Journal of Global Optimization (2021) 80:177–193
https://doi.org/10.1007/s10898-021-01016-z

Necessary conditions for weak minima and for strict minima of order two in nonsmooth constrained multiobjective optimization

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Received: 9 March 2020 / Accepted: 20 March 2021 / Published online: 21 April 2021

Abstract
In this paper, we give necessary conditions for the existence of a strict local minimum of order two for multiobjective optimization problems with equality and inequality constraints. We suppose that the objective function and the active inequality constraints are only locally Lipschitz. We consider both regular equality constraints and degenerate equality constraints. This article could be considered as a continuation of [E. Constantin, Necessary Conditions for Weak Efficiency for Nonsmooth Degenerate Multiobjective Optimization Problems, J. Global Optim, 75, 111-129, 2019]. We introduce a constraint qualification and a regularity condition, and we show that under each of them, the dual necessary conditions for a weak local minimum of the aforementioned article become of Kuhn-Tucker type.

Keywords Weak local minimum · Strict local minimum of order two · Nonsmooth multiobjective optimization · Degenerate equality constraints · Locally Lipschitz optimization problems · Kuhn-Tucker dual necessary optimality conditions

Mathematics Subject Classification 90C29 · 49K27 · 90C30 · 90C48

1 Introduction

We consider the following optimization problem

$$\text{Minimize } f(x) \text{ subject to } x \in D, \text{ and } g_i(x) \le 0, \; i = 1, 2, ..., m, \tag{P}$$

where X is a Banach space, $f = (f_1, ..., f_p) : U \to \mathbb{R}^p$, the functions f_k, $k = 1, ..., p$, $g_i : U \to \mathbb{R}$, $i = 1, ..., m$, are locally Lipschitz on an open set U, and D is an arbitrary set, $D \subseteq U \subseteq X$.

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Also we consider problem (P_1), which is a particular case of problem (P), obtained from (P) by taking $D = D_h = \{z \in X;\ h(z) = 0\}$, where $h : X \to Y$, and Y is a Banach space.

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0,\ i = 1, 2, ..., m, \text{ and } h(x) = 0. \qquad (P_1)$$

We suppose that the objective function and the active inequality constraint functions are only locally Lipschitz. We consider both regular and degenerate equality constraints. An equality constraint given as the null-set of a Fréchet differentiable operator is said to be regular at a point in this set, if the derivative of the operator at that point is onto. Otherwise, the equality constraint is said to be degenerate (irregular, abnormal).

In the literature there exist only a few papers concerning degenerate optimization problems with both inequality and equality constraints, namely Ledzewicz-Schattler, [21,22], which deal with scalar problems with sufficiently often continuously differentiable data, and Constantin, [12], which deals with nonsmooth multiobjective problems.

In [12], we established dual necessary conditions (Fritz-John type Lagrange multiplier rules) for a weak local minimum to problem (P_1) with inequality constraints, and degenerate equality constraints that are 2-regular in the sense of Tret'yakov, [33]. We presented a constraint qualification under which our dual necessary optimality conditions became of Kuhn-Tucker type (i.e., at least one of the multipliers associated to the components of the objective function is nonzero). Also, in [12], we gave a necessary condition for a degenerate equality constraint function to be 2-regular.

In this paper, we complete the work from [12]. We introduce one more constraint qualification and a regularity condition, and we prove that, under each of them, our dual Fritz-John necessary conditions, established in [12] for the existence of a weak local minimum for problem (P_1), become of Kuhn-Tucker type. We show that our necessary condition of [12] for a degenerate function to be 2-regular is sufficient too. We extend some results from Sect. 5, [25].

Moreover, we continue the investigations from [12]. We develop necessary conditions for a strict local minimum of order two for multiobjective optimization problems with inequality constraints, and with an arbitrary set constraint or with equality constraints. This type of minimum is useful for studying the convergence of iterative numerical procedures and for providing stability conditions in optimization problems. Optimality conditions for such minima for nonsmooth scalar optimization problems have been derived by many authors including Ginchev-Ivanov, [14], Ivanov, [16–18], Constantin, [7,8,11]. Sufficient conditions for a higher-order strict local minimum for nonsmooth multiobjective problems have been given very recently in [11,18,19]. Necessary conditions for a strict local minimum of order two for nonsmooth constrained multiobjective optimization problems have been presented in [13,20,23,24]. The paper [13] deals with locally Lipschitz multiobjective problems with only inequality constraints. The results in [20,23,24] concern multiobjective optimization problems with an arbitrary set constraint and/or inclusion constraints, and are formulated in terms of first-order tangent cones to the constraint set. None of the existing papers give necessary conditions for a strict local minimum of order two for optimization problems with equality constraints. Our necessary conditions for such minima involve the second-order tangent cones to the arbitrary constraint set. Our characterizations of the second-order tangent cones [6,9,12], allow us to deal with regular equality constraints and also with degenerate equality constraints.

In Sect. 2, we give some preliminaries, and a sufficient condition for a function to be 2-regular (Lemma 1). In Sect. 3, we introduce a constraint qualification and a regularity condition for (P_1), and we show that, under each of them, the dual necessary conditions for a weak local minimum of [12] become of Kuhn-Tucker type. In Sect. 4, we give some primal necessary conditions for a strict local minimum of order two for (P) and (P_1).

2 Preliminaries

We begin with some preliminary definitions and notations.

In this paper, we accept $0 \times (-\infty) = 0$ and $0 \times \infty = 0$.

For a subset A, $cl\ A$ denotes the closure of A and $conv\ A$ denotes the convex hull of A.

Denote $C := \{x \in U;\ g_i(x) \leq 0,\ i = 1, 2, .., m\}$, and the feasible set of problem (P) by $S := C \cap D$.

Let us recall that a point $\bar{x} \in S$ is a weak local minimum to problem (P), if there exists a neighborhood V of $\bar{x}$ such that no $x \in V \cap S$ satisfies $f_i(x) < f_i(\bar{x})$ for all $i = 1, \ldots, p$. The notion of local weak minimum is the concept of local minimum when $f : X \to \mathbb{R}$ in problem (P).

A point $\bar{x} \in S$ is a strict local minimum of order two for (P) (Jiménez, [20]), if there exists a constant $\alpha > 0$ and a neighborhood V of $\bar{x}$ such that

$(f(x) + \mathbb{R}^p_+) \cap B(f(\bar{x}), \alpha\|x - \bar{x}\|^2) = \emptyset,\ \forall x \in S \cap V,\ x \neq \bar{x},$

where $B(f(\bar{x}), \alpha\|x - \bar{x}\|^2)$ denotes the open ball of center $f(\bar{x})$ and radius $\alpha\|x - \bar{x}\|^2$, $\mathbb{R}^p$ is the p-dimensional Euclidean space, and $\mathbb{R}^p_+ = \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p :\ x_i \geq 0,\ i = 1, \ldots, p\}$.

