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On the maxima of motzkin-straus programs and cliques of graphs

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Abstract
In this paper, we establish a connection between the local maximizers (global maximizers) of a Motzkin-Straus quadratic program and a specific type of regular multipartite cliques. Our work extends a remarkable connection between the maximum cliques and the global maximizers of the Motzkin-Straus quadratic program. This connection and its extensions can be successfully employed in optimization to provide heuristics for the maximal cliques in graphs. We provide two counterexamples to the results from previous work about the global and local maximizers of the Motzkin-Straus quadratic program. We then amend the previous theorems by introducing a new structure. We also answer two questions raised by Pelillo and Jagota about the maxima of the Motzkin-Straus quadratic program.

Keywords Cliques of graphs · Lagrangian of graphs · Quadratic program · Motzkin-Straus

Mathematics Subject Classification 05C35 · 05C65 · 05D99 · 90C27 · 90C30

1 Introduction

A simple undirected graph G consists of a vertex set $V(G) = \{1, 2, 3, \dots, n\}$ and an edge set $E(G) \subseteq \binom{V(G)}{2}$. In this paper, we use $[n]$ to represent the set $\{1, 2, 3, \dots, n\}$. An edge $\{a_1, a_2\}$ in G will be simply denoted by $a_1 a_2$. Throughout the paper, we assume that $G =$

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(V, E) contains no isolated vertices. For any non-empty set $S \subseteq [n]$, we let $\mathbf{x}^S$ denote the characteristic vector of S (defined as an n-dimensional column vector satisfying $x_i^S = 1/|S|$ whenever $i \in S$ and $x_i^S = 0$ otherwise, where x_i^S stands for the ith component of $\mathbf{x}^S$ and $|S|$ stands for the cardinality of S). A subset of vertices $C \subseteq V(G)$ is called a clique of a graph G if every pair of vertices in C is adjacent in G. A clique is said to be maximum if it has maximum cardinality. The order of a maximum clique in a graph G is called the clique number of G, denoted by $\omega(G)$. A clique C is said to be maximal if it is not contained in any strictly larger clique in G, that is, if there does not exist a clique D such that $C \subset D$. A maximal clique C is said to be strictly maximal if there do not exist vertices $i \in C$ and $j \notin C$ such that $C \cup \{j\} \backslash \{i\}$ is a clique, or equivalently, if, for all $j \in V \setminus C$, j is adjacent to at most $|C| - 2$ vertices from C. For a subset $C \subseteq V$, we use $G[C]$ to represent the subgraph induced by the vertex set C. In this paper, we consider the classical Maximum Clique Problem (MCP). The MCP asks to find a clique $C \subseteq V(G)$ such that $|C|$ is maximum.

Given a graph G, the Motzkin-Straus formulation of the MCP is a quadratic program (QP) formed from the adjacency matrix of the graph G over the standard simplex. For a graph $G = ([n], E(G))$ and a vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in R^n$, define the polynomial form $L(G, \mathbf{x}) : R^n \to R$ as

$$L(G, \mathbf{x}) := \frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x},$$

where $\mathbf{A} = (a_{ij})_{i,j \in [n]}$ is the adjacency matrix for G defined by

$$a_{ij} = \begin{cases} 1, & ij \in E(G), \\ 0, & ij \notin E(G), \end{cases} \quad \forall i, j \in [n].$$

Let $\Delta_n := \{\mathbf{x} = (x_1, x_2, \ldots, x_n)^T : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i \in [n]\}$ be the n-dimensional standard simplex. The Motzkin-Straus formulation of the MCP is a quadratic program (QP) over the standard simplex as follows.

$$\begin{aligned} &\text{maximize } L(G, \mathbf{x}), \\ &\text{subject to } \mathbf{x} \in \Delta_n. \end{aligned} \tag{1}$$

A vector $\mathbf{x}^* \in \Delta_n$ is said to be a global maximizer of (1), if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Delta_n$ and $f(\mathbf{x}^*)$ is said to be a global maximum value of (1). A vector $\mathbf{x}^* \in \Delta_n$ is said to be a local maximizer of (1), if there exists an $\epsilon > 0$ such that $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Delta_n$ satisfying $||\mathbf{x} - \mathbf{x}^*|| < \epsilon$ and $f(\mathbf{x}^*)$ is said to be a local maximum value of (1). A vector $\mathbf{x}^* \in \Delta_n$ is said to be a strict local maximizer of (1), if there exists an $\epsilon > 0$ such that $f(\mathbf{x}^*) > f(\mathbf{x})$ for all $\mathbf{x} \in \Delta_n$ satisfying $||\mathbf{x} - \mathbf{x}^*|| < \epsilon$ and $f(\mathbf{x}^*)$ is said to be a strict local maximum value of (1). In particular, the global maximum value of (1), denoted by $\lambda(G)$, is often called the Lagrangian[1] of G in extremal graph theory.

It is well-known that the Lagrangian has applications in both combinatorics and optimization. Motzkin and Straus' result says that the Lagrangian of a graph corresponds to its clique number (the precise statement is given in Theorem 1). This result provides a solution to the optimization problem for a class of homogeneous quadratic bilinear functions over the standard simplex of a Euclidean space. The Motzkin-Straus result and its extensions were successfully employed in studying the MCP [3–5, 7]. It has also been generalized to

[1] While the Lagrangian is typically used as a function, in extremal graph theory, the Lagrangian for a graph is the maximum value of a polynomial function over the standard simplex. This definition was first used by Motzkin and Straus in the classical paper: *Maxima for graphs and a new proof of a theorem of Turán*.

vertex-weighted graphs [7] and edge-weighted graphs with applications in computer vision and pattern recognition, especially in graph matching and related problems [3–5, 7, 13–17, 19, 23].

In this paper, we explore the relationship between the local (global) maximizers and the maximal (maximum) cliques of graphs. The contributions of this work are mainly in three aspects.

First, we study the connection between the global maximizers of the Motzkin-Straus QP and the MCP. In [12], Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph. In [18], Pelillo and Jagota further studied the connection between the local maximizers of the Motzkin-Straus QP and the maximal cliques in the graph. Pardalos and Phillips [14] and Pelillo and Jagota [18] both showed that not all the global maximizers of the Motzkin-Straus QP correspond to maximum cliques. In [18], Pelillo and Jagota found that if a global maximizer is a characteristic vector of a subset S of vertices, then S is a maximum clique ([18], Proposition 2). However, in our paper we find a counterexample that proves that this claim does not always hold. Instead, we prove that if a characteristic vector of a subset C is a global maximizer of the Motzkin-Straus QP, then C is a Turán graph with the partition that all parts have the same cardinality.

