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Variational rationality, variational principles and the existence of traps in a changing environment

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Abstract
This paper has two aspects. Mathematically, in the context of global optimization, it provides the existence of an optimum of a perturbed optimization problem that generalizes the celebrated Ekeland variational principle and equivalent formulations (Caristi, Takahashi), whenever the perturbations need not satisfy the triangle inequality. Behaviorally, it is a continuation of the recent variational rationality approach of stay (stop) and change (go) human dynamics. It gives sufficient conditions for the existence of traps in a changing environment. In this way it emphasizes even more the striking correspondence between variational analysis in mathematics and variational rationality in psychology and behavioral sciences.

Keywords Quasi-metric space · The Ekeland variational principle · Variational rationality · Changing environment · Traps

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1 Introduction and preliminaries

This paper continues to explore, both in psychology/ behavioral sciences and in mathematics/ variational analysis, the variational rationality approach of stay and change human dynamics initiated by Antoine Soubeyran in psychology and behavioral sciences [23–28] and followed by some researchers in mathematics; see for example [2,4,5,12,20–22]. To save space, for a more complete list of applications with other mathematical aspects, see Antoine SOUBEYRAN—Google Sites. Thus, this paper has two complementary aspects, a mathematical aspect and a behavioral aspect.

1.1 Mathematical aspect

A quasi-metric q on a nonempty set X is a bifunction $q : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of non-negative real numbers, such that

(i) $q(x, y) = 0$ if and only if $x = y$,
(ii) $q(x, y) \leq q(x, z) + q(z, y)$, for all $x, y, z \in X$.

A set X equipped with a quasi-metric q is said to be a quasi-metric space and it is denoted by (X, q). Mathematical aspect starts from the classical Ekeland variational principle (EVP) [11].

EVP problem Let (X, q) be a quasi-metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an objective function. Suppose that $-\infty < f_* = \inf\{f(y) : y \in X\}$. Let $\varepsilon > 0$, $\lambda > 0$ and a status quo $x \in X$ be given such that

(E0) $f(x) < f_* + \varepsilon$.
Then, the EVP is the problem of finding $x_* = y \in X$ such that
(E1) $q(x, y) \leq \lambda$;
(E2) $f(x) - f(y) \geq \frac{\varepsilon}{\lambda} q(x, y)$;
(E3) $f(y) - f(z) < \frac{\varepsilon}{\lambda} q(y, z) \quad \forall z \neq y$.

For $\varepsilon > 0$, we set $E_\varepsilon = \{x \in X :$ condition E_0 is satisfied$\}$, and we consider $S_E(f(\cdot), \lambda, \varepsilon, x)$ the set of all elements $y \in X$ which satisfy the conditions (E1), (E2) and (E3); that is, the set of solutions of the EVP problem.

Let (X, q) be a quasi-metric space. A sequence $\{x_n\}$ in X is said to be left convergent to $x \in X$, if $\lim_{n\to\infty} q(x_n, x) = 0$. The sequence $\{x_n\}$ is said to be left Cauchy if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m \geq n \geq N_\varepsilon$. A quasi-metric space (X, q) is said to be left-complete if every left Cauchy sequence is left convergent. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be sequentially lower monotone (denoted by slm) if for any sequence $\{x_n\}$ left convergent to an element $x_0 \in X$ and such that $f(x_{n+1}) \leq f(x_n)$ we have that $f(x_0) \leq \liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n)$.

Theorem 1 [22, Theorem 3.1] *Let (X, q) be left-complete quasi-metric space, Y be a real linear space, $K \subset Y$ be a convex cone and $k_0 \in K \setminus -vcl(K)$, such that K is k_0-closed. Let $F : X \times X \to 2^Y \setminus \{\emptyset\}$ be a set-valued bimap satisfying the following conditions:*

(i) $\exists x_0 \in X$ such that $F(x_0, x_0) \subset -K$;
(ii) $\exists \alpha \in \mathbb{R}, \exists s_0 \in (-\infty, +\infty)k_0 - K$ such that $F(x_0, X) \cap (-s_0 + \alpha k_0 - K) = \emptyset$;
(iii) $F(x, z) \subset F(x, y) + F(y, z) - K, \forall x, y, z \in X$;
(iv) F is left K-slm in (X, q).

Assume that ψ is F-decreasing (that is $\psi(x) \leq \psi(x')$ whenever $F(x, x') \subset -k$). Then, there exists $\bar{x} \in X$ such that

(a) $F(x_0, \bar{x}) + \psi(x_0)q(x_0, \bar{x})k_0 \subset -K$;
(b) $\forall x \in X \setminus \{\bar{x}\}$, $F(\bar{x}, x) + \psi(\bar{x})q(\bar{x}, x)k_0 \not\subset -K$.

According to the above theorem we have the following result which shows the set of solutions of the EVP problem is nonempty.

Theorem 2 (Replication of EVP) *Let (X, q) be a left-complete quasi-metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper, bounded from below and slm. For $\varepsilon > 0$, $\lambda > 0$ and $x \in E_\varepsilon$, we have $S_E(f, \lambda, \varepsilon, x) \neq \emptyset$.*

Proof Assume that $F : X \times X \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is defined by $F(u, v) := \{f(v) - f(u)\}$ and we consider $K := [0, \infty)$. Suppose that $\psi(x) := \frac{1}{\lambda}$ for all $x \in X$ and $k_0 = \varepsilon$. Then, conditions (iii), (iv) of Theorem 1 are satisfied and ψ is F-decreasing. By choosing $s_0 = 0$, $x_0 = x$ and $\alpha = -1$, it is clear that conditions (i) and (ii) of Theorem 1 are satisfied too. Therefore, by Theorem 1 there exists $\bar{x} \in X$ such that

(a) $f(\bar{x}) - f(x_0) + \frac{1}{\lambda}q(x_0, \bar{x})\varepsilon \in -K$;
(b) $\forall z \in X \setminus \{\bar{x}\}$, $f(z) - f(\bar{x}) + \frac{1}{\lambda}q(\bar{x}, z)\varepsilon \notin -K$.

If we put $\bar{x} = y$, then conditions (a) and (b) are equivalent to conditions (E2) and (E3), respectively. Since $f(x) < f_* + \varepsilon$ and by (E2) we have

$$\varepsilon = \varepsilon + f_* - f_* > f(x) - f(y) \geq \frac{\varepsilon}{\lambda} q(x, y).$$

Hence, $q(x, y) \leq \lambda$. □

At the mathematical level, the goal of this paper is to extend the EVP as follows. Let $\lambda : X \ni x \longmapsto \lambda(x) \in \mathbb{R}^{++}$ ($\mathbb{R}^{++}$ denotes the set of positive real numbers) be a nonconstant, and positive resource function, $\varepsilon > 0$ and a status quo $x \in X$ be given such that $x \in E_\varepsilon$. Then, the generalized Ekeland variational principle (GEVP) problem is to find $x_* = y \in X$ such that

(VR1) $q(x, y) \leq \lambda(x)$;
(VR2) $f(x) - f(y) \geq \frac{\varepsilon}{\lambda(x)} q(x, y)$;
(VR3) $f(y) - f(z) < \frac{\varepsilon}{\lambda(y)} q(y, z)$ for all $z \neq y$.

