Journal of Global Optimization (2021) 81:323–350
https://doi.org/10.1007/s10898-021-01025-y

A variational approach to the alternating projections method

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Received: 2 November 2020 / Accepted: 10 April 2021 / Published online: 23 April 2021

Abstract
The 2-sets convex feasibility problem aims at finding a point in the nonempty intersection of two closed convex sets A and B in a Hilbert space H. The method of alternating projections is the simplest iterative procedure for finding a solution and it goes back to von Neumann. In the present paper, we study some stability properties for this method in the following sense: we consider two sequences of closed convex sets $\{A_n\}$ and $\{B_n\}$, each of them converging, with respect to the Attouch-Wets variational convergence, respectively, to A and B. Given a starting point a_0, we consider the sequences of points obtained by projecting on the "perturbed" sets, i.e., the sequences $\{a_n\}$ and $\{b_n\}$ given by $b_n = P_{B_n}(a_{n-1})$ and $a_n = P_{A_n}(b_n)$. Under appropriate geometrical and topological assumptions on the intersection of the limit sets, we ensure that the sequences $\{a_n\}$ and $\{b_n\}$ converge in norm to a point in the intersection of A and B. In particular, we consider both when the intersection $A \cap B$ reduces to a singleton and when the interior of $A \cap B$ is nonempty. Finally we consider the case in which the limit sets A and B are subspaces.

Keywords Convex feasibility problem · Stability · Set-convergence · Alternating projections method

Mathematics Subject Classification Primary: 47J25; Secondary: 90C25 · 90C48

1 Introduction

The 2-sets convex feasibility problem is the classical problem of finding a point in the nonempty intersection of two closed and convex sets A and B in a Hilbert space H (see [6, Section 4.5] for some basic results on this subject). Many efforts have been devoted to the study of algorithmic procedures to solve convex feasibility problems, both from a theoretical and from a computational point of view (see, e.g., [1,4,5,9,19] and the references therein). The method of alternating projections is the simplest iterative procedure for finding a solution

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and it goes back to von Neumann [27]. Let us denote by P_A and P_B the projections onto the sets A and B, respectively, and introduce the following definition.

Definition 1.1 Given a starting point $c_0 \in H$, the *alternating projections sequences* $\{c_n\}$ and $\{d_n\}$ with starting point a_0, are defined inductively by

$$d_n := P_B(c_{n-1}) \quad \text{and} \quad c_n := P_A(d_n) \qquad (n \in \mathbb{N}).$$

In the case in which the sequences $\{c_n\}$ and $\{d_n\}$ converge in norm to a point in the intersection of A and B, we say that the method of alternating projections converges.

Many concrete problems in applications can be formulated as a convex feasibility problem. As typical examples, we mention solution of convex inequalities, partial differential equations, minimization of convex nonsmooth functions, medical imaging, computerized tomography and image reconstruction. For some details and other applications see, e.g., [1] and the references therein.

In the present paper we investigate some "stability" properties of the alternating projections method. We deem that two motivations to develop this study are especially relevant. First, often the data in concrete applications are affected by some uncertainties, hence the "stability" of solutions of a convex feasibility problem may be a useful property in the development of computational method. On the other hand, the other (and the main) motivation of the paper is theoretical and consists in proving that some conditions ensuring the convergence of the classical alternating projection method (and well-known in literature, see e.g. [1,5]) also imply the "stability" of the same convex feasibility problem in the sense described below.

Let us suppose that $\{A_n\}$ and $\{B_n\}$ are two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the Attouch-Wets variational convergence (see Definition 2.2) and let us introduce the definition of *perturbed alternating projections sequences*.

Definition 1.2 Given $a_0 \in H$, the *perturbed alternating projections sequences* $\{a_n\}$ and $\{b_n\}$, w.r.t. $\{A_n\}$ and $\{B_n\}$ and with starting point a_0, are defined inductively by

$$b_n := P_{B_n}(a_{n-1}) \quad \text{and} \quad a_n := P_{A_n}(b_n) \qquad (n \in \mathbb{N}).$$

In the sequel, the notations $\{a_n\}$ and $\{b_n\}$ always refer to the perturbed alternating projections sequences, whereas $\{c_n\}$ and $\{d_n\}$ refer to the standard alternating projections sequences.

Our aim is to find some conditions on the limit sets A and B that guarantee, for each choice of the sequences $\{A_n\}$ and $\{B_n\}$ and for each choice of the starting point a_0, the convergence in norm of the corresponding perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$. If this is the case, we say that the couple (A, B) is *stable*.

The results reported in this paper can be seen as a continuation of the research considered in [11]. However, compared with the notion of stability studied in that paper, the approach developed here seems to be more interesting also from a computational point of view since it does not require to find an exact solution of the "perturbed problems" (i.e. the problems given by the sets A_n and B_n) but only to consider projections on the "perturbed" sets A_n and B_n. Moreover, the techniques used in the proofs are completely different from those of [11].

Clearly, in order that the couple (A, B) is stable, it is necessary that the alternating projections sequences $\{c_n\}$ and $\{d_n\}$ converge in norm (indeed, we can consider the particular case in which the sequences of sets $\{A_n\}$ and $\{B_n\}$ are given by $A_n = A$ and $B_n = B$, whenever $n \in \mathbb{N}$). Since, in general, this is not the case (see [19,24]), we shall restrict our attention to those situations in which the method of alternating projections converges. After some preliminaries, contained in Sect. 2, we consider, in Sects. 3, 4 and 5, respectively, the following three cases:

(i) A and B are separated by a strongly exposing functional for the set A, i.e., there exist $x_0 \in A \cap B$ and a linear continuous functional f such that $\inf f(A) = f(x_0) = \sup f(B)$ and such that $(-f)$ strongly exposes A at x_0 (see Definition 2.6);
(ii) the intersection between A and B has nonempty interior;
(iii) A and B are closed subspaces.

The structure of the paper is based on the study of the three classes of problems presented above. Let us point out that these three cases are not exhaustive, i.e., there are situation, not included in (i)–(iii), in which the alternating projections sequences converge in norm (as a typical example see [2, Section 5]).

In Sect. 3, we deal with (i). First, it is useful to recall why if (i) is satisfied then the method of alternating projections for the couple (A, B) converges: by [6, Lemma 4.5.11] or by [21, Theorem 1.4], the alternating projections sequences $\{c_n\}$ and $\{d_n\}$ satisfy $\|c_n - d_n\| \to 0$. Then it is easy to verify that $f(c_n), f(d_n) \to f(x_0)$ and hence, since f strongly exposes A at x_0, we have that $c_n, d_n \to x_0$ in norm. Similar assumption on the limit sets has been considered by the authors and E. Molho in the recent paper [11], in which they proved, among other things, that if (i) is satisfied and if $x_n \in A_n,\ y_n \in B_n$ are such that $\|x_n - y_n\|$ coincides with the distance between A_n and B_n then $x_n, y_n \to x_0$ in norm (see the proof of [11, Theorem 4.5]). In Sect. 3 of the present paper, we prove that if A and B are separated by a strongly exposing functional f for the set A then, for each choice of sequences $\{A_n\},\ \{B_n\}$ and starting point a_0, the corresponding perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$ converge in norm to x_0 (cf. Theorem 3.4 below). In this case, our approach is essentially based on suitable approximations of the sets A_n and B_n by convex and non-convex cones, respectively.

In Sect. 4, we investigate to what extent it is possible to guarantee convergence of the perturbed alternating projections in the case $A \cap B$ is nonempty but does not reduce to a singleton. Example 4.4 show that, in general, even in the finite-dimensional setting and even if $A \cap B$ is bounded, the couple (A, B) may be not stable. On the other hand, Theorem 4.2 ensures that the couple (A, B) is stable whenever $\operatorname{int}(A \cap B) \neq \emptyset$. We point out that boundedness of $A \cap B$ is not required. Moreover, we apply the results of this section to study a typical mathematical programming problem. Indeed, we investigate the convergence of perturbed alternating projections for the inequality constraints problem.

Finally, Sect. 5 is devoted to case (iii) where A and B are closed subspaces. The convex feasibility problem where A and B are subspaces is the original problem studied by von Neumann. In his, now classical, theorem (see [27]), he proved that the alternating projections sequences $\{c_n\}$ and $\{d_n\}$ converge in norm to $P_{A\cap B}(a_0)$. This theorem was rediscovered by several authors and many alternative proofs were provided (see, e.g., [21,22] and the references therein). In Sect. 5, we study the problem of convergence of perturbed alternating projections sequences in the case in which A and B are subspaces. Example 5.1 below shows that even in the finite-dimensional setting it is conceivable that the perturbed projections sequences are unbounded in the case $A \cap B \neq \{0\}$. For this, in Sect. 5, we focus on the situation in which A and B are closed subspaces such that $A \cap B = \{0\}$. It turns out that if $A + B$ is a closed subspace then the couple (A, B) is stable (Theorem 5.2). On the other hand, in Theorem 5.9, we provide a couple (A, B) of closed subspaces such that $A \cap B = \{0\}$ and such that there exist sequences of sets $\{A_n\},\ \{B_n\}$ and starting point a_0 such that the corresponding perturbed projections sequences are unbounded. Our construction is based on the example, contained in [16], of two subspaces of a Hilbert space with non-closed sum such that the convergence of the corresponding alternating projections method is not geometric (for the definition of geometric convergence see [16], see also [26] for some results concerning the convergence rate of the alternating projection algorithm for the case of n subspaces).

2 Notations and preliminaries

In this section we introduce some preliminary notions and related results, valid in real normed spaces. Let X be a real normed space with the topological dual X^*. We denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. For $x, y \in X$, $[x, y]$ denotes the closed segment in X with endpoints x and y. For a subset K of X, $\alpha > 0$, and a functional $f \in S_{X^*}$ bounded on K, let

$$S(f, \alpha, K) := \{x \in K;\ f(x) \geq \sup f(K) - \alpha\}$$

be the closed slice of K given by α and f.

For $f \in S_{X^*}$ and $\alpha \in (0, 1)$, we denote

$$C(f, \alpha) := \{x \in X;\ f(x) \geq \alpha\|x\|\},\ \ V(f, \alpha) := \{x \in X;\ f(x) \leq \alpha\|x\|\}.$$

It is easy to see that $C(f, \alpha)$ and $V(f, \alpha)$ are nonempty closed cones and that $C(f, \alpha)$ is convex.

For a subset A of X, we denote by $\mathrm{int}\,(A)$, ∂A, $\mathrm{conv}\,(A)$ and $\overline{\mathrm{conv}}\,(A)$ the interior, the boundary, the convex hull and the closed convex hull of A, respectively. We denote by

$$\mathrm{diam}(A) := \sup_{x,y\in A} \|x - y\|,$$

the (possibly infinite) diameter of A. For $x \in X$, let

$$\mathrm{dist}(x, A) := \inf_{a\in A} \|a - x\|.$$

Moreover, given A, B nonempty subsets of X, we denote by $\mathrm{dist}(A, B)$ the usual "distance" between A and B, that is,

$$\mathrm{dist}(A, B) := \inf_{a\in A} \mathrm{dist}(a, B).$$

Let us now introduce some definitions and basic properties concerning convergence of sets. By $\mathrm{c}(X)$ we denote the family of all nonempty closed subsets of X. Let us introduce the (extended) Hausdorff metric h on $\mathrm{c}(X)$. For $A, B \in \mathrm{c}(X)$, we define the excess of A over B as

$$e(A, B) := \sup_{a\in A} \mathrm{dist}(a, B).$$

Moreover, if $A \neq \emptyset$ and $B = \emptyset$ we put $e(A, B) = \infty$, if $A = \emptyset$ we put $e(A, B) = 0$. For $A, B \in \mathrm{c}(X)$, we define

$$h(A, B) := \max\big\{e(A, B), e(B, A)\big\}.$$

Definition 2.1 A sequence $\{A_j\}$ in $\mathrm{c}(X)$ is said to Hausdorff converge to $A \in \mathrm{c}(X)$ if

$$\lim_j h(A_j, A) = 0.$$

Next we recall the definition of the so called Attouch-Wets convergence (see, e.g., [23, Definition 8.2.13]), which can be seen as a localization of the Hausdorff convergence. If $N \in \mathbb{N}$ and $A, C \in \mathrm{c}(X)$, define

$$e_N(A, C) := e(\widetilde{A}, C) \in [0, \infty),$$

where $\widetilde{A} = A \cap NB_X$, and define

$$h_N(A, C) := \max\{e_N(A, C), e_N(C, A)\}.$$

Definition 2.2 A sequence $\{A_j\}$ in $\mathrm{c}(X)$ is said to Attouch-Wets converge to $A \in \mathrm{c}(X)$ if, for each $N \in \mathbb{N}$,

$$\lim_j h_N(A_j, A) = 0.$$

In the sequel, we shall use the following two results several times.

Theorem 2.3 *(see, e.g., [23, Theorem 8.2.14]) The sequence of sets $\{A_n\}$ Attouch-Wets converges to A iff*

$$\sup_{\|x\|\leq N} |\mathrm{dist}(x, A_n) - \mathrm{dist}(x, A)| \to 0 \quad (n \to \infty),$$

whenever $N \in \mathbb{N}$.

As an easy consequence of the theorem above we have the following fact.