Secondly, we consider the local maximizers of the Motzkin-Straus QP and the maximal cliques in the graph. In [18], Pelillo and Jagota studied the correspondence between strictly maximal cliques and strict local maximizers of the Motzkin-Straus QP. They showed that C is a strictly maximal clique of G if and only if $\mathbf{x}^C$ is a strict local maximizer of the Motzkin-Straus QP. They concluded their paper by asking whether the “strictly" condition can be removed. We find a counterexample in Sect. 2(see Example 2) and this counterexample shows that the “strictly” condition cannot be removed. We also show that a result from [7], which states that “if C is a maximal clique in G, then its characteristic vector is a local maximizer of the Motzkin-Straus QP" ([7], Corollary 2), is not correct. Because Corollary 2 in [7] is not correct, we amend it using several new concepts including the strongly maximal clique. We prove that the characteristic vector of a subset M is a local maximizer of the Motzkin-Straus QP if and only if M is a regular multipartite clique of G satisfying specific conditions (see Theorem 5 in Sect. 2).

We also answer another question raised by [18]. In Corollary 8 of their paper, the authors provided a powerful condition under which every convex combination of the characteristic vectors of the maximum cliques is a global maximizer of the Motzkin-Straus QP. Then, they asked if the result also holds for maximal cliques and local maximizers. We show that these broader results hold for strongly maximal cliques, that is, in Theorem 6, we find that every convex combination of the characteristic vectors of the strongly maximal cliques is a local maximizer of the Motzkin-Straus QP.

The rest of the paper is organized as follows: In Sect. 2, we introduce notations and definitions, give two counterexamples that show that Proposition 2 in [18] and Corollary 2 in [7] do not always hold, and state our main results, Theorems 3, 5 and 6. We prove the theorems in Sect. 3. In Sect. 4, we provide final remarks about our results.

2 Preliminary and Main Results

In [12], Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph.

Theorem 1 (Motzkin and Straus [12], Theorem 1) *If G is a graph with clique number ω then* $\lambda(G) = \frac{1}{2}\left(1 - \frac{1}{\omega}\right)$.

Using Theorem 1, Motzkin and Straus gave a new proof of the following celebrated Turán's theorem [25].

Theorem 2 ([25]) *A graph with n vertices which contains no complete subgraph of order k has number of edges no more than*

$$e(n,k) = m^2\binom{k-1}{2} + m(k-2)r + \binom{r}{2}$$

where $n = (k-1)m + r$, $0 \le r < k-1$. *This maximum is attained only for a graph in which the vertices are divided into* $k-1$ *classes of which r contain* $m+1$ *vertices and the remainder contains m vertices with two vertices connected if and only if they belong to different classes. Note that the extremal graph is called Turán graph and is denoted by* $T(n, k-1)$. [2]

Pardalos and Phillips [14] and Pelillo and Jagota [18] both showed that not all global maximizers of (1) correspond to maximum cliques. In [18], Pelillo and Jagota explored the connection between the local (global) maximizers of (1) and the maximal (maximum) cliques. They proved the following proposition.

Proposition 1 ([18], Proposition 2) *Let* $G = (V, E)$ *be a graph and let C be a subset of V. Then C is a maximum clique of G if and only if* $\mathbf{x}^C$ *is a global maximizer of (1).*

Proposition 1 implies that if a global maximizer is a characteristic vector of a subset C of V, then C is indeed a maximum clique of G. However, Proposition 1 does not always hold. In fact, the following counterexample shows that $\mathbf{x}^C$ is a global maximizer of (1) doesn't mean C is a maximum clique of G.

Example 1 Let $G = (V, E)$ be the graph with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{13, 14, 23, 24\}$. Let $C = V$. Clearly, the clique number of G is 2. So by Theorem 1, $\lambda(G) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$. Thus the global maximum value of $L(G, \mathbf{x})$ on Δ_4 is $\frac{1}{4}$. On the other hand, $L(G, \mathbf{x}^C) = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$. So $\mathbf{x}^C$ is a global maximizer of (1). However, C is not a clique of G.

In this paper, we explore the correspondence between the local (global) maximizers of (1) and the maximal (maximum) cliques of G. First, we consider the global maximum value of (1). We need the following definition.

Definition 1 A vertex set M is called a regular multipartite clique of graph G if $G[M]$ (the subgraph induced by M) is a complete multipartite subgraph of G with each part having the same order.

Definition 1 implies that, if a vertex set M is a regular multipartite clique of graph G, then $G[M]$ is a Turán subgraph of G with each part having the same order. Note that if each part in the partition of a regular multipartite clique consists of a single vertex, Definition 1 is the

[2] In the more standard terminology (and that adopted here), the (n, k)-Turán graph, denoted $T(n, k)$, is the extremal graph on n graph vertices that contains no $(k+1)$-clique for $1 \le k \le n$ (Diestel 1997 [6], p. 149; Bollobás 1998, p. 108 [2], and Gross and Yellen 2006, p. 476 [8]). Unfortunately, some authors, including Skiena (1990, pp. 143-144 [24]), Aigner (1995) [1], and Pemmaraju and Skiena (2003, pp. 247-248 [20]), use the convention that the (n, k)-Turán graph contains no k-clique (instead of $(k+1)$-clique), meaning that the $T(n, k)$-Turán graph of these authors is the $(n, k-1)$-Turán graph as defined above.

same as the clique definition. Also note that a regular multipartite clique is defined by two parameters: the number of parts in the partition and the number of vertices in each part of the partition. Given a regular multipartite clique M with l parts in the partition, $G[M]$ has $\frac{l(l-1)}{2}\frac{|M|^2}{l^2}$ edges and

$$L(G, \mathbf{x}^M) = \frac{l(l-1)}{2}\frac{|M|^2}{l^2}\frac{1}{|M|^2} = \frac{1}{2}\left(1 - \frac{1}{l}\right). \tag{2}$$

Note $L(G, \mathbf{x}^M)$ is only related to the parameter l. So, when we discuss whether $\mathbf{x}^M$ is the local (global) maximizer of (1), we only need the information of the number of the parts in the partition of a regular multipartite clique. This leads to Definition 2.

Definition 2 Let $G = (V, E)$ be a graph with clique number ω. A regular multipartite clique M of G is called a maximum regular multipartite clique of G if $G[M]$ is a complete ω-partite subgraph of G.

We have the following theorem.

Theorem 3 *Let $G = (V, E)$ be a graph and let M be a subset of V. Then M is a maximum regular multipartite clique of G if and only if $\mathbf{x}^M$ is a global maximizer of (1).*

Next, we consider the local maximizers of (1). In [18], Pelillo and Jagota provided a correspondence between the strictly maximal cliques and strict local maximizers of (1).

Theorem 4 ([18], Theorem 5) *Let $G = (V, E)$ be a graph and let C be a subset of V. Then C is a strictly maximal clique of G if and only if $\mathbf{x}^C$ is a strict local maximizer of (1).*

In [7], Gibbons et al. proved the following result which characterizes local optimality of the Motzkin-Straus QP.