We also set

$$S_{GE}(f(\cdot), \lambda(\cdot), \epsilon, x) := \{y \in X : \text{conditions (VR1)–(VR3) are satisfied}\},$$

that is, $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$ is the set of solutions of the GEVP. The main problem in this paper, at the mathematical level, is to give sufficient conditions for the existence of solution of the GEVP problem. This generalization is important because it gives conditions of existence of solution for the GEVP problem and its equivalent formulations when there are several perturbation functions $p_x(x, y) = \frac{\varepsilon}{\lambda(x)} q(x, y)$ and $p_y(y, z) = \frac{\varepsilon}{\lambda(y)} q(y, z)$ that do not satisfy the triangle inequality.

Related literature Several papers dealing with the EVP already considered various types of perturbations related to the generalized metric q of the underlying space; see for example [3,6,8,9,13–16,19]. But none are status quo dependent as ours.

1.2 Behavioral aspect

The behavioral aspect motivates our interest for the GEVP. At the behavioral level this paper gives conditions for the existence of variational traps, rather easy to reach and difficult to leave, in a changing environment, when available resources are changing each period. Thus, this paper is a continuation of the recent (VR) variational rationality approach of stay (stop) and change (go) human dynamics driven by changing needs and moving goals in a changing environment.

The VR approach This approach [23–28] is highly multidisciplinary because it can model and unify a lot of stay and change human dynamics in behavioral sciences. See [23] for different aspects in psychology, economics, management sciences, sociology, political sciences, decision theory, game theory, artificial intelligence and mathematics. For example, stays refer to exploitation phases, temporary repetitions of the same action, temporary habits, routines, rules and norms, while changes refer to exploration phases, learning and innovations processes, forming and breaking habits and routines, changing doings (actions), havings and beings.

The spatial and unifying VR "traveler perspective" The VR approach models human dynamics in a locomotion space where individuals change or stay for a while. It models each individual as a traveler and a position as a city. Traveler's behavior means "doing a bundle of situated activities", which represents "living in a city". Doing the same situated bundle in the previous and current periods is like "living in a city for two periods". Changing a behavior means moving to another city. His/Her short-term problem is to choose, at the beginning of each period, between to live again in the status quo city or to live in a new city. The long-term problem is to reach a final (desired) city. The presence of costs of moving generates inconveniences (hence resistances) to change. Then, it may be not worthwhile to move even if there are advantages of moving. This defines a variational trap " rather worthwhile to reach, but not worthwhile to quit". Thus, the VR approach focuses the attention on the existence of traps which prevent individuals from reaching their desired ends (desires). The VR approach is unifying. It can provide a general theory of motivation, moving goals and intentions as well as a general theory of self regulation. Its traveler perspective helps to get closer to the Lewin's dream of "Topological psychology" (see [17,18]). Moreover, its formulation exhibits a striking and an almost one to one correspondence between famous principles in psychology and most of the famous variational principles and leading optimizing algorithms in the recent developments of variational analysis, see [23].

1.3 Main topics of the VR approach

The VR approach starts with the idea that all things are changing at two levels. First, in our internal environment (body and head), our needs and the goals, thoughts, feelings, emotions, preferences and utilities that help us to choose how much of each unsatisfied need must be satisfied, how and when. Second, in our external environment, all the related means (things, objects, peoples, and landscapes). The difficulty is that, most of the time, internal and external environments are changing endogenously, because doing something changes the environment where this thing has been done.

The main topic of the VR approach: stop (go)—go (stop) human dynamics The VR approach has modeled life as a succession of stays and changes, "stop and go", "go and stop" dynamics where, each period, individuals, i) stay, i.e., continue doing some situated activities and, ii) change, i.e., stop and start doing other situated activities. This is in accordance with Lewin's view of human behaviors (see [17,18]), one of the giants that can be compared to Freud, among all the psychologists of the last century. He saw life as "a constant interplay between completing old situations and opening up new ones".

The main VR question The VR approach is driven by the following question: when to start, continue, and stop changing in a changing internal and external environment, for an individual, an organization or several interacting individuals (games)? As such, it is a theory of starts, ends and transitions, where ends can be desired or not (desires or traps). More concretely, the VR approach considers the following main questions that each individual can pose: i) I am there, in an undesirable position, ii) I want to be elsewhere, sooner or later, in a better position, iii) I am unable to reach in one move this momentary desired position within the current period. Hence, iv), I must accept to follow a transition, that is, a succession of acceptable moves (changes or stays) over a succession of periods. The way the VR approach poses this question provides the main verbal and spatial model that drives the VR approach. This basic model must be decomposed in two submodels: i) a model and its variants for proximal (short run) dynamics, and, ii) a model for distal (long run) dynamics, including also a lot of variants.

2 Generalized variational principles

In this section, we study the existence of solutions for the GEVP problem, where the positive resource factor $\lambda : X \ni x \longmapsto \lambda(x) \in \mathbb{R}^{++}$ is not constant. Then, perturbation p can be redefined as $p(x, y) = \frac{1}{\lambda(x)} q(x, y)$, where q is a quasi-metric. This apparently small change does not provide a generalization if p itself is a quasi-metric. A quasi-metric space (X, q) is called bounded if for some (equivalently for all) $x \in X$, $\sup\{q(x, z) : z \in X\} < \infty$.

In the following result, we show that if λ is not constant, then p does not have the triangle property.

Proposition 1 *Suppose that (X, q) is a quasi-metric space, q is unbounded and $\lambda(\cdot)$ is a positive function on X. If $p(x, y) := \frac{1}{\lambda(x)} q(x, y)$, for every $x, y \in X$ and p satisfies the triangle inequality, then $\lambda(\cdot)$ is a constant function.*

Proof Let $x, y \in X$ and $x \neq y$. Since q is unbounded, there exist $k \in X$ and a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} q(k, z_n) = +\infty$. By triangle inequality we have

$$\lim_{n\to\infty} q(x, z_n) = \lim_{n\to\infty} q(y, z_n) = +\infty.$$

Since $\frac{q(x,z_n)}{q(y,z_n)} \leq \frac{q(x,y)}{q(y,z_n)} + 1$, then $\limsup_{n\to\infty} \frac{q(x,z_n)}{q(y,z_n)} \leq 1$.