Fact 2.4 *Let A be a nonempty closed convex set in X. Suppose that $\{A_n\}$ is a sequence of closed convex sets such that $A_n \to A$ for the Attouch-Wets convergence. Then, if $\{a_n\}$ is a bounded sequence in X such that $a_n \in A_n$ ($n \in \mathbb{N}$), we have that* $\mathrm{dist}(a_n, A) \to 0$.

Proof Since $\{a_n\}$ is bounded, there exists $N \in \mathbb{N}$ such that $\|a_n\| \leq N$, whenever $n \in \mathbb{N}$. By Theorem 2.3, we have

$$\mathrm{dist}(a_n, A) \leq \sup_{\|x\|\leq N} |\mathrm{dist}(x, A_n) - \mathrm{dist}(x, A)| \to 0 \quad (n \to \infty),$$

and the proof is concluded. □

The Attouch-Wets convergence is widely used to study approximation and optimization problems and it is a natural variation of the Hausdorff convergence. Moreover, Theorem 2.3 shows that it coincides with uniform convergence on bounded sets of the functions $\mathrm{dist}(\cdot, A_n)$ ($n \in \mathbb{N}$). The use of such convergence may be more appropriate than the Hausdorff convergence especially when we work with unbounded sets. To see this, consider the following example: take closed hyperplanes $A_n, A \subset X$ being the kernels of functionals $x_n^*, x^* \in S_{X^*}$ ($n \in \mathbb{N}$), respectively. It is easy to see that $\{A_n\}$ Hausdorff converges to A iff eventually $A_n = A$. On the other hand, if $x_n^* \to x^*$ in norm then $\{A_n\}$ Attouch-Wets converges to A. For more details on the Attouch-Wets convergence see [23] and the references therein.

Let us recall that, given a normed space Z, the *modulus of convexity of* Z is the function $\delta_Z : [0, 2] \to [0, 1]$ defined by

$$\delta_Z(\eta) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|; x, y \in B_X, \|x - y\| \geq \eta\right\}.$$

It is clear that $\delta_Z(\eta_1) \leq \delta_Z(\eta_2)$, whenever $0 \leq \eta_1 \leq \eta_2 \leq 2$. Moreover, if $r > 0$ and $\eta \in [0, 2r]$, we have

$$r\delta_Z\left(\tfrac{\eta}{r}\right) = \inf\left\{r - \left\|\frac{x+y}{2}\right\|; x, y \in rB_X, \|x - y\| \geq \eta\right\}.$$

In particular, if $r, M > 0$ and $x, y \in rB_X$ are such that $\|x - y\| \geq M$ then we have

$$\left\|\frac{c_1 + c_2}{2}\right\| \leq r\left[1 - \delta_Z\left(\tfrac{M}{r}\right)\right]. \tag{1}$$

We say that Z is *uniformly rotund* if $\delta_Z(\eta) > 0$, whenever $\eta \in (0, 2]$. It is well known that Hilbert spaces are uniformly rotund.

Lemma 2.5 *Let Z be a uniformly rotund normed space. Let $K, M > 0$, then there exists $\varepsilon' > 0$ such that, if $\rho \in [0, K]$ and if C is a convex set such that $\rho - \varepsilon' \leq \|c\| \leq \rho + \varepsilon'$, whenever $c \in C$, then* $\operatorname{diam}(C) \leq M$.

Proof Suppose without any loss of generality that $M \leq 2$. We claim that any $\varepsilon' \in (0, 1)$ such that $\varepsilon'\left[2 - \delta_Z\left(\frac{M}{K+1}\right)\right] < \frac{M}{4}\delta_Z\left(\frac{M}{K+1}\right)$ works. Let $\rho \in [0, K]$ and let C be a convex set such that $\rho - \varepsilon' \leq \|c\| \leq \rho + \varepsilon'$, whenever $c \in C$. First, observe that, since δ_Z assumes values in [0, 1], we have $\varepsilon' < \frac{M}{4}$. Hence, if $\rho < \frac{M}{4}$, we have

$$\operatorname{diam}(C) \leq 2(\rho + \varepsilon') \leq M.$$

Now, suppose that $\rho \geq \frac{M}{4}$ and let us prove that $\operatorname{diam}(C) \leq M$. Suppose on the contrary that there exist $c_1, c_2 \in C$ satisfying $\|c_1 - c_2\| > M$. Put $r := \rho + \varepsilon'$. By (1) and since $\frac{c_1+c_2}{2} \in C$, we have

$$\rho - \varepsilon' \leq \left\|\frac{c_1 + c_2}{2}\right\| \leq r\left[1 - \delta_Z\left(\tfrac{M}{r}\right)\right] \leq r\left[1 - \delta_Z\left(\tfrac{M}{K+1}\right)\right].$$

Therefore, we have $\varepsilon'\left[2 - \delta_Z\left(\frac{M}{K+1}\right)\right] \geq \rho\delta_Z\left(\frac{M}{K+1}\right) \geq \frac{M}{4}\delta_Z\left(\frac{M}{K+1}\right)$, a contradiction. □

Let us recall the notions of strongly exposed point and strongly exposing functional. This notions, and the corresponding dual versions (see, e.g., [12, Definition 6.2]), play an important role in the theory of Banach spaces.

Definition 2.6 (see, e.g., [15, Definition 7.10]) Let A be a nonempty subset of a normed space X, $a \in A$, and $f \in X^* \setminus \{0\}$. We say that f *strongly exposes* A at a if f is a support functional for A at a $\big($i.e., $f(a) = \sup f(A)\big)$ and $x_n \to a$, whenever $\{x_n\}$ is a sequence in A such that $\lim_n f(x_n) = \sup f(A)$. If this is the case, we say that a is a *strongly exposed* point of A.

Let us recall the geometrical meaning of the notion of strongly exposing: $f \in S_{X^*}$ strongly exposes A at a iff $f(a) = \sup f(A)$ and

$$\operatorname{diam}\big(S(f, \alpha, A)\big) \to 0 \text{ as } \alpha \to 0^+;$$

that is, $f \in S_{X^*}$ strongly exposes A at a iff f is a support functional for A at a and the diameter of the slice of A given by the functional f at level α goes to 0 as α goes to 0.
Let us recall that a *body* in X is a closed convex set in X with nonempty interior.

Definition 2.7 (see, e.g., [20, Definition 1.3] or [14]) Let $A \subset X$ be a body. We say that $x \in \partial A$ is an *LUR (locally uniformly rotund) point* of A if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in A$ and $\operatorname{dist}(\partial A, (x + y)/2) < \delta$ then $\|x - y\| < \varepsilon$. We say that A is an *LUR body* if each point in ∂A is an LUR point of A.

If $A = B_X$, the previous definition coincides with the standard definition of local uniform rotundity of the norm at x. Hence, B_X is an LUR body iff the norm of X is LUR. The notion of LUR norm is a natural generalization of uniform rotundity and plays an important role in the theory of Banach spaces (see, e.g., [15] for the definition of and the main results on LUR norms; see also [13,14] for some recent results involving this notion). Moreover, it is easy to see that, in the case X is finite-dimensional, a body is LUR iff it is strictly convex (i.e., its boundary does not contain nontrivial segments).

Lemma 2.8 *Let A be a body in X and suppose that $a \in \partial A$ is an LUR point of A. Then, if $f \in S_{X^*}$ is a support functional for A in a, f strongly exposes A at a.*

The lemma is well-known in the case the body is a ball (see, e.g., [15, Exercise 8.27]) and in the general case the proof is similar (see, e.g., [11, Lemma 4.3]).

The next lemma gives a characterization of those functionals f that strongly expose a set A in terms of containment of A in translations of cones of the form $C(f, \alpha)$.

Lemma 2.9 *Let A be a convex set in X such that $0 \in A$. Let $f \in S_{X^*}$ be such that $f(0) = \inf f(A)$ and let $x_0 \in S_X$ be such that $f(x_0) = 1$. Let us consider $\varepsilon : (0, 1) \to [0, \infty]$ defined by*

$$\varepsilon(\alpha) := \inf\{\lambda > 0;\ A \subset C(f, \alpha) - \lambda x_0\} \quad (0 < \alpha < 1).$$

Then $\varepsilon(\alpha)$ is $o(\alpha)$ as $\alpha \to 0^+$ iff $(-f)$ strongly exposes A at 0.

Remark 2.10 Observe that if $\alpha \in (0, 1)$ is such that $\varepsilon(\alpha)$ is finite then, in the definition of the function ε, the infimum is actually a minimum. Hence, in this case, we have that $A \subset C(f, \alpha) - \varepsilon(\alpha)x_0$.

Proof of Lemma 2.9 On the contrary, suppose that $\varepsilon(\alpha)$ is not $o(\alpha)$ as $\alpha \to 0^+$, then there exist $M > 0$ and $\alpha_n \to 0^+$ such that $\varepsilon(\alpha_n) > M\alpha_n$. For $n \in \mathbb{N}$, let $z_n \in A \setminus [C(f, \alpha_n) - M\alpha_n x_0]$ and observe that

$$f(z_n) + M\alpha_n = f(z_n + M\alpha_n x_0) < \alpha_n \|z_n + M\alpha_n x_0\|.$$

Hence it holds

$$0 \leq f(z_n) < \alpha_n \|z_n + M\alpha_n x_0\| - M\alpha_n = \alpha_n(\|z_n + M\alpha_n x_0\| - M).$$

Since $\alpha_n > 0$, from the previous inequality we have $\|z_n + M\alpha_n x_0\| > M$. Hence, by the continuity of the norm, eventually it holds $\|z_n\| > \frac{M}{2}$. So, eventually we have

$$0 \leq f(\tfrac{z_n}{\|z_n\|}) < \alpha_n \tfrac{\|z_n + M\alpha_n x_0\| - M}{\|z_n\|} \leq \alpha_n \tfrac{\|z_n\| + M\alpha_n - M}{\|z_n\|} \leq \alpha_n.$$

In particular, we have $f(\frac{Mz_n}{2\|z_n\|}) \to 0^+$ as $n \to \infty$. Since A is convex, $0 \in A$, and eventually $\frac{M}{2\|z_n\|} < 1$, we have that eventually $\frac{Mz_n}{2\|z_n\|} \in A$. Hence, by the definition of strongly exposing functional, we have that $(-f)$ does not strongly expose A at 0.

For the other implication, suppose that $\varepsilon(\alpha)$ is $o(\alpha)$ as $\alpha \to 0^+$. We have that eventually (for $\alpha \to 0^+$) $\varepsilon(\alpha)$ is finite and, by Remark 2.10,

$$A \subset C(f, \alpha) - \varepsilon(\alpha)x_0.$$

Let $x \in A \cap \{z \in X;\ f(z) \leq \alpha^2\}$, then eventually

$$\alpha\|x + \varepsilon(\alpha)x_0\| \leq f(x + \varepsilon(\alpha)x_0) = f(x) + \varepsilon(\alpha)f(x_0) \leq \alpha^2 + \varepsilon(\alpha)$$

and hence

$$\|x\| \leq \|x + \varepsilon(\alpha)x_0\| + \|\varepsilon(\alpha)x_0\| \leq \|x + \varepsilon(\alpha)x_0\| + \varepsilon(\alpha) \leq \alpha + \frac{\varepsilon(\alpha)}{\alpha} + \varepsilon(\alpha).$$

This proves that

$$\operatorname{diam}\big(S(-f, \alpha^2, A)\big) \to 0 \qquad (\alpha \to 0^+),$$

and hence that $(-f)$ strongly exposes A at 0. □

In the following two lemmas we analyse some relations between the Attouch-Wets convergence of a sequence of sets and the containment of the sets of the sequence in a cone of the form $V(f,\alpha)$ or $C(f,\alpha)$. Roughly speaking, the first lemma says that, if the limit set B is contained in a half plane given by a functional $f \in S_{X^*}$, then the sets B_n, Attouch-Wets converging to B, are eventually contained in a translation of a cone of the form $V(f,\alpha)$.

Lemma 2.11 *Let B, B_n ($n \in \mathbb{N}$) be closed convex sets in X such that $B_n \to B$ for the Attouch-Wets convergence, and $f \in S_{X^*}$. Suppose that $x_0 \in S_X$ is such that $f(x_0) = 1$ and suppose that $0 \in B \subset \{x \in X;\ f(x) \le 0\}$. Then, for each $\alpha \in (0,1)$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $B_n \subset V(f,\alpha) + \varepsilon x_0$, whenever $n \ge n_0$.*

Proof On the contrary, suppose that there exists a sequence of integers $\{n_k\}$ such that, for each $k \in \mathbb{N}$, there exists

$$b_{n_k} \in B_{n_k} \setminus [V(f,\alpha) + \varepsilon x_0].$$

Since

$$\|x - b\| \ge f(x - b) \ge \varepsilon,$$

whenever $x \in C(f,\alpha) + \varepsilon x_0$ and $b \in B$, we have

$$\operatorname{dist}(B, C(f,\alpha) + \varepsilon x_0) > 0.$$

Hence, by Fact 2.4, $\{b_{n_k}\}$ is unbounded and we can suppose without any loss of generality that $\|b_{n_k}\| \ge 1$ ($k \in \mathbb{N}$). Since $b_{n_k} \notin V(f,\alpha) + \varepsilon x_0$, we have

$$f(b_{n_k}) = f(b_{n_k} - \varepsilon x_0) + \varepsilon > \alpha\|b_{n_k} - \varepsilon x_0\| + \varepsilon \ge \alpha\|b_{n_k}\| - \varepsilon\alpha\|x_0\| + \varepsilon \ge \alpha\|b_{n_k}\|. \quad (2)$$

Let $\delta = \min\{\varepsilon, \alpha/2\}$, since $0 \in B$ and $B_n \to B$ for the Attouch-Wets convergence, we can suppose without any loss of generality that, for each $k \in \mathbb{N}$, there exists $d_k \in (\delta B_X) \cap B_{n_k}$. For $k \in \mathbb{N}$, let us consider the convex combination of b_{n_k} and d_k

$$w_k := \tfrac{1}{\|b_{n_k}\|} b_{n_k} + \tfrac{\|b_{n_k}\| - 1}{\|b_{n_k}\|} d_k \in B_{n_k},$$

and observe that $\|w_k\| \le 1 + \|d_k\| \le 1 + \delta$. Moreover, by (2), we have

$$f(w_k) \ge f(b_{n_k}) \tfrac{1}{\|b_{n_k}\|} - \|d_k\| \ge \alpha - \|d_k\| \ge \alpha - \delta \ge \tfrac{\alpha}{2}.$$

Since $\{w_k\}$ is a bounded sequence, by Fact 2.4, $\operatorname{dist}(w_k, B) \to 0$. Hence we get a contradiction since $\{w_k\} \subset \{x \in X;\ f(x) \ge \alpha/2\}$ and

$$\operatorname{dist}(B, \{x \in X;\ f(x) \ge \alpha/2\}) > 0.$$

□

The next lemma is the counterpart of the previous result for cones of the form $C(f,\alpha)$. In its proof we shall need the following fact.