Lemma 1 ([7], Theorem 2) *Let $\mathbf{x} \in \Delta_n$. Then $\mathbf{x}$ is a local maximizer of (1) if and only if there exists an integer l such that, with*

$$\tau := 1 - \frac{1}{l},\ \ S := \{u|x_u > 0\},\ \ T := \{u|x_u = 0, (Ax)_u = \tau\},$$

we have

(1) $(A\mathbf{x})_u \le \tau$ for $u \in [n]$, where $(\mathbf{Ax})_u$ denotes the uth component of $\mathbf{Ax}$.
(2) $(A\mathbf{x})_u = \tau,\ \ \forall u \in S$.
(3) There exist partitions of S and T,

$$S = S_1 \bigcup \cdots \bigcup S_l,\ \ T = T_1 \bigcup \cdots \bigcup T_l$$

such that

(a) $S_i \neq \varnothing,\ \ \forall i \in [l]$.
(b) $S_i \bigcup T_i$ is independent for every i.
(c) If $i \neq j$, $u \in S_i \bigcup T_i$ and $v \in S_j$ then vertex u is adjacent to vertex v.

Remark 1 Lemma 1 implies that, if $\mathbf{x}$ is a local (global) maximizer of (1), then the graph induced by the support set of $\mathbf{x}$, (i.e., $\sigma(\mathbf{x}) = \{i|x_i > 0\}$) is a complete l-partite subgraph of G for some integer l. In this paper, we assume that $G = (V, E)$ contains no isolated vertices. So $l \geq 2$.

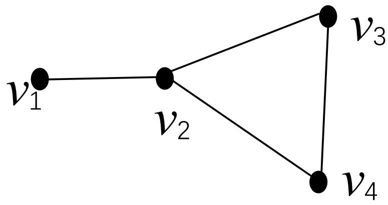

Fig. 1 $\{v_1, v_2\}$ is a maximal clique but not a strongly maximal clique of the above graph. $M = \{v_1, v_2\}$ is a regular multipartite clique of the graph not satisfying Property (P)

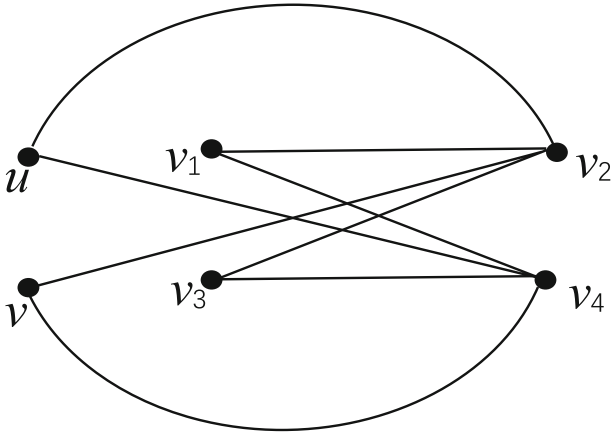

Fig. 2 The graph above has a regular multipartite clique $M' = \{v_1, v_2, v_3, v_4\}$ satisfying Property (P) with partitions $\{v_1, v_3\}$ and $\{v_2, v_4\}$

Then they obtained the following proposition as a corollary of Lemma 1.

Proposition 2 ([7], Corollary 2) *Let C be a maximal clique in G. Then $\mathbf{x}^C$ is a local maximizer of (1).*

We now provide a counterexample to Proposition 2.

Example 2 Let $G = (V, E)$ be the graph in Fig.1 Let $C = \{v_1, v_2\}$. Clearly, C is a maximal clique of G. And

$$L(G, \mathbf{x}^C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Let ε be any positive number small enough and $\mathbf{y} = (\frac{1}{2} - 2\varepsilon, \frac{1}{2}, \varepsilon, \varepsilon)$. Clearly, $\mathbf{y}$ is also a feasible solution of (1). And

$$\begin{aligned} L(G, \mathbf{y}) - L(G, \mathbf{x}^C) &= \left[\left(\frac{1}{2} - 2\varepsilon\right) \times \frac{1}{2} + 2 \times \frac{1}{2} \times \varepsilon + \varepsilon^2\right] - \frac{1}{4} \\ &= \varepsilon^2 > 0. \end{aligned}$$

So $\mathbf{x}^C$ is not a local maximizer of (1).

In fact, though Lemma 1 is correct, it may not be used to deduce Proposition 2. The reason is that the characteristic vector of a maximal clique doesn't always satisfy the conditions in Lemma 1. In Example 2, $C = \{v_1, v_2\}$ is a maximal clique of G, but $\mathbf{x}^C$ doesn't satisfy the conditions in Lemma 1. Here is the reason: since $\mathbf{x}^C = \{\frac{1}{2}, \frac{1}{2}, 0, 0\}$, then clearly we have S in Lemma 1 is $\{v_1, v_2\}$. Because $(\mathbf{Ax})_1 = \cdots = (\mathbf{Ax})_4 = \frac{1}{2}$, then we have $T = \{v_3, v_4\}$ by Lemma 1. Note that $S = \{v_1, v_2\}$ is a clique. To satisfy Lemma 1(3)(a) and (b), S must be divided into $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$. If we divide T into $T_1 = \{v_3, v_4\}$ and $T_2 = \phi$, then $S_1 \bigcup T_1$ is not an independent set since v_3 and v_4 are adjacent to each other. If T_2 is not an empty set, then $S_2 \bigcup T_2$ is not an independent set because both v_3 and v_4 are adjacent to v_2. Thus the condition of Lemma 1(3)(b) does not hold.

Remark 2 One of the open questions Pelillo and Jagota asked in the paper (Section 4 [18]) is whether "strictly" can be removed in Theorem 4. The counterexample (Example 2) shows, that C is a maximal clique of G does not imply $\mathbf{x}^C$ is a local maximizer of (1). Thus "strictly" cannot be removed in Theorem 4.

Remark 3 We note that another counterexample to Proposition 2 was reported in a recent paper [9]. Hence, there is no any necessarily relationship between the local maxima of (1) and the maximal cliques in G.

In Example 2, the reason that the condition of Lemma 1 does not hold is that, there exist $|C| - 1$ vertices of the maximal clique C contained in a clique with order larger than $|C|$. To amend Proposition 2, we may exclude such a case and introduce the definition of strongly maximal cliques as follows.

Definition 3 A maximal clique C is called a strongly maximal clique if any $|C| - 1$ vertices of C are not contained in a clique with order larger than $|C|$.

In Fig. 1, $C = \{v_1, v_2\}$ is maximal clique of the graph with order 2. However it is not a strongly maximal clique of the graph since v_2 is contained in the clique $\{v_2, v_3, v_4\}$ with order 3. Clearly, the following chains of inclusions hold:

$$\{strictly\ maximal\ cliques\} \subseteq \{strongly\ maximal\ cliques\} \\ \subseteq \{maximal\ cliques\}$$

and

$$\{maximum\ cliques\} \subseteq \{strongly\ maximal\ cliques\}$$

We also need the following definition.

Definition 4 A regular multipartite clique M with l ($l \geq 2$) parts in the partition satisfies Property (P) if M satisfies the following conditions:

(a) Each vertex in $V(G) \setminus M$ is adjacent to at most $\frac{l-1}{l}|M|$ vertices of M.
(b) Each vertex in $V(G) \setminus M$ is not adjacent to $\frac{l-1}{l}|M|$ vertices from l different parts in the partition of M.
(c) Any pair adjacent vertices in $V(G) \setminus M$ are not adjacent to $\frac{l-1}{l}|M|$ vertices in the same $l-1$ parts in the partition of M.

