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Scalar and vector equilibrium problems with pairs of bifunctions

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Abstract
In this paper, existence results for scalar and vector equilibrium problems involving two bifunctions are established. To this aim, a new concept of generalized pseudomonotonicity for a pair of bifunctions is introduced. It leads to existence criteria different from the ones encountered in the literature. The given applications refer to minimax inequalities and variational inequality problems.

Keywords Equilibrium problem · Generalized pseudomonotonicity · Mimimax inequality · Variational inequality

1 Introduction and preliminaries

Various optimization problems can be brought to the following format:

$$\text{find } x_0 \in X \text{ such that } f(x_0, y) \geq 0, \ \forall y \in X, \tag{EP}$$

where X is a nonempty convex subset of a Hausdorff topological vector space and f is a real bifunction defined on $X \times X$.

This problem was called by Muu and Oettli [30] an equilibrium problem and, after the appearance of the seminal article of Blum and Oettli [13], the term "equilibrium problem"has been adopted by all the researchers working on this topic. The study of the problem (EP) actually started before the papers mentioned above with the works of Nikaido and Isoda [32] and of Ky Fan [21]. Over time a considerable number of existence criteria of the solution has been obtained, both in finite and infinite dimensional settings, under various assumptions on the set X and on the bifunction f. The bulk of the literature on problem (EP) assumes that the bifunction f is monotone or it satisfies various types of generalized monotonicity. We need to recall two of them, used in the paper:

Definition 1 A bifunction $f : X \times X \to \mathbb{R}$ is said to be

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(i) pseudomonotone, if

$$x, y \in X,\ f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0;$$

(ii) properly quasimonotone, if for any nonempty finite subset A of X and all $y \in \operatorname{conv} A$, $\min_{x \in A} f(x, y) \leq 0$.

In some works, in order to establish existence results of the solutions, besides the bifunction f, is considered a second bifunction $g : X \times X \to \mathbb{R}$, the pair (f, g) satisfying a certain type of generalized monotonicity(see [23, 33, 35]). The present paper fits in this group of articles. Its goal is to study problem (EP) when the bifunctions f and g satisfy the following condition:

$$\begin{aligned} &\text{for each } y \in X,\ \text{there exists } z \in X \text{ such that} \\ &x \in X,\ \ g(z, x) \leq 0 \Longrightarrow f(x, y) \geq 0. \end{aligned} \tag{1}$$

It is easy to see that (1) holds whenever the condition below is fulfilled:

$$\text{for every } y \in X,\ \text{there exists } z \in X \text{ such that} - g(z, x) \leq f(x, y)\ \text{ for all } x \in X.$$

But the above condition is only sufficient and not necessary. For instance, if all the values of f are nonnegative, then (1) is satisfied for any real bifunction g.

If we wish to relate condition (1) by others encountered in literature, we need to recall the concept of pseudomonotone (ordered) pair of bifunctions. Given two bifunctions $p, q : X \times X \to \mathbb{R}$, it is said that p is pseudomonotone with respect to q (see [9, 33]) if

$$x, y \in X,\ \ p(y, x) \geq 0 \Longrightarrow q(x, y) \leq 0,$$

This fact motivates us to introduce a more general concept:

Definition 2 We say that p is generalized pseudomonotone with respect to q if there exists a function $h : X \to \mathbb{R}$ such that for $x, y \in X$

$$p(h(y), x) \geq 0 \Longrightarrow q(x, y) \leq 0.$$

Note that in the above definition the order of the two bifunctions is relevant and that the pseudomonotonicity implies the generalized pseudomonotonicity. Condition (1) can be now restated as follows: $-g$ is generalized pseudomonotone with respect to $-f$.

In Sect. 2, we establish existence results of the solution for problem (EP) when X is a convex subset of Hausdorff topological vector space and $-g$ is generalized pseudomonotone with respect to $-f$. As usual, the absence of compactness of X will impose a coercivity condition.

In Sect. 3, as applications of the aforementioned results we obtain generalizations of Ky Fan minimax inequality and Sion minimax theorem.

To the best of our knowledge, Oettli ([33]) was the first who observed that a vector equilibrium problem can be reduced to a scalar equilibrium problem and his method has been then used in many papers (see, for instance, [6, 9, 20, 23]). Oettli's approach will be used in Sect. 4 to derive existence results for two type of vector equilibrium problems. In the same section we study the existence of solution for a vector variational inequality problem of Stampacchia type.

From now on, for a subset A of a topological vector space E, the standard notations $\operatorname{conv} A$, $\operatorname{cl} A$, $\operatorname{int} A$ designate respectively, the convex hull, the closure and the interior of A. If $A \subseteq C \subseteq E$, we denote by $\operatorname{cl}_C A$ the closure of A with respect to C.

2 Scalar equilibrium problems

Definition 3 A subset Y of a topological space X is said to be compactly closed if for each compact subset K of X, the set $Y \cap K$ is closed in K.

It is worth mentioning that the class of compactly closed sets is larger than the one of closed sets. A Hausdorff space X is called c-space (see, [19, p. 152], [27, p. 230]) if every compactly closed subset of X is closed. Necessary and sufficient conditions in order that a Hausdorff space to be c-space can be found in [34] and [37].