Fact 2.12 *Let $0 < \beta < \alpha < 1$ and let $0 < \varepsilon < \varepsilon'$. Then*

$$\operatorname{dist}\big(C(f,\alpha) - \varepsilon x_0, V(f,\beta) - \varepsilon' x_0\big) > 0.$$

Proof Let $0 < \delta < \frac{(1-\alpha)(\varepsilon'-\varepsilon)}{1+\alpha}$. Let us prove that, if $x \in C(f, \alpha) - \varepsilon x_0$ and $b \in B_X$, then $z := x + \delta b \notin V(f, \beta) - \varepsilon' x_0$. To do this, it is sufficient to check that $f(z + \varepsilon' x_0) > \beta \|z + \varepsilon' x_0\|$. Since $(1-\alpha)(\varepsilon' - \varepsilon) - \delta(1+\alpha) > 0$, we have

$$\begin{aligned} f(z + \varepsilon' x_0) &= f(x + \delta b + \varepsilon' x_0) = f(x + \varepsilon x_0) + f(\delta b) + f(\varepsilon' x_0 - \varepsilon x_0) \\ &\geq \alpha \|x + \varepsilon x_0\| - \delta + (\varepsilon' - \varepsilon) \\ &\geq \alpha \|x + \varepsilon' x_0\| - \alpha \|\varepsilon' x_0 - \varepsilon x_0\| - \delta + (\varepsilon' - \varepsilon) \\ &\geq \alpha \|z + \varepsilon' x_0\| - \alpha\delta - \delta + (1-\alpha)(\varepsilon' - \varepsilon) \\ &> \alpha \|z + \varepsilon' x_0\| \geq \beta \|z + \varepsilon' x_0\|, \end{aligned}$$

and the proof is concluded. □

Lemma 2.13 *Let A, A_n $(n \in \mathbb{N})$ be closed convex sets in X such that $A_n \to A$ for the Attouch-Wets convergence, $f \in S_{X^*}$, $\alpha \in (0, 1)$, and $\varepsilon > 0$. Suppose that $x_0 \in S_X$ is such that $f(x_0) = 1$ and suppose that $0 \in A \subset C(f, \alpha) - \varepsilon x_0$. Then, for each $\beta \in (0, \alpha)$ and $\varepsilon' > \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that $A_n \subset C(f, \beta) - \varepsilon' x_0$, whenever $n \geq n_0$.*

Proof Suppose on the contrary that there exists a sequence of integers $\{n_k\}$ such that, for each $k \in \mathbb{N}$, there exists

$$a_{n_k} \in A_{n_k} \setminus [C(f, \beta) - \varepsilon' x_0].$$

Since $a_{n_k} + \varepsilon' x_0 \notin C(f, \beta)$, we have

$$f(a_{n_k} + \varepsilon' x_0) = f(a_{n_k}) + \varepsilon' < \beta \|a_{n_k} + \varepsilon' x_0\|. \tag{3}$$

Fix any $\gamma \in (\beta, \alpha)$ and let $M \geq 1$ be such that $M > \frac{2\varepsilon'}{\alpha-\gamma}$. Finally, let $\theta \in (0, 1)$ satisfy $\theta < \min\{M - \frac{2\varepsilon'}{\alpha-\gamma}, \frac{M(\gamma-\beta)}{1+\gamma}\}$. Therefore, it follows:

(a) $M - \theta > \frac{2\varepsilon'}{\alpha-\gamma}$;
(b) $\frac{\beta M + \theta}{M - \theta} \leq \gamma$.

Since, by Fact 2.12,

$$\operatorname{dist}\big(A, V(f, \beta) - \varepsilon' x_0\big) \geq \operatorname{dist}\big(C(f, \alpha) - \varepsilon x_0, V(f, \beta) - \varepsilon' x_0\big) > 0,$$

applying Fact 2.4, we have that $\{a_{n_k}\}$ is unbounded. Hence we can suppose without any loss of generality that $\|a_{n_k}\| \geq M$ $(k \in \mathbb{N})$. Moreover, since $0 \in A$ and $A_n \to A$ for the Attouch-Wets convergence, we can suppose without any loss of generality that, for each $k \in \mathbb{N}$, there exists $c_k \in A_{n_k} \cap \theta B_X$. For each $k \in \mathbb{N}$, consider the convex combination of a_{n_k} and c_k

$$b_k := \frac{M}{\|a_{n_k}\|} a_{n_k} + \frac{\|a_{n_k}\| - M}{\|a_{n_k}\|} c_k \in A_{n_k},$$

and observe that $M - \theta \leq \|b_k\| \leq M + \theta$. Now, by (3), we have

$$f(a_{n_k}) < \beta \|a_{n_k} + \varepsilon' x_0\| - \varepsilon' \leq \beta \|a_{n_k}\| + \beta\varepsilon' - \varepsilon' \leq \beta \|a_{n_k}\|,$$

and hence

$$f(b_k) \leq \frac{M f(a_{n_k})}{\|a_{n_k}\|} + f(c_k) \leq M\beta + \theta \leq \|b_k\| \frac{M\beta + \theta}{\|b_k\|} \leq \frac{M\beta + \theta}{M - \theta} \|b_k\| \leq \gamma \|b_k\|,$$

where the last inequality holds by (b). Moreover, since $\{b_k\}$ is bounded and $A \subset C(f, \alpha) - \varepsilon x_0$, by Fact 2.4, we have that

$$\operatorname{dist}\big(b_k, C(f, \alpha) - \varepsilon x_0\big) \to 0 \qquad (k \to \infty).$$

Since $f(w) \geq \alpha\|w\| - 2\varepsilon$, whenever $w \in C(f, \alpha) - \varepsilon x_0$, and since $\varepsilon' > \varepsilon$, by the continuity of f and of the norm, we have that eventually $f(b_k) \geq \alpha\|b_k\| - 2\varepsilon'$. Hence we have that

$$\alpha\|b_k\| - 2\varepsilon' \leq f(b_k) \leq \gamma\|b_k\|.$$

By (a), we have that eventually $\|b_k\| \leq \frac{2\varepsilon'}{\alpha-\gamma} < M - \theta$, a contradiction since $\|b_k\| \geq M - \theta$. □

In the sequel, we shall use the following elementary fact.

Fact 2.14 *Let $n_1 \in \mathbb{N}$, $0 < M_1 < M_2$, and $\xi \in (0, 1)$. Let $\{t_n\}$ be a sequence of nonnegative numbers such that, for each $n \geq n_1$, we have:*

(i) if $t_n, t_{n+1} > M_1$ then $t_{n+1} \leq \xi t_n$;
(ii) if $t_n \leq M_1$ then $t_{n+1} \leq M_2$.

Then eventually we have $t_n \leq M_2$.

Proof Clearly, (i) implies that there exists $n_0 \geq n_1$ such that $t_{n_0} \leq M_1$. Let us prove that $t_n \leq M_2$, whenever $n \geq n_0$. Suppose on the contrary that the set $E = \{n \in \mathbb{N};\ n > n_0,\ t_n > M_2\}$ is nonempty and let $n_2 = \min E$. By (ii), we have that $M_1 < t_{n_2-1} \leq M_2$. By (i), we get a contradiction considering the couple t_{n_2-1}, t_{n_2}. □

3 The case where the intersection of limits sets is a singleton

In the sequel of the paper, we restrict our attention to Hilbert spaces, since we need some additional geometrical structure. From now on, let H be a real Hilbert space. If $u, v \in H\setminus\{0\}$, we denote as usual

$$\cos(u, v) := \tfrac{\langle u,v\rangle}{\|u\|\|v\|},$$

where $\langle u, v\rangle$ denotes the inner product between u and v.

If K is a nonempty closed convex subset of H, let us denote by P_K the projection onto the set K, i.e. the map sending any point in H to its nearest point in K. Several times without mentioning it, we shall use the variational characterization of best approximation from a convex set in Hilbert spaces: let K be as above, $x \in H$ and $y_0 \in K$, then $y_0 = P_K(x)$ if and only if

$$\langle x - y_0, y - y_0\rangle \leq 0 \quad \text{whenever } y \in K. \tag{4}$$

It is easy to see that, if $x \notin K$, (4) is equivalent to the following condition:

$$\|y - y_0\| \leq \|x - y\| \cos(y_0 - y, x - y) \quad \text{whenever } y \in K \setminus \{y_0\}. \tag{5}$$

Moreover, if K is a subspace of H then (4) becomes

$$\langle x - y_0, y - y_0\rangle = 0 \quad \text{whenever } y \in K. \tag{6}$$

Let us recall the definition of stability for a couple (A, B) of subsets of H.

Definition 3.1 Let A and B be closed convex subsets of H such that $A \cap B$ is nonempty. We say that the that the couple (A, B) is *stable* if for each choice of sequences $\{A_n\}, \{B_n\} \subset \mathrm{c}(H)$ converging for the Attouch-Wets convergence to A and B, respectively, and for each choice of the starting point a_0, the corresponding perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$ (defined as in Definition 1.2) converge in norm.

Remark 3.2 We remark that in the above definition we can equivalently require that there exists $c \in A \cap B$ such that $a_n, b_n \to c$ in norm.

To prove the remark we shall need the following lemma, whose proof is probably known but for which we did not find any reference.

Lemma 3.3 *Let X be a Hilbert space. Suppose that a sequence $\{A_n\}$ in $\mathrm{c}(X)$ Attouch-Wets converges to $A \in \mathrm{c}(X)$. Then the corresponding sequence of projections $\{P_{A_n}\}$ uniformly converges on bounded set to P_A.*

Proof Without any loss of generality we can suppose that $0 \in A$. Let us prove that, for each $K, M > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x \in K B_X} \|P_{A_n} x - P_A x\| \leq M,$$

whenever $n \geq n_0$. By Lemma 2.5, there exists $\varepsilon' \in (0, K)$ such that, if $\rho \in [0, K]$ and if C is a convex set such that $\rho - \varepsilon' \leq \|c\| \leq \rho + \varepsilon'$, whenever $c \in C$, then $\mathrm{diam}(C) \leq M$. Since $\{A_n\}$ Attouch-Wets converges to A, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, we have

(i) $A_n \cap 3K B_X \subset A + \varepsilon' B_X$;
(ii) $A \cap 3K B_X \subset A_n + \varepsilon' B_X$;

Let $x \in K B_X$, $y = P_A x$, $n \geq n_0$ and $y_n = P_{A_n} x$. Put $\rho = \|x - y\|$ and observe that $\rho \leq K$. By (ii), $\|x - y_n\| \leq \rho + \varepsilon'$ and hence $\|y_n\| \leq \|x\| + \|x - y_n\| \leq 3K$. Hence, by (i), y_n belongs to the convex set

$$C := (A + \varepsilon' B_X) \cap [x + (\rho + \varepsilon') B_X].$$

Moreover, since $\mathrm{dist}(x, A) = \rho$, we have $\mathrm{dist}(x, C) \geq \rho - \varepsilon'$. By Lemma 2.5, $\mathrm{diam} C \leq M$ and hence $\|y_n - y\| = \|P_{A_n} x - P_A x\| \leq M$. By the arbitrariness of $x \in K B_X$, the proof is concluded. □

Proof of Remark 3.2 It suffices to prove that if the perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$ converge in norm then they both converge to the same point belonging to $A \cap B$. Without any loss of generality, we can suppose that $0 \in B$. Let us start by proving that if $a_n \to c$ then $c \in A \cap B$.

