyset$ so the ℓ-weak minimality of $\overline{x}$ means that

$$F(\overline{x}) \not\subset F(x) + (0, +\infty), \ \forall x \in X.$$

So, for all $x \in X$ there is $y_x \in F(\overline{x})$ such that for all $y \in F(x)$ one has the inequality $y_x \leq y$. Then for all $\varepsilon > 0$, and $x \in X$,

$$F(x) \subset y_x + \|x - \overline{x}\| (-\varepsilon, \varepsilon) + [0, +\infty),$$

whence

$$F(x) + [0, +\infty) \subset F(\overline{x}) + \|x - \overline{x}\| (-\varepsilon, \varepsilon) + [0, +\infty),$$

so $0 \in \widehat{\partial} F(\overline{x})$. □

Example 5.7 Consider $F : X \rightrightarrows \mathbb{R}$ given by

$$F(x) := [f(x), g(x)], \quad \forall x \in \mathbb{R},$$

where $f, g : \mathbb{R} \to \mathbb{R}$ ar arbitrary functions such that $f(x) \leq g(x)$ for any $x \in \mathbb{R}$. Suppose $\overline{x}$ is a ℓ-weak minimum for F and f is lower bounded. This means that

$$[f(\overline{x}), g(\overline{x})] \not\subset [f(x), g(x)] + (0, +\infty), \quad \forall x \in X,$$

or

$$f(\overline{x}) \leq f(x), \quad \forall x \in X,$$

i.e., $\overline{x}$ is a global scalar minimum for f.

Now, since Epi $F(x) =$ Epi $f(x)$ for any x, the relation (5.4) is obviously satisfied for $T = 0$, which means that $0 \in \widehat{\partial} F(\overline{x}) \subset \partial_- F(\overline{x})$.

Trying to get sufficient optimality conditions one arrives at the next results.

Proposition 5.8 *Suppose that X is finite dimensional,* WMin $(F(\overline{x}), K) \neq \emptyset$. *If*

$$0 \in \partial_- \left(y^* \circ F\right)(\overline{x}) = \widehat{\partial}\left(y^* \circ F\right)(\overline{x})$$

for some $y^* \in K^+ \setminus \{0\}$ for which the set $y^* \circ F(\overline{x})$ has a minimum, then for any $\theta > 0$, and $e \in \operatorname{int} K$, $\overline{x}$ is a ℓ-weak minimum for $x \rightrightarrows F(x) + \theta \|x - \overline{x}\| e$.

Proof Suppose, by way of contradiction, that there are $\theta > 0$, $e \in \operatorname{int} K$, and $(x_n) \to \overline{x}$ such that for all n

$$F(\overline{x}) \subset F(x_n) + \theta \|x_n - \overline{x}\| e + \operatorname{int} K.$$

Clearly, $x_n \neq \overline{x}$ for all n. Then, noting $(t_n) = (\|x_n - \overline{x}\|)$, and $(u_n) = \left(t_n^{-1}(x_n - \overline{x})\right)$, one can suppose, without loss of generality, that $u_n \to u \in X \setminus \{0\}$. Therefore, we get as well that for all $\varepsilon \in (0, \theta y^*(e))$ and n large enough,

$$\begin{aligned}\left(y^* \circ F\right)(\overline{x}) &\subset \left(y^* \circ F\right)(x_n) + \theta \|x_n - \overline{x}\| y^*(e) + (0, +\infty) \\ &= \left(y^* \circ F\right)(\overline{x} + t_n u_n) + \theta t_n y^*(e) + (0, +\infty) \\ &\subset \left(y^* \circ F\right)(\overline{x}) + t_n \cdot 0 + t_n(-\varepsilon, \varepsilon) + \theta t_n y^*(e) + (0, +\infty) \\ &= \left(y^* \circ F\right)(\overline{x}) + t_n \left((-\varepsilon, \varepsilon) + \theta y^*(e)\right) + (0, +\infty) \\ &\subset \left(y^* \circ F\right)(\overline{x}) + (0, +\infty).\end{aligned}$$

This is a contradiction, with the fact that the set $y^* \circ F(\overline{x})$ has a minimum. □

In the constrained case, one easily gets, on similar lines, the next result.