Note that if each part in the partition of a regular multipartite clique consists of a single vertex, Definition 4 is the same as the strongly maximal clique definition. We illustrate Definition 4 by the examples in Fig. 1 and Fig. 2, respectively. In Fig. 1, let $M = \{v_1, v_2\}$, we verify that M is not a multipartite clique satisfying Property (P). To illustrate that M does not satisfy Definition 4, we divide M into two parts: $Q_1 = \{v_1\}$, $Q_2 = \{v_2\}$. Then M is a regular multipartite clique with two parts. Note that $l = 2$ and $|M| = 2$, $\frac{l-1}{l}|M| = 1$, also, both v_3 and v_4 are adjacent to v_2 in $G[M]$ and v_3 and v_4 are adjacent in Fig. 1, thus M does not meet the condition (c) in Definition 4. In Fig. 2, let M' be the vertex set $\{v_1, v_2, v_3, v_4\}$. We divide M' into two parts: $P_1 = \{v_1, v_3\}$ and $P_2 = \{v_2, v_4\}$. Then we see that M' is a regular multipartite clique with $l = 2$ and $\frac{l-1}{l}|M'| = 2$. Notice that u is adjacent to two vertices in P_2 and and v is adjacent to two vertices in P_2, respectively, and both u and v are not adjacent in the graph, thus M' is a multipartite clique satisfying Property (P).

The next result states that a connection exists between the local maximizers of (1) and the regular multipartite cliques of G satisfying Property (P).

Theorem 5 *Let M be a subset of V in graph G. Then, $\mathbf{x}^M$ is a local maximizer of (1) if and only if M is a regular multipartite clique of G satisfying Property (P).*

Theorem 5 implies Corollary 1 which amends Proposition 2.

Corollary 1 *Let C be a strongly maximal clique of graph G. Then $\mathbf{x}^C$ is a local maximizer of (1).*

Theorem 6 answers another question posed by Pelillo and Jagota (Section 4 in [18]).

Theorem 6 *Let $C_1, \ldots, C_k$ be strongly maximal cliques of G, $\alpha_1 > 0, \ldots, \alpha_k > 0$, where $\alpha_1 + \cdots + \alpha_k = 1$, and $\mathbf{y} = \alpha_1 \mathbf{x}^{C_1} + \cdots + \alpha_k \mathbf{x}^{C_k}$.*

i) Assume that $|C_1| = \cdots = |C_k| = l$ and $|C_i \backslash C_j| = |C_j \backslash C_i| = m_{ij}$, where $1 \le i < j \le k$ and $l \ge 2$. Then $\mathbf{y}$ is a local maximizer of (1) if and only if the number of edges crossing $C_i \setminus C_j$ and $C_j \setminus C_i$ is exactly $m_{ij}(m_{ji} - 1)$, where $1 \le i < j \le k$.

ii) Assume that there exist C_i and C_j satisfying $|C_i| \ne |C_j|$, where $1 \le i, j \le k$. Then $\mathbf{y}$ is not a local maximizer of (1).

3 Proofs of Theorems 3, 5 and 6

In this section, we prove our main results.

We know that if a vector $\mathbf{x}$ is a local (global) maximizer of (1), then it satisfies the first order Karush-Kuhn-Tucker (KKT) [11] necessary conditions. Stated below are the first order KKT necessary conditions for (1).

Let $\mathbf{x}$ be a local (global) maximizer of (1). Note that the partial derivative $\frac{\partial L(G,\mathbf{x})}{\partial x_i} = (\mathbf{A}x)_i$ for any $i \in [n]$. So there should exist some real constant τ such that for all $j \in [n]$,

$$(\mathbf{A}\mathbf{x})_j \begin{cases} = \tau, & x_j > 0, \\ \le \tau, & x_j = 0. \end{cases} \tag{3}$$

We need the following lemma.

Lemma 2 *Let M be a subset of $V(G)$ in graph G. If $\mathbf{x}^M$ is a local (global) maximizer of (1) then M is a regular multipartite clique of G.*

Proof Assume that $\mathbf{x}^M$ is a local maximizer of (1). By the definition of $\mathbf{x}^M$, we have that $M = \sigma(\mathbf{x}^M)$. So by Remark 1, there exists some integer l such that $G[M]$ is a complete l-partite graph. Assume the partition of M is

$$M = M_1 \bigcup \cdots \bigcup M_l.$$

For any $i, j \in [l]$, take $u \in M_i$ and $v \in M_j$ then $(\mathbf{A}\mathbf{x}^M)_u = (\mathbf{A}\mathbf{x}^M)_v$ by (3), where $\mathbf{A}$ is the adjacency matrix of G. Since $G[M]$ is a complete l-partite graph, then $(\mathbf{A}\mathbf{x}^M)_u = 1-\sum_{w\in M_i} x_w^M = 1-\sum_{w\in M_i} \frac{1}{|M|} = 1-\frac{|M_i|}{|M|}$. Similarly, $(\mathbf{A}\mathbf{x}^M)_v = 1-\frac{|M_j|}{|M|}$. So $|M_i| = |M_j|$ and M is a regular multipartite clique of G. □

3.1 Proof of Theorem 3

Proof of Theorem 3 Assume that the clique number of G is ω. (⇒)If M is a maximum regular multipartite clique of G, then the number of edges in $G[M]$ is $\frac{\omega-1}{2\omega}|M|^2$ and $L(G, \mathbf{x}^M) = \frac{\omega-1}{2\omega}|M|^2\frac{1}{|M|^2} = \frac{1}{2}\left(1-\frac{1}{\omega}\right)$. On the other side, by Theorem 1, $\lambda(G) = \frac{1}{2}\left(1-\frac{1}{\omega}\right)$. This implies $\mathbf{x}^M$ is a global maximizer of (1).

(⇐) Conversely, assume that $\mathbf{x}^M$ is a global maximizer of (1). By Lemma 2, M is a regular multipartite clique with l parts for some integer l. So $L(G, \mathbf{x}^M) = \frac{l-1}{2l}|M|^2\frac{1}{|M|^2} = \frac{1}{2}\left(1-\frac{1}{l}\right)$. Recall that $\lambda(G) = \frac{1}{2}\left(1-\frac{1}{\omega}\right)$ and $\mathbf{x}^M$ is a global maximizer of (1), thus we have $l = \omega$. So M is a maximum regular multipartite clique of G. □

3.2 Proof of Theorem 5

Proof of Theorem 5 (⇐) Let M be a regular multipartite clique with the partition $M_1, \ldots, M_l$ satisfying Property (P). In the following, we verify that $\mathbf{x}^M$ satisfies the three conditions of Lemma 1. Thus $\mathbf{x}^M$ is a local maximizer of (1).