We claim that the sequence $\left\{P_{A_n} P_{B_n}\right\}$ uniformly converges on the bounded sets to $P_A P_B$. To see this observe that:

- Since $0 \in B$, we have $\|P_B x\| \leq \|x\|$, whenever $x \in X$;
- Since projections are nonexpansive, we have

$$\|P_{A_n} P_{B_n} x - P_{A_n} P_B x\| \leq \|P_{B_n} x - P_B x\|,$$

  whenever $x \in X$ and $n \in \mathbb{N}$;
- For each $x \in X$ and $n \in \mathbb{N}$, we have

$$\|P_{A_n} P_{B_n} x - P_A P_B x\| \leq \|P_{A_n} P_{B_n} x - P_{A_n} P_B x\| + \|P_{A_n} P_B x - P_A P_B x\|.$$

The previous observation implies that, for $N > 0$, , we have

$$\begin{aligned}
&\sup_{\|x\| \leq N} \|P_{A_n} P_{B_n} x - P_A P_B x\| \leq \\
&\sup_{\|x\| \leq N} \|P_{B_n} x - P_B x\| + \sup_{\|x\| \leq N} \|P_{A_n} P_B x - P_A P_B x\| \leq \\
&\quad \sup_{\|x\| \leq N} \|P_{B_n} x - P_B x\| + \sup_{\|y\| \leq N} \|P_{A_n} y - P_A y\|.
\end{aligned}$$

Since $A_n \to A$, $B_n \to B$ for the Attouch-Wets convergence, by Lemma 3.3, $\{P_{A_n}\}$ uniformly converges on bounded set to P_A and $\{P_{B_n}\}$ uniformly converges on bounded set to P_B. The claim follows by the previous inequality.

Since $\{a_n\}$ is bounded and

$$a_{n+1} = P_{A_n} P_{B_n} a_n = P_A P_B a_n + (P_{A_n} P_{B_n} - P_A P_B) a_n,$$

passing to the limit as $n \to \infty$, and using the claim, we obtain $c = P_A P_B c$. By [2, Facts 1.1, (ii)], we have that $c \in A \cap B$. Similarly, we have

$$b_{n+1} = P_{B_n} a_n = P_B a_n + (P_{B_n} - P_B) a_n \to P_B c = c,$$

and the proof is concluded. □

The main aim of this section is to prove that under the assumption that the sets A and B are separated by a strongly exposing functional f for the set A (i.e. condition (i) in the introduction) the couple (A, B) is stable. The following theorem is the main result of this section.

Theorem 3.4 *Let H be a Hilbert space and A, B nonempty closed convex subsets of H. Suppose that there exist $y \in A \cap B$ and a linear continuous functional $f \in S_{H^*}$ such that* $\inf f(A) = f(y) = \sup f(B)$ *and such that $(-f)$ strongly exposes A at y. Then the couple (A, B) is stable, i.e., if we let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the Attouch-Wets convergence, then, for each $a_0 \in H$, the corresponding perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$ (with starting point a_0), converge to y in norm.*

Before starting with the proof of the theorem we need some preliminary work. First of all, let us observe that without any loss of generality we can suppose that $y = 0$ and hence that

$$\inf f(A) = f(0) = \sup f(B).$$

Suppose that $x_0 \in S_H$ is such that $f(x_0) = 1$, i.e., f is represented by x_0, in the sense that $f(\cdot) = \langle x_0, \cdot \rangle$. Then it is straightforward to give the following representation of the cones $C(f, \alpha)$ and $V(f, \alpha)$, introduced at the beginning of Sect. 2: if we define

$$C(\theta) := \{x \in H \setminus \{0\};\ \cos(x, x_0) \ge \sin(\theta)\} \cup \{0\} \quad (\theta \in (0, \tfrac{\pi}{2})),$$

then the set $C(\theta)$ coincides with $C(f, \alpha)$, where $\alpha = \sin\theta$. Similarly, if we define

$$V(\theta) := \{x \in H \setminus \{0\};\ \cos(x, x_0) \le \sin(\theta)\} \cup \{0\} \quad (\theta \in (0, \tfrac{\pi}{2})),$$

then the set $V(\theta)$ coincides with $V(f, \alpha)$, where $\alpha = \sin\theta$. In the proof of Theorem 3.4, we shall need the following fact, stating that, if $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$, then the "angle" between non-null vectors in $C(\theta_2)$ and $V(\theta_1)$, respectively, is uniformly bounded away from the origin.

Fact 3.5 *Suppose that $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$ are such that $\theta_1 < \theta_2$. If $x \in C(\theta_2) \setminus \{0\}$ and $y \in V(\theta_1) \setminus \{0\}$ then* $\cos(x, y) \le \cos(\theta_2 - \theta_1)$.

Proof For $z \in H \setminus \{0\}$ define

$$\theta(z) := \frac{\pi}{2} - \arccos\cos(z, x_0) = \arcsin\cos(z, x_0).$$

Observe that

$$z \in C(\theta_2) \Leftrightarrow \cos(z, x_0) \ge \sin\theta_2 \Leftrightarrow \arcsin\cos(z, x_0) \ge \theta_2 \Leftrightarrow \theta(z) \ge \theta_2. \tag{7}$$

Similarly, we have

$$z \in V(\theta_1) \iff \theta(z) \leq \theta_1. \tag{8}$$

Let us define $x_1 := x - f(x)x_0$ and $y_1 := y - f(y)x_0$ (where x and y are as in the statement above), observe that

$$\tfrac{\|x_1\|}{\|x\|} = \sqrt{\tfrac{\|x\|^2 - \langle x, x_0\rangle^2}{\|x\|^2}} = \sqrt{1 - \tfrac{\langle x, x_0\rangle^2}{\|x\|^2}} = \sqrt{1 - [\cos(x, x_0)]^2} = \cos[\theta(x)].$$

Similarly, we have $\frac{\|y_1\|}{\|y\|} = \cos[\theta(y)]$. Taking into account the fact that

$$\frac{f(z)}{\|z\|} = \frac{\langle z, x_0\rangle}{\|z\|} = \cos(z, x_0) = \sin[\theta(z)] \qquad (z \in H \setminus \{0\}),$$

we have

$$\begin{aligned}\cos(x, y) &= \frac{f(x)f(y)}{\|x\|\,\|y\|} + \frac{\langle x_1, y_1\rangle}{\|x\|\,\|y\|} \leq \frac{f(x)f(y)}{\|x\|\|y\|} + \frac{\|x_1\|\|y_1\|}{\|x\|\|y\|}\\ &= \sin[\theta(x)]\sin[\theta(y)] + \cos[\theta(x)]\cos[\theta(y)]\\ &= \cos[\theta(x) - \theta(y)] \leq \cos(\theta_2 - \theta_1),\end{aligned}$$

where the last inequality holds since, by (7) and (8), we have $\theta(x) - \theta(y) \geq \theta_2 - \theta_1$. □

Proof of Theorem 3.4 Fix $M > 0$, it suffices to prove that the sequences $\{a_n\}$ and $\{b_n\}$ are eventually contained in $2MB_H$. Let $f \in S_{H^*}$ and $x_0 \in H$ be as above. Let $\varepsilon : (0, 1) \to [0, \infty]$ be the function defined by

$$\varepsilon(\alpha) := \inf\{\lambda > 0;\ A \subset C(f, \alpha) - \lambda x_0\}, \qquad \alpha \in (0, 1).$$

By Lemma 2.9, $\varepsilon(\alpha)$ is $o(\alpha)$ as $\alpha \to 0^+$. Observe that

- $\frac{1}{2}\arcsin(2\alpha) = \alpha + o(\alpha)$ as $\alpha \to 0^+$;
- $2\varepsilon(3\alpha) = o(\alpha)$ as $\alpha \to 0^+$.

Hence, eventually, as $\alpha \to 0^+$, we have

(A) $2\varepsilon(3\alpha) \leq M/2$;
(B) $\sin[\frac{1}{2}\arcsin(2\alpha)] + \frac{8}{M}2\varepsilon(3\alpha) \leq \sin[\frac{2}{3}\arcsin(2\alpha)]$;
(C) $2\alpha - \frac{8}{M}2\varepsilon(3\alpha) \geq \sin[\frac{5}{6}\arcsin(2\alpha)]$;
(D) $\cos[\frac{1}{6}\arcsin(2\alpha)] + \frac{2}{M}[2\varepsilon(3\alpha)]^2 \leq \cos[\frac{1}{12}\arcsin(2\alpha)]$.

In particular, we can fix $\beta \in (0, 1/3)$ such that if $\theta := \frac{1}{2}\arcsin(2\beta)$ and $\varepsilon' := 2\varepsilon(3\beta)$ then $\varepsilon' < 1$ and

(a) $\varepsilon' \leq M/2$;
(b) $\sin\theta + \frac{8}{M}\varepsilon' \leq \sin(\frac{4}{3}\theta)$;
(c) $\sin(2\theta) - \frac{8}{M}\varepsilon' \geq \sin(\frac{5}{3}\theta)$;
(d) $\cos(\frac{1}{3}\theta) + \frac{2}{M}(\varepsilon')^2 \leq \cos(\frac{1}{6}\theta)$.

Since, by Remark 2.10, $0 \in A \subset C(f, 3\beta) - \varepsilon(3\beta)x_0$, by Lemma 2.13, we have that eventually

$$A_n \subset C(f, 2\beta) - 2\varepsilon(3\beta)x_0 = C(2\theta) - \varepsilon' x_0.$$

Since, $0 \in B \subset \{x \in H;\ f(x) \leq 0\}$, by Lemma 2.11, we have that eventually

$$B_n \subset V(\theta) + \varepsilon' x_0.$$

Since $0 \in A \cap B$, $A_n \to A$ and $B_n \to B$ for the Attouch-Wets convergence, eventually there exist $x_n \in A_n \cap (\varepsilon')^2 B_H$ and $y_n \in B_n \cap (\varepsilon')^2 B_H$. □

Claim Eventually, if $a_n, b_n, b_{n+1} \notin M B_H$, the following conditions hold:

(i) $a_n - x_n \in C(\frac{5}{3}\theta)$;
(ii) $b_n - x_n \in V(\frac{4}{3}\theta)$;
(iii) $a_n - y_{n+1} \in C(\frac{5}{3}\theta)$;
(iv) $b_{n+1} - y_{n+1} \in V(\frac{4}{3}\theta)$.

Proof of the claim Let us prove (i) and (ii), the proof of (iii) and (iv) is similar. To prove (i), observe that, since $\|x_n\| \leq (\varepsilon')^2 \leq \varepsilon'$, we have

$$f(a_n - x_n) \geq f(a_n + \varepsilon' x_0) - 2\varepsilon',$$

since $a_n \in A_n \subset C(2\theta) - \varepsilon' x_0$ and using the triangle inequality, we have

$$\begin{aligned} f(a_n + \varepsilon' x_0) - 2\varepsilon' &\geq \sin(2\theta)\|a_n + \varepsilon' x_0\| - 2\varepsilon' \\ &\geq \sin(2\theta)\left(\|a_n - x_n\| - 2\varepsilon'\right) - 2\varepsilon' \\ &= \|a_n - x_n\|\left(\sin(2\theta) - \tfrac{2\varepsilon'\sin(2\theta)+2\varepsilon'}{\|a_n - x_n\|}\right), \end{aligned}$$

finally, by (c) and since $\|a_n - x_n\| \geq \|a_n\| - \|x_n\| \geq \frac{M}{2}$, we obtain

$$\begin{aligned} \|a_n - x_n\|\left(\sin(2\theta) - \tfrac{2\varepsilon'\sin(2\theta)+2\varepsilon'}{\|a_n - x_n\|}\right) &\geq \|a_n - x_n\|\left(\sin(2\theta) - \tfrac{8}{M}\varepsilon'\right) \\ &\geq \|a_n - x_n\|\sin(\tfrac{5}{3}\theta), \end{aligned}$$

To prove (ii), we proceed similarly: since $\|x_n\| \leq \varepsilon'$, we have

$$f(b_n - x_n) \leq f(b_n - \varepsilon' x_0) + 2\varepsilon'$$

since $b_n \in B_n \subset V(\theta) + \varepsilon' x_0$ and using the triangle inequality, we have

$$\begin{aligned} f(b_n - \varepsilon' x_0) + 2\varepsilon' &\leq \sin(\theta)\|b_n - \varepsilon' x_0\| + 2\varepsilon' \\ &\leq \sin(\theta)\left(\|b_n - x_n\| + 2\varepsilon'\right) + 2\varepsilon' \\ &= \|b_n - x_n\|\left(\sin\theta + \tfrac{2\varepsilon'\sin\theta+2\varepsilon'}{\|b_n - x_n\|}\right) \\ &\leq \|b_n - x_n\|\left(\sin\theta + \tfrac{8}{M}\varepsilon'\right) \\ &\leq \|b_n - x_n\|\sin(\tfrac{4}{3}\theta), \end{aligned}$$

where the last two inequalities hold by (b) and since $\|b_n - x_n\| \geq \|b_n\| - \|x_n\| \geq \frac{M}{2}$. The claim is proved. □

Now, since $a_n = P_{A_n} b_n$ and $x_n \in A_n$, by (5), it holds

$$\|a_n - x_n\| \leq \|b_n - x_n\| \cos(a_n - x_n, b_n - x_n). \tag{9}$$

Then we can observe that, by (i) and (ii) in our claim and by Fact 3.5, we have that eventually, if $a_n, b_n \notin M B_H$, it holds $\|a_n - x_n\| \leq \|b_n - x_n\| \cos(\frac{1}{3}\theta)$. Since $\|x_n\| \leq (\varepsilon')^2$, we have

$$\begin{aligned} \|a_n\| \leq \|a_n - x_n\| + (\varepsilon')^2 &\leq [\|b_n\| + (\varepsilon')^2]\cos(\tfrac{1}{3}\theta) + (\varepsilon')^2 \\ &\leq \|b_n\|[\cos(\tfrac{1}{3}\theta) + \tfrac{2}{M}(\varepsilon')^2] \leq \|b_n\|\cos(\tfrac{1}{6}\theta), \end{aligned}$$

where the last inequality holds by (d). Similarly, since $b_{n+1} = P_{B_n} a_n$ and $y_{n+1} \in B_n$, it holds $\|b_{n+1} - y_{n+1}\| \le \|a_n - y_{n+1}\| \cos(b_{n+1} - y_{n+1}, a_n - y_{n+1})$. By (iii) and (iv) in our claim and by Fact 3.5, we have that eventually, if $a_n, b_{n+1} \notin MB_H$, it holds $\|b_{n+1} - y_{n+1}\| \le \|a_n - y_{n+1}\| \cos(\frac{1}{3}\theta)$. Since $\|y_{n+1}\| \le (\varepsilon')^2$, we have

$$\|b_{n+1}\| \le [\|a_n\| + (\varepsilon')^2] \cos(\tfrac{1}{3}\theta) + (\varepsilon')^2 \le \|a_n\| [\cos(\tfrac{1}{3}\theta) + \tfrac{2}{M}(\varepsilon')^2] \le \|a_n\| \cos(\tfrac{1}{6}\theta),$$

where the last inequality holds by (d).