Theorem 1 *Let X be a convex subset of a Hausdorff topological vector space and f, g be two real bifunctions defined on $X \times X$ such that*

(i) for each $y \in X$, the set $\{x \in X : f(x, y) \geq 0\}$ is compactly closed;
(ii) g is properly quasimonotone;
(iii) -g is generalized pseudomonotone with respect to -f;
(iv) there exist a nonempty compact subset K of X and a point $y_0 \in X$ such that $f(x, y_0) < 0$ for all $x \in X \setminus K$.

Then, there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in X$.

Proof Denote by $\mathcal{A}$ the family of all finite subsets of X that contain the point y_0 and choose arbitrarily an $A \in \mathcal{A}$. From (iii), for each $y \in A$, there exists $z_y \in X$ such that

$$\{x \in X : g(z_y, x) \leq 0\} \subseteq \{x \in X : f(x, y) \geq 0\}. \tag{2}$$

Set

$$C := \text{ conv } \{z_y : y \in A\}.$$

Clearly, C is compact. Consider the set-valued mapping $G : \{z_y : y \in A\} \rightrightarrows C$ defined by

$$G(z) = \{x \in C : g(z, x) \leq 0\}.$$

Since g is properly quasimonotone, it follows easily that G is a KKM mapping. By Ky Fan's lemma [21, Lemma 1],

$$\emptyset \neq \bigcap_{y \in A} \text{cl}_C G(z_y).$$

From (2) and (i), we infer that

$$\bigcap_{y \in A} \text{cl}_C G(z_y) = \bigcap_{y \in A} \text{cl}_C\big[\{x \in X : g(z_y, x) \leq 0\} \cap C\big] \subseteq \bigcap_{y \in A} \text{cl}_C\big[\{x \in X : f(x, y) \geq 0\} \cap C\big]$$
$$= \bigcap_{y \in A}\{x \in X : f(x, y) \geq 0\} \cap C = \bigcap_{y \in A}\{x \in C : f(x, y) \geq 0\}.$$

Consequently,

$$\bigcap_{y \in A}\{x \in C : f(x, y) \geq 0\} \neq \emptyset.$$

From (iv), $\{x \in C : f(x, y_0) \geq 0\} \subseteq K$, hence any point in the intersection above belongs to K. Therefore, the set

$$\mathbb{S}(A) = \{x \in K : f(x, y) \geq 0 \text{ for all } y \in A\}$$

is nonempty and closed in K.

Let now A_1, A_2 be two subsets from $\mathcal{A}$. As $\mathbb{S}(A_1 \cup A_2) \subseteq \mathbb{S}(A_1) \cap \mathbb{S}(A_2)$, it follows that the family $\{\mathbb{S}(A) : A \in \mathcal{A}\}$ has the finite intersection property. Since K is compact, we infer that $\bigcap_{A\in\mathcal{A}} \mathbb{S}(A) \neq \emptyset$. It is straightforward to see that any point $x_0 \in \bigcap_{A\in\mathcal{A}} \mathbb{S}(A)\}$ satisfies the conclusion of the theorem.

Remark 1 Clearly, when X is compact, the coercivity condition (iv) is automatically satisfied taking $K = X$.

It is easy to see that Theorem 1 remains true if condition (iv) is replaced with the following one:

there exist a nonempty compact subset K of X and a finite set $F \subset X$, such that for every $x \in X \setminus K$, $\min_{y\in F} f(x, y) < 0$.

It could be of interest to compare Theorem 1 with [35, Theorem 2.2], that offers sufficient conditions for the solvability of the dual equilibrium problem. Replacing $f(x, y)$ with $-f(y, x)$, the mention theorem from [35] becomes a particular case of Theorem 1. The coercivity condition is the same in the both results but the other assumptions are different. Thus, in [35] is imposed the condition that $-g(y, x) \le f(x, y)$ for all $(x, y) \in X \times X$, and as we have seen in Sect. 1, this condition is stronger than assumption (iii) of Theorem 1. Then, in the mentioned paper is required as for every $y \in X$, the function $f(\cdot, y)$ to be upper semicontinuous on each nonempty compact subset of X, and again this assumption is stronger than condition (i) of Theorem 1.

We give below a simple example in which Theorem 1 works but other existence results encountered in the literature are not applicable.

Example 1 Consider the bifunctions $f, g : [-1, 1] \times [-1, 1] \to \mathbb{R}$ defined by

$$f(x, y) = y^2 - x, \quad g(x, y) = y - x.$$

Obviously g is properly quasimonotone and, since

$$g(y^2, x) = x - y^2 = -f(x, y),$$

condition (iii) of Theorem 1 is fulfilled, taking $h(x) = x^2$. Hence, Theorem 1 is applicable. One can see that $x_0 = 0$ is the unique point that satisfies the conclusion of the theorem. On the other hand, the bifunction f does not satisfy a standard condition, encountered in many papers (see, for instance, [1, 12, 14, 20, 24, 25]), namely, $f(x, x) \ge 0$ for all $x \in [-1, 1]$. Observe also that f is not properly quasimonotone because, for $A = \{-\frac{1}{2}, 0\}$ and $y = -\frac{1}{3}$, we have $\min\{f(-\frac{1}{2}, -\frac{1}{3}), f(0, -\frac{1}{3})\} > 0$. Consequently, [10, Proposition 2.2] and [11, Proposition 2.1] are not useful for this example, because in the mentioned results the proper quasimonotonicity of f is required.