First, let $S = M$. We have $S = \sigma(\mathbf{x}^M)$. In addition, if $u \in M$ then $(\mathbf{A}\mathbf{x}^M)_u = \frac{l-1}{l}|M|\frac{1}{|M|} = 1-\frac{1}{l} := \tau$ since u is adjacent to the vertices in exactly $l-1$ parts of M.

Second, let T be the set of vertices in $V(G) \setminus M$ that are adjacent to exactly $\frac{l-1}{l}|M|$ vertices of M. Then for any $u \in T$, $x_u^M = 0$ and $(\mathbf{A}\mathbf{x}^M)_u = 1-\frac{1}{l}$.

For any vertex $u \notin T\bigcup S$, $(\mathbf{A}\mathbf{x}^M)_u < 1-\frac{1}{l}$ since u is adjacent to fewer than $\frac{l-1}{l}|M|$ vertices of M. Thus, $\mathbf{x}^M$ satisfies the conditions in Lemma 1(1) and Lemma 1(2).

Third, let $S_i = M_i$ for $i \in [l]$. Then $S = S_1\bigcup\cdots\bigcup S_l$ and $S_i\bigcap S_j = \phi$ for $i \neq j$, thus $\{S_1\cdots S_l\}$ is a partition of S. Let T_i be the set of vertices in T that are not adjacent to the vertices in S_i for $i \in [l]$. Clearly, $T \supseteq T_1\bigcup\cdots\bigcup T_l$. On the other side, if $v \in T$ then v is adjacent to $\frac{l-1}{l}|M|$ vertices of M. So v is adjacent to all the vertices in some $l-1$ parts of $S_1, \ldots, S_l$ but not adjacent to any vertices in S_i for some i by Definition 4(b). Therefore $v \in T_i$ and $T \subseteq T_1\bigcup\cdots\bigcup T_l$. So $T = T_1\bigcup\cdots\bigcup T_l$. Since any vertex in T_i is adjacent to all the vertices in $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_l$ but not adjacent to the vertices in S_i, and any vertex in S_j is adjacent to all the vertices in $S_1, \ldots, S_{j-1}, S_{j+1}, \ldots, S_l$ but not adjacent to the vertices in S_j by Definition 4(b), then $T_i\bigcap T_j = \phi$ for $i \neq j$. So $\{T_1, \ldots, T_l\}$ is a partition of T. In addition, we have

(a) Clearly, S_i is not empty for $i \in [l]$. Thus, the condition (3)(a) in Lemma 1 holds.

(b) If both $u, v \in S_i$, then u, v are not adjacent to each other, since u, v are in the same part of a l-partite graph. If $u \in S_i$ and $v \in T_i$ then u, v are not adjacent to each other since T_i is the set of vertices in T that are not adjacent to the vertices in S_i. Assume that u, v are in T_i. Note that u, v are adjacent to $\frac{l-1}{l}|M|$ vertices of M, respectively. By Definition 4(b), u, v are adjacent to all the vertices in some $l-1$ parts in the partition of M, respectively. Since both u and v are not adjacent to the vertices in S_i, then both u and v are adjacent to the vertices in $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_l$. So u, v are not adjacent to each other by Definition 4(c). Therefore, $S_i \bigcup T_i$ is an independent set for every i. Thus, the condition (3)(b) in Lemma 1 holds.

(c) Let $i \neq j$, $u \in S_i \bigcup T_i$ and $v \in S_j$. If $u \in S_i$ and $v \in S_j$, then u, v are adjacent since u, v are in different parts of a complete l-partite graph $G[M]$. If $u \in T_i$ and $v \in S_j$, then u, v are adjacent by the definition of S_i and T_j for $i \neq j$. Therefore, if $i \neq j$, $u \in S_i \bigcup T_i$, $v \in S_j$, then vertex u is adjacent to vertex v. Thus, the condition (3)(c) in Lemma 1 holds.

Hence by Lemma 1, $\mathbf{x}^M$ is a local maximizer of (1).

($\Rightarrow$) Conversely, assume that $\mathbf{x}^M$ is a local maximizer of (1). We know that M is a regular multipartite clique of G by Lemma 2. We only need to show that M is regular multipartite clique of G satisfying Property (P). Assume that $G[M]$ is a complete l-partite graph and M consists of l parts $M_1, \ldots, M_l$ in the partition,where $l \geq 2$. We will verify that M satisfies Definition 4, that is we only need to show that M satisfies:

(a) Each vertex in $V(G) \setminus M$ is adjacent to at most $\frac{l-1}{l}|M|$ vertices of M.

(b) Each vertex in $V(G) \setminus M$ is not adjacent to $\frac{l-1}{l}|M|$ vertices from l different parts in the partition of M.

(c) Any two adjacent vertices in $V(G) \setminus M$ are not adjacent to $\frac{l-1}{l}|M|$ vertices in the same $l-1$ parts in the partition of M.

According to the cases (a), (b), and (c), we divide the proof into three claims as follows.

Claim 1 *Each vertex in $V(G) \setminus M$ is adjacent to at most $\frac{l-1}{l}|M|$ vertices of M.*

Proof By contradiction, there exists a vertex $u \in V \setminus M$ adjacent to at least $\frac{l-1}{l}|M| + 1$ vertices in M. Then

$$(\mathbf{A}\mathbf{x}^M)_u \geq \left(\frac{l-1}{l}|M| + 1\right)\frac{1}{|M|} = 1 - \frac{1}{l} + \frac{1}{|M|}, \tag{4}$$

and, for every vertex $v \in M$,

$$(\mathbf{A}\mathbf{x}^M)_v = \frac{l-1}{l}|M|\frac{1}{|M|} = 1 - \frac{1}{l}. \tag{5}$$

Both (4) and (5) altogether violate the first order KKT necessary condition (3). This contradicts to the fact that $\mathbf{x}^M$ is a local maximizer of (1). □

Claim 2 *Any vertex in $V(G) \setminus M$ is not adjacent to $\frac{l-1}{l}|M|$ vertices from l different parts in the partition of M.*

Proof If $|M_1| = \cdots = |M_l| = 1$, this claim clearly holds. So we assume that $|M_1| = \cdots = |M_l| \geq 2$. Assume that there exists a vertex u adjacent to $\frac{l-1}{l}|M|$ vertices from l different parts in the partition of M. By Claim 1, u is adjacent to at most $\frac{l-1}{l}|M|$ vertices of M. Thus there must be at least two vertices, v and w, and these two vertices belong to two different parts M_i and M_j such that u is not adjacent to either v or w. Let ε be any positive number that is small enough and $\mathbf{e}_i \in \mathbf{R}^n$ be the unit vector with the ith entry equals 1. Let $\mathbf{y} = \mathbf{x}^M + 2\varepsilon\mathbf{e}_u - \varepsilon\mathbf{e}_v - \varepsilon\mathbf{e}_w$. Clearly, $\mathbf{y}$ is a feasible solution of (1). By means of the Taylor expansion of L, we have that

$$\begin{aligned} L(G, \mathbf{y}) - L(G, \mathbf{x}^M) = {} & 2\varepsilon\frac{\partial L(G, \mathbf{x}^M)}{\partial x_u^M} - \varepsilon\frac{\partial L(G, \mathbf{x}^M)}{\partial x_v^M} \\ & - \varepsilon\frac{\partial L(G, \mathbf{x}^M)}{\partial x_w^M} - \varepsilon^2\frac{\partial^2 L(G, \mathbf{x}^M)}{\partial x_u^M \partial x_v^M} \end{aligned}$$

$$- \varepsilon^2 \frac{\partial^2 L(G, \mathbf{x}^M)}{\partial x_u^M \partial x_w^M} + \varepsilon^2 \frac{\partial^2 L(G, \mathbf{x}^M)}{\partial x_v^M \partial x_w^M}. \tag{6}$$