A point $\bar{x} \in S$ is a strict local minimum for (P) (Jimenez, [20]), if there exists a neighborhood V of $\bar{x}$ such that $f(x) - f(\bar{x}) \notin -\mathbb{R}^p_+,\ \forall x \in S \cap V,\ x \neq \bar{x}$.

If $\bar{x}$ is a strict local minimum of order two, then $\bar{x}$ is a strict local minimum, and thus, $\bar{x}$ is a weak local minimum.

Definition 1 Let F be a real-valued locally Lipschitz mapping on an open set U of X, and $x \in U$. Then,

i) ([5]) Clarke's generalized derivative of F at x is defined by

$$F^\circ(x; v) = \limsup_{(u,t)\to(x,0^+)} \frac{F(u + tv) - F(u)}{t},\ v \in X.$$

ii) ([29]) Páles and Zeidan's second-order upper generalized directional derivative of F at x is defined by

$$F^{\circ\circ}(x; v) = \limsup_{t\to 0^+} 2\frac{F(x + tv) - F(x) - tF^\circ(x; v)}{t^2},\ v \in X.$$

Here, $h : X \to Y, X,\ Y$ linear normed spaces, is said to be Fréchet differentiable at x ([1]), if given an arbitrary $\epsilon > 0$, there is $\delta > 0$ for which the inequality $\|h(x+u)-h(x)-\Lambda u\| \leq \epsilon\|u\|$ holds for some operator $\Lambda \in \mathrm{L}(X, Y)$ and for all $u \in X$ such that $\|u\| < \delta$. Here, $\Lambda \in \mathrm{L}(X, Y)$ stands for the space of linear continuous operators from X to Y. The operator Λ is called the Fréchet derivative of h at x and is denoted by $h'(x)$. A function $h : X \to Y$ is twice Fréchet differentiable at x, if $h'(u)$ exists for all u in a neighborhood of x, and $h''(x) := (h')'(x) \in L(X, L(X, Y))$ exists. The higher-order derivatives $h^{(l)}(x)$, $l \geq 3$, are defined by induction. If $h^{(l)}(x)$ exists at each point x in an open set $U \subseteq X$ and the mapping $x \to h^{(l)}(x)$ is continuous in the uniform topology of the space $L(X, \ldots, L(X, Y), \ldots)$ (generated by the norm), then h is said to be l-times continuously differentiable on U (or of class $C^{(l)}(U)$).

If $F : X \to \mathbb{R}$ is Fréchet differentiable at x, then F is Gâteaux differentiable at x and $F'(x) = \nabla F(x)$. If $F : X \to \mathbb{R}$ is continuously differentiable on an open set U, then F is locally Lipschitz on U and $F^{\circ}(x; v) = \nabla F(x)(v) = F'(x)(v)$, for all $v \in X$, $x \in U$. If F is twice Fréchet differentiable on U, and locally Lipschitz on U, then $F^{\circ\circ}(x; v) = F''(x)(v)(v)$, for all $v \in X$, $x \in U$.

Definition 2 ([33]) Let $h : X \to Y$ be twice Fréchet differentiable at $x \in X$. Then, h is said to be 2-regular at x if, given any $v \in X$, $v \neq 0$ with $h''(x)(v)(v) = 0$, we have $h''(x)(v)X = Y$.

We give a necessary and sufficient condition for a function to be 2-regular.

Lemma 1 *Let $h = (h_1, \ldots, h_r) : X \to \mathbb{R}^r$, r positive integer, X an Hilbert space, and $v \in \mathbb{R}^q$, $v \neq 0$. The mapping $h''(\bar{x})(v)(\cdot) : X \to \mathbb{R}^r$ is onto, if and only if the vectors $h_1''(x)(v)$,..., $h_r''(x)(v)$ are linearly independent.*

Then, h is 2-regular at x, if and only if, given any $v \in X$, $v \neq 0$ with $h''(x)(v)(v) = 0$, the vectors $h_1''(x)(v)$,..., $h_r''(x)(v)$ are linearly independent.

Proof In Lemma 2, [12], we have shown that if $h''(\bar{x})(v)(\cdot)$ is onto, then the vectors $h_1''(x)(v)$,..., $h_r''(x)(v)$ must be linearly independent. This implication holds even if X is a general linear normed space.

Conversely, we prove that if X is a Hilbert space and the vectors $h_1''(x)(v)$, ..., $h_r''(x)(v)$ are linearly independent, then the mapping $h''(\bar{x})(v)(\cdot) : X \to \mathbb{R}^r$ must be onto.

We will show that given $w = (w_1, \ldots, w_r) \in \mathbb{R}^r$, we can find real numbers $\lambda_1, \ldots, \lambda_r$ such that $u = \lambda_1 h_1''(x)(v) + \ldots + \lambda_r h_r''(x)(v)$ satisfies

$$h''(x)(v)(u) = w, \tag{1}$$

that is,

$$\begin{cases} < h_1''(x)(v), \lambda_1 h_1''(x)(v) + \ldots + \lambda_r h_r''(x)(v) >= w_1 \\ \vdots \\ < h_r''(x)(v), \lambda_1 h_1''(x)(v) + \ldots + \lambda_r h_r''(x)(v) >= w_r, \end{cases} \tag{2}$$

where $<, >$ denotes the dot product in X.

The above system can be rewritten as

$$\begin{cases} < h_1''(x)(v), h_1''(x)(v) > \lambda_1 + \ldots + < h_1''(x)(v), h_r''(x)(v) > \lambda_r = w_1 \\ \vdots \\ < h_r''(x)(v), h_1''(x)(v) > \lambda_1 + \ldots + < h_r''(x)(v), h_r''(x)(v) > \lambda_r = w_r \end{cases} \tag{3}$$

We will show that the determinant

$$\Delta = \det \begin{bmatrix} < h_1''(x)(v), h_1''(x)(v) > \ldots < h_1''(x)(v), h_r''(x)(v) > \\ \vdots \\ < h_r''(x)(v), h_1''(x)(v) > \ldots < h_r''(x)(v), h_r''(x)(v) > \end{bmatrix} \neq 0. \tag{4}$$

We have $\Delta \neq 0$ because its columns are linearly independent. Indeed, for any $j = 1, \ldots, r$, set $C_j = \begin{bmatrix} < h_1''(x)(v), h_j''(x)(v) > \\ \vdots \\ < h_r''(x)(v), h_j''(x)(v) > \end{bmatrix}$.