Taking $g = f$ and $h(x) = x$ for all $x \in X$, Theorem 1 reduces to the following corollary:

Corollary 2 *Let X be a convex subset of a Hausdorff topological vector space, f be a real bifunction defined on $X \times X$, K be a compact subset of X and y_0 a point in X. Assume that*

(i) $-f$ is pseudomonotone and f is properly quasimonotone;
(ii) for each $y \in X$, the set $\{x \in X : f(x, y) \ge 0\}$ is compactly closed;
(iii) $f(x, y_0) < 0$, for all $x \in X \setminus K$.

Then, there exists $x_0 \in K$ such that $f(x_0, y) \ge 0$ for all $y \in X$.

In Corollary 2, the pseudomonotonicity of $-f$ can be replaced by other conditions. To do this we need to recall two concepts encountered in the literature.

(a) a function $h : X \to \mathbb{R}$ is said to be semistrictly quasiconvex, if it is quasiconvex and, for any $x, y \in X$ with $h(x) \neq h(y)$,

$$h(\lambda x + (1-\lambda)y) < \max\{h(x), h(y)\}, \ \forall \lambda \in]0, 1[.$$

(b) a bifunction $f : X \times X \to \mathbb{R}$ is said to have the upper sign property if, for any $x, y \in X$ the following implication holds:

$$f((1-\lambda)x + \lambda y, x) \leq 0, \ \forall \lambda \in]0, 1[\Longrightarrow f(x, y) \geq 0.$$

Remark 2 It can be seen immediately that if a bifunction f satisfies assumption (i) of Corollary 2, then it vanishes on the diagonal of $X \times X$. For a properly quasimonotone bifunction f that vanishes on the diagonal of $X \times X$, the pseudomonotonicity of $-f$ is implied (see [15, Proposition 3.3 and Remark 3]) by any of the following conditions:

- $-f(\cdot, y)$ is semistrictly quasiconvex, for all $y \in X$;
- f has the upper sign property on X and $f(\cdot, y)$ is quasiconvex for all $y \in X$.

The next corollary extends, to noncompact sets, [16, Corollary 4.2] and [31, Theorem 2.3].

Corollary 3 *Let X be a convex subset of a Hausdorff topological vector space, f be a real bifunction defined on $X \times X$, K be a compact subset of X and y_0 a point in X. Suppose that the following assumptions hold:*

(i) for each $y \in X$, the set $\{x \in X : f(x, y) \geq 0\}$ is compactly closed;
(ii) for every nonempty finite subset A of X and each $x \in$ conv A, $\max_{y \in A} f(x, y) \geq 0$;
(iii) $f(x, y_0) < 0$, for all $x \in X \setminus K$.

Then, there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in X$.

Proof Consider the bifunction $g : X \times X \to \mathbb{R}$ defined by

$$g(x, y) = -f(y, x).$$

Obviously, $-g$ is pseudomonotone with respect to $-f$ and assumption (ii) is equivalent to the proper quasimonotonicity of g. Thus, the conclusion follows from Theorem 1.

3 Applications

The origin of the next result goes back to the Ky Fan's minimax inequality. It can be regarded as a version of [18, Corollary 2], [35, Corollary 2.5] and [39, Theorem 1].

Theorem 4 *Let X be a nonempty convex subset of a Hausdorff topological vector space and $\varphi, \psi : X \times X \to \mathbb{R}$ two bifunctions that satisfy the following conditions:*

(i) for each $y \in Y$, the set $\{x \in X : \varphi(x, y) \leq \sup_{u \in X} \psi(u, u)\}$ is compactly closed;
(ii) for any nonempty finite subset A of X and all $y \in$ conv$A$, $\min_{x \in A} \psi(x, y) \leq \psi(y, y)$;
(iii) for each $y \in X$, there exists $z \in X$ such that

$$\varphi(x, y) \leq \psi(z, x), \quad \textit{for all} \ \ x \in X \ \ \textit{satisfying} \ \psi(z, x) \leq \sup_{u \in X} \psi(u, u);$$

(iv) there exist a nonempty compact subset K of X and a point y_0 in X such that for any $x \in X \setminus K$, $\varphi(x, y_0) < \sup_{u\in X} \psi(u, u)$.

Then, there exists $x_0 \in K$ such that $\varphi(x_0, y) \leq \sup_{u\in X} \psi(u, u)$ for all $y \in X$.

Proof We may assume that $\sup_{u\in X} \psi(u, u) < \infty$. Consider the bifunctions $f, g : X \times X \to R$ defined as follows:

$$f(x, y) = \sup_{u\in X} \psi(u, u) - \varphi(x, y),$$
$$g(x, y) = \psi(x, y) - \sup_{u\in X} \psi(u, u).$$

If A is a nonempty finite subset of X and $y \in \operatorname{conv} A$, then

$$\min_{x\in A} g(x, y) = \min_{x\in A} \psi(x, y) - \sup_{u\in X} \psi(u, u) \leq \min_{x\in A} \psi(x, y) - \psi(y, y) \leq 0,$$

hence g is properly quasimonotone.