Similarly, $\limsup_{n\to\infty} \frac{q(y,z_n)}{q(x,z_n)} \leq 1$, so that

$$\lim_{n\to\infty} \frac{q(x, z_n)}{q(y, z_n)} = \lim_{n\to\infty} \frac{q(y, z_n)}{q(x, z_n)} = 1,$$

because, for any sequence $\{\alpha_n\}$ of positive real numbers

$$\limsup_n \frac{1}{\alpha_n} = \frac{1}{\liminf_n \alpha_n}.$$

From the triangle inequality property of p, we get $p(y, z_n) \leq p(y, x) + p(x, z_n)$, for all $n \in \mathbb{N}$, and so

$$\frac{1}{\lambda(y)} \frac{q(y, z_n)}{q(x, z_n)} \leq \frac{1}{\lambda(y)} \frac{q(y, x)}{q(x, z_n)} + \frac{1}{\lambda(x)}.$$

Letting $n \to \infty$, one obtains, $\frac{1}{\lambda(y)} \leq \frac{1}{\lambda(x)}$, or equivalently, $\lambda(y) \geq \lambda(x)$. Since x, y are arbitrary points, $\lambda(x) \geq \lambda(y)$, and so $\lambda(x) = \lambda(y)$. Hence, $\lambda(\cdot)$ is a constant function. □

In continuation of our work, we discuss the necessity of investigating the GEVP is independent of the EVP. In fact, we show that we may have $S_{GE} = \emptyset$ and $S_E \neq \emptyset$ simultaneously; while on the other hand, $S_{GE} \neq \emptyset$ and $S_E = \emptyset$ may be true at the same time. The following example shows that S_{GE} for a suitable quasi-metric space (X, q), a positive function $\lambda(\cdot)$ and a real valued function f can be nonempty, but S_E is a empty set.

Example 1 Let $X = \mathbb{R}^+$ and $f : X \to \mathbb{R}^+$ be defined as follow:

$$f(x) := \begin{cases} \sqrt{x} & x > 0, \\ 1 & x = 0. \end{cases}$$

We have $f_* = 0$ so that the following conditions must be satisfied by x and y:

(E0) $0 < \varepsilon < \frac{1}{2}, 0 < \sqrt{x} < \varepsilon$;
(E1) $|x - y| \leq 1$;
(E2) $f(x) - f(y) \geq \varepsilon|x - y|$;
(E3) $f(y) - f(z) < \varepsilon|y - z| \quad \forall z \in X \setminus \{y\}$.

We distinguish 3 cases.
Case I. Suppose that $y = 0$. Then, (E3) is equivalent to

$$\varepsilon z + \sqrt{z} - 1 > 0 \quad \forall z \neq 0,$$

which obviously fails for sufficiently small z.

Case II. Suppose that $y = x$. Then, by (E3)

$$\varepsilon\sqrt{x} > 1 - \varepsilon\sqrt{z}, \quad \forall z \in (0, x),$$

which implies $\sqrt{x} \geq \frac{1}{\varepsilon} > 2$ in contradiction to (E0).

Case III. Assume that $y \in (0, \infty) \setminus \{x\}$. Then, (E2) implies $0 < y < x$ and

$$\varepsilon(\sqrt{x} + \sqrt{y}) \leq 1. \tag{1}$$

On the other hand, by (E3), we have

$$\varepsilon\sqrt{y} > 1 - \varepsilon\sqrt{z} \quad \forall z \in (0, y).$$

Therefore, $\varepsilon\sqrt{y} \geq 1$, and so $\varepsilon(\sqrt{x} + \sqrt{y}) > 1$ which contradicts to (1). Hence, $S_E(f(\cdot), 1, \varepsilon, x) = \emptyset$.

Let now $\lambda : X \to \mathbb{R}^{++}$ be given by

$$\lambda(x) := \begin{cases} x & x > 0, \\ 1 & x = 0. \end{cases}$$

The conditions that must be satisfied are

(E0) $\varepsilon > 0, 0 < f(x) < \varepsilon$;
(VR1) $|x - y| \leq \lambda(x)$;
(VR2) $f(x) - f(y) \geq \frac{\varepsilon}{\lambda(x)}|x - y|$;
(VR3) $f(y) - f(z) < \frac{\varepsilon}{\lambda(y)}|y - z|$ for all $z \in X \setminus \{y\}$.

Case I. If $x = 0$, then $\varepsilon > 1$. For $y > 0$, (VR2) is equivalent to

$$\varepsilon y + \sqrt{y} - 1 \leq 0.$$

The inequalities

$$y > 0 \text{ and } \varepsilon y + \sqrt{y} - 1 \leq 0$$

hold if and only if

$$0 < \sqrt{y} \leq \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon},$$

or equivalently

$$0 < y \leq \frac{1+2\varepsilon-\sqrt{1+4\varepsilon}}{2\varepsilon^2}.$$

We show that (VR3) holds for all these y. If $z > 0$ and $z \neq y$, (VR3) becomes

$$\sqrt{y} - \sqrt{z} < \frac{\varepsilon}{y}|y - z|.$$

If $z > y$, then this inequality is trivially true. If $0 < z < y$, then it is equivalent to

$$y - \varepsilon\sqrt{y} < \varepsilon\sqrt{z}.$$

Since $\sqrt{y} \leq \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon^2} < \varepsilon$, it follows

$$y - \varepsilon\sqrt{y} = \sqrt{y}(\sqrt{y} - \varepsilon) < 0 < \varepsilon\sqrt{z}.$$

If $z = 0$, then (VR3) becomes

$$\sqrt{y} - 1 < \varepsilon,$$

which is true, because $\sqrt{y} < \varepsilon < 1 + \varepsilon$.

If $y = 0$, then by (VR3) we must have $\varepsilon z + \sqrt{z} - 1 > 0$, for all $z > 0$, which fails for sufficiently small z. Consequently

$$S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, 0) = \{y \in \mathbb{R} : 0 < y \leq \frac{1+2\varepsilon-\sqrt{1+4\varepsilon}}{2\varepsilon^2}\}.$$

Case II. If $x > 0$, then $0 < \sqrt{x} < \varepsilon$.

In this case, take $y = x$. Then (VR1), (VR2) are obviously satisfied. If $z = 0$, then (VR3) is equivalent to

$$\sqrt{x} - 1 < \varepsilon,$$

which is true, because $\sqrt{x} < \varepsilon$.

If $0 < z < x$, then it is equivalent to

$$x - \varepsilon\sqrt{x} < \varepsilon\sqrt{z},$$

which holds, because $x - \varepsilon\sqrt{x} = \sqrt{x}(\sqrt{x} - \varepsilon) < 0$.

If $z > x$, then (VR3) is trivially true.