By (9) and by the observations above, there exists $n_1 \in \mathbb{N}$ such that if $n \ge n_1$ then the following conditions hold:

(α) if $a_n, b_n \notin MB_H$ then $\|a_n\| \le \|b_n\| \cos(\frac{1}{6}\theta)$, and if $a_n, b_{n+1} \notin MB_H$ then $\|b_{n+1}\| \le \|a_n\| \cos(\frac{1}{6}\theta)$;

(β) if $b_n \in MB_H$ then $\|a_n\| \le \|b_n\| + 2\varepsilon' \le 2M$, and if $a_n \in MB_H$ then $\|b_{n+1}\| \le \|a_n\| + 2\varepsilon' \le 2M$.

By (α), (β), and applying Fact 2.14, with $\xi = \cos(\frac{1}{6}\theta) < 1$, to the sequence $\{t_n\}$ given by

$$\{\|b_1\|, \|a_1\|, \|b_2\|, \|a_2\|, \ldots\},$$

it follows that eventually $a_n, b_n \in 2MB_H$. □

Corollary 3.6 *Let H be a Hilbert space, B a nonempty closed convex subset of H, A a body in H and $y \in \partial A$ an LUR point of A such that $A \cap B = \{y\}$. Then the couple (A, B) is stable.*

Proof Since $(\text{int } A) \cap B = \emptyset$, by the Hahn-Banach separation theorem, there exists $f \in S_{H^*}$ such that

$$\inf f(A) = f(y) = \sup f(B).$$

Since y is an LUR point of A, by Lemma 2.8, $(-f)$ strongly exposes A at y. The thesis follows by Theorem 3.4. □

Remark 3.7 It is worth noting the following facts about the results presented above.

(i) In the recent paper [18], a result concerning the convergence of iterates of nonexpansive mapping has been obtained under a geometrical condition involving LUR points.

(ii) In Theorem 3.4, the assumption that $(-f)$ strongly exposes A at y cannot be removed. Indeed, the celebrated example of Hundal (see [19]) provides a couple of sets (A, B) such that: (a) $A \cap B = \{0\}$; (b) there exists $f \in S_{H^*}$ such that $\inf f(A) = 0 = \sup f(B)$; (c) there exists a starting point whose corresponding alternating projections sequences do not norm converge (and hence the couple (A, B) is not stable).

(iii) The additional assumption that "$(-f)$ *strongly exposes* A *at* y" contained in Theorem 3.4, geometrically represents the fact that, not only the sets A and B have to "touch" at the point y and are separated by using the functional f, but also that the diameter of the slice of A at level α given by the functional $(-f)$ goes to 0 as α goes to 0.

4 The case where the interior of the intersection of limits sets is nonempty

The main aim of this section is to prove that, under the assumption that the interior of $A \cap B$ is nonempty, the couple (A, B) is stable.

We start by the following two dimensional fact. Even if the argument used is elementary, we include a sketch of a proof for the sake of completeness.

Fact 4.1 *Let H be a Hilbert space and $K \geq \varepsilon > 0$. Then there exists a constant $\mu > 0$ such that, whenever C is a closed convex subset of H containing εB_H and $x \in K B_H$, we have*

$$\|x - P_C x\| \leq \mu(\|x\| - \|P_C x\|). \tag{10}$$

Proof We claim that $\mu = K/\varepsilon$ works. Let us denote $y = P_C x$. We can (and do) assume that y and x are not proportional, if else (10) trivially holds, since $\|x - P_C x\| = \|x\| - \|P_C x\| \leq \mu(\|x\| - \|P_C x\|)$. Hence, since $\varepsilon B_H \subset C$, we have that $\varepsilon < \|y\| < \|x\|$ (the strict inequality $\varepsilon < \|y\|$ holds since x and y are not proportional). In the sequel of the proof we work in the 2-dimensional subspace $Y := \operatorname{span}\{x, y\}$. Let $w \in \varepsilon S_Y$ be such that:

(i) The line containing $\{y, w\}$ is tangent to εB_Y;
(ii) The segment $[y, w]$ intersects the segment $[0, x]$.

The existence of such an element w is guaranteed by the fact that

$$\|x - y\| = \|x - P_C x\| \leq \|x - \varepsilon \tfrac{x}{\|x\|}\| = \|x\| - \varepsilon, \tag{11}$$

where the inequality holds since $\varepsilon \frac{x}{\|x\|} \in C$. Indeed, since $\|y\| > \varepsilon$, there are two points $w_1, w_2 \in \varepsilon S_X$ satisfying (i). Moreover, at least one of the segments $[y, w_1]$, $[y, w_2]$ intersects the segment $[0, x]$ iff y is contained in the closed half plane T containing x and determined by the tangent line to εB_X at the point $\varepsilon \frac{x}{\|x\|}$. Finally, observe that $\{y \in Y;\ \|x - y\| \leq \|x\| - \varepsilon\} \subset T$. Hence, (11) implies that one of the two points w_1, w_2 satisfies (ii).

Let us denote by $\theta(u, v)$ the angle between two not null vectors u and v. Since the vectors w and $w - y$ are orthogonal, we clearly have

$$\sin\theta(-y, w - y) = \tfrac{\|w\|}{\|y\|} \geq \tfrac{\|w\|}{\|x\|} \geq \varepsilon/K. \tag{12}$$

Let us denote $z = \frac{\|y\|}{\|x\|} x$, by the variational characterization of best approximations from convex sets in Hilbert spaces and by the fact that $\|z\| = \|y\|$, respectively, we have:

(i) $\theta(x - y, w - y) \geq \pi/2$;
(ii) $\theta(-y, z - y) \leq \pi/2$.

It follows that $\theta(x - y, z - y) \geq \theta(-y, w - y)$ and hence that

$$\|x - y\| \leq \tfrac{1}{\sin\theta(x-y,z-y)} \|x - z\| \leq \tfrac{1}{\sin\theta(-y,w-y)} \|x - z\| \leq \tfrac{K}{\varepsilon} \|x - z\| = \tfrac{K}{\varepsilon}(\|x\| - \|y\|),$$

where the last inequality holds by (12). □

The following theorem is the main result of this section and it is an application of the previous argument.

Theorem 4.2 *Let H be a Hilbert space and A, B nonempty closed convex subsets of H. Suppose that* $\operatorname{int}(A \cap B) \neq \emptyset$, *then the couple (A, B) is stable.*

Proof Without any loss of generality, we can suppose that $0 \in \operatorname{int}(A \cap B)$. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the Attouch-Wets convergence. Suppose that $\{a_n\}$ and $\{b_n\}$ are the corresponding perturbed alternating projections sequences with respect to a given starting point a_0.

By [25, Proposition 27] we have that $A_n \cap B_n \to A \cap B$ for the Attouch-Wets convergence. Hence, by [3, Theorem 7.4.2], we can suppose without any loss of generality that there exists $\varepsilon > 0$ such that $\varepsilon B_H \subset A_n \cap B_n$, whenever $n \in \mathbb{N}$. Since $0 \in A_n \cap B_n$, we have that $\|a_n\| \leq \|b_n\|$, $\|b_n\| \leq \|a_{n-1}\|$ and hence there exists $K > \varepsilon$ such that $\{a_n\}, \{b_n\} \subset K B_H$.

By Fact 4.1, we have that there exists $\mu > 0$ such that $\|a_n - b_n\| \leq \mu(\|b_n\| - \|a_n\|)$ and $\|b_n - a_{n-1}\| \leq \mu(\|a_{n-1}\| - \|b_n\|)$. Hence

$$\begin{aligned}\textstyle\sum_{n=1}^{N}(\|a_n - a_{n-1}\|) &\leq \textstyle\sum_{n=1}^{N}(\|a_n - b_n\| + \|b_n - a_{n-1}\|)\\ &\leq \textstyle\sum_{n=1}^{N} \mu(\|a_{n-1}\| - \|a_n\|) = \mu(\|a_0\| - \|a_N\|) \leq \mu K.\end{aligned}$$

This proves that the series $\sum_{n\in\mathbb{N}}(a_n - a_{n-1})$ is absolutely convergent and hence convergent. Hence, the sequence $\{a_n\}$ is convergent. Similarly, we have that also the sequence $\{b_n\}$ is convergent and the proof is complete. □

By combining the results contained in Sect. 3 and the previous theorem we have the following corollary, describing the stability property for the couple (A, B) where A and B are bodies.

Corollary 4.3 *Let H be a Hilbert space, suppose that at least one of the following conditions holds.*

(i) A is a closed convex set with nonempty interior, $f \in H^* \setminus \{0\}$ is such that f strongly exposes A at the origin, and $B = \{x \in H;\ f(x) \geq \alpha\}$, where $\alpha \leq 0$.
(ii) A, B are bodies in H such that A is LUR and $A \cap B \neq \emptyset$.
(iii) A is a closed ball in H and B is a body such that $A \cap B \neq \emptyset$.

Then the couple (A, B) is stable.

Proof (i) If $\alpha < 0$ then $\operatorname{int}(A \cap B) \neq \emptyset$ and we can apply Theorem 4.2. If $\alpha = 0$ apply Theorem 3.4.
(ii) If $\operatorname{int}(A \cap B) \neq \emptyset$ we can apply Theorem 4.2. If $\operatorname{int}(A \cap B) = \emptyset$, since A and B are bodies, we have that $\operatorname{int}(A) \cap B = \emptyset$. Since A is an LUR body, there exists $y \in \partial A$ such that $A \cap B = \{y\}$. Apply Corollary 3.6.
(iii) follows by (ii), since H is uniformly rotund and hence each closed ball in H is in particular an LUR body.

□

The following simple 2-dimensional example shows that it is not possible to avoid the assumptions in Corollary 4.3.

Example 4.4 Let $H = \mathbb{R}^2$ and let us consider, for each $h \in \mathbb{N}$, the following subsets of H:

$$\begin{aligned}A &= \operatorname{conv}\{(1,1), (-1,1), (1,0), (-1,0)\};\\ C_{2h} &= \operatorname{conv}\{(1,1), (-1,1), (1,\tfrac{1}{h}), (-1,0)\};\\ C_{2h-1} &= \operatorname{conv}\{(1,1), (-1,1), (1,0), (-1,\tfrac{1}{h})\};\\ B &= \operatorname{conv}\{(1,-1), (-1,-1), (1,0), (-1,0)\};\\ D_{2h} &= \operatorname{conv}\{(1,-1), (-1,-1), (1,-\tfrac{1}{h}), (-1,0)\};\\ D_{2h-1} &= \operatorname{conv}\{(1,-1), (-1,-1), (1,0), (-1,-\tfrac{1}{h})\}.\end{aligned}$$

It is easy to see that $C_h \to A$ and $D_h \to B$ for the Attouch-Wets convergence. We claim that the couple (A, B) is not stable. To prove this, let us consider the starting point $z_0 = (0, 0)$ and observe that, if we consider the points $a_k^1 = (P_{C_1} P_{D_1})^k z_0$, then $a_k^1 \to (1, 0)$ and hence there exists $N_1 \in \mathbb{N}$ such that

$$\|a_{N_1}^1 - (1,0)\| < \tfrac{1}{4}.$$

Define $A_n = C_1$ and $B_n = D_1$ whenever $1 \leq n \leq N_1$. Similarly, if we consider the points $a_k^2 = (P_{C_2} P_{D_2})^k a_{N_1}^1$, then $a_k^2 \to (-1, 0)$ and hence there exists $N_2 \in \mathbb{N}$ such that

$$\|a_{N_2}^2 - (-1, 0)\| < \tfrac{1}{4}.$$

Define $A_n = C_2$ and $B_n = D_2$ whenever $N_1 + 1 \leq n \leq N_1 + N_2$. Then, proceeding inductively, it is easy to construct sequences $\{A_n\}$ and $\{B_n\}$ converging respectively to A and B for the Attouch-Wets convergence and such that the perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$, w.r.t. $\{A_n\}$ and $\{B_n\}$ and with starting point z_0, do not converge.

Inequality constraints

Inequality constraints are a typical example of problem that can be solved by projections and reflections methods (see, e.g., [5, Remark 3.17]). This problem appears very in often in mathematical programming theory and reveals to be a stable problem under mild assumptions. Indeed, in the rest of this section, we will show that under suitable additional hypotheses also the method of perturbed alternating projections sequences can be applied to deal with such a problem.