Let $y \in X$ and z the point associated by assumption (iii). If, for some $x \in X$, $g(z, x) \leq 0$, then $\psi(z, x) \leq \sup_{u\in X} \psi(u, u)$. From here, we have

$$f(x, y) \geq \psi(z, x) - \varphi(x, y) \geq 0.$$

Consequently, $-g$ is generalized pseudomonotone with respect to $-f$. By Theorem 1, it follows that there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in X$. Clearly, this means that the conclusion of the theorem holds.

Remark 3 In our opinion, some comments about the hypotheses of Theorem 4 worth be done.

A bifunction ψ satisfying condition (ii) of the previous theorem is often called in the literature, diagonally quasiconcave in the first variable (see, for instance [40]).

The condition of coercivity (iv) is the same as that from [35, Corollary 2.5], but in [35] it is required the convexity of the sets $\{x \in X : \psi(x, y) > \sup_{u\in X} \psi(u, u)\}$, for all $y \in X$.

In all the results mentioned at the beginning of this section, instead of (iii), appears the condition as $\varphi(x, y) \leq \psi(x, y)$, for all $x, y \in X$. The two conditions are incomparable. Indeed, let us consider the real bifunctions $\varphi_1, \varphi_2, \psi$ defined on $[0, 1] \times [0, 1]$ by $\varphi_1(x, y) = x - y$, $\varphi_2(x, y) = \psi(x, y) = y - x$. It can be verified easily that for φ_1 and ψ condition (iii) holds, but $\varphi_1(x, y) > \psi(x, y)$ when $y < x$. On the other hand, φ_2 and ψ satisfy the condition $\varphi_2(x, y) \leq \psi(x, y)$ but they don't satisfy (iii).

The next theorem is closely related by [36, Corollary 2] and [7, Theorem 3.5].

Theorem 5 *Let X_1, X_2 be nonempty convex subsets of two Hausdorff topological vector spaces, K a compact subset of $X_1 \times X_2$ and $(y_1^0, y_2^0) \in X_1 \times X_2$. Let $\varphi, \psi : X_1 \times X_2 \to \mathbb{R}$ be two bifunctions such that*

- *(i) φ is lower semicontinuous in the first variable and upper semicontinuous in the second variable;*
- *(ii) ψ is convex in the first variable and concave in the second variable;*
- *(iii) for every $y_1 \in X_1$, there exists $z_1 \in X_1$ such that*

$$\psi(z_1, x_2) \leq \varphi(y_1, x_2) \ \textit{for all } x_2 \in X_2;$$

- *(iv) for every $y_2 \in X_2$, there exists $z_2 \in X_2$ such that*

$$\varphi(x_1, y_2) \leq \psi(x_1, z_2) \ \textit{for all } x_1 \in X_1;$$

(v) $\varphi(y_1^0, x_2) < \varphi(x_1, y_2^0)$*, for all* $(x_1, x_2) \in X_1 \times X_2 \setminus K$.

Then, φ *has a saddle point, i.e. there is* $(x_1^0, x_2^0) \in K$ *such that* $\varphi(x_1^0, y_2) \leq \varphi(x_1^0, x_2^0) \leq \varphi(y_1, x_2^0)$*, for all* $(y_1, y_2) \in X_1 \times X_2$.

Proof For each $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ set

$$f\big((x_1, x_2), (y_1, y_2)\big) = \varphi(y_1, x_2) - \varphi(x_1, y_2),$$
$$g\big((x_1, x_2), (y_1, y_2)\big) = \psi(y_1, x_2) - \psi(x_1, y_2).$$

From (i), it follows easily that for each $(y_1, y_2) \in X_1 \times X_2$, the set $\{(x_1, x_2) \in X_1 \times X_2 : f\big((x_1, x_2), (y_1, y_2)\big) \geq 0\}$ is closed in $X_1 \times X_2$.

We claim that g is properly quasimonotone. Suppose to the contrary that there exist a finite subset $\{(x_1^1, x_2^1) \ldots, (x_1^n, x_2^n)\}$ of $X_1 \times X_2$ and (y_1, y_2) in its convex hull, such that

$$\min_{1 \leq i \leq n} g\big((x_1^i, x_2^i), (y_1, y_2)\big) > 0.$$

Consequently:

$$\psi(y_1, x_2^i) > \psi(x_1^i, y_2), \quad \text{for all index } i \in \{1, \ldots, n\}.$$

If $(y_1, y_2) = \sum_{i=1}^n \lambda_i (x_1^i, x_2^i)$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$, then taking into account condition (ii) we obtain the following contradiction:

$$\psi(y_1, y_2) \leq \sum_{i=1}^n \lambda_i \psi(x_1^i, y_2) < \sum_{i=1}^n \lambda_i \psi(y_1, x_2^i) \leq \psi(y_1, y_2).$$

Let $(y_1, y_2) \in X_1 \times X_2$ and $z_1 \in X_1$, $z_2 \in X_2$ the points associated by conditions (iii) and (iv). If $g((z_1, z_2), (x_1, x_2)) \leq 0$, then

$$\psi(x_1, z_2) \leq \psi(z_1, x_2).$$

Using conditions (iii) and (iv), we obtain

$$\varphi(x_1, y_2) \leq \psi(x_1, z_2) \leq \psi(z_1, x_2) \leq \varphi(y_1, x_2)$$

whence, $f\big((x_1, x_2), (y_1, y_2)\big) = \varphi(y_1, x_2) - \varphi(x_1, y_2) \geq 0$. Consequently, -$g$ is generalized pseudomonotone with respect to -f.