Suppose

$$\alpha_1 C_1 + \ldots + \alpha_r C_r = 0, \tag{5}$$

for some real numbers $\alpha_1, \ldots, \alpha_r$. We will prove that $\alpha_1 = \ldots = \alpha_r = 0$.

Equation (5) means

$$\begin{cases} \alpha_1 < h_1''(x)(v), h_1''(x)(v) > + \ldots + \alpha_r < h_1''(x)(v), h_r''(x)(v) >= 0 \\ \vdots \\ \alpha_1 < h_r''(x)(v), h_1''(x)(v) > + \ldots + \alpha_r < h_r''(x)(v), h_r''(x)(v) >= 0. \end{cases} \tag{6}$$

Equivalently, the system (6) can be written as

$$\begin{cases} < h_1''(x)(v), \alpha_1 h_1''(x)(v) + \ldots + \alpha_r h_r''(x)(v) >= 0 \\ \vdots \\ < h_r''(x)(v), \alpha_1 h_1''(x)(v) + \ldots + \alpha_r h_r''(x)(v) >= 0. \end{cases} \tag{7}$$

We denote $b = \alpha_1 h_1''(x)(v) + \ldots + \alpha_r h_r''(x)(v)$.

We multiply the j-th equation of (7) by α_j, $1 \le j \le r$, to get

$$\begin{cases} < \alpha_1 h_1''(x)(v), b >= 0 \\ \vdots \\ < \alpha_r h_r''(x)(v), b >= 0. \end{cases} \tag{8}$$

Adding the equations of system (8), we obtain $< b, b >= 0$, so $b = 0$.

Since the vectors $h_1''(x)(v), \ldots, h_r''(x)(v)$ are assumed to be linearly independent, it follows that $\alpha_j = 0$, $1 \le j \le r$. Thus, the columns $C_1, \ldots, C_r$ of the determinant Δ are linearly independent, which implies $\Delta \neq 0$.

Therefore, system (3) has a unique solution $\lambda_1, \ldots, \lambda_r$. Since system (2) and equation (1) are each equivalent to system (3), we have showed that, given $w \in \mathbb{R}^r$, there exists a vector $u = \lambda_1 h_1''(x)(v) + \ldots + \lambda_r h_r''(x)(v) \in X$, which satisfies $h''(x)(v)(u) = w$, that is, the mapping $h''(\bar{x})(v) : X \to \mathbb{R}^r$ is onto. □

Definition 3 i) (Ursescu, [32]) An element $v \in X$ is called a tangent vector to D at x, if

$$\lim_{t \to 0^+} \frac{1}{t} d(x + tv; D) = 0. \tag{9}$$

ii) (Pavel-Ursescu, [31]) An element $w \in X$ is called a second-order tangent vector to D at $x \in D$, if there is $v \in X$ such that

$$\lim_{t \to 0^+} \frac{1}{t^2} d(x + tv + \frac{t^2}{2} w; D) = 0, \tag{10}$$

where $d(x; D) = \inf\{\|x - y\|;\ y \in D\}$.

The vector v is said to be associated to w. The tangent cone is also known as the adjacent cone or Ursescu cone.

The sets of all first and second-order tangent vectors to D at $x \in D$ are denoted by $T_x D$ and $T_x^2 D$, respectively. It is known that $T_x D$ is a closed cone in X (Proposition 1.2, [28]), and $T_x^2 D$ is a cone in X (Proposition 1.8 ii), [28]). If $w \in T_x^2 D$ with associated vector v, then $v \in T_x D$ (Proposition 1.8, i), [28]).

It is obvious that if x is an interior point of D, then $T_x D = T_x^2 D = X$.

Proposition 1 *[30] i) A vector v belongs to $T_x D$, if and only if there exists a function $\gamma_1 : (0, \infty) \to X$ such that $\gamma_1(t) \to 0$ as $t \to 0^+$, and*

$$x + t\,(v + \gamma_1(t)) \in D,\ \forall t > 0. \tag{11}$$

ii) A vector w belongs to $T_x^2 D$ with the corespondent vector $v \in X$, if and only if there exists a function $\gamma_2 : (0, \infty) \to X$ with $\gamma_2(t) \to 0$ as $t \to 0^+$, and

$$x + tv + \frac{t^2}{2}(w + \gamma_2(t)) \in D, \ \forall t > 0. \tag{12}$$

It can easily be seen that $0 \in T_x D$ (take $\gamma_1 \equiv 0$), and $0 \in T_x^2 D$ (take $\gamma_2 \equiv 0, \ v = 0$).

There are known several situations when the tangent cones to the null-set of a mapping $h : X \to Y$, i.e., $D = D_h = \{z \in X; h(z) = 0\}$, can be determined (see [1], Pavel-Ursescu, [31], Constantin, [6,9,12]). We recall the known characterization of the first and second-order tangent vectors because they will be used to analyze some examples and to formulate necessary conditions for a strict local minimum of order two for problem (P_1).

Theorem 1 (Lyusternik's Theorem, [1]) *Let X, Y be Banach spaces, let U be a neighborhood of a point $x \in X$, and let $h : U \to Y$, $h(x) = 0$.*

If h is strictly differentiable at x and $h'(x)$ is onto, then the tangent space to the set $D_h = \{z \in X; \ h(z) = 0\}$ at the point x is given by

$$T_x D_h = \left\{v \in X; \ h'(x)(v) = 0\right\}.$$

Here, a mapping $h : X \to Y$ is said to be strictly differentiable at a point x ([1]), if there exists a linear continuous operator $\Lambda \in L(X, Y)$ with the property that, for any $\epsilon > 0$, there is $\delta > 0$ such that for all x_1 and x_2 satisfying the inequalities $\|x_1 - x\| < \delta$ and $\|x_2 - x\| < \delta$, the inequality $\|h(x_1) - h(x_2) - \Lambda(x_1 - x_2)\| \le \epsilon \|x_1 - x_2\|$ holds.

If $h : X \to Y$ is strictly differentiable at a point x, then h is Fréchet differentiable at x.

Theorem 2 (Constantin, Theorem 3.5, [9]) *Let X and Y be Banach spaces, let U be a neighborhood of a point x in X.*

Assume that $h : U \to Y$ is strictly differentiable at $x \in U$ with $h(x) = 0$, its derivative $h^{'}(x) : X \to Y$ is onto, and h is twice Fréchet differentiable at $x \in D_h = \{z \in X; \ h(z) = 0\}$.