Note that $\mathbf{x}^M$ is a local maximizer of (1) and $x_u^M = x_v^M = x_w^M = \frac{1}{M} > 0$. So $\frac{\partial L(G,\mathbf{x}^M)}{\partial x_u^M} = \frac{\partial L(G,\mathbf{x}^M)}{\partial x_v^M} = \frac{\partial L(G,\mathbf{x}^M)}{\partial x_w^M}$ by the KKT condition. And $\frac{\partial^2 L(G,\mathbf{x}^M)}{\partial x_u^M \partial x_v^M} = \frac{\partial^2 L(G,\mathbf{x}^M)}{\partial x_u^M \partial x_w^M} = 0$ since u is not adjacent to v and w, and $\frac{\partial^2 L(G,\mathbf{x}^M)}{\partial x_v^M \partial x_w^M} = 1$ since v is adjacent to w. So $L(G, \mathbf{y}) - L(G, \mathbf{x}^M) = \varepsilon^2$. This contradicts to the fact that $\mathbf{x}^M$ is a local maximizer of (1). □

Claim 3 *Any of two adjacent vertices in $V(G) \setminus M$ are not adjacent to $\frac{l-1}{l}|M|$ vertices in the same $l-1$ parts in the partition of M.*

Proof Assume that there are two adjacent vertices u, v in $V(G) \setminus M$ adjacent to $\frac{l-1}{l}|M|$ vertices in the same $l-1$ parts in the partition of M. By Claim 1, u and v are adjacent to at most $\frac{l-1}{l}|M|$ vertices of M, respectively. Thus there must exist a vertex $w \in M$ adjacent to neither u nor v. Let ε be any positive number that is small enough. Let $\mathbf{y} = \mathbf{x}^M + \varepsilon \mathbf{e}_u + \varepsilon \mathbf{e}_v - 2\varepsilon \mathbf{e}_w$. Clearly, $\mathbf{y}$ is a feasible solution of (1). In a similar way to (6), we have

$$\begin{aligned} L(G, \mathbf{y}) - L(G, \mathbf{x}^M) &= \varepsilon \frac{\partial L(G, \mathbf{x}^M)}{\partial x_u^M} + \varepsilon \frac{\partial L(G, \mathbf{x}^M)}{\partial x_v^M} - 2\varepsilon \frac{\partial L(G, \mathbf{x}^M)}{\partial x_w^M} \\ &\quad + \varepsilon^2 \frac{\partial^2 L(G, \mathbf{x}^M)}{\partial x_u^M \partial x_v^M} - \varepsilon^2 \frac{\partial^2 L(G, \mathbf{x}^M)}{\partial x_u^M \partial x_w^M} - \varepsilon^2 \frac{\partial^2 L(G, \mathbf{x}^M)}{\partial x_v^M \partial x_w^M} \\ &= \varepsilon^2 > 0. \end{aligned} \tag{7}$$

This contradicts to the fact that $\mathbf{x}^M$ is a local maximizer of (1). □

Claims 1-3 complete the proof of Theorem 5. □

3.3 Proof of Theorem 6

We first prove Lemmas 3 and 4, which are then used to prove Theorem 6.

Lemma 3 *i) Let C and D be two maximal cliques of G and $l = max(|C|, |D|)$, then $G[C \bigcup D]$ is a l-partite graph.*

ii) Let C and D be two maximal cliques of G. Assume that $|C| = |D| = l$ and $|C \backslash D| = |D \backslash C| = m$. Then $G[C \bigcup D]$ is a complete l-partite graph if and only if the number of edges crossing $C \setminus D$ and $D \setminus C$ is exactly $m(m-1)$.

iii) If $|C| \neq |D|$ then $G[C \bigcup D]$ is not a complete multipartite graph even if both C and D are strongly maximal cliques of G.

Proof i) Let $H_1 = C \setminus D$ and $H_2 = D \setminus C$. Then, the complement graph of $G[H_1 \bigcup H_2]$, denoted by $\overline{G[H_1 \bigcup H_2]}$, is a bipartite graph with vertex partition H_1 and H_2 since both H_1 and H_2 are cliques of G. Without loss of generality, assume that $m = |H_1| \leq |H_2|$. We claim that H_1 is a minimum vertex cover of $\overline{G[H_1 \bigcup H_2]}$. Clearly H_1 is a vertex cover of $\overline{G[H_1 \bigcup H_2]}$ since $\overline{G[H_1 \bigcup H_2]}$ is a bipartite graph with vertex partition H_1 and H_2. If H_1 is not a minimum cover of $\overline{G[H_1 \bigcup H_2]}$, then there is a vertex v in H_1 not adjacent to any vertex of H_2 in $\overline{G[H_1 \bigcup H_2]}$. So v is adjacent to every vertex of D in G. This contradicts to the fact that D is a maximal clique of G. Thus H_1 is a minimum vertex cover of $\overline{G[H_1 \bigcup H_2]}$ and there is a matching with m edges in $\overline{G[H_1 \bigcup H_2]}$ that saturates H_1 by König's theorem [21].

Assume that $H_1 = C \setminus D = \{u_1, \ldots, u_m\}$. We list these m matching edges in $\overline{G[H_1 \bigcup H_2]}$ as $u_1v_1, \ldots, u_mv_m$, where $v_1, v_2, \ldots, v_m \in H_2$. Then $G[C \bigcup D]$ is a l-partite graph with vertex partition $V_1 = \{u_1, v_1\}, \ldots, V_m = \{u_m, v_m\}, V_{m+1} = \{v_{m+1}\}, \ldots, V_l = \{v_l\}$, where $v_{m+1} \in D, \ldots, v_l \in D$ and some of them perhaps also belong to C.

ii) Assume that $|C| = |D| = l$. Let $C \setminus D = \{u_1, \ldots, u_m\}$ and $D \setminus C = \{v_1, \ldots, v_m\}$. In i), we have proved that $G[C \bigcup D]$ is a l-partite graph. Without loss of generality, assume that the partition of $G[C \bigcup D]$ is $V_1 = \{u_1, v_1\}, \ldots, V_m = \{u_m, v_m\}, V_{m+1} = \{v_{m+1}\}, \ldots, V_l = \{v_l\}$, where $v_{m+1} \in D \bigcap C, \ldots, v_l \in D \bigcap C$. So u_i is not adjacent to v_i for $i \in [m]$ and the number of the edges crossing $C \setminus D$ and $D \setminus C$ is at most $m \times m - m = m(m-1)$.