We must also note that taking into account assumption (v), the condition of coercivity (ii) of Theorem 1 is fulfilled. By Theorem 1, there exists $(x_1^0, x_2^0) \in K$ such that

$$f\big((x_1^0, x_2^0), (y_1, y_2)\big) \geq 0, \text{ for all } (y_1, y_2) \in X_1 \times X_2.$$

Therefore,

$$\varphi(x_1^0, y_2) \leq \varphi(y_1, x_2^0), \ \forall (y_1, y_2) \in X_1 \times X_2.$$

Taking in the above inequality, first $y_2 = x_2^0$ and then $y_1 = x_1^0$, we get $\varphi(x_1^0, x_2^0) \leq \varphi(y_1, x_2^0)$, respectively $\varphi(x_1^0, y_2) \leq \varphi(x_1^0, x_2^0)$. This means nothing but that (x_1^0, x_2^0) is a saddle point for f.

Remark 4 If $K = K_1 \times K_2$, where $K_1 \subseteq X_1$, $K_2 \subseteq X_2$ are compact sets, then under the assumptions of Theorem 5, the following minimax inequality of Sion type holds:

$$\min_{x_1 \in K_1} \sup_{y_2 \in X_2} \varphi(x_1, y_2) \leq \inf_{y_1 \in X_1} \max_{x_2 \in K_2} \varphi(y_1, x_2).$$

4 Vector equilibrium problems, vector variational inequalities

Let E be a locally convex Hausdorff topological vector space and E^* be its topological dual. Consider a closed convex cone $C \subset E$ with $\operatorname{int} C \neq \emptyset$. For $e \in E$, we define the relations $\succeq, \succ, \not\succeq, \not\succ, \preceq$ as follows

$$\begin{aligned} e \succeq 0 &\Longleftrightarrow e \in C; \\ e \succ 0 &\Longleftrightarrow e \in \operatorname{int} C; \\ e \not\succeq 0 &\Longleftrightarrow e \notin C; \\ e \not\succ 0 &\Longleftrightarrow e \notin \operatorname{int} C; \\ e \preceq 0 &\Longleftrightarrow e \in -C. \end{aligned}$$

Similarly may be defined the order relations $\prec, \not\preceq, \not\prec$.

Given a nonempty convex subset X of a Hausdorff topological vector space and a vector bifunction $F : X \times X \to E$ we can formulate two types of vector equilibrium problems, as follows:

$$\text{(VEP-1) find } x_0 \in X \text{ such that } F(x_0, y) \succeq 0, \ \forall y \in X;$$

and, respectively

$$\text{(VEP-2) find } x_0 \in X \text{ such that } F(x_0, y) \not\prec 0, \ \forall y \in X.$$

Actually, in what follows, the solutions of the two problems will be localized in a compact subset of X. Though in many papers, the cone defining the order in the space E moves along with the point x, for the sake of simplicity, we preferred the above formats. Any case, as techniques of proof, there is no major difference. Clearly, when $E = \mathbb{R}$ and $C = [0, \infty[$, both problems reduces to the scalar equilibrium problem.

We will show further that the existence of solutions for problems (VEP-1) and (VEP-2) can be established by means of Theorem 1. Before moving on, let us recall that the dual cone of C, denoted by C^*, is defined as follows:

$$C^* = \{e^* \in E^* : \langle e^*, e\rangle \geq 0, \quad \forall e \in C\}.$$

A subset B of the cone C^* is called a base of C^*, if $0 \notin \operatorname{cl} B$ and $C^* = \bigcup_{\lambda > 0} \lambda B$. It is known (see [26]) that, since C has nonempty interior, the dual cone C^* has a weak* compact base B, and

$$e \succeq 0 \ \Leftrightarrow \ \langle e^*, e\rangle \geq 0, \ \forall e^* \in B, \tag{3}$$

respectively,

$$e \succ 0 \ \Leftrightarrow \ \langle e^*, e\rangle > 0, \ \forall e^* \in B. \tag{4}$$

Now, we are ready to formulate the first existence result of this section.

Theorem 6 *Let X be a convex subset of a Hausdorff topological vector space, C be a closed convex cone with nonempty interior in a locally convex Hausdorff topological vector space E and $F, G : X \times X \to E$. Assume that:*

(i) for each $y \in X$, the set $\{x \in X : F(x, y) \succeq 0\}$ is compactly closed;
(ii) for any nonempty finite subset A of X and all $y \in \operatorname{conv} A$, there exists $x \in A$ such that $G(x, y) \not\succ 0$;
(iii) for each $y \in X$, there exists $z \in X$ such that

$$x \in X, G(z, x) \not\succ 0 \Longrightarrow F(x, y) \not\succeq 0;$$

(iv) *there exist a nonempty compact subset K of X and a point $y_0 \in X$ such that $F(x, y_0) \not\succeq 0$ for all $x \in X \setminus K$.*

Then, there exists $x_0 \in K$ such that $F(x_0, y) \succeq 0$ for all $y \in X$.