Then, $w \in T_x^2 D_h$ with associated $v \in T_x D_h$, if and only if

$$h^{'}(x)(v) = 0, \text{ and}$$
$$h^{''}(x)(v)(v) + h^{'}(x)(w) = 0.$$

Theorem 3 *(i) (Pavel-Ursescu, Corollary 3.1, [31]) Assume that X is a linear normed space, Y is a finite dimensional normed space, $h : X \to Y$ is Fréchet differentiable at x, $h(x) = 0$, $h^{'}(x)$ is onto, and h is continuous near x. Let $D_h = \{z \in X; h(z) = 0\}$.*

Then, $v \in T_x D_h$, if and only if $h'(x)(v) = 0$.

(ii) (Pavel-Ursescu, Corollary 3.2, [31]) Assume that X is a linear normed space, Y is a finite dimensional normed space, $h : X \to Y$ is twice Fréchet differentiable at x and continuous near x, $h(x) = 0$, and $h^{'}(x)$ is onto.

Then, $w \in T_x^2 D_h$ with associated vector $v \in T_x D_h$, if and only if

$$h'(x)(v) = 0,$$
$$h'(x)(w) + h''(x)(v)(v) = 0.$$

The following result is well-known.

Lemma 2 *If $h = (h_1, \ldots, h_r) : \mathbb{R}^q \to \mathbb{R}^r$, r, q positive integers, then $h'(\bar{x})$ is onto, if and only if the gradient vectors $h_1'(\bar{x}), \ldots, h_r'(\bar{x})$ are linearly independent.*

Theorem 4 (Constantin, Theorem 1, [12]) *Assume that X is a linear normed space, Y is a finite dimensional normed space, $h : X \to Y$ is continuous near $x \in D_h$ and three times Fréchet differentiable at x, and h is 2-regular at x.*

Then, $w \in T_x^2 D_h$ with associated vector $v \neq 0$, if and only if

$$h^{''}(x)(v)(v) = 0, \ \ v \neq 0, \text{and}$$
$$h^{'''}(x)(v)(v)(v) + 3h^{''}(x)(v)(w) = 0.$$

The following result is needed for the proof of Theorem 9.

Theorem 5 (Jimenez, Theorem 3.7, a), [20]) *Let $f : \Omega \to \mathbb{R}^p$ be a function and $\bar{x} \in S \subseteq \Omega \subseteq X$, where Ω is an open set. Then, $\bar{x}$ is a strict local minimum of order two of f on an arbitrary set S, if and only if there exist $\alpha > 0$, $\bar{U}$ a neighborhood of $\bar{x}$, and at most p sets V_k, $k \in K' \subset K$, such that $\{V_k \ : \ k \in K'\}$ is a covering of $S \cap \bar{U} \setminus \{\bar{x}\}$, and $f_k(x) > f_k(\bar{x}) + \alpha\|x - \bar{x}\|^2, \forall x \in \bar{S}_k \setminus \{\bar{x}\}$, where $\bar{S}_k = (S \cap \bar{U} \cap V_k) \cup \{\bar{x}\}$.*

Theorem 6 (Theorem 5, Constantin, [12]) *Let U be an open set in $\mathbb{R}^q$, q positive integer, and the functions f_k, $k \in K = \{1, \ldots, p\}$ and g_i, $i \in I = \{1, \ldots, m\}$ be defined on U. Suppose that $\bar{x} \in S$ is a local weak minimum for problem (P_1), the functions g_i, $i \notin I(\bar{x})$ are continuous at $\bar{x}$, and the functions f_k, $k \in K$ and g_i, $i \in I(\bar{x})$ are locally Lipschitz near $\bar{x}$, Gâteaux differentiable, and regular in the sense of Clarke at $\bar{x}$.*

Suppose that the mapping $h : \mathbb{R}^q \to \mathbb{R}^r$, r positive integer, is continuous near $\bar{x}$ and three times Fréchet differentiable at $\bar{x}$, $h'(\bar{x}) = 0$, and h is 2-regular at $\bar{x}$.

Then, corresponding to every critical direction $v \neq 0$ with $h''(\bar{x})(v)(v) = 0$, $f_k^{\circ\circ}(\bar{x}; v) < \infty$, $k \in K(\bar{x}; v)$ and $g_i^{\circ\circ}(\bar{x}; v) < \infty$, $i \in I(\bar{x}; v)$, there are real numbers $e \geq 0$, ν_j, $j \in J = \{1, \ldots, r\}$, $\lambda_k \geq 0$, $k \in K$, and $\mu_i \geq 0$, $i \in I$, $\{\lambda_k, \ \mu_i \ : \ k \in K, \ i \in I(\bar{x})\}$ not all equal to zero, such that

$$\mu_i g_i(\bar{x}) = 0, \ \ i \in I, \tag{13}$$
$$\sum_{k \in K} \lambda_k \nabla f_k(\bar{x}) + \sum_{i \in I(\bar{x})} \mu_i \nabla g_i(\bar{x}) + \sum_{j \in J} 3\nu_j h_j''(\bar{x})(v) = 0, \tag{14}$$
$$\lambda_k \nabla f_k(\bar{x})(v) = 0, \ k \in K, \ \ \mu_i \nabla g_i(\bar{x})(v) = 0, \ i \in I(\bar{x}), \tag{15}$$
$$\sum_{k \in K} \lambda_k f_k^{\circ\circ}(\bar{x}; v) + \sum_{i \in I(\bar{x})} \mu_i g_i^{\circ\circ}(\bar{x}; v) + \sum_{j \in J} \nu_j h_j^{(3)}(\bar{x})(v)(v)(v) = e \geq 0. \tag{16}$$

Denote $I := \{1, 2, \ldots, m\}$, $K := \{1, 2, \ldots, p\}$.

For every point $\bar{x} \in C = \{x \in U : \ g_i(x) \leq 0, \ i = 1, 2, \ldots, m\}$, let $I(\bar{x})$ be the set of active constraints $I(\bar{x}) := \{i \in \{1, 2, \ldots, m\} : \ g_i(\bar{x}) = 0\}$.

The functions f_k, $k \in K$, and g_i, $i \in I(\bar{x})$, are assumed to be locally Lipschitz on U. For fixed vectors $\bar{x} \in U$ and $v \in X$, let the set $I(\bar{x}; v)$ be defined as $I(\bar{x}; v) := \{i \in I(\bar{x}) : g_i^{\circ}(\bar{x}; v) = 0\}$, and the set $K(\bar{x}; v)$ be defined as $K(\bar{x}; v) := \{k \in K : \ f_k^{\circ}(\bar{x}; v) = 0\}$.