In what follows we assume that (X, q) is a quasi-metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper bounded from below function and $\lambda(\cdot)$ is a positive function on X. We also set $\arg\min f := \{x \in X : f(x) = \inf_{y \in X} f(y)\}$. We consider $B[a, r] := \{x \in X : q(a, x) \leq r\}$, $B(a, r) := \{x \in X : q(a, x) < r\}$, where $a \in X$ and $r > 0$. We now obtain relation among the sets $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$ and $S_E(f(\cdot), \lambda(x), \varepsilon, x)$. It is easy to see that if $\varepsilon > 0$, $x \in E_\varepsilon$ and $f(\cdot)$ or $\lambda(\cdot)$ is constant, then $S_E(f(\cdot), \lambda(x), \varepsilon, x) = S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$.

Theorem 3 *Suppose that* $\varepsilon > 0$, $x \in E_\varepsilon$, $S_E(f(\cdot), \lambda(x), \varepsilon, x) \neq \emptyset$ *and the following condition holds*

$$\lambda(x) \geq \lambda(y), \quad \forall y \in S_E(f(\cdot), \lambda(x), \varepsilon). \tag{2}$$

Then $S_E(f(\cdot), \lambda(x), \varepsilon, x) \subset S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$, *and so* $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \neq \emptyset$.

Proof Since $S_E(f(\cdot), \lambda(x), \varepsilon, x) \neq \emptyset$, then conditions (VR1) and (VR2) hold. From condition (E2), we have $f(y) \leq f(x)$ for all $y \in S_E(f(\cdot), \lambda(x), \varepsilon, x)$. Therefore, from (2) we deduce that $\lambda(x) \geq \lambda(y)$. Hence, to obtain condition (VR3), it is enough to replace $\lambda(x)$ in condition (E3) by $\lambda(y)$. Thus, $S_E(f(\cdot), \lambda(x), \varepsilon, x) \subset S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$. □

The function $\lambda(\cdot)$ is said to be f-increasing iff $\lambda(x_1) \leq \lambda(x_2)$ if $f(x_1) \leq f(x_2)$, where $x_1, x_2 \in X$; see [22]. Motivated by this notion, we introduce the concept of f-proper increasing which is weaker than the notion of f-increasing.

Definition 1 The function $\lambda(\cdot)$ is said to be f-proper increasing iff $\lambda(x_1) \leq \lambda(x_2)$ if $f(x_1) \leq f(x_2) < +\infty$, where $x_1, x_2 \in X$.

It is clear that if $\lambda(\cdot)$ is f-increasing, then it is f-proper increasing, but the following example shows that the converse of this fact does not necessarily hold in general.

Example 2 Let

$$f(x) := \begin{cases} e^x & x \in \mathbb{Z}, \\ +\infty & x \notin \mathbb{Z}. \end{cases}$$

If $\lambda(x) := e^x$, then it is clear that λ is f-proper increasing, but it is not f-increasing.

As a consequence of Theorem 3, we obtain the following result.

Corollary 1 *Suppose that* $\varepsilon > 0$, $x \in E_\varepsilon$ *and* $S_E(f, \lambda(x), \varepsilon, x) \neq \emptyset$. *If* $\lambda(\cdot)$ *is* f*-proper increasing, then* $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \neq \emptyset$.

Proof It is enough to show that condition (2) holds. For this purpose, suppose that $y \in S_E(f(\cdot), \lambda(x), \varepsilon, x)$. From condition (E2), we have $f(y) \leq f(x)$. On the other hand $x \in E_\varepsilon$, thus $f(x) < +\infty$. Therefore, $\lambda(y) \leq \lambda(x)$, and so condition (2) in Theorem 3 holds. □

Remark 1 (a) If (X, q) is left-complete and f is slm, then by Theorem 2 we have $S_E(f(\cdot), \lambda(x), \varepsilon, x) \neq \emptyset$.
(b) Corollary 1 is a similar result to Theorem 1 for real valued functions.

The following example shows that condition (2) is weaker than the f-proper increasing condition relative to $\lambda(\cdot)$.

Example 3 Suppose that $k \geq 1$, $f(x) := e^x$ and $\lambda(x) := \frac{1}{k+e^x}$ for all $x \in \mathbb{R}$. It is clear that λ is not f-proper increasing and $S_E(f(\cdot), \lambda(x), \varepsilon, x) = \{x\}$, for every $\varepsilon > 0$ and $x \in E_\varepsilon$. Hence, condition (2) is satisfied.

We now skim conditions under which the GEVP does not hold. However, according to Theorem 2, one can see that if $\lambda(\cdot)$ is a positive constant function, $\varepsilon > 0$ and $x \in E_\varepsilon$, then $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) = S_E(f(\cdot), \lambda(x), \varepsilon, x) \neq \emptyset$.

Proposition 2 *Suppose that* $\varepsilon > 0$, $x \in E_\varepsilon$ *and* $\inf_X f < \inf_A f$, *for every bounded subset* A *of* X. *Then there exists* $\lambda : X \to \mathbb{R}^{++}$ *such that* $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) = \emptyset$.

Proof Since the set $A := \{x\}$ is bounded, then $\inf_X f < \inf_A f = f(x)$. Therefore, there exists $x_1 \in X$ such that $f(x_1) < f(x)$. If we define

$$\lambda(x) := \frac{2\varepsilon q(x, x_1)}{f(x) - f(x_1)},$$

we have

$$f(x_1) + \frac{\varepsilon}{\lambda(x)} q(x, x_1) < f(x_1) + \frac{2\varepsilon}{\lambda(x)} q(x, x_1) = f(x),$$

and so $x \notin S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$. On the other hand, from $\inf_X f < \inf_{B[x,\lambda(x)]} f$, there exists $x_2 \in X$ such that $f(x_2) < \inf_{B[x,\lambda(x)]} f$. Now, for $z \neq x$, we consider $\lambda(\cdot)$ as follows:

$$\lambda(z) := \begin{cases} \frac{2\varepsilon q(z,x_2)}{\inf_{B[x,\lambda(x)]} f - f(x_2)} & z \in B[x, \lambda(x)] \text{ and } z \neq x \\ 1 & o.w. \end{cases}$$

Hence, $f(x_2) + \frac{\varepsilon}{\lambda(z)} q(z, x_2) < f(z)$ for any $x \neq z \in B[x, \lambda(x)]$, and so $z \notin S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$. Therefore, $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) = \emptyset$. □

Remark 2 (a) It is clear that for every bounded subset A of X, $\inf_X f < \inf_A f$ iff there is no bounded minimizing sequence. Furthermore, there exists a bounded subset A of X such that $\inf_X f = \inf_A f$, whenever $\arg\min f \neq \emptyset$.