Proof Let B be a weak* compact base of C^*. If we consider the bifunctions $f, g : X \times X \to \mathbb{R}$ defined by

$$f(x, y) = \min_{e^* \in B} \langle e^*, F(x, y)\rangle \text{ and } g(x, y) = \min_{e^* \in B} \langle e^*, G(x, y)\rangle$$

then, by (3) and (4) we infer that

$$f(x, y) \geq 0 \Longleftrightarrow F(x, y) \succeq 0, \quad f(x, y) < 0 \Longleftrightarrow F(x, y) \not\succeq 0,$$
$$g(x, y) \leq 0 \Longleftrightarrow G(x, y) \not\succ 0.$$

Based on these equivalences one can easily check that each of the conditions (i)–(iv) is nothing else than the condition similarly noted in Theorem 1. Applying Theorem 1 we obtain the desired conclusion.

Theorem 7 *Let X, C, F and G as in the previous theorem. Assume that:*

(i) *for each $y \in X$, the set $\{x \in X : F(x, y) \not\prec 0\}$ is compactly closed;*
(ii) *for any nonempty finite subset A of X and all $y \in$ conv A, there exists $x \in A$ such that $G(x, y) \preceq 0$;*
(iii) *for each $y \in X$, there exists $z \in X$ such that*

$$x \in X, G(z, x) \preceq 0 \Longrightarrow F(x, y) \not\prec 0;$$

(iv) *there exist a nonempty compact subset K of X and a point $y_0 \in X$ such that $F(x, y_0) \prec 0$ for all $x \in X \setminus K$;*

Then, there exists $x_0 \in K$ such that $F(x_0, y) \not\prec 0$ for all $y \in X$.

Proof Define the bifunctions $f, g : X \times X \to \mathbb{R}$ by

$$f(x, y) = \max_{e^* \in B} \langle e^*, F(x, y)\rangle \text{ and } g(x, y) = \max_{e^* \in B} \langle e^*, G(x, y)\rangle,$$

where B be a weak* compact base of C^*. From (3) and (4) we can easily establish the following equivalences

$$f(x, y) \geq 0 \Longleftrightarrow F(x, y) \not\prec 0, \quad f(x, y) < 0 \Longleftrightarrow F(x, y) \prec 0,$$
$$g(x, y) \leq 0 \Longleftrightarrow G(x, y) \preceq 0.$$

Based on these equivalences, Theorem 7 reduces to Theorem 1.

Example 2 Let $X =]-\infty, 1]$ and $C = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. Consider the bifunctions $F, G :]-\infty, 1] \times]-\infty, 1] \to \mathbb{R}^2$ defined by

$$F(x, y) = (F_1(x, y), F_2(x, y)) = (x - y, 2x - y - 1),$$
$$G(x, y) = (G_1(x, y), G_2(x, y)) = (x^3 - 3y^2 + 3y - 1, x - y).$$

For each $y \in]-\infty, 1]$, the set

$$\{x \in]-\infty, 1] : F(x, y) \not\prec 0\} = \{x \in]-\infty, 1] : x - y \geq 0\} \cup \{x \in]-\infty, 1] : 2x - y - 1 \geq 0\}$$

is closed in $]-\infty, 1]$.

If A is a nonempty finite subset of $]-\infty, 1]$ and $x_0 = \min A$, one can easily see that for every $y \in \operatorname{conv} A$, $G_1(x_0, y) \le 0$, $G_2(x_0, y) \le 0$, hence $G(x_0, y) \preceq 0$.

Since $G_2(\frac{1+y}{2}, x) = -\frac{F_2(x,y)}{2}$, we infer that assumption (iii) of Theorem 7 is satisfied, if we take $z = \frac{1+y}{2}$. Indeed,

$$G\left(\frac{1+y}{2}, x\right) \preceq 0 \Rightarrow G_2\left(\frac{1+y}{2}, x\right) \le 0 \Rightarrow F_2(x, y) \ge 0 \Rightarrow F(x, y) \not\prec 0.$$

The verification of the coercive condition from Theorem 7 is almost trivial. If for $a < 1$, arbitrarily chosen, we take $K = [a, 1]$ and y_0 any point from $[a, 1]$, then for every $x < a$ we have

$$F_1(x, y_0) < a - y_0 \le 0 \text{ and } F_2(x, y_0) < 2a - y_0 - 1 = (a - y_0) + (a - 1) < 0,$$

hence $F(x, y_0) \prec 0$.

By Theorem 7, problem (VEP-2) has at lest a solution. It can be easily verify that $x_0 = 1$ is the unique solution.

For problem (VEP-2), there are in the literature various existence theorems of solutions, in which condition (iii) of Theorem 7 is replaced by some conditions of generalized monotonicity. Thus, in [20, Theorem 2.2] the existence of solutions is established when, for all $x, y \in X$ the following implication holds:

$$F(x, y) \not\preceq 0 \Longrightarrow G(y, x) \not\prec 0.$$

Note that this condition is not satisfied in the given example, because $F(1, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) \not\preceq 0$, but $G(\frac{1}{2}, 1) = (-\frac{7}{8}, -\frac{1}{2}) \prec 0$, hence the mention theorem is not applicable.

As in other papers, some assumptions of Theorems 6 and 7 can be replaced by stronger conditions that imply certain properties of cone continuity and cone concavity. Recall that (see [4]) a function $\varphi : X \to E$ is said to be:

(i) C-upper semicontinuous at $x_0 \in X$, if for any neighborhood $V \subseteq E$ of $\varphi(x_0)$ there exists a neighborhood U of x_0 in X such that $\varphi(x) \in V - C$, for all $x \in U$. Furthermore, φ is said to be C-upper semicontinuous on X if it is C-upper semicontinuous at each $x \in X$;
(ii) C-concave, if $\varphi(\lambda x_1 + (1-\lambda)x_2) \in \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) + C$, for all $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$.
(iii) C-quasiconcave, if for every $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ there is an index $i \in \{1, 2\}$ such that $\varphi(\lambda x_1 + (1-\lambda)x_2) \in \varphi(x_i) + C$.