A direction v is called critical at the point $\bar{x} \in C$, if $f_k^{\circ}(\bar{x}; v) \leq 0$ for all $k \in K$ and $g_i^{\circ}(\bar{x}; v) \leq 0$ for all $i \in I(\bar{x})$.

3 Kuhn-Tucker necessary conditions for a weak local minimum for (P_1) with degenerate equality constraints

In this sect., we complete the work from [12]. We introduce a constraint qualification and a regularity condition, and we show that, under each of them, the Fritz-John necessary conditions of Theorem 6 for problem (P_1) with degenerate equality constraints become of Kuhn-Tucker type.

First, we introduce a constraint qualifications (C_1) for problem (P_1) with the functions $f_k, k \in K, g_i, i \in I(\bar{x})$ locally Lipschitz on U, and the function $h = (h_1, \ldots, h_r) : X \to \mathbb{R}^r$ three times Fréchet differentiable at $\bar{x}$.
(C_1) The constraint qualification (C_1) is verified at $\bar{x} \in X$ in the direction $v \in X$, if there exists a vector $w \in X$ such that

$$g_i^{\circ}(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v) < 0, \ \forall i \in I(\bar{x}; v),$$
$$3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v) = 0, \ \forall j \in J.$$

We give next an example where our constraint qualification (C_1) is verified, but several known constraint qualifications and regularity conditions for multiobjective optimization problems with locally Lipschitz data do not hold.

Example 1 Let $f = (f_1, f_2)$ with $f_1(x_1, x_2) = |x_1|$, $f_2(x_1, x_2) = -x_1 + x_2^2$, subject to $g_1(x) = x_1 + .25x_2^2 + x_2^4 \le 0$, and $h(x) = x_1^2 + 2x_1x_2 + x_2^3 = 0$, $f_1, \ f_2, \ h, \ g_1 : \mathbb{R}^2 \to \mathbb{R}$.

We examine the point $\bar{x} = (0, 0) \in D \cap D_h$, which is a minimum point of F on $D \cap D_h$. Clearly, f_1 and f_2, g_1 are locally Lipschitz near $\bar{x}$, and f_1 is not differentiable at $\bar{x}$.

The set of critical directions at $\bar{x} \in D$ is the set of all $v = (v_1, v_2) \in \mathbb{R}^2$ with $v_1 = 0$. For a critical direction v, $K(\bar{x}; v) = \{1, 2\}$ and $I(\bar{x}; v) = \{1\}$.

Any nonzero critical direction v satisfies $h''(\bar{x})(v)(v) = 0$, $f_k^{\circ\circ}(\bar{x}; v) < \infty, k \in K(\bar{x}; v)$, and $g_i^{\circ\circ}(\bar{x}; v) < \infty$, $i \in I(\bar{x}; v)$ as $v = (0, v_2) \in \mathbb{R}^2$, $v_2 \ne 0$, and then, $v_1^2 + 2v_1v_2 = 0$. We will show that (C_1) is verified at $\bar{x}$ in any nonzero critical direction v. For any such a direction v, there exists $w \in \mathbb{R}^2$ such that $g_1^{\circ}(\bar{x}; w) + g_1^{\circ\circ}(\bar{x}; v) < 0$, and $3h''(\bar{x})(v)(w) + h'''(\bar{x})(v)(v)(v) = 0$, that is, such that $0.5v_2^2 + w_1 < 0$ and $6(v_1w_1 + v_2w_1 + v_1w_2) + 6v_2^3 = 0$. Indeed, any vector $w = (-v_2^2, w_2)$, $w_2 \in \mathbb{R}$, is a solution. So (C_1) holds.

Let us find the sets Q^i, $i = 1, 2$ considered in [15] and the sets M^i, $i = 1, 2$ considered in [10].
$Q^2 = M^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ f_1(\bar{x}) \le 0, \ g_1(x) \le 0\} = \{\bar{x}\}$ and
$Q^1 = M^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ f_2(\bar{x}) \le 0, \ g_1(x) \le 0\} = \{\bar{x}\}$. Then, $T(M^1 \cap C_h, \bar{x}) = T(M^2 \cap C_h, \bar{x}) = \{\bar{x}\}$. We have $Q = M = M_1 \cap M_2$, $L(M, \bar{x}) = \{v \in \mathbb{R}^2 : \ f_1^{\circ}(\bar{x}; v) \le 0, \ f_2^{\circ}(\bar{x}; v) \le 0, \ g_1^{\circ}(\bar{x}; v) \le 0\}$,
$L(M, \bar{x}) = \{v = (v_1, v_2) \in \mathbb{R}^2 : \ |v_1| \le 0, \ -v_1 \le 0, \ v_1 \le 0\}$.
So $L(Q, \bar{x}) = L(M, \bar{x}) = \{(0, v_2) : \ v_2 \in \mathbb{R}\}$. Also, $Ker\, h'(\bar{x}) = \mathbb{R}^2$ as $h'(\bar{x}) = 0$.

It follows $L(M, \bar{x}) \cap Ker\, h'(\bar{x}) = \{(0, v_2) : \ v_2 \in \mathbb{R}\} \not\subseteq cl\, conv\, T(M^1 \cap C_h, \bar{x})$ and $L(M, \bar{x}) \cap Ker\, h'(\bar{x}) \not\subseteq cl\, conv\, T(M^2 \cap C_h, \bar{x})$, so the Guignard regularity condition (GRC) we introduced in [10] does not hold at $\bar{x}$. Since $L(M, \bar{x}) \cap Ker\, h'(\bar{x}) \not\subseteq \cap_{i=1}^{i=2} T(M^i \cap C_h, \bar{x}) = \{\bar{x}\}$, the generalized Abadie regularity condition (GARC) we introduced in [10] does not hold at $\bar{x}$.

Since $L(Q, \bar{x}) \cap Ker\, h'(\bar{x}) \not\subseteq \cap_{i=1}^{2} T(Q^i \cap C_h, \bar{x}) = \{\bar{x}\}$, the generalized Abadie constraint qualification $(GACQ)$ of Giorgi et.al. [15] is not satisfied at $\bar{x}$. As $L(Q, \bar{x}) \cap Ker\, h'(\bar{x}) \not\subseteq \cap_{i=1}^{2} cl\, conv\, T(Q^i \cap C_h, \bar{x}) = \{\bar{x}\}$, the generalized Guignard constraint qualification $(GGCQ)$ of Giorgi et al. [15] is not verified.