(b) If $\arg\min f = \emptyset$ and X is a finite dimensional normed space, then $\inf_X f < \inf_A f$ for every bounded subset A of X. Suppose that $A \subseteq X$ is bounded, then cl$A$ (the closure of A) is compact and so there exists $\bar{x} \in \text{cl}A$ such that

$$f(\bar{x}) = \inf_{\text{cl}A} f.$$

Hence,

$$\inf_X f < f(\bar{x}) = \inf_{\text{cl}A} f \leq \inf_A f.$$

(c) It is well known that the closure of a bounded subset A of a reflexive Banach space X with respect to the weak topology is weakly compact. Hence, by the same proof as that of the part (b), one can obtain that $\inf_X f < \inf_A f$, for every bounded subset A of X, in the case where f is lower semicontinuous with respect to the weak topology and $\arg\min f = \emptyset$.

The following example shows that condition $\inf_X f < \inf_A f$, for every bounded subset A of X, in the above result cannot be omitted.

Example 4 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f(x) :=| x | \quad \forall x \in \mathbb{R}.$$

Then $\inf_{\mathbb{R}} f = \inf_{[-a,a]} f$ for each $a \in \mathbb{R}^+$. We shall show that $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \neq \emptyset$, for every positive function $\lambda(\cdot)$, $\varepsilon > 0$ and for each $x \in E_\varepsilon$.

If $x \in E_\varepsilon$ and $x = 0$, then it is clear that $x \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$. So, we suppose that $x \neq 0$. If $\frac{\varepsilon}{\lambda(x)} < 1$, then we have $| 0 | + \frac{\varepsilon}{\lambda(x)} | x - 0 | \leq | x |$. In this case it is easy to see that $y = 0 \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$. If $\frac{\varepsilon}{\lambda(x)} \geq 1$, then $x \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$.

In order to characterize the non-emptiness of $S_{GE}(f(.), \lambda(.), \varepsilon, x)$, where $x \in E_\varepsilon$, we need the following concept. A quasi-metric space (X, q) is said to have the property (P), if for each $u \neq v$ and for every positive real number r with $q(u, v) < r$, there exists $w \in X$

such that $r = q(u, w) = q(u, v) + q(v, w)$. Now, we show that the quasi-metric induced by an asymmetric norm on a real vector space (see [10]) has the property (P). An asymmetric norm on a real vector space E is a functional $p : E \to [0, +\infty)$ satisfying the following conditions:

(AN1) $p(u) > 0, \quad$ whenever $\; u \neq 0$

(AN2) $p(\alpha u) = \alpha p(u), \quad$ for all $u \in E$ and $\alpha \geq 0$;

(AN3) $p(u + v) \leq p(u) + p(v) \quad$ for all $u, v \in E$.

Proposition 3 *Let p be an asymmetric norm on a real vector space E and q_p be a quasi metric on E defined by $q_p(u, v) = p(v - u)$ for all $u, v \in E$. Then, (E, q_p) has the property (P).*

Proof Let $u, v \in E$, $u \neq v$ and r be a positive real number with $q_p(u, v) < r$. If $w = \frac{r}{q_p(u,v)}(v - u) + u$, then

$$q_p(u, w) = p(w - u) = p(\frac{r}{p(v - u)}(v - u)) = r,$$
$$q_p(v, w) = p(w - v) = p(\frac{r - p(v - u)}{p(v - u)}(v - u)) = r - p(v - u).$$

Hence, $q_p(u, w) = q_p(u, v) + q_p(v, w)$. □

Theorem 4 *Let (X, q) be a left-complete quasi-metric space, then the following statements hold.*

(i) If f is slm, $\varepsilon > 0$ and $x \in E_\varepsilon \cap \arg\max_{B[x,\lambda(x)]}\lambda$, then $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \neq \emptyset$.

(ii) If (X, q) has the property (P) and $x \notin \arg\max_{B(x,\lambda(x))}\lambda$, then for every $\varepsilon > 0$ there exists a slm function f on X such that $x \in E_\varepsilon$ and $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) = \emptyset$.

Proof (i) Suppose that $\varepsilon > 0$, f is slm and $x \in \arg\max_{B[x,\lambda(x)]} \lambda \cap E_\varepsilon$. From Theorem 2, there exists $y \in S_E(f, \lambda(x), \varepsilon, x)$. Hence, conditions (VR1), (VR2) hold. By condition (E3) we have $f(z) + \frac{\varepsilon}{\lambda(x)} q(y, z) > f(y)$, for all $z \neq y$. Since $y \in B[x, \lambda(x)]$, then $\lambda(y) \leq \lambda(x)$, and so

$$f(z) + \frac{\varepsilon}{\lambda(y)} q(y, z) \geq f(z) + \frac{\varepsilon}{\lambda(x)} q(y, z) > f(y), \quad \forall z \neq y.$$

Thus, $y \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$ and $S_E(f(\cdot), \lambda(x), \varepsilon, x) \subset S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$.

(ii) If $x \notin \arg\max_{B(x,\lambda(x))} \lambda$, then there exists $y \in B(x, \lambda(x))$ with $0 < \lambda(x) < \lambda(y)$. For a given $\varepsilon > 0$, we choose $n \in \mathbf{N}$ sufficiently large such that

$$k_1 := \left(\frac{\lambda(x)}{\lambda(y)} + \frac{q(x, y)}{\lambda(x)} - \frac{q(x, y)}{\lambda(y)}\right)\varepsilon + \frac{3}{n} < \varepsilon,$$

and let

$$k_2 := \left(\frac{\lambda(x)}{\lambda(y)} - \frac{q(x, y)}{\lambda(y)}\right)\varepsilon + \frac{2}{n}, \quad \text{and} \quad k_3 := \frac{1}{n}.$$

It is clear that $0 < k_3 < k_2 < k_1 < \varepsilon$. Now, we define

$$f(t) := \begin{cases} k_1(1 + q(x, t)) & t \in B(x, \lambda(x)) \quad \text{and} \quad t \neq y, \\ k_2 & t = y, \\ k_3 \exp(\lambda(x) - q(x, t)) & o.w. \end{cases}$$

It is obvious that f is slm and $x \in E_\varepsilon$. We shall show that $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) = \emptyset$. By the property (P), there exists $w \in X$ such that $\lambda(x) = q(x, w) = q(x, y) + q(y, w)$. Hence, we have $f(x) = k_1$, $f(y) = k_2$ and $f(s) = k_3$, for each $s \in \{t \in X : q(x,t) = \lambda(x)\}$. Form conditions (VR1), (VR2) and $x \in E_\varepsilon$, we have $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \subset B[x, \lambda(x)] \cap \{t \in X : f(t) \leq f(x)\}$. Hence,

$$S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \subset \{x, y\} \cup \{t \in X : q(x,t) = \lambda(x)\}.$$

If $x \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$, condition (VR3)(for $z = y$) gives

$$k_2 + \frac{\varepsilon}{\lambda(x)} q(x,y) \geq k_1.$$

Thus,

$$k_1 - k_2 = \frac{\varepsilon}{\lambda(x)} q(x,y) + \frac{1}{n} \leq \frac{\varepsilon}{\lambda(x)} q(x,y).$$