Let $\{x_1, \ldots, x_n\}$ be a finite subset of X and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$. By mathematical induction it is shown that:

(i) if φ is C-concave, then $\varphi(\sum_{i=1}^n \lambda_i x_i) \in \sum_{i=1}^n \lambda_i \varphi(x_i) + C$;
(ii) if φ is C-quasiconcave, then there is an index $i \in \{1, \ldots, n\}$ such that $\varphi(\sum_{i=1}^n \lambda_i x_i) \in \varphi(x_i) + C$.

Proposition 8 *Each of conditions (i) and (ii) from below implies the assumption similarly noted in Theorem 6*

(i) for every $y \in X$, the function $F(\cdot, y)$ is C-upper semicontinuous on X.
(ii) for every $y \in X$, $G(y, y) \not\succ 0$ and the function $G(\cdot, y)$ is C-concave.

Proof Let y be an arbitrarily chosen point in X. We want to prove that under condition (i), the set

$$M = \{x \in X : F(x, y) \succeq 0\}$$

is closed in X. Let $x \in \text{ cl } M$ and $\{x_t\}$ be a net in M converging to x. Then, $F(x_t, y) \in C$ for all indices t. If V is an arbitrary neighborhood of the origin of E, since $F(\cdot, y)$ is C-upper semicontinuous, there exists an t_0 such that for every $t \geq t_0$, $F(x_t, y) \in F(x, y) - V - C$. Then, for any such index t,

$$F(x, y) \in F(x_t, y) + V + C \subseteq C + V + C = V + C.$$

Hence, $F(x, y) \in \text{ cl } C = C$ and thus $x \in M$.

Suppose now that (ii) holds. Let y be a point in the convex set of a finite set $\{x_1, x_2, \ldots, x_n\}$. Then, $y = \sum_{i=1}^{n} \lambda_i x_i$, for some nonnegative λ_i, with $\sum_{i=1}^{n} \lambda_i = 1$. Assume, by way of contradiction, that for each $i \in \{1, \ldots, n\}$, $G(x_i, y) \succ 0$, that is, $G(x_i, y) \in \text{ int } C$. As the function $G(\cdot, y)$ is C-concave, we have

$$G(y, y) = G(\sum_{i=1}^{n} \lambda_i x_i, y) \in \sum_{i=1}^{n} \lambda_i G(x_, y) + C \in \text{ int } C + C = \text{ int } C.$$

Thus, $G(y, y) \prec 0$; a contradiction.

Similarly, one can establish the next proposition.

Proposition 9 *Conditions (i) and (ii) of Theorem 7 are fulfilled whenever the following assumptions hold:*

(i) for every $y \in X$, the function $F(\cdot, y)$ is C -upper semicontinuous on X.
(ii) for every $y \in X$, $G(y, y) \preceq 0$ and the function $G(\cdot, y)$ is C-quasiconcave.

In the last part of this section we study the existence of solutions for a vector variational inequality problem introduced by Giannessi [22] and subsequently studied by many other authors (see, for instance, [17, 20, 23, 29]). Let E_1 and E_2 be two normed vector spaces and $L(E_1, E_2)$ the set of all linear continuous operators from E_1 into E_2. In the sequel, the space $L(E_1, E_2)$ will be endowed with the strong operator topology (recall that this is the weakest topology for which the functions $L(E_1, E_2) \ni A \to A(x) \in E_2$ are continuous for every $x \in E_1$.)

In order to formulate the studied problem, we need to consider:

- a nonempty convex subset X of E_1;
- a set-valued mapping $T : X \rightrightarrows L(E_1, E_2)$ with nonempty values;
- a convex closed cone C in E_2, with nonempty interior.

The vector variational inequality problem read as follows:

$$\text{find } x_0 \in X \text{ such that } \forall y \in X, \exists A \in T(x_0) : A(y - x_0) \not\prec 0.$$

In the proving of Theorem 11 we need the following lemma, due to Berge [8, p. 116]:

Lemma 10 *Let X and Y be topological spaces. If $\phi : X \times Y \to \mathbb{R}$ is a upper semicontinuous bifunction and $\Gamma : X \rightrightarrows Y$ is a upper semicontinuous set-valued mapping, with nonempty compact values, then the function $M : X \to \overline{\mathbb{R}}$ defined by*

$$M(x) = \sup\{\phi(x, y) : y \in \Gamma(x)\}$$

is upper semicontinuous.

As we already mentioned, in the next theorem, $L(E_1, E_2)$ is endowed with the strong operator topology.