In this example, the basic regularity condition introduced by Chandra et. al. ((4) in [4]) is not satisfied at $\bar{x}$ as for any $i \in \{1, 2\} = I$, there exist nonzero $\tau_r \ge 0$, $r \in \{1, 2\}$, $r \ne i$, $\mu_1 \ge 0$, $\beta \in \mathbb{R}$ such that
$0 \in \sum_{r \in I, r \ne i} \tau_r \partial_C f_r(\bar{x}) + \mu_1 g_1'(\bar{x}) + \beta h'(\bar{x})$. For $i = 1$, the equation
$\tau_2 f_2'(\bar{x}) + \mu_1 g_1'(\bar{x}) + \beta h'(\bar{x}) = 0$ has solution $\tau_1 = \mu_1 = 1 > 0$ and $\beta \in \mathbb{R}$. Similarly, for

$i = 2$, the inclusion $0 \in \tau_1 \partial_C f_1(\bar{x}) + \mu_1 g_1'(\bar{x}) + \beta h'(\bar{x})$ is satisfied with $\tau_1 = \mu_1 = 1 > 0$ and $\beta \in \mathbb{R}$.

Since $h'(\bar{x}) = 0$, the regularity condition (RC2) used in the dual necessary conditions due to Luu, [25] does not hold in this example. In [25], (RC2) assumes that $\sum_{j\in J} \nu_j h'(\bar{x})(v) \geq 0$, for all $v \in T_{\bar{x}} B$, implies $\nu_j = 0$, $j \in J$, where B is a convex constraint set.

Thus, we have shown that our constraint qualification (C_1) does not imply any of the above mentioned constraint qualifications and regularity conditions.

In the following theorem, we show that under the constraint qualification (C_1), our Fritz-John necessary conditions of Theorem 5, [12] become Kuhn-Tucker necessary conditions.

Theorem 7 *Suppose that all the hypotheses of Theorem 6 (Theorem 5, [12]) hold. If (C_1) is assumed at $\bar{x}$ in the nonzero critical direction v with $f_k^{\circ\circ}(\bar{x}; v) < \infty$, $k \in K(\bar{x}; v)$, $g_i^{\circ\circ}(\bar{x}; v) < \infty$, $i \in I(\bar{x}; v)$, and $h''(\bar{x})(v)(v) = 0$, then the multipliers λ_k, $k \in K$, are not all equal to zero.*

Proof Suppose by contradiction that there exists a nonzero critical direction v at which (C_1) holds, $f_k^{\circ\circ}(\bar{x}; v) < \infty$, $k \in K(\bar{x}; v)$, $g_i^{\circ\circ}(\bar{x}; v) < \infty$, $i \in I(\bar{x}; v)$, $h''(\bar{x})(v)(v) = 0$, and (13)–(16) in Theorem 6 are verified with all λ_k, $k \in K$, equal to zero.

Since $h_j''(\bar{x})(v)$, $j \in J$ are linearly independent because $h''(\bar{x})(v)(\cdot)$ is onto as h is 2-regular at $\bar{x}$, from (17) in the proof of Theorem 6 (Theorem 5, [12]), it follows that $\{\lambda_k, \ \mu_i : k \in K(\bar{x}; v),\ i \in I(\bar{x}; v)\}$ are not all equal to zero. Thus, at least one μ_i, $i \in I(\bar{x}; v)$ must be positive. Also, from the proof of Theorem 6, we have that $\mu_i = 0$ if $i \in I(\bar{x}) \setminus I(\bar{x}; v)$. Let $w \in \mathbb{R}^q$ be the vector guaranteed by (C_1). From (14) and (16) we get

$$\sum_{k\in K} \lambda_k \left[\nabla f_k(\bar{x})(w) + f_k^{\circ\circ}(\bar{x}; v)\right] + \sum_{i\in I(\bar{x})} \mu_i \left[\nabla g_i(\bar{x})(w) + g_i^{\circ\circ}(\bar{x}; v)\right] + \sum_{j\in J} \nu_j [3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v)] \geq 0. \tag{17}$$

Hence, if we assume that $\lambda_k = 0$, for all $k \in K$, from (17) we have

$$\sum_{i\in I(\bar{x})} \mu_i [\nabla g_i(\bar{x})(w) + g_i^{\circ\circ}(\bar{x}; v)] + \sum_{j\in J} \nu_j [3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v)] \geq 0. \tag{18}$$

On the other hand, due to the constraint qualification (C_1), we obtain

$$\sum_{i\in I(\bar{x})} \mu_i [\nabla g_i(\bar{x})(w) + g_i^{\circ\circ}(\bar{x}; v)] + \sum_{j\in J} \nu_j [3h_j''(x)(\bar{v})(w) + h_j'''(\bar{x})(v)(\bar{v})(v)] < 0.$$

This contradicts (18). Therefore, the conclusion follows. □

Next, we introduce a regularity condition (R) for problem (P_1) with the functions f_k, $k \in K$, g_i, $i \in I(\bar{x})$ locally Lipschitz on U, and the function $h = (h_1, \ldots, h_r) : X \to \mathbb{R}^r$ three times Fréchet differentiable at $\bar{x}$.

(R): The regularity condition (R) is verified at $\bar{x}$ in the direction $v \in X$, if there exists a vector $w \in X$ and an index $s \in K(\bar{x}; v)$ such that

$$f_k^{\circ}(\bar{x}; w) + f_k^{\circ\circ}(\bar{x}; v) < 0,\ \forall k \in K(\bar{x}; v),\ \ k \neq s,$$
$$g_i^{\circ}(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v) < 0,\ \forall i \in I(\bar{x}; v),$$
$$3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v) = 0,\ \forall j \in J.$$

Theorem 8 *Suppose that all the hypotheses of Theorem 6 (Theorem 5, [12]) hold. If* (*R*) *is assumed at* $\bar{x}$ *in the nonzero critical direction* v *with* $f_k^{\circ\circ}(\bar{x}; v) < \infty$, $k \in K(\bar{x}; v)$, $g_i^{\circ\circ}(\bar{x}; v) < \infty$, $i \in I(\bar{x}; v)$, *and* $h''(\bar{x})(v)(v) = 0$, *then the multipliers* λ_k, $k \in K$, *are not all equal to zero. More precisely,* $\lambda_s > 0$.