If the number of edges crossing $C \setminus D$ and $D \setminus C$ is exactly $m(m-1)$, then u_i must be adjacent to every vertex of D in G except v_i for $i \in [m]$. So $G[C \bigcup D]$ is a complete l-partite graph. Conversely, if $G[C \bigcup D]$ is a complete l-partite graph, then u_i must be adjacent to every vertex of D in G except v_i for $i \in [m]$. Thus the number of edges crossing $C \setminus D$ and $D \setminus C$ is exactly $m(m-1)$.

iii) Assume that C and D are strongly maximal cliques of G such that $|C| \neq |D|$. Without loss of generality, assume that $|C| < |D| = l$. If $G[C \bigcup D]$ is a complete multipartite graph, then $G[C \bigcup D]$ must be a complete l-partite graph since each vertex in D must be in different parts in the partition. So C is contained in a larger clique with order l since the vertices in C are in different parts of the complete l-partite graph. This contradicts to the fact that C is a strongly maximal clique of G.

This completes the proof of Lemma 3. □

Lemma 4 *i) Let $C_1, \ldots, C_k$ be strongly maximal cliques of G satisfying $|C_1| = \cdots = |C_k| = l$ and $|C_i \setminus C_j| = |C_j \setminus C_i| = m_{ij}$, where $1 \leq i < j \leq k$. Then $G[C_1 \bigcup \cdots \bigcup C_k]$ is a complete l-partite graph if and only if the number of edges crossing $C_i \setminus C_j$ and $C_j \setminus C_i$ is exactly $m_{ij}(m_{ji} - 1)$, where $1 \leq i < j \leq k$.*

ii) Let $C_1, \ldots, C_k$ be strongly maximal cliques of G. If there exist C_i and C_j such that $|C_i| \neq |C_j|$, where $1 \leq i, j \leq k$, then $G[C_1 \bigcup \cdots \bigcup C_k]$ is not a complete multipartite graph.

Proof i) ($\Rightarrow$) If $G[C_1 \bigcup \cdots \bigcup C_k]$ is a complete l-partite graph, then $G[C_i \bigcup C_j]$ is a complete l-partite graph for $1 \leq i < j \leq k$. This implies that the number of edges crossing $C_i \setminus C_j$ and $C_j \setminus C_i$ is exactly $m_{ij}(m_{ji} - 1)$ for $1 \leq i < j \leq k$ by Lemma 3 ii).

($\Leftarrow$)Assume that the number of edges crossing $C_i \setminus C_j$ and $C_j \setminus C_i$ is exactly $m_{ij}(m_{ji} - 1)$ for $1 \leq i < j \leq k$. We prove that $G[C_1 \bigcup \cdots \bigcup C_k]$ is a complete l-partite graph by mathematical induction on k. By Lemma 3 ii), this result holds for $k = 2$.

Our inductive assumption is: if the number of edges crossing $C_i \setminus C_j$ and $C_j \setminus C_i$ is exactly $m_{ij}(m_{ji} - 1)$ for $1 \leq i < j \leq k$, then $G[C_1 \bigcup \cdots \bigcup C_k]$ is a complete l-partite graph.

Now we show that $G[C_1 \bigcup \cdots \bigcup C_k \bigcup C_{k+1}]$ is a l-partite graph. Using induction, assume that the l-partition of $G[C_1 \bigcup \cdots \bigcup C_k]$ consists of vertex sets $V_1, \ldots, V_l$. By Lemma 3 ii), $G[C_1 \bigcup C_{k+1}]$ is a complete l-partite graph. Thus, we assume that the l-partition of $G[C_1 \bigcup C_{k+1}]$ consists of $\{v_1, w_1\}, \ldots, \{v_p, w_p\}, \ldots, \{v_l, w_l\}$, where $v_p \in C_1 \bigcap V_p$, $w_p \in C_{k+1}$, and it is possible that vertex w_p is the same vertex v_p, for $p \in [l]$ in the vertex partition. For $p \in [l]$, if w_p is adjacent to a vertex u in V_p then $\{u, v_1, \ldots, v_{p-1}, w_p, v_{p+1} \ldots v_l\}$ is a clique with order $l + 1$. This contradicts to the fact that C_1 is a strongly maximal clique of G. Hence w_p is not adjacent to any vertex in V_p for $p \in [l]$. Therefore, $G[C_1 \bigcup \cdots \bigcup C_{k+1}]$ is a l-partite graph with partition $V_1 \bigcup \{w_1\}, \ldots, V_l \bigcup \{w_l\}$. Next we show that $G[C_1 \bigcup \cdots \bigcup C_{k+1}]$ is a complete l-partite graph. Otherwise, there must be two nonadjacent vertices u and v, both of them belong to different parts in the partition of $G[C_1 \bigcup \cdots \bigcup C_{k+1}]$. Assume that $u \in C_s$ and $v \in C_t$. Then $G[C_s \bigcup C_t]$ is a complete

l-partite graph by Lemma 3 ii) and thus u must be adjacent to v. This is a contradiction. Therefore $G[C_1 \bigcup \cdots \bigcup C_k \bigcup C_{k+1}]$ is also a complete l-partite graph and Lemma 4 i) holds by the induction principle.

ii) The result holds clearly by Lemma 3 iii). □

Now we prove Theorem 6 by verifying the conditions in Lemma 1.

Proof of Theorem 6 i) ($\Leftarrow$) Assume that $C_1, \ldots, C_k$ are strongly maximal cliques of G satisfying $|C_1| = \cdots = |C_k| = l$ and $|C_i \backslash C_j| = |C_j \backslash C_i| = m_{ij}$ for $1 \leq i < j \leq k$. Then by Lemma 4, $G[C_1 \bigcup \cdots \bigcup C_k]$ is a complete l-partite graph. Assume that the vertex partition of $G[C_1 \bigcup \cdots \bigcup C_k]$ consists of vertex sets $V_1, \ldots, V_l$. Let $\mathbf{y} = \alpha_1 \mathbf{x}^{C_1} + \cdots + \alpha_k \mathbf{x}^{C_k}$. If u belongs to r cliques of $C_1, \ldots, C_k$, that is, $u \in C_{i_1} \bigcap \cdots \bigcap C_{i_r}$, then $y_u = \alpha_{i_1} x_u^{C_{i_1}} + \cdots + \alpha_{i_r} x_u^{C_{i_r}} = \frac{1}{l}(\alpha_{i_1} + \cdots + \alpha_{i_r})$. And note that each clique of $C_1, \ldots, C_k$ has exactly one vertex in V_i for $i \in [l]$. So

$$\sum_{u \in V_i} y_u = \alpha_1 \cdot \frac{1}{l} + \cdots + \alpha_k \cdot \frac{1}{l} = \frac{1}{l} \tag{8}$$

for $i \in [l]$.

Next we verify that $\mathbf{y}$ satisfies the conditions in Lemma 1.