But this is impossible.
If $y \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$, condition (VR3)(for $z = w$) gives

$$k_3 + \frac{\varepsilon}{\lambda(y)} q(y,w) \geq k_2.$$

Therefore,

$$k_2 - k_3 = \left(\frac{\lambda(x)}{\lambda(y)} - \frac{q(x,y)}{\lambda(y)}\right)\varepsilon + \frac{1}{n} \leq \frac{\varepsilon}{\lambda(y)} q(y,w) = \frac{\varepsilon}{\lambda(y)}(\lambda(x) - q(x,y)),$$

which is absurd.
If $s \in \{t \in X : q(x,t) = \lambda(x)\} \cap S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$, then from condition (VR2) we have

$$\varepsilon = \frac{\varepsilon}{\lambda(x)} q(x,s) \leq f(x) - f(s) = k_1 - k_3 < k_1 < \varepsilon,$$

which is a contradiction. □

In what follows we give two examples of a positive function $\lambda(\cdot)$ such that $x \in \arg\max_{B[x,\lambda(x)]} \lambda$.

Example 5 (I) Suppose that $a, b > 0$ with $a \neq b$ and consider the following quasi-metric on $\mathbb{R}^+$.

$$q(x,y) := \begin{cases} a(y-x) & \text{if } y \geq x, \\ b(x-y) & \text{if } y < x. \end{cases}$$

Then, the set $B[x, \lambda(x)] = \{y \in \mathbb{R}^+ : q(x,y) \leq \lambda(x)\}$ is the closed interval $[x - \frac{\lambda(x)}{b},\ x + \frac{\lambda(x)}{a}]$. Suppose that $\lambda : \mathbb{R}^+ \to \mathbb{R}^{++}$ is defined as follows:

$$\lambda(x) := \begin{cases} 2- \mid x - 1 \mid & x \in [0,2], \\ 1 & o.w. \end{cases}$$

Since $\lambda(1) \geq \lambda(x)$ for all $x \in \mathbb{R}^+$, then $1 \in \arg\max_{B[1,\lambda(1)]} \lambda$.
(II) Let p be an asymmetric norm on a real vector space E, and let $\lambda : E \to \mathbb{R}^{++}$ be defined as follows:

$$\lambda(x) := \begin{cases} 2 - p(x) & p(x) < 2, \\ 1 & p(x) \geq 2. \end{cases}$$

Then $\lambda(0) = 2 \geq \lambda(x)$ for all $x \in E$, and so $0 \in \arg\max_{B[0,\lambda(0)]} \lambda$.

In the sequel, from the GEVP we deduce two general forms of Caristi's theorem [7] and Takahashi's theorem [29]. We also show that these results are equivalent.

Theorem 5 (Caristi) *Suppose that* $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \neq \emptyset$ *for every* $\varepsilon > 0$ *and* $x \in E_\varepsilon$, *then the following statements hold.*

(i) If $T : X \rightrightarrows X$ *is a set-valued mapping with nonempty values satisfying*

$$\frac{1}{\lambda(u)} q(u, v) \leq f(u) - f(v), \quad \forall u \in X, \quad \forall v \in T(u), \tag{3}$$

then there exists $w \in X$ *such that* $T(w) = \{w\}$.

(ii) If $T : X \rightrightarrows X$ *is a set-valued mapping with nonempty values satisfying*

$$\forall u \in X, \quad \exists v \in T(u) \quad \text{such that} \quad \frac{1}{\lambda(u)} q(u, v) \leq f(u) - f(v), \tag{4}$$

then T *has a fixed point in* X.

Proof Let $\varepsilon = 1$ and $y \in S_{GE}(f(\cdot), \lambda(\cdot), 1, x)$. Then, by condition (VR3) we have

$$f(y) - f(z) < \frac{1}{\lambda(y)} q(y, z) \quad \forall z \neq y. \tag{5}$$

(i) We show that $T(y) = \{y\}$. Assume on the contrary that there exists $y \neq z \in T(y)$. Then by (3), we get

$$\frac{1}{\lambda(y)} q(y, z) \leq f(y) - f(z),$$

which contradicts with (5).
(ii) We show that $y \in T(y)$. Assume on the contrary that $y \notin T(y)$. Hence, from (4) there exists $y \neq z \in T(y)$ such that

$$\frac{1}{\lambda(y)} q(y, z) \leq f(y) - f(z),$$

which contradicts (5). □

Theorem 6 (Takahashi) *Assume that* $S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x) \neq \emptyset$ *and the following condition holds.*

$$\forall x \in X \ \text{ with } \ \inf_X f < f(x) \ \exists u \neq x \ : \ f(u) + \frac{\varepsilon}{\lambda(x)} q(x, u) \leq f(x). \tag{6}$$

Then $\arg\min f \neq \emptyset$.

Proof Suppose that $y \in S_{GE}(f(\cdot), \lambda(\cdot), \varepsilon, x)$, then by (VR3) we have

$$f(y) - f(z) < \frac{\varepsilon}{\lambda(y)} q(y, z) \quad \forall z \neq y \tag{7}$$

We show that $y \in \arg\min f$. If $y \notin \arg\min f$, then from (6) there exists $u \neq y$ such that

$$f(u) + \frac{\varepsilon}{\lambda(y)} q(y, u) \leq f(y). \tag{8}$$

From Eqs. (7) and (8), we have $f(y) < f(y)$ which is a contradiction. □

Now, we show that Caristi's theorem and Takahashi' theorem are equivalent.

Theorem 7 *Part (i) of Theorem 5 and the following statements are equivalent.*

(EV) There exists $y \in X$ *such* $f(y) < f(z) + \frac{1}{\lambda(y)} q(y, z), \ \ \forall z \neq y$.

(TA) If for every $x \in X$ *with* $\inf_X f < f(x)$ *there exists* $u \neq x$ *such that*

$$f(u) + \frac{1}{\lambda(x)} q(x, u) \leq f(x),$$

then $\arg\min f \neq \emptyset$.

Proof By Theorems 5 and 6, condition (EV) imply Theorem 5(i) and (TA), respectively.

(TA)⇒(EV). If (EV) does not hold, then

$$\forall y \in X \quad \exists y \neq z \in X \quad \text{suchthat} \ \ f(z) + \frac{1}{\lambda(y)} q(y, z) \leq f(y). \tag{9}$$

Therefore, by condition (TA) there is a $\bar{y} \in X$ such that $f(\bar{y}) = \inf_X f$. On the other hand, by (9) there exists $\bar{y} \neq \bar{z} \in X$ such that

$$f(\bar{z}) + \frac{1}{\lambda(\bar{y})} q(\bar{y}, \bar{z}) \leq f(\bar{y}).$$

Hence,

$$\frac{1}{\lambda(\bar{y})} q(\bar{y}, \bar{z}) \leq f(\bar{y}) - f(\bar{z}) \leq 0.$$

Thus, $q(\bar{y}, \bar{z}) = 0$, and so $\bar{y} = \bar{z}$ which is a contradiction.