Proof Suppose by contradiction that there exists a nonzero critical direction v at which (R) holds, $f_k^{\circ\circ}(\bar{x}; v) < \infty$, $k \in K(\bar{x}; v)$, $g_i^{\circ\circ}(\bar{x}; v) < \infty$, $i \in I(\bar{x}; v)$, $h''(\bar{x})(v)(v) = 0$, and (13)-(16) in Theorem 6 are verified with $\lambda_s = 0$.

Since $h_j''(\bar{x})(v)$, $j \in J$ are linearly independent because $h''(\bar{x})(v)$ is onto as h is 2-regular at $\bar{x}$, from (17) in the proof of Theorem 6 (Theorem 5, [12]), it follows that $\{\lambda_k,\ \mu_i\ :\ k \in K(\bar{x}; v),\ i \in I(\bar{x}; v)\}$ are not all equal to zero. Thus, at least one of $\{\lambda_k,\ \mu_i\ :\ k \in K(\bar{x}; v),\ i \in I(\bar{x}; v)\}$ must be positive. Also, from the proof of Theorem 6, we have that $\mu_i = 0$ if $i \in I(\bar{x}) \setminus I(\bar{x}; v)$ and $\lambda_k = 0$ if $k \in K \setminus K(\bar{x}; v)$.

Let $w \in \mathbb{R}^q$ be the vector guaranteed by (R).

As in the proof of Theorem 7, (17) holds for the directions v and w.

Then, we have

$$\sum_{k\in K(\bar{x};v)} \lambda_k \left[\nabla f_k(\bar{x})(w) + f_k^{\circ\circ}(\bar{x}; v)\right] + \sum_{i\in I(\bar{x};v)} \mu_i[\nabla g_i(\bar{x})(w) + g_i^{\circ\circ}(\bar{x}; v)] + \sum_{j\in J} \nu_j[3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v)] \geq 0.$$

Thus, if $\lambda_s = 0$, then

$$\sum_{k\in K(\bar{x};v),\, k\neq s} \lambda_k \left[\nabla f_k(\bar{x})(w) + f_k^{\circ\circ}(\bar{x}; v)\right] + \sum_{i\in I(\bar{x};\bar{v})} \mu_i[\nabla g_i(\bar{x})(w) + g_i^{\circ\circ}(\bar{x}; v)] + \sum_{j\in J} \nu_j[3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v)] \geq 0. \tag{19}$$

In view of (R), since $\{\lambda_k,\ \mu_i\ :\ k \in K(\bar{x}; v),\ i \in I(\bar{x}; v)\}$ are not all equal to zero, we get

$$\sum_{k\in K(\bar{x};v),\, k\neq s} \lambda_k \left[\nabla f_k(\bar{x})(w) + f_k^{\circ\circ}(\bar{x}; v)\right] + \sum_{i\in I(\bar{x};v)} \mu_i \left[\nabla g_i(\bar{x})(w) + g_i^{\circ\circ}(\bar{x}; v)\right] + \sum_{j\in J} \nu_j \left[3h_j''(\bar{x})(v)(w) + h_j'''(\bar{x})(v)(v)(v)\right] < 0,$$

which contradicts (19). Therefore, $\lambda_s > 0$. □

Remark 1 Theorems 7 and 8 extend to locally Lipschitz multiobjective problems with inequality constraints and degenerate equality constraints some results from Sect. 5, Luu, [25] for such problems with inequality constraints and regular equality constraints, and some results from Maciel et al, [26,27] for twice continuously differentiable multiobjective problems with inequality constraints and equality constraints. In Theorems 6, 7 and 8, as in Theorem 6, [12], we consider degenerate equality constraints that have the first derivatives at $\bar{x}$ equal to zero, so they are not linearly independent, and they do not verify the condition (RC2) used in [25]. Also our degenerate equality constraints do not verify the positive regularity condition (PLIRC) used in Theorem 4.2, [26], or the constant rank constraint qualification (CRCQ) used in Theorem 4.3, [26], and Theorem 1.1, [27].

Except for the dual necessary conditions of Theorems 6, 7, 8, and Theorem 6, [12], and for the primal necessary conditions given in [12], in the literature there exist no other optimality conditions for locally Lipschitz multiobjective problems with inequality constraints and degenerate equality constraints. The exiting papers on extremum problems with degenerate equality constraints deal with sufficiently often continuously differentiable scalar optimization problems (Tret'yakov, [33], Bednarczuk-Tret'yakov, [2], Brezhneva-Tret'yakov, [3], Ledzewicz-Schattler, [21,22]), and with the exception of [21,22], do not consider any other types of constraints.

4 Necessary conditions for strict local minimum of order two for problems (P) and (P_1)

We provide some primal second-order necessary conditions for a point $\bar{x}$ to be a strict local minimum of order two for problem (P) and also for its particular case problem (P_1) (with inequality and equality constraints).

Theorem 9 *Let U be an open set in the Banach space X, and the functions f_k, $k \in K = \{1, \dots, p\}$ and g_i, $i \in I = \{1, \dots, m\}$ be defined on U. Suppose that $\bar{x} \in S$ is a strict local minimum of order two for problem (P), the functions g_i, $i \notin I(\bar{x})$ are continuous at $\bar{x}$, and the functions f_k, $k \in K$ and g_i, $i \in I(\bar{x})$ are locally Lipschitz on U.*

Then, for every nonzero critical direction $v \in T_{\bar{x}}D$, it follows that there is no $w \in X$ which solves the system

$$\begin{cases} f_k^\circ(\bar{x}; w) + f_k^{\circ\circ}(\bar{x}; v) \leq 0, \ k \in K(\bar{x}; v), \\ g_i^\circ(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v) < 0, \ i \in I(\bar{x}; v), \\ w \in T_{\bar{x}}^2 D \text{ with associated vector } v. \end{cases} \tag{20}$$

Proof Suppose the contrary that there exists a nonzero critical direction $v \in T_{\bar{x}}D$ such that system (20) has a solution $w \in X$.

By Corollary 3.1, [12], $K(\bar{x}; v) \cup I(\bar{x}; v) \neq \emptyset$ because $\bar{x}$ is a strict local minimum of order two and thus, $\bar{x}$ is a local weak minimum for problem (P).

Since $w \in T_{\bar{x}}^2 D$ with associated vector v, there exists a mapping $\gamma_2 : (0, +\infty) \to X$ such that $\bar{x} + tv + \dfrac{t^2}{2}(w + \gamma_2(t)) \in D$ for all $t > 0$ and $\gamma_2(t) \to 0$ as $t \to 0^+$.