Let $\tau := 1 - 1/l$, $S = C_1 \bigcup \cdots \bigcup C_k$, and let T be the set of vertices that belong to $V \setminus (C_1 \bigcup \cdots \bigcup C_k)$ and are adjacent to all the vertices in some $l-1$ parts of $V_1, \ldots, V_l$. Then

(A) For each vertex $u \in S$. Assume that $u \in V_i$ for some i. Then u is not adjacent to the vertices in V_i but adjacent to all the vertices in $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_l$. So $(\mathbf{Ay})_u = 1 - \sum_{u \in V_i} y_u = 1 - \frac{1}{l}$ by (8).

(B) For each vertex $u \in T$. Since $u \in T$ then u is not adjacent to the vertices in V_i for some i but adjacent to all the vertices in $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_l$. So $(\mathbf{Ay})_u = 1 - \sum_{u \in V_i} y_u = 1 - \frac{1}{l}$ by (8).

(C) For each vertex $u \notin S \bigcup T$. Clearly, u is adjacent to at most $l-1$ vertices in C_i since C_i is a strongly maximal clique of G for $i \in [k]$. We claim that, for some j, u is adjacent to at most $l-2$ vertices in C_j.

Assume that u is adjacent to $l-1$ vertices in each C_i for $i \in [k]$. Let the vertices in C_1 that are adjacent to u be $v_1, \ldots, v_{l-1}$ and $v_1 \in V_1$, ..., $v_{l-1} \in V_{l-1}$. If u is not adjacent to any vertex in V_l, then u must be adjacent to at most $l-2$ vertices in C_j for some j. Otherwise, u must be adjacent to all the vertices in V_1, ..., V_{l-1} and so $u \in T$. This contradicts to the fact that $u \notin S \bigcup T$. If u is adjacent to some vertex v in V_l, then $\{u, v, v_1, \ldots, v_{l-1}\}$ would be a clique since $G[V_1 \bigcup \cdots \bigcup V_l]$ is a complete l-partite graph. This contradicts to the fact that C_1 is a strongly maximal clique of G. So u must be adjacent to at most $l-2$ vertices in C_j for some j.

Thus u is adjacent to at most $l-1$ vertices in C_i for $i \in [k]$ and u is adjacent to at most $l-2$ vertices in C_j for some j. This leads to $(\mathbf{Ay})_u < (l-1)(\alpha_1 \cdot \frac{1}{l} + \cdots + \alpha_k \cdot \frac{1}{l}) = 1 - \frac{1}{l}$.

Therefore, S satisfies the condition $S = \sigma(\mathbf{y})$, and T satisfies the condition $T = \{u | y_u = 0, (\mathbf{Ay})_u = \tau\}$ in Lemma 1. Since $(\mathbf{Ay})_u = \tau$ for $u \in S \bigcup T$ and $(\mathbf{Ay})_u < \tau$ for $u \notin S \bigcup T$, thus, the conditions (1) and (2) of Lemma 1 hold.

We now verify the condition (3) in Lemma 1. Let $S_i = V_i$. Then $S = S_1 \bigcup \cdots \bigcup S_l$ and $S_i \bigcap S_j = \phi$ for $i \neq j$, thus $\{S_1, \ldots, S_l\}$ forms a partition of S. Also let T_i be the set of vertices in T that are not adjacent to the vertices in S_i for $i \in [l]$. Clearly, $T \supseteq T_1 \bigcup \cdots \bigcup T_l$. For any $u \in T$, u is adjacent to all the vertices in some $l-1$ parts of $V_1, \ldots, V_l$ but is not adjacent to all the vertices in V_i for some i. So $u \in T_i$ and $T \subseteq T_1 \bigcup \cdots \bigcup T_l$. Therefore $T = T_1 \bigcup \cdots \bigcup T_l$. Since T_i is the set of vertices in T that are not adjacent to the vertices in S_i, then T_i consists of the vertices adjacent to all the vertices in $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_l$ but not adjacent to the vertices in S_i. Similarly, T_j consists of the vertices adjacent all the vertices in $S_1, \ldots, S_{j-1}, S_{j+1}, \ldots, S_l$ but not adjacent to the vertices in V_j. So $T_i \bigcap T_j = \phi$ for $i \neq j$ and $\{T_1, \ldots, T_l\}$ is a partition of T. In addition, we have

(a) Clearly, $S_i \neq \varnothing$ for $i \in [l]$. So the condition (3)(a) in Lemma 1 holds.

(b) For $i \in [l]$, let $u, v \in S_i \bigcup T_i$. If $u, v \in S_i$ or $u \in S_i$ and $v \in T_i$, respectively, then u is not adjacent to v by the definition of S_i and T_i. Assume that $u, v \in T_i$ for some i. Note that u, v are adjacent to all the vertices in some $l-1$ parts of $V_1, \ldots, V_l$ in the l-partition. Since both u and v are not adjacent to the vertices in S_i, then both u and v are adjacent to the vertices in $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_l$. If u is adjacent to v, then each vertex set $\{u, v, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_l\}$, where $v_1 \in C_j, \ldots, v_{i-1} \in C_j, v_{i+1} \in C_j, \ldots, v_l \in C_j$ for $j \in [k]$, would be a clique of G with order greater than l. This contradicts to the fact that $C_1, \ldots, C_k$ are strongly maximal cliques of G. So, u and v are not adjacent to each other. Thus, the condition (3)(b) in Lemma 1 holds.

(c) Assume that $i \neq j$, $u \in S_i \bigcup T_i$, $v \in S_j$. Then u is adjacent to v by the definition of S_i and T_i. So the condition (3)(c) in Lemma 1 holds.

Thus by Lemma 1, $\mathbf{y}$ is a local maximizer of G.

($\Rightarrow$) To prove the other part of i) in Theorem 6, we assume that the number of edges crossing $C_i \backslash C_j$ and $C_j \backslash C_i$ is fewer than $m_{ij}(m_{ji}-1)$. Then by Lemma 3, $G[C_1 \bigcup \cdots \bigcup C_k]$ is a l-partite graph but not a complete l-partite graph. However, by Remark 1, the support set of a local maximizer of (1) must induce a complete multipartite graph. Thus, $\mathbf{y}$ is not a local maximizer of (1).

Proof of Theorem 6 ii) Assume that there exists $i \in [k]$ and $j \in [k]$ satisfying $|C_i| \neq |C_j|$ then $G[C \bigcup D]$ is not a complete multipartite graph. Thus $\mathbf{y}$ is not a local maximizer of (1) by Remark 1.

This completes the proof of Theorem 6. □

4 Concluding Remarks

In this paper, we establish a connection between the local maximizers (global maximizers) of a Motzkin-Straus quadratic program and a specific type of regular multipartite cliques. We find a necessary and sufficient condition for the characteristic vector of a vertex set to be a local maximizer (global maximizer) of a Motzkin-Straus quadratic program. We also derive a necessary and sufficient condition under which the convex combination of the characteristic vectors of strongly maximal cliques is a local maximizer (global maximizer) of a Motzkin-Straus quadratic program.

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