Theorem 5(i)⇒ (EV). Let $T : X \rightrightarrows X$ be defined as follows:

$$T(u) = \{v \in X : f(v) + \frac{1}{\lambda(u)} q(u, v) \leq f(u)\}.$$

Hence, $\frac{1}{\lambda(u)} q(u, v) \leq f(u) - f(v)$ for any $u \in X$ and $v \in T(u)$. Part (i) of Theorem 5 implies that there exists $y \in X$ such that $T(y) = \{y\}$. Therefore, if $z \in X$ and $z \neq y$, then $z \notin T(y)$, and so (EV) holds. □

3 Application: how the size of a monopoly changes in a changing environment

3.1 The VR approach: a short summary

To save space we present very briefly the VR approach [23–28] on a leading example relative to the limit size of a monopoly. This reminder is mandatory if we want the reader to understand the interest of the mathematical part of the paper.

The need for change The following example is borrowed from [23]. It is one of the leading models that drive the VR approach in economics and management sciences. Consider the simplest model of an organization, that is, a monopoly that produces, within a given period, $\psi(x) = x \in \mathbb{R}^+$ units of a final good, using $x \in X = \mathbb{R}^+$ workers. This monopoly sells $x \in \mathbb{R}^+$ units of a final good at the unit price $p = p_s(x) = a - bx \geq 0$, $a, b > 0$ and pays a wage $w > a$ to each worker. Thus, production costs are wage costs wx. Let s be the environment of the monopoly in the current period (see below its content). Then, the per period profit of this firm is $g_s(x) = p(x)x - wx$. That is, $g_s(x) = (a - w)x - bx^2$. The size of this monopoly is, either the number of workers x it uses or, how much units x of the final good it produces and sells. Let $g_s^* = \sup\{g_s(y) : y \in X\} < +\infty$ be the highest profit the entrepreneur can hope to satisfy within a period. Then, the optimal size x^* of this monopoly solves the equation $g_s(x^*) = g_s^*$. This example gives its ideal size $x_s^* = (a - w)/2b$ and the ideal profit level $g_s^* = (a - w)^2/4b$. Starting from the status quo x, the discrepancy $f_s(x) = g_s^* - g_s(x) \geq 0$ models the unrealized profit of the monopoly when its size is x at the status quo. When it exists, an unrealized profit $f_s(x) > 0$ in the previous period generates an unsatisfaction/frustration feeling that pushes the monopoly, in the current period, to change from having employed x workers in the previous period to employ $y \neq x$ workers in the current period. Thus, moving means, either, i) to change, i.e., to hire $y - x > 0$ workers, or to fire $x - y > 0$ workers or, ii) to stay, stopping to hire or to fire workers if $y = x$.

The environment Let $s = (x, \lambda(x), \varepsilon, a, b, w, \alpha_+, \alpha_\searrow)$ be the environment of the monopoly in the current period. In this simple example it includes the status quo x, the size $\lambda(x) > 0$ of a pool of workers that the monopoly can hire or fire, $\varepsilon > 0$, the level a and the elasticity of the inverse demand b, the per worker wage w, and the unit cost to fire or to hire a worker $\alpha_+ > 0, \alpha_\searrow > 0$. These unit costs have material, financial, psychological and social aspects. Notice that ε is a upper bound of the unrealized profit: $0 \leq f_s(x) \leq \varepsilon$.

Advantages to move (change rather than stay) The entrepreneur will have some advantages to move from x to y, i.e., $A_s(y/x) = g_s(y) - g_s(x) = f_s(x) - f_s(y) > 0$, if he can improve his profit from $g_s(x)$ to $g_s(y) > g_s(x)$ or decrease his frustration feelings from $f_s(x)$ to $f_s(y) < f_s(x)$. In the example, $A_s(y/x) = g_s(y) - g_s(x) = (a - w)(y - x) - b(y^2 - x^2)$.

Inconveniences to move from the status quo To save space, we limit the complex definition of inconveniences to move $I_s(y/x)$ to the monopoly example. In this case they represent costs to hire $y - x \geq 0$ workers $I_s^+(y/x) = \alpha_+(y - x)$, if $y \geq x$, or costs to fire $x - y > 0$ workers $I_s^\searrow(y/x) = \alpha_\searrow(x - y)$, if $y < x$. For the general case; see [23]. Thus, we will suppose that limited knowledge relative to the inverse demand function $p(x) = a - bx$, and limited time and resources forbid the entrepreneur to reach the optimal size of the firm x^* within the current period.

Motivation and resistance to move Motivation to move (change rather than stay) $M_s(y/x) = U_s[A_s(y/x)]$ is the utility of advantages to move. Resistance to move $R_s(y/x) = D_s[I_s(y/x)]$ is the disutility of inconveniences to move. The utility and disutility functions $U_s[.] : X \longmapsto R_+$, $D_s[.] : X \longmapsto R_+$ are, i) zero at zero, i.e., $U_s[0] = 0 = D_s[0]$ and, ii) strictly increasing. The balance $B_s(y/x) = M_s(y/x) - \xi_s R_s(y/x)$ between motivation $M_s(y/x)$ and resistance to move $R_s(y/x)$ weights the difference between them. The weight $\xi_s > 0$ models how much an individual gives importance to resistance to move compared to motivation. This is a way to model the status quo bias.

Worthwhile moves A move $m = x \curvearrowright y$ is worthwhile if the balance $B_s(y/x) = M_s(y/x) - \xi_s R_s(y/x) \geq 0$ between motivation and resistance to move (change rather than stay) is non negative. That is, if motivation to move is high enough relative to resistance to move, i.e., if $M_s(y/x) \geq \xi_s R_s(y/x)$.

Desires A position $x^* \in X$ is a desire (desired end) if $A_s(z/x^*) \leq 0$ for all $z \in X$. In the example the equilibrium condition $A_s(z/x^*) = g_s(z) - g_s(x^*) \leq 0$ for all $z \in X$ defines a maximum of the profit function $g_s(.)$.

Stationary traps The position $x_* \in X$ is a (strict) stationary trap if, starting from the status quo x_*, there is no way to find a worthwhile change $x_* \curvearrowright z \neq x_*$. That is, if $B_s(z/x_*) = M_s(z/x_*) - \xi_s R_s(z/x_*) < 0$, i.e., if $M_s(z/x_*) < \xi_s R_s(z/x_*)$ for all $z \in X \setminus \{x_*\}$. The equilibrium condition $B_s(z/x_*) < 0$ for all $z \in X$, $z \neq x_*$, defines a (strict) maximum x_* of the worthwhile balance function $B_s(./x_*)$ on X.