Proposition 5.9 *Suppose that X is finite dimensional, $\overline{x} \in M$ and* WMin $(F(\overline{x}), K) \neq \emptyset$. *If*

$$0 \in D^- \text{Epi}\left(y^* \circ F\right)(\overline{x})(u)$$

for some $y^* \in K^+ \setminus \{0\}$ for which the set $y^* \circ F(\overline{x})$ has a minimum, and for all $u \in T_B(M, \overline{x}) \setminus \{0\}$, then for any $\theta > 0$, and $e \in \operatorname{int} K$, $\overline{x}$ is a ℓ-weak minimum point for $x \rightrightarrows F(x) + \theta \|x - \overline{x}\| e$.

We further observe that, in fact, other known subdifferentials for functions can be extended to general set-valued maps on the same line of reasoning. We illustrate this assertion for the Clarke subdifferential. Recall that, for $f : X \to \mathbb{R} \cup \{+\infty\}$ a Lipschitz function, $\overline{x} \in \operatorname{dom} f$ and $u \in X$, the Clarke directional derivative of f at $\overline{x}$ in the direction u is

$$d_{+}^{C} f(\overline{x}, u) = \limsup_{t \downarrow 0, x \to \overline{x}} \frac{f(x+tu) - f(x)}{t}.$$

Define

$$D_{+}^{C} F(\overline{x})(u) = \{v \in Y \mid \forall \varepsilon > 0, \exists (t_n) \downarrow 0, \exists (x_n) \to \overline{x}, \forall n \in \mathbb{N}, \\ F(x_n + t_n u) \subset F(x_n) + t_n v + t_n B(0, \varepsilon)\}.$$

Then, as above, for $F = f$, a real-valued function, for all u

$$D_{+}^{C} \operatorname{Epi} f(\overline{x})(u) = (-\infty, d_{+}^{C} f(\overline{x}, u)].$$

Therefore, if $F : X \rightrightarrows Y$, the construction

$$\partial^{C} F(\overline{x}) = \left\{T \in \mathcal{L}(X, Y) \mid T(u) \in D_{+}^{C} \operatorname{Epi} F(\overline{x})(u), \ \forall u \in X\right\}$$

generalizes the Clarke subdifferential of a locally Lipschitz function. It is easy to observe that $\partial_{-} F(\overline{x}) \subset \partial^{C} F(\overline{x})$, again, a relation that generalizes the classical case. Therefore, following Proposition 5.6, if $\overline{x}$ is a ℓ-weak minimum for $F : X \rightrightarrows \mathbb{R}$ and $F(\overline{x})$ is lower bounded with respect to $[0, +\infty)$, then $0 \in \partial^{C} F(\overline{x})$.

Actually, similarly one can introduce

$$D_{*}^{C} F(\overline{x})(u) = \{v \in Y \mid \forall \varepsilon > 0, \forall (x_n) \to \overline{x}, \exists (t_n) \to 0, \forall n, \\ F(x_n + t_n u) \subset F(x_n) + t_n v + t_n B(0, \varepsilon)\}.$$

Therefore, if $F : X \rightrightarrows Y$, the construction

$$\partial_{*}^{C} F(\overline{x}) = \left\{T \in \mathcal{L}(X, Y) \mid T(u) \in D_{*}^{C} \operatorname{Epi} F(\overline{x})(u), \ \forall u \in X\right\}$$

represents another generalized subdifferential for which a Fermat rule holds, as follows.

Proposition 5.10 *Suppose that $\overline{x}$ is a ℓ-weak minimum for $F : X \rightrightarrows \mathbb{R}$ and $F(\overline{x})$ is lower bounded with respect to $[0, +\infty)$. If F is locally Lipschitz around $\overline{x}$, then $0 \in \partial_{*}^{C} F(\overline{x})$.*