Given a closed convex cone K in a Hilbert space H (recall that a subset K of H is called cone if $\lambda k \in K$, whenever $\lambda \in [0, \infty)$ and $k \in K$), we denote by K^- its *negative polar cone*, i.e., the closed convex cone defined by

$$K^- := \{x \in H;\ \langle x, k\rangle \leq 0,\ \text{whenever } k \in K\}.$$

Let us suppose that $a \in H \setminus \{0\}$, $b \in \mathbb{R}$, and define $A := \{x \in H;\ \langle a, x\rangle \geq b\}$. Then the following assertions hold true.

- If $\operatorname{int} K \neq \emptyset$, $a_1, \ldots, a_n \in H$, $b_1, \ldots, b_n > 0$ and
$$B := \{x \in H;\ \langle a_i, x\rangle \leq b_i,\ i = 1, \ldots, n\}$$
then $\operatorname{int}(B \cap K) \neq \emptyset$. (It follows since B contains a neighbourhood of the origin.)
- If $\operatorname{int} K \neq \emptyset$ and $a \notin K^-$ then $\operatorname{int}(A \cap K) \neq \emptyset$. (Since there exists $k \in \operatorname{int}(K)$ such that $\langle a, k\rangle > 0$, there exists $\lambda > 0$ such that $\langle a, \lambda k\rangle > b$. This implies that $\lambda k \in \operatorname{int}(A \cap K)$.)
- If $a \in \operatorname{int}(K^-)$ and $b = 0$ then A and K are separated by a strongly exposing functional for the set K. (It follows by [8, Theorem 3.4].)

Hence, by combining the previous observation, Theorem 4.2, and Theorem 3.4, we obtain the following result about the convergence of perturbed projections for the inequality constraints problem.

Theorem 4.5 *Let K be a closed convex cone in a Hilbert space H. Suppose that at least one of the following conditions holds true.*

(i) $\operatorname{int} K \neq \emptyset$, $a_1, \ldots, a_n \in H$, $b_1, \ldots, b_n > 0$, *and*
$$B := \{x \in H;\ \langle a_i, x\rangle \leq b_i,\ i = 1, \ldots, n\}.$$

(ii) $\operatorname{int} K \neq \emptyset$, $a \notin K^-$, $b \in \mathbb{R}$, *and*
$$B := \{x \in H;\ \langle a, x\rangle \geq b\}.$$

(iii) $a \in \operatorname{int}(K^-)$ *and*
$$B := \{x \in H;\ \langle a, x\rangle \geq 0\}.$$

Then the couple (K, B) *is stable.*

As a corollary, we obtain the following finite-dimensional result, where the cone K is the standard nonnegative lattice cone in $\mathbb{R}^N$, a very common assumption in many application.

Corollary 4.6 *Let* $H = \mathbb{R}^N$ *and* $K = \{(x_1, \ldots, x_N) \in \mathbb{R}^N;\ x_k \geq 0, k = 1, \ldots, N\}$. *Suppose that at least one of the following conditions holds true.*

(i) $a_1, \ldots, a_n \in H$, $b_1, \ldots, b_n > 0$, *and*

$$B := \{x \in H;\ \langle a_i, x\rangle \leq b_i,\ i = 1, \ldots, n\}.$$

(ii) $a \notin K^-$, $b \in \mathbb{R}$, *and*

$$B := \{x \in H;\ \langle a, x\rangle \geq b\}.$$

(iii) $a \in \operatorname{int}(K^-)$ *and*

$$B := \{x \in H;\ \langle a, x\rangle \geq 0\}.$$

Then the couple (K, B) *is stable.*

5 Perturbed alternating projections sequences for subspaces

In this section, we study the convergence of the perturbed alternating projections sequences in the case where the limit sets are subspaces. The following elementary example shows that if the intersection of the subspaces is non-trivial, in general, convergence does not hold.

Example 5.1 Let $H = \mathbb{R}^2$ and let us consider $A_n = A = B = \{(x, y) \in \mathbb{R}^2;\ y = 0\}$ $(n \in \mathbb{N})$. For each $h \in \mathbb{N}$, let us consider the line $C_h := \{(x, y) \in \mathbb{R}^2;\ y = \frac{1}{h} - \frac{1}{h^2}x\}$ passing through the points $(0, \frac{1}{h})$ and $(h, 0)$. Let us consider the starting point $z_0 = (0, 0)$ and observe that, if we consider the points $a^1_k := (P_A P_{C_1})^k z_0$, then $a^1_k \to (1, 0)$. Hence, there exists $N_1 \in \mathbb{N}$ such that $\|a_{N_1}\| > \frac{1}{2}$. Define $B_n := C_1$ whenever $1 \leq n \leq N_1$. Similarly, if we consider the points $a^2_k := (P_A P_{C_2})^k a^1_{N_1}$ then $a^2_k \to (2, 0)$. Hence, there exists $N_2 \in \mathbb{N}$ such that $\|a^2_{N_2}\| > 1$. Define $B_n := C_2$ whenever $N_1 + 1 \leq n \leq N_1 + N_2$. Then, proceeding inductively, it is easy to construct a sequence $\{B_n\}$ such that the perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$, w.r.t. $\{A_n\}$ and $\{B_n\}$ and with starting point z_0, are unbounded.

In order to avoid such a situation we consider the case in which the intersection of the subspaces reduces to the origin. We have the following theorem.

Theorem 5.2 *Let* H *be a Hilbert space and suppose that* $U, V \subset H$ *are closed subspaces such that* $U \cap V = \{0\}$ *and* $U + V$ *is closed. Then, the couple* (U, V) *is stable.*

In the proof of the theorem above we shall use the following notation: if W is a subspace of H and $\varepsilon \in (0, 1)$, let $W(\varepsilon) \subset H$ be the set defined by

$$W(\varepsilon) := (\varepsilon B_H) \cup \{w \in H \setminus \{0\};\ \exists u \in W \setminus \{0\} \text{ such that } \cos(u, w) \geq 1 - \varepsilon\}.$$

Hence, $W(\varepsilon)$ is the union of the ball of radius ε centred at the origin and an "approximation" of the subspace W by a non-convex cone, more precisely, the cone containing all the vectors whose angle with W is less that $\arccos(1 - \varepsilon)$. We have the following fact.

Fact 5.3 *Let W be a subspace of H and $\varepsilon \in (0, 1)$. We have*

$$W(\varepsilon) = (\varepsilon B_H) \cup \{w \in H \setminus \{0\};\ \exists u \in W \cap \|w\| S_H \text{ such that } \|u - w\|^2 \leq 2\varepsilon \|w\|^2\}. \tag{13}$$

Moreover, it holds

$$\operatorname{dist}(H \setminus W(\varepsilon), W) > 0.$$

Proof Observe that the set

$$\{w \in H \setminus \{0\};\ \exists u \in W \setminus \{0\} \text{ such that } \cos(u, w) \geq 1 - \varepsilon\}$$

coincides with

$$\{w \in H \setminus \{0\};\ \exists u \in W \cap \|w\| S_H \text{ such that } \cos(u, w) \geq 1 - \varepsilon\}.$$

Now, if $\|u\| = \|w\| \neq 0$, the law of cosines, implies that

$$\|u - v\|^2 = 2\|w\|^2 \left(1 - \cos(u, w)\right),$$

and hence (13) holds.

Let us prove that $\operatorname{dist}(H \setminus W(\varepsilon), W) > 0$. Suppose on the contrary that there exist sequences $\{x_n\} \subset W$ and $\{y_n\} \subset H \setminus W(\varepsilon)$ such that $\|x_n - y_n\| \to 0$. Since $\varepsilon B_X \subset W(\varepsilon)$, we have that $\|y_n\| > \varepsilon$. Since $\|x_n\| \geq \|y_n\| - \|x_n - y_n\|$, eventually we have $\|x_n\| \geq \frac{\varepsilon}{2}$. Moreover, we have $\frac{\|x_n\|}{\|y_n\|} \to 1$ and $\frac{\|y_n - x_n\|}{\|y_n\|} \to 0$, as $n \to \infty$. Then, we can observe that eventually

$$\cos(x_n, y_n) = \tfrac{\langle x_n, x_n \rangle}{\|x_n\|\,\|y_n\|} + \tfrac{\langle y_n - x_n, x_n \rangle}{\|x_n\|\,\|y_n\|} \geq \tfrac{\|x_n\|}{\|y_n\|} - \tfrac{\|y_n - x_n\|}{\|y_n\|}.$$

Hence, $\cos(x_n, y_n) \to 1$, as $n \to \infty$. A contradiction with the fact that $\{y_n\} \subset H \setminus W(\varepsilon)$. □

In the proof of the theorem above we shall need the following two lemmas.

Lemma 5.4 *Let H be a Hilbert space and U a subspace of H. Let $\{A_n\}$ be a sequence of closed convex sets such that $A_n \to U$ for the Attouch-Wets convergence. Then, for each $\varepsilon \in (0, 1)$, it eventually holds that $A_n \subset U(\varepsilon)$.*

Proof On the contrary, suppose that there exist $\varepsilon \in (0, 1)$ and a sequence $\{n_k\}$ of integers such that, for each $k \in \mathbb{N}$, there exists $x_{n_k} \in A_{n_k} \setminus U(\varepsilon)$. By Fact 5.3, we have $\operatorname{dist}\big(U, H \setminus U(\varepsilon)\big) > 0$. By Fact 2.4 and since $A_n \to U$ for the Attouch-Wets convergence, the sequence $\{x_{n_k}\}$ is unbounded, and hence we can suppose, without any loss of generality, that $\|x_{n_k}\| > 1$, whenever $k \in \mathbb{N}$. Let $\gamma \in (0, 1)$ be such that $\frac{(1-\varepsilon)(1+\frac{\gamma}{1-\varepsilon})}{(1-\frac{\varepsilon}{2})(1-\gamma)} \leq 1$ and let $k \in \mathbb{N}$ be such that there exists $z_k \in A_{n_k} \cap \gamma B_H$. Consider the convex combination

$$w_k := \lambda x_{n_k} + (1 - \lambda) z_k \in A_{n_k},$$

where $\lambda = \frac{1}{\|x_{n_k}\|}$, and observe that by the triangular inequality we have $1 - \gamma \leq \|w_k\| \leq 1 + \gamma$. For each $u \in U$, taking into account that $x_{n_k} \notin U(\varepsilon)$ $(k \in \mathbb{N})$, we have

$$\begin{aligned}
\langle w_k, u \rangle &= \lambda \langle x_{n_k}, u \rangle + (1 - \lambda)\langle z_k, u \rangle \leq \|u\|(1 - \varepsilon)\|\lambda x_{n_k}\| + \|z_k\|\|u\| \\
&\leq \|u\|(1 - \varepsilon)\|\lambda x_{n_k}\| + \gamma\|u\| = \|u\|(1 - \varepsilon)(1 + \tfrac{\gamma}{1-\varepsilon}) \\
&= (1 - \tfrac{\varepsilon}{2})\|u\|\|w_k\| \tfrac{(1-\varepsilon)(1+\frac{\gamma}{1-\varepsilon})}{(1-\frac{\varepsilon}{2})\|w_k\|} \\
&\leq (1 - \tfrac{\varepsilon}{2})\|u\|\|w_k\| \tfrac{(1-\varepsilon)(1+\frac{\gamma}{1-\varepsilon})}{(1-\frac{\varepsilon}{2})(1-\gamma)} \leq (1 - \tfrac{\varepsilon}{2})\|u\|\|w_k\|.
\end{aligned}$$

The previous chain of inequalities implies that $\cos(w_k, u) \le 1 - \frac{\varepsilon}{2}$; hence, $w_k \in A_{n_k} \setminus U(\frac{\varepsilon}{2})$, whenever $k \in \mathbb{N}$. Since $\{w_k\}$ is a bounded sequence, by Fact 2.4, $\operatorname{dist}(w_k, U) \to 0$. We get a contradiction since, by Fact 5.3,

$$\operatorname{dist}\big(U, H \setminus U(\tfrac{\varepsilon}{2})\big) > 0.$$

□

Lemma 5.5 *Let U, V be closed subspace of a Hilbert space H such that $U \cap V = \{0\}$ and $U + V$ is closed. Let $M \in (0, 1)$, then there exist $\varepsilon \in (0, M)$ and $\eta \in (0, 1)$ such that, for each $x \in U(\varepsilon) \setminus MB_H$, $y \in V(\varepsilon) \setminus MB_H$ and $z \in \varepsilon B_H$, we have* $\cos(x - z, y - z) \le \eta$.