Resistance to change constraint It looks like $0 \leq R_s(y/x) \leq \overline{R}_s < +\infty$. It comes from a lot of different reasons; for example resource constraints. Given that $R_s(y/x) = D_s[I_s(y/x)]$

and if $D_s\,[.]$ is invertible, it is equivalent to the resource constraint $I_s(y/x) \leq \lambda_s$. In the monopoly example $\lambda_s = \lambda(x) > 0$ models the size of a pool of workers that the entrepreneur can hire or fire.

Variational traps in a changing environment The VR approach [23] makes a clear distinction between a proximal dynamic, which runs over a small number of periods, and a distal (longuer) dynamic. More precisely, it models a proximal dynamic as a three periods model, including the previous period, the current period (first effective period) and the next period (second effective period). One of its main discovery has been to show how the EVP is a specific instance of a VR proximal dynamic, where an entity (for example, a monopoly),

i) period 1: starts moving, making a worthwhile move $x \curvearrowright y = x_*$ in the initial environment s:

$$M_s(y/x) \geq \xi_s R_s(y/x) \quad \text{with } \; 0 \leq R_s(y/x) \leq \lambda(x);$$

ii) period 2: stops moving, having reached a stationary trap, in a new (or the same) environment s':

$$M_{s'}(z/y) < \xi_{s'} R_{s'}(z/y) \;\; \text{forall } \; z \neq y = x_*.$$

This worthwhile proximal dynamic defines a variational trap, worthwhile to reach (period 1), but not worthwhile to leave (period 2).

3.2 Interpretations of the results: when to stop changing when the environment changes

Behavioral results This paper shows the existence of traps in a changing environment in a specific context. This context considers resources (a pool of workers) as changing from $\lambda_s = \lambda(x)$ to $\lambda_{s'} = \lambda(y)$, when the size of the monopoly changes from x to y. That is,

A) **In the first period, the monopoly starts moving**, **making a worthwhile move**. The environment of the monopoly in the first period is $s = (x, \lambda(x), \varepsilon, a, b, w, \alpha_+, \alpha_\searrow)$. To be concrete, consider the following environment in the first period: $s = (x, \lambda(x), \varepsilon > 0, a = 2, b = 1/2, w = 1, \alpha_+ = 1, \alpha_\searrow = 2)$. Then, in the previous period, the profit function at x was $g_s(x) = g(x) = x - (1/2)x^2$, the optimal size of the monopoly was $x_s^* = x^* = 1$, the highest profit was $g_s^* = g^* = 1/2$ and unrealized profits at x has been $f_s(x) = f(x) = g^* - g(x)$.

In this first period advantages to move are $A_s(y/x) = A(y/x) = g(y) - g(x) = f(x) - f(y) = (y - x) - (1/2)(y^2 - x^2)$. Inconveniences to move are $I_s(y/x) = I(y/x)$, where $I(y/x) = y - x$ if $y \geq x$ and $I(y/x) = 2(x - y)$, if $y < x$. Then, $q : X \times X \ni (x, y) \longmapsto q(x, y) = I(y/x) \in \mathbb{R}^+$ is a quasi-metric. The resource constraint in the first period is $I(y/x) = q(x, y) \leq \lambda(x)$. We consider a linear formulation of motivation and resistance to move, $M_s = U_s\,[A_s] = A_s$ and $R_s = D_s\,[I_s] = \delta_s I_s$ where $\delta_s = \frac{\varepsilon}{\lambda(x)} > 0$ and $\varepsilon > 0$. Then, a worthwhile move $x \curvearrowright y$ from the previous period to the current (first) period satisfies condition(VR1) $g(y) - g(x) = f(x) - f(y) \geq \frac{\varepsilon}{\lambda(x)} q(x, y)$. Notice that the formulation $R_s = D_s\,[I_s] = \delta_s I_s$ with $\delta_s = \frac{\varepsilon}{\lambda(x)} > 0$ means that when the pool of workers $\lambda(x)$ increases, resistance to move R_s decreases because it is easier to hire or to fire workers.

B) **In the second period, the monopoly stops moving because it reaches a stationary trap.** Condition

$$\text{(VR3)} \;\; g(z) - g(y) = f(y) - f(z) < \frac{\varepsilon}{\lambda(y)} q(y, z) \;\; \text{for all } \; z \neq y,$$

means that, in the second period, the monopoly has reached a variational trap, worthwhile to reach, but not worthwhile to leave. At the same time the environment moved from $s = (x, \lambda(x), \varepsilon, a, b, w, \alpha_+, \alpha_\searrow)$ in the current period (first period) to $s' = (y, \lambda(y), \varepsilon, a, b, w, \alpha_+, \alpha_\searrow)$ in the next period (second period). In this context, the resource constraint $q(y, z) \leq \lambda(y)$ plays no effective role in the second period.

Interpretation of Proposition 2 We remind that, in the monopoly example, i) $g(x)$ is the profit of the monopolist when he produces x units of the final good, that is, when he employs x workers, ii) $g^* = g^*_X = \sup\{g(z) : z \in X\} < +\infty$ is his aspiration level, that is, the highest profit he can hope to get and, iii) $f(x) = g^* - g(x) \geq 0$ is his unrealized profit at the status quo x. Then, $\inf_X f = g^* - g^* = 0$. Let $g^*_A = \sup\{g(z) : z \in X\} \leq g^*_X$ be the highest profit the monopolist can hope to reach if he produces a bounded quantity $x \in A \subset X = \mathbb{R}^+$ of the final good (a mild hypothesis !). Then, condition $\inf_X f < \inf_A f$ for each bounded subset A of X is equivalent to $g^*_A < g^*_X$ for each bounded set $A \subset X$. Thus, Proposition 2 means that a monopolist will not reach his idealistic size g^*_X as long as he produces a bounded number of units of the final good. This is not the case in our example where the optimal size $0 < x^* = (a - w)/2b < +\infty$ of the monopoly is finite and exists.

Interpretation of Theorem 4 Condition

$$x \in \arg\max_{B[x,\lambda(x)]} \lambda = \max\{\lambda(y) : y \in X, q(x, y) \leq \lambda(x)\}, \tag{10}$$

means that, away from the status quo, for each $y \in B[x, \lambda(x)]$ that satisfies the resource constraint $q(x, y) \leq \lambda(x)$ the resource constraint at y will be more severe than at the status quo x, that is, $\lambda(y) \leq \lambda(x)$. Theorem 4 shows that under this condition (and the given other ones), starting from the status quo x, a variational trap will exist. This condition is beautiful and intuitive, because it supposes that resistance to change will be not lower at any $y \in B[x, \lambda(x)]$ than at x. Thus, this is in favor of the existence of a trap at some $y \in B[x, \lambda(x)]$.

The behavioral interpretations of the Caristi theorem [7] and the Takahashi theorem [29] follow easily. But their interpretations require too much space and other examples. They will be examined elsewhere.

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