Proof Again, as before, one has that $\operatorname{WMin}(F(\overline{x}), [0, +\infty)) \neq \emptyset$ so the ℓ-weak minimality of $\overline{x}$ means that

$$F(\overline{x}) \not\subset F(x) + (0, +\infty), \ \forall x \in X.$$

So, for all $x \in X$ there is $y_x \in F(\overline{x})$ such that for all $y \in F(x)$ one has the inequality $y_x \leq y$. Take $u \in X$, $\varepsilon > 0$, $(x_n) \to \overline{x}$ and define $(t_n) = (\|x_n - \overline{x}\|)$. For all $n \in \mathbb{N}$ one has

$$F(x_n + t_n u) \subset y_{x_n + t_n u} + t_n \cdot 0 + t_n(-\varepsilon, \varepsilon) + [0, +\infty),$$

whence

$$\begin{aligned} F(x_n + t_n u) + [0, +\infty) &\subset F(\overline{x}) + t_n \cdot 0 + t_n(-\varepsilon, \varepsilon) + [0, +\infty) \\ &\subset F(x_n) + L t_n[-1, 1] + t_n \cdot 0 + t_n(-\varepsilon, \varepsilon) + [0, +\infty) \\ &= F(x_n) + t_n \cdot 0 + t_n(-\varepsilon - L t_n, \varepsilon + L t_n) + [0, +\infty), \end{aligned}$$

and this confirms that $0 \in \partial_{*}^{C} F(\overline{x})$. □

6 Concluding remarks

The generalized differentiation objects for set-valued maps we introduce in this paper seem to have a promising potential in dealing with set optimization problems in a more direct manner than other approaches in literature. We illustrate this point of view by showing several basic properties and consequences that are naturally expected from such kind of constructions, and from this perspective the present study is a promising one. Now, the subsequent question is to what extent these methods and ideas could be successful in order to cover other topics related to set optimization (see the Introduction) and/or could be explored in other directions. This is an open theme that we intend to pursue in future research.

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References

1. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
2. Ansari, Q.H., Bao, T.Q.: A limiting subdifferential version of Ekeland's variational principle in set optimization. Optim. Lett. **15**, 1537–1551 (2021)
3. Bao, T.Q., Mordukhovich, B.S.: Variational principles for set-valued mappings with applications to multiobjective optimization. Control. Cybern. **36**, 531–562 (2007)
4. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers for multiobjective problems: existence and optimality conditions. Math. Program. **122**, 301–347 (2010)
5. Ben-Tal, A., Zowe, J.: Directional Derivatives in Nonsmooth Optimization. J. Optim. Theory Appl. **47**, 483–490 (1985)
6. Burlică, M., Durea, M., Strugariu, R.: New concepts of directional derivatives for set-valued maps and applications to set optimization. Optimization (2022). https://doi.org/10.1080/02331934.2022.2088368
7. Durea, M., Strugariu, R.: On some Fermat rules for set-valued optimization problems. Optimization **60**, 575–591 (2011)
8. Gutiérrez, C., López, R., Martínez, J.: Generalized $\varepsilon-$quasi solutions of set optimization problems. J. Global Optim. **82**, 559–576 (2022)
9. Hamel, A.H., Heyde, F., Löhne, A., Rudloff, B., Schrage, C.: Set Optimization and Applications - The State of the Art. Springer, Berlin (2015)
10. Han, Y., Huang, N.J.: Well-posedness and stability of solutions for set optimization problems. Optim. **66**, 17–33 (2017)
11. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**, 1–18 (2007)
12. Ioffe, A.D.: Calculus of Dini subdifferentials of functions and contingent coderivatives of set-valued maps. Nonlinear Anal. Theory Methods Appl. **8**, 517–539 (1984)
13. Jiménez, B., Novo, V., Vílchez, A.: Characterization of set relations through extensions of the oriented distance. Math. Methods Oper. Res. **91**, 89–115 (2020)
14. Khan, A.A., Tammer, C., Zălinescu, C.: Set-valued Optimization. Springer, An Introduction with Applications (2015)
15. Mordukhovich, B.S.: Variational Analysis and Applications. Springer, Cham (2018)
16. Penot, J.-P.: Differentiability of relations and differential stability of perturbed optimization problems. SIAM J. Control. Optim. **22**, 529–551 (1984)
17. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Springer, Berlin (2009)
18. Som, K., Vetrivel, V.: On robustness for set-valued optimization problems. J. Global Optim. **79**, 905–925 (2021)