Proof By [16, Lemma 3.5], we have that

$$\Omega := \sup\{< a, b >;\ a \in V \cap S_H, b \in U \cap S_H\} < 1.$$

That is, the angle between two nonnull vectors in U and V, respectively, is uniformly bounded away from the origin. Fix any $\eta \in (\Omega, 1)$ and take $\varepsilon \in (0, M)$ such that

$$\big(\tfrac{M}{M-\varepsilon}\big)^2\big(\Omega + \tfrac{15\sqrt{\varepsilon}}{M^2}\big) \le \eta.$$

Suppose that $x \in U(\varepsilon) \setminus MB_H$, $y \in V(\varepsilon) \setminus MB_H$ and $z \in \varepsilon B_H$. By (13), there exist $u \in U \cap \|x\| S_H$ and $v \in V \cap \|y\| S_H$ such that $\|x - u\| \le \sqrt{2\varepsilon}\|x\|$ and $\|y - v\| \le \sqrt{2\varepsilon}\|y\|$. Hence, $x' := x - u - z \in 3\sqrt{\varepsilon} B_H$ and $y' := y - v - z \in 3\sqrt{\varepsilon} B_H$. Then we have:

$$\begin{aligned}
\langle x - z, y - z\rangle &= \langle u + x', v + y'\rangle \\
&= \langle u, v\rangle + \langle u, y'\rangle + \langle x', v\rangle + \langle x', y'\rangle \\
&\le \Omega\|x\|\|y\| + 3\sqrt{\varepsilon}\|x\| + 3\sqrt{\varepsilon}\|y\| + 9\varepsilon \\
&\le \|x\|\|y\|(\Omega + \tfrac{3\sqrt{\varepsilon}}{\|x\|} + \tfrac{3\sqrt{\varepsilon}}{\|y\|} + \tfrac{9\varepsilon}{\|x\|\|y\|}) \\
&\le \|x\|\|y\|(\Omega + \tfrac{6\sqrt{\varepsilon}}{M} + \tfrac{9\varepsilon}{M^2}) \\
&\le \|x\|\|y\|(\Omega + \tfrac{15\sqrt{\varepsilon}}{M^2}) \\
&\le \|x - z\|\|y - z\| \tfrac{\|x\|}{\|x\|-\varepsilon} \tfrac{\|y\|}{\|y\|-\varepsilon} (\Omega + \tfrac{15\sqrt{\varepsilon}}{M^2}).
\end{aligned}$$

Since the function $t \mapsto \frac{t}{t-\varepsilon}$ is decreasing on the interval (ε, ∞) and since $\|x\|, \|y\| \in [M, \infty) \subset (\varepsilon, \infty)$, we have

$$\begin{aligned}
\langle x - z, y - z\rangle &\le \|x - z\|\|y - z\| \tfrac{\|x\|}{\|x\|-\varepsilon} \tfrac{\|y\|}{\|y\|-\varepsilon} (\Omega + \tfrac{15\sqrt{\varepsilon}}{M^2}) \\
&\le \|x - z\|\|y - z\| \big(\tfrac{M}{M-\varepsilon}\big)^2 (\Omega + \tfrac{15\sqrt{\varepsilon}}{M^2}) \\
&\le \eta\|x - z\|\|y - z\|.
\end{aligned}$$

□

We are now ready to prove our theorem.

Proof of Theorem 5.2 Let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to U$ and $B_n \to V$, for the Attouch-Wets convergence, and let $a_0 \in H$. Let us consider the corresponding perturbed alternating projections sequences $\{a_n\}$ and $\{b_n\}$, with starting point a_0.

Fix an arbitrary $M \in (0, 1)$, then it suffices to prove that eventually $a_n, b_n \in 3MB_H$. Let $\varepsilon \in (0, M)$ and $\eta \in (0, 1)$ be given by Lemma 5.5. Let us consider the sets $U(\varepsilon)$, $V(\varepsilon)$

and observe that, by Lemma 5.4, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $A_n \subset U(\varepsilon)$ and $B_n \subset V(\varepsilon)$. Let us fix $\varepsilon' \in (0, \varepsilon)$ such that $\eta + \frac{2\varepsilon'}{M} \leq \frac{\eta+1}{2}$, then there exists an integer $n_1 \geq n_0$ such that, for each $n \geq n_1$, there exist $x_n \in A_n \cap \varepsilon' B_H$ and $y_n \in B_n \cap \varepsilon' B_H$.

Suppose that $n \geq n_1$, we can observe that:

- by the equivalent formulation of the variational characterization of best approximation from a convex set in Hilbert spaces (5), and by Lemma 5.5, if $a_n, b_n \notin M B_H$, it holds $\|a_n - x_n\| \leq \|b_n - x_n\|\eta$ (remember that $a_n = P_{A_n} b_n$) and hence

$$\|a_n\| \leq \|a_n - x_n\| + \varepsilon' \leq \eta(\|b_n\| + \varepsilon') + \varepsilon' \leq \|b_n\|(\eta + \tfrac{2\varepsilon'}{M}) \leq \tfrac{\eta+1}{2}\|b_n\|;$$

- similarly, if $a_n, b_{n+1} \notin M B_H$, it holds

$$\|b_{n+1}\| \leq \tfrac{\eta+1}{2}\|a_n\|;$$

- by (5), if $b_n \in M B_H$ then

$$\|a_n\| \leq \|a_n - x_n\| + \varepsilon' \leq \|b_n - x_n\| + \varepsilon' \leq \|b_n\| + 2\varepsilon' \leq 3M$$

  and, similarly, if $a_n \in M B_H$ then $\|b_{n+1}\| \leq 3M$.

By the items above and applying Fact 2.14, with $\xi = \frac{\eta+1}{2} < 1$, to the sequence $\{t_n\}$ given by

$$\left\{\|b_1\|, \|a_1\|, \|b_2\|, \|a_2\|, \ldots\right\},$$

it follows that eventually $a_n, b_n \in 3M B_H$. □

The remaining part of this section is devoted to proving that the assumption on the closedness of the sum of the subspaces, in Theorem 5.2, cannot be removed. This result is contained in Theorem 5.9 below and is inspired by the construction contained in [16, Section 4], in which the authors considered two closed subspaces U, V of a Hilbert space such that $U \cap V = \{0\}$ and $U + V$ is not closed.

Notation 5.6 In the sequel of the present section, we adopt the following notation.

- $H := \ell_2$.
- If, for each $h \in \mathbb{N}$, x^h is an element of H, we denote by $\{x^h\}$ the corresponding sequence in H. Moreover, if $h \in \mathbb{N}$ is fixed, we can consider x^h as a sequence of real numbers and we write $x^h = \{x^h_n\}_n$.
- Suppose that $\{\theta_n\} \subset \mathbb{R}$ is a bounded sequence and let us consider the linear continuous operator $D : H \to H$ given by $Dx = D\{x_n\} := \{\theta_n x_n\}$ $(x = \{x_n\} \in H)$.
- $Z := H \oplus H$ is endowed with the inner product $\langle \cdot, \cdot \rangle_Z$ (denoted in the sequel simply by $\langle \cdot, \cdot \rangle$) defined by

$$\langle z_1, z_2 \rangle_Z = \langle (x_1, y_1), (x_2, y_2) \rangle_Z = \langle x_1, x_2 \rangle_H + \langle y_1, y_2 \rangle_H,$$

  where $z_i = (x_i, y_i) \in Z$ $(i = 1, 2)$.
- Suppose that $b = \{b_n\} \in H$ and consider the closed convex subsets of Z defined as follows:

$$A := \{(x, y) \in Z;\ y = 0\} \quad \text{and} \quad V := \{(x, y) \in Z;\ y = b + Dx\}.$$

  Observe that A is a subspace of Z and V is an affine set in Z.

The following remark concerns the computation of projections onto sets A and V, defined as above.

Remark 5.7 If $(\alpha, \beta) \in Z$ then we obtain immediately that $P_A(\alpha, \beta) = (\alpha, 0)$. Now, let us suppose that $(\alpha, 0) \in A$ and let us compute $P_V(\alpha, 0)$. If we denote $P_V(\alpha, 0) = (\{x_n\}, \{b_n + \theta_n x_n\})$, by the characterization of best approximation in Hilbert space, we have, for each $\{y_n\} \in H$,

$$\Big\langle (\{x_n - \alpha_n\}, \{b_n + \theta_n x_n\}), (\{y_n\}, \{\theta_n y_n\}) \Big\rangle = 0.$$

Hence, we must have $x_n - \alpha_n + b_n\theta_n + x_n\theta_n^2 = 0$, whenever $n \in \mathbb{N}$. That is, for each $n \in \mathbb{N}$, it holds $x_n = \frac{\alpha_n - \theta_n b_n}{1+\theta_n^2}$. Hence,

$$P_A P_V(\{\alpha_n\}, 0) = (\{\tfrac{\alpha_n - \theta_n b_n}{1+\theta_n^2}\}, 0).$$

Repeating N times the same argument yields:

$$(P_A P_V)^N(\{\alpha_n\}, 0) = (\{\tfrac{\alpha_n - \theta_n b_n \sum_{l=0}^{N-1}(1+\theta_n^2)^l}{(1+\theta_n^2)^N}\}, 0). \tag{14}$$

In the sequel we shall need the following result concerning Attouch-Wets convergence of certain sequences of sets.

Lemma 5.8 *Let Z be defined as above. Let $\{b^n\} \subset H$ be a norm null sequence (i.e., $\|b^n\| \to 0$). Let $D, D^n : H \to H$ ($n \in \mathbb{N}$) be linear bounded operators such that $D^n \to D$ in the operator norm. Then, if we define*

$$W := \{(x, Dx) \in Z;\ x \in H\} \quad \textit{and} \quad W_n := \{(x, b^n + D^n x) \in Z;\ x \in H\} \quad (n \in \mathbb{N}),$$

we have that $W_n \to W$ for the Attouch-Wets convergence.

Proof Let us fix $N \in \mathbb{N}$. If $z = (x, Dx) \in W \cap NB_Z$ then we can consider $z' = (x, b^n + D^n x) \in W_n$ and observe that

$$\|z - z'\|_Z = \|Dx - D^n x - b^n\|_H \leq N\|D - D^n\| + \|b^n\|_H.$$

Similarly, if $w = (y, b^n + D^n y) \in W_n \cap NB_Z$ then we can consider $w' = (y, Dy) \in W$ and observe that

$$\|w - w'\|_Z = \|Dy - D^n y - b^n\|_H \leq N\|D - D^n\| + \|b^n\|_H.$$

Hence, $h_N(W, W_n) \leq N\|D - D^n\| + \|b^n\| \to 0$ ($n \to \infty$), and the proof is concluded. □

Theorem 5.9 *Let Z be defined as above and $A := \{(x, y) \in Z;\ y = 0\}$, then there exist*

(a) B a closed subspace of Z,
(b) $z_0 \in Z$,
(c) $\{A_n\}, \{B_n\}$ two sequences of nonempty closed convex sets Attouch-Wets converging to A and B, respectively,

such that the perturbed alternating projections sequences (w.r.t. $\{A_n\}$ and $\{B_n\}$ and with starting point z_0), are unbounded.

Proof Let us consider the sequence $\{\xi_n\} \subset \mathbb{R}$, given by $\xi_n = 4^{-n}$, and let us consider the operator $D : H \to H$, given by $D\{x_n\} = \{\xi_n x_n\}$. Then define $B = \{(x, y) \in Z;\ y = Dx\}$ and, for each $n \in \mathbb{N}$, put $A_n = A$. Now, consider any $z_0 = (\{\alpha_n\}, 0) \in A$ such that $\alpha_n > 0$ ($n \in \mathbb{N}$) and $\|z_0\| < 1$.

Let us put, $N_0 = 1$ and, for each $n \in \mathbb{N}$, $\alpha_n^{0,1} = \alpha_n$. We shall define inductively (with respect to $h \in \mathbb{N}$) positive integers N_h, sequences of elements of H

$$\{\alpha_n^{h,1}\}_n, \{\alpha_n^{h,2}\}_n, \{\alpha_n^{h,3}\}_n \dots,$$

positive real numbers M_h, and sets $C_h \subset Z$ such that:

(i) $2^h + h > (1 + M_h)^2 \sum_{n=h+1}^{\infty} (\alpha_n^{h-1,N_{h-1}})^2 > 2^h$
(ii) $C_h = \{(x, b^h + D^h x) \in Z;\ x \in H\}$, where $D^h : H \to H$ is given by $D^h\{x_n\} := \{\theta_n^h x_n\}$ and where $b^h = \{b_n^h\}_n \in H$ and $\theta_n^h \in \mathbb{R}$ are given by

$$b_n^h := \begin{cases} 0 & \text{if } n \le h \\ \alpha_n^{h-1,N_{h-1}} \xi_n \frac{1+M_h}{M_h} & \text{if } n > h \end{cases} \quad \text{and} \quad \theta_n^h := \begin{cases} \xi_n & \text{if } n \le h \\ -\frac{1}{M_h}\xi_n & \text{if } n > h \end{cases};$$

(iii) $(\{\alpha_n^{h,1}\}_n, 0) = P_A P_{C_h}(\{\alpha_n^{h-1,N_{h-1}}\}_n, 0)$;
(iv) $(\{\alpha_n^{h,t+1}\}_n, 0) = P_A P_{C_h}(\{\alpha_n^{h,t}\}_n, 0)$, $t \in \mathbb{N}$;
(v) $2^h + h > \sum_{n=1}^{\infty} (\alpha_n^{h,N_h})^2 \ge \sum_{n=h+1}^{\infty} (\alpha_n^{h,N_h})^2 > 2^h$;
(vi) $\alpha_n^{h,t} > 0$, whenever $n, t \in \mathbb{N}$.

Let us show that this is possible. Let $h \in \mathbb{N}$ and suppose we already have $N_{h-1} \in \mathbb{N}$ and elements

$$\{\alpha_n^{h-1,1}\}_n, \dots, \{\alpha_n^{h-1,N_{h-1}}\}_n \in H$$

such that the following conditions hold:

- $2^{h-1} + h - 1 > \sum_{n=1}^{\infty} (\alpha_n^{h-1,N_{h-1}})^2$;
- $\alpha_n^{h-1,N_{h-1}} > 0$, whenever $n \in \mathbb{N}$.