Theorem 11 *Assume that*

(i) *T is upper semicontinuous with compact values;*
(ii) *there exist a nonempty compact subset K of X and a point $y_0 \in X$ such that $A(y_0 - x) \prec 0$ for all $x \in X \setminus K$ and $A \in T(x)$.*

Then, there exists $x_0 \in K$ such that

$$\forall y \in X, \exists A \in T(x_0) : A(y - x_0) \not\prec 0.$$

Proof We intend to apply Corollary 3, when the bifunction $f : X \times X \to \mathbb{R}$ is defined by

$$f(x, y) = \max_{\substack{e^* \in B \\ A^* \in T(x)}} \langle e^*, A(y - x)\rangle$$

where, as above, B is a weak* compact base of C^*. From (4), it follows easily the following equivalence:

$$f(x, y) \geq 0 \Longleftrightarrow \exists A \in T(x) : \; A(y - x) \not\prec 0.$$

Fix an $y \in X$ and consider the bifunction $\phi : X \times (B \times L(E_1, E_2)) \to \mathbb{R}$, the set-valued mapping $\Gamma : X \rightrightarrows B \times L(E_1, E_2)$ and the function $M : X \to \mathbb{R}$ defined by

$$\begin{aligned} &\phi(x, (e^*, A)) = \langle e^*, A(y - x)\rangle, \\ &\Gamma(x) = B \times T(x) = \{(e^*, A) : e^* \in B, A \in T(x)\}, \\ &M(x) = \max_{(e^*, A) \in \Gamma(x)} \phi(x, (e^*, A)). \end{aligned}$$

Since B is weak* compact, by Alaoglu's theorem [3, Theorem 6.21], it is a norm bounded subset of E_2^*. Then, according to [5, Lemma 4.3] (or [3, Corrolary 6.40]), the duality pairing $\langle \cdot, \cdot \rangle$ restricted to $B \times E_2$ is jointly continuous, when B is endowed with its weak* topology and E_2 has its norm topology. Consequently, ϕ is continuous and Γ is compact-valued and upper semicontinuous, if X, B and $L(E_1, E_2)$ are equipped with the norm topology, the weak* topology and the strong operator topology, respectively. By Lemma 10, the function M is upper semicontinuous, and consequently the set

$$\{x \in X : M(x) \geq 0\} = \{x \in X : f(x, y) \geq 0\}$$

is closed in X.

Let $\{y_1, \ldots, y_n\}$ be a finite subset of X and x a point in its convex hull, say $x = \sum_{i=1}^n \lambda_i y_i$, ($\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$.) Take an arbitrary pair $(e_0^*, A_0) \in B \times T(x)$. Since

$$0 = \sum_{i=1}^n \lambda_i \langle e_0^*, A_0(y_i - x)\rangle,$$

there is $i_0 \in \{1, \ldots, n\}$ such that $\langle e_0^*, A_0(y_{i_0} - x)\rangle \geq 0$. Then,

$$\max_{1 \leq i \leq n} f(x, y_i) \geq f(x, y_{i_0}) \geq \langle e_0^*, A_0(y_{i_0} - x)\rangle \geq 0.$$

By Corollary 3, there exists $x_0 \in K$ such that for all $y \in X$, $f(x_0, y) \geq 0$, that is, exactly the conclusion of the theorem.

Remark 5 (i) In view of [2, Proposition 2.4], T is upper semicontinuous on each compact subset of X if and only if, the lower inverse of every closed subset F of $L(E_1, E_2)$ by T (that is the set $T^-(F) = \{x \in X : T(x) \cap F \neq \emptyset\}$) is compactly closed.

(ii) It is worthwhile noticing that unlike other existence results encountered in the literature (as, [17, Theorem 1], [23, Theorem 4.1], [29, Theorem 3.1]), in Theorem 11, the pseudomonotonicity (with respect to C) of the set-valued mapping T is not required.

When $E_2 = \mathbb{R}$, the strong operator topology reduces to the weak* topology on the dual E_1^*. If in addition, $C = \mathbb{R}_+$, from Theorem 11 we obtain the following corollary:

Corollary 12 *Let X be a convex subset of a normed space E, $T : X \rightrightarrows E^*$ be a set-valued mapping with nonempty weak* compact values. Assume that:*

(i) for every compact subset C of X, T is norm-to-weak upper semicontinuous on C;*
(ii) there exist a compact subset K of X and $y_0 \in X$ such that for any $x \in X \setminus K$,

$$\max_{x^* \in T(x)} \langle x^*, y - x\rangle < 0.$$

Then, there exists $x_0 \in K$ such that

$$\max_{x^* \in T(x_0)} \langle x^*, y - x_0\rangle \geq 0, \quad \textit{for all} \quad y \in X.$$

Moreover, if $T(x_0)$ is a convex set, then there exists $x_0^* \in T(x_0)$ such that

$$\langle x_0^*, y - x_0\rangle \geq 0, \quad \textit{for all} \quad y \in X.$$

Proof The fist part follows from Theorem 11.

If $T(x_0)$ is convex, Kneser minimax theorem [28] applied to the bifunction $X \times T(x_0) \ni (y, x^*) \to \langle x^*, x_0 - y\rangle$, leads to the following relation

$$0 \geq \sup_{y \in X} \min_{x^* \in T(x_0)} \langle x^*, x_0 - y\rangle = \min_{x^* \in T(x_0)} \sup_{y \in X} \langle x^*, x_0 - y\rangle,$$

whence it follows the desired conclusion.

Remark 6 Corollary 12 has some similarities with [38, Theorem 2.3]. But, in [38], the existence result is established in a reflexive Banach space and the set-valued mapping T is assumed pseudomonotone.

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