(Observe that for $h = 1$ the two conditions above are trivially satisfied since $\alpha_n^{0,N_0} = \alpha_n > 0$ and $\sum_{n=1}^{\infty} (\alpha_n^{0,N_0})^2 = \|z_0\|^2 < 1$.)

By combining these two relations, we obtain that

$$2^h + h > 2^{h-1} + h - 1 > \sum_{n=1}^{\infty} (\alpha_n^{h-1,N_{h-1}})^2 > \sum_{n=h+1}^{\infty} (\alpha_n^{h-1,N_{h-1}})^2 > 0.$$

Hence there exists a positive real number M_h such that (i) holds true. Now, let us consider C_h defined as in (ii). Then, by the relations in (iii) and (iv), we define $\{\alpha_n^{h,t}\}_n$ ($t \in \mathbb{N}$). We just have to prove that there exists $N_h \in \mathbb{N}$ such that (v) is satisfied and that (vi) holds true. Let $N \in \mathbb{N}$. By (14) in Remark 5.7, since $(\{\alpha_n^{h,N}\}_n, 0) = (P_A P_{C_h})^N(\{\alpha_n^{h-1,N_{h-1}}\}_n, 0)$, and taking into account the definitions of $b^h \in H$ and $\{\theta_n^h\}_n \subset \mathbb{R}$ contained in (ii), we have that, for each $n > h$,

$$\alpha_n^{h,N} = \alpha_n^{h-1,N_{h-1}} \frac{1 + \frac{1+M_h}{M_h^2}\xi_n^2 \sum_{l=0}^{N-1}(1+\frac{1}{M_h^2}\xi_n^2)^l}{(1+\frac{1}{M_h^2}\xi_n^2)^N}.$$

Similarly, for each $n \le h$, we have

$$\alpha_n^{h,N} = \alpha_n^{h-1,N_{h-1}} \frac{1}{(1+\xi_n^2)^N}.$$

Since

$$0 < \frac{1 + \frac{1+M_h}{M_h^2}\xi_n^2 \sum_{l=0}^{N-1}(1+\frac{1}{M_h^2}\xi_n^2)^l}{(1+\frac{1}{M_h^2}\xi_n^2)^N} = \frac{-M_h + (1+M_h)(1+\frac{1}{M_h^2}\xi_n^2)^N}{(1+\frac{1}{M_h^2}\xi_n^2)^N} \nearrow 1 + M_h \ \ (N \to \infty)$$

and

$$\frac{1}{(1+\xi_n^2)^N} \to 0 \ \ (N \to \infty),$$

we have that, for $n > h$,

$$0 < \alpha_n^{h,N} \nearrow (1 + M_h)\alpha_n^{h-1,N_{h-1}} \qquad (N \to \infty),$$

and that, for $n \leq h$,

$$0 < \alpha_n^{h,N} \to 0 \qquad (N \to \infty).$$

As an application of Lebesgue monotone convergence theorem, we have that

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} (\alpha_n^{h,N})^2 = \lim_{N\to\infty} \sum_{n=h+1}^{\infty} (\alpha_n^{h,N})^2 = (1 + M_h)^2 \sum_{n=h+1}^{\infty} (\alpha_n^{h-1,N_{h-1}})^2.$$

Hence, by (i), there exists $N_h \in \mathbb{N}$ such that

$$2^h + h > \textstyle\sum_{n=1}^{\infty} (\alpha_n^{h,N_h})^2 \geq \sum_{n=h+1}^{\infty} (\alpha_n^{h,N_h})^2 > 2^h,$$

and (v) is satisfied. Moreover, it follows immediately by our construction that condition (vi) is satisfied.

Now, if $\sum_{k=0}^{h-1} N_k \leq n < \sum_{k=0}^{h} N_k$, put $B_n = C_h$. By our construction, it holds that $a_N = (\{\alpha_n^{h,N_h}\}, 0)$ where $N = \sum_{k=1}^{h} N_k$. In particular,

$$\|b_N\|^2 \geq \|P_A b_N\|^2 = \|P_{A_N} b_N\|^2 = \|a_N\|^2 \geq \sum_{n=h+1}^{\infty} (\alpha_n^{h,N_h})^2 > 2^h$$

and hence the the sequences $\{a_n\}$ and $\{b_n\}$ are unbounded.

It remains to prove that $B_n \to B$ for the Attouch-Wets convergence or, equivalently, that $C_h \to B$ for the Attouch-Wets convergence. In view of Lemma 5.8, it suffices to prove that the sequence $\{b^h\}$ is norm null (i.e., $\|b^h\| \to 0$) and that $D^h \to D$ in the operator norm.

By the inequalities in (i) and (v), we have

$$(1 + M_h)^2(2^{h-1} + h - 1) \geq (1 + M_h)^2 \textstyle\sum_{n=h+1}^{\infty} (\alpha_n^{h-1,N_{h-1}})^2 > 2^h,$$

and hence

$$(1 + M_h)^2 > \frac{2^h}{2^{h-1}+h-1}.$$

Therefore the sequence $\{M_h\}$ is bounded away from 0. Hence, the sequences $\{\frac{1}{M_h}\}$ and $\{\frac{1+M_h}{M_h}\}$ are bounded above by a positive constant K. Then, by the definition of b^h in (ii), we have

$$\|b^h\| \leq K\xi_h \|\{\alpha_n^{h-1,N_{h-1}}\}\|_H \leq \frac{K}{4^h} \|\{\alpha_n^{h-1,N_{h-1}}\}\|_H \leq \frac{K}{4^h}\sqrt{2^{h-1} + h - 1},$$

where the last inequality holds by (v). Moreover, by the definition of θ_n^h in (ii), we have that

$$\|(D - D^h)x\|^2 \leq \textstyle\sum_{n=h+1}^{\infty} (\xi_n - \frac{1}{M_h}\xi_n)^2 x_n^2 \leq (1 + K)^2 \xi_{h+1}^2 \|x\|^2 \quad (x = \{x_n\} \in H).$$

Therefore, finally we obtain that

$$\|D - D^h\| \leq (1 + K)\xi_{h+1} \to 0.$$

□

6 Conclusion and final remarks

In this paper, we introduced a notion of stability for the alternating projection method related to a couple (A, B) of closed convex subsets of a Hilbert space H. Namely, we consider two sequences of closed convex sets $\{A_n\}$ and $\{B_n\}$, each of them converging, with respect to the Attouch-Wets variational convergence, respectively, to A and B. Given a starting point $a_0 \in H$, we consider the sequences of points obtained by projecting on the "perturbed" sets, i.e., the sequences $\{a_n\}$ and $\{b_n\}$ given by $b_n = P_{B_n}(a_{n-1})$ and $a_n = P_{A_n}(b_n)$. The main results of the paper, summarized in the next theorem, show that some classical assumptions implying norm convergence of the standard alternating projections method also guarantee its stability, i.e., norm convergence of $\{a_n\}$ and $\{b_n\}$. Moreover, we provided some examples showing that we cannot omit the assumptions contained in our results.

Theorem 6.1 *Let H be a Hilbert space and A, B nonempty closed convex subsets of H. Then the couple (A, B) is stable if at least one of the following conditions is satisfied.*

(a) *There exist $y \in A \cap B$ and a linear continuous functional $f \in S_{H^*}$ such that* $\inf f(B) = f(y) = \sup f(A)$ *and such that f strongly exposes A at y.*
(b) *A is a body and $y \in \partial A$ is an LUR point of A such that $A \cap B = \{y\}$.*
(c) $\operatorname{int}(A \cap B) \neq \emptyset$.
(d) *A, B are bodies in H such that A is LUR and $A \cap B \neq \emptyset$.*
(e) *H is finite-dimensional and A, B are bodies in H such that A is strictly convex and $A \cap B \neq \emptyset$.*
(f) *A, B are closed subspace of H such that $A \cap B = \{0\}$ and $A + B$ is closed.*
(g) *H is finite-dimensional and A, B are closed subspace of H such that $A \cap B = \{0\}$.*

In the next remark, we collect some interesting problems and some possible subjects of further studies.

Remark 6.2 (i) In the present paper, we focused on the stability of the alternating projections method. It is clear that a similar study can be considered also for other projection and reflection method, such as the Douglas-Rachford method.

(ii) A natural question is to what extent our results can be generalized to Banach spaces in which a notion of projection is defined (see [17] for some results in this setting concerning the alternating projection method). Even if some of the arguments contained in our paper work in general Banach spaces (for example, those contained in Sect. 2), the proofs of the main results presented above rely on the geometrical structure of Hilbert spaces and on the peculiar properties of projections in such spaces.

(iii) Another natural question is whether it is possible to extend our approach to the much more general setting of fixed point algorithms. For some result in this direction, but under different assumptions, see e.g., [7,10,18].

(iv) A subject of further study will be the application of the perturbed alternating projection method to concrete examples (such as the so called *noiseless phase retrieval problem*). Indeed, we think that the notion of stability, introduced in our work, may be a useful property also in the study of concrete applications.

Acknowledgements Open access funding provided by Università Cattolica del Sacro Cuore within the CRUI-CARE Agreement. The research of the authors is partially supported by GNAMPA-INdAM, Project GNAMPA 2018. The research of the second author is partially supported by the Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) (Spain) and Fondo Europeo de Desarrollo Regional (FEDER) under project PGC2018-096899-B-I00 (MCIU/AEI/FEDER, UE). The authors thank S. Reich and E. Molho for useful remarks that helped them in preparing this paper. Data sharing not applicable to this article as datasets were generated or analysed during the current study.

References

1. Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**, 367–426 (1996)
2. Bauschke, H.H., Borwein, J.M.: On the convergence of von Neumann's alternating projection algorithm for two sets. Set-Valued Anal. **1**, 185–212 (1993)
3. Beer, G.: Topologies on closed and closed convex sets. Mathematics and its applications. Kluwer Academic Publishers Group, Dordrecht (1993)
4. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham (2017)
5. Borwein, J.M., Sims, B., Tam, M.K.: Norm convergence of realistic projection and reflection methods. Optimization **64**, 161–178 (2015)
6. Borwein, J.M., Zhu, Q.J.: Techniques of Variational Analysis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer-Verlag, New York (2005)
7. R.I. Boţ and D. Meier, *A strongly convergent Krasnosel'skiĭ-Mann-type algorithm for finding a common fixed point of a countably infinite family of nonexpansive operators in Hilbert spaces*, arXiv preprint arXiv:1911.11656, (2019)
8. Casini, E., Miglierina, E.: The geometry of strict maximality. SIAM J. Optim. **20**, 3146–3160 (2010)
9. Censor, Y., Cegielski, A.: Projection methods: an annotated bibliography of books and reviews. Optimization **64**, 2343–2358 (2015)
10. Combettes, P.L.: Solving monotone inclusions via compositions of nonexpansive averaged operators. Optimization **53**, 475–504 (2004)
11. De Bernardi, C.A., Miglierina, E., Molho, E.: Stability of a convex feasibility problem. J. Global Optim. **75**(4), 1061–1077 (2019)
12. De Bernardi, C.A., Veselý, L.: Intersection properties of the unit ball. J. Math. Anal. Appl. **475**, 1108–1129 (2019)
13. De Bernardi, C.A., Somaglia, J., Veselý, L.: Star-finite coverings of Banach spaces. J. Math. Anal. Appl. **491**(2), 124384 (2020)
14. De Bernardi, C.A., Veselý, L.: Tilings of normed spaces. Can. J. Math. **69**, 321–337 (2017)
15. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory. The Basis for Linear and Nonlinear Analysis CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York (2011)
16. Franchetti, C., Light, W.: On the von Neumann alternating algorithm in Hilbert space. J. Math. Anal. Appl. **114**, 305–314 (1986)
17. Franchetti, C., Light, W.: The alternating algorithm in uniformly convex spaces. J. Lond. Math. Soc. **29**, 545–555 (1984)
18. Grzesik, A., Kaczor, W., Kuczumow, T., Reich, S.: Convergence of iterates of nonexpansive mappings and orbits of nonexpansive semigroups. J. Math. Anal. Appl. **475**, 519–531 (2019)
19. Hundal, H.S.: An alternating projection that does not converge in norm. Nonlinear Anal. **57**, 35–61 (2004)
20. Klee, V., Veselý, L., Zanco, C.: Rotundity and smoothness of convex bodies in reflexive and nonreflexive spaces. Stud. Math. **120**, 191–204 (1996)
21. Kopecká, E., Reich, S.: A note on alternating projections in Hilbert space. J. Fixed Point Theory Appl. **12**, 41–47 (2012)
22. Kopecká, E., Reich, S.: A note on the von Neumann alternating projections algorithm. J. Nonlinear Conv. Anal. **5**, 379–386 (2004)
23. Lucchetti, R.: Convexity and Well-Posed Problems. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York (2006)
24. Matoušková, E., Reich, S.: The Hundal example revisited. J. Nonlinear Conv. Anal. **4**, 411–427 (2003)

25. Penot, J.-P., Zălinescu, C.: Continuity of usual operations and variational convergences. Set-Valued Anal. **11**, 225–256 (2003)
26. Pustylnik, E., Reich, S., Zaslavski, A.J.: Inner inclination of subspaces and infinite products of orthogonal projections. J. Nonlinear Conv. Anal. **14**, 423–436 (2013)
27. von Neumann, J.: On rings of operators. Reduction theory. Ann. Math. **50**, 401–485 (1949)