Consider the following cases concerning the inequality constraints:

(1) For every $i \in \{1, 2, ..., m\} \setminus I(\bar{x})$, we have $g_i(\bar{x}) < 0$. Hence, by continuity, there exists $\epsilon_i > 0$ such that $g_i(\bar{x} + tv + \dfrac{t^2}{2}(w + \gamma_2(t))) < 0$, for all $t \in [0, \epsilon_i)$.

(2) For every $i \in I(\bar{x}) \setminus I(\bar{x}; v)$, we have $g_i^\circ(\bar{x}; v) < 0$. Then, there exists $\epsilon_i > 0$ such that for all $t \in (0, \epsilon_i)$,

$$g_i\left(\bar{x} + tv + \frac{t^2}{2}\left(w + \gamma_2(t)\right)\right) < g_i(\bar{x}) = 0.$$

To show the above inequality, we suppose by contradiction that, for any $\epsilon > 0$, there is $0 < t(\epsilon) < \epsilon$ such that $g_i(\bar{x} + t(\epsilon)v + \dfrac{t^2(\epsilon)}{2}(w + \gamma_2(t(\epsilon)))) \geq g_i(\bar{x})$.

Let $\epsilon_n > 0$ be a sequence convergent to 0 as $n \to \infty$, and $t_n \in (0, \epsilon_n)$ such that $g_i(\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n))) - g_i(\bar{x}) \geq 0$. Then,

$$\begin{aligned} 0 &\leq \limsup_{t_n \to 0} \frac{g_i\left(\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n))\right) - g_i(\bar{x})}{t_n} \\ &\leq \limsup_{t_n \to 0} \frac{1}{t_n}\left[g_i(\bar{x} + t_n v) - g_i(\bar{x})\right] \\ &+ \limsup_{t_n \to 0+} \frac{1}{t_n}\left[g_i\left(\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n))\right) - g_i(\bar{x} + t_n v)\right], \\ 0 &\leq g_i^\circ(\bar{x}; v) + \limsup_{t_n \to 0+} \frac{1}{2} L_i t_n \|w + \gamma_2(t_n)\| \end{aligned}$$

as g_i is locally Lipschitz of constant $L_i > 0$. Therefore, $g_i^\circ(\bar{x}; v) \geq 0$, which contradicts $i \notin I(\bar{x}; v)$.

(3) For every $i \in I(\bar{x}; v)$, using the fact that $g_i^\circ(\bar{x}; v) = 0$, we can show that there exists $\epsilon_i > 0$ such that for all $t \in (0, \epsilon_i)$,

$$g_i\left(\bar{x} + tv + \frac{t^2}{2}(w + \gamma_2(t))\right) < g_i(\bar{x}) = 0.$$

Assume by contradiction that for any $\epsilon > 0$, there is $0 < t(\epsilon) < \epsilon$ such that $g_i(\bar{x} + t(\epsilon)v + \frac{t(\epsilon)^2}{2}(w + \gamma_2(t(\epsilon)))) \geq g_i(\bar{x}) = 0$. Let $\{\epsilon_n\}_{n \geq 0}$ be a positive sequence convergent to 0 and $t_n \in (0, \epsilon_n)$ such that $g_i(\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n))) \geq g_i(\bar{x}) = 0$. Then,

$$\begin{aligned} 0 &\leq g_i\left(\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n))\right) - g_i(\bar{x}) \\ &= \frac{t_n^2}{2}\left[\frac{2}{t_n^2}\left(g_i\left(\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n))\right) - g_i(\bar{x} + t_n v)\right)\right] \\ &+ \frac{t_n^2}{2}\left[\frac{2}{t_n^2}\left(g_i(\bar{x} + t_n v) - g_i(\bar{x}) - t_n g_i^\circ(\bar{x}; v)\right)\right] \end{aligned}$$

After dividing the above inequality by $t_n^2/2$ and taking the upper limit as $t_n \to 0^+$, we obtain

$$0 \leq g_i^\circ(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v),$$

which contradicts the assumption that w is a solution of system (20).

Cases (1), (2), and (3) lead us to the conclusion that $\bar{x} + tv + \frac{t^2}{2}(w + \gamma_2(t)) \in S$ for all $t \in (0, \tilde{\epsilon})$, where $\tilde{\epsilon} = \min_{i \in \{1,2,\ldots,m\}} \epsilon_i$, because for all $t \in (0, \tilde{\epsilon})$ we have $\bar{x} + tv + \frac{t^2}{2}(w + \gamma_2(t)) \in D$ and $g_i(\bar{x} + tv + \frac{t^2}{2}(w + \gamma_2(t))) < 0$, for all $i \in I$.

Since $\bar{x}$ is a strict local minimum of order two of f on S, by Theorem 5 (Theorem 3.7, a), [20]), there exist $\alpha > 0$, $\bar{U}$ a neighborhood of $\bar{x}$, and at most p sets V_k, $k \in K' \subset K$ such that

$\{V_k:\ k \in K'\}$ is a covering of $S \cap \bar{U} \setminus \{\bar{x}\}$, and $f_k(x) > f_k(\bar{x}) + \alpha\|x - \bar{x}\|^2, \forall\, x \in \bar{S}_k \setminus \{\bar{x}\}$, where $\bar{S}_k = (S \cap \bar{U} \cap V_k) \cup \{\bar{x}\}$.

For every positive sequence $\{t_n\}_{n\geq 1}$, $t_n \to 0^+$ as $n \to \infty$, there exist an index $k \in K' \subset K$, and an infinite subsequence, which we can denote again by $\{t_n\}_{n\geq 1}$, such that $\bar{x} + t_n v + \frac{t_n^2}{2}(w + \gamma_2(t_n)) \in \bar{S}_k$ for any positive integer $n \geq \bar{n}$ for some $\bar{n} \geq 1$ because $\{V_k:\ k \in K'\}$ is a covering of $S \cap \bar{U} \setminus \{\bar{x}\}$.

Then, for all $n \geq \bar{n}$, we have

$$f_k\left(\bar{x} + t_n^k v + \frac{t_n^2}{2}\left(w + \gamma_2\left(t_n\right)\right)\right) > f_k(\bar{x}) + \alpha\|t_n v + \frac{t_n^2}{2}\left(w + \gamma_2\left(t_n\right)\right)\|^2. \tag{21}$$

Consider the two possible cases concerning a function f_k, $k \in K'$:

(i) If $k \in K'$ and $k \notin K(\bar{x}; v)$, then there exists $\tilde{n} \geq \bar{n}$ such that for all $n \geq \tilde{n}$

$$f_k\left(\bar{x} + t_n^k v + \frac{\left(t_n^k\right)^2}{2}\left(w + \gamma_2\left(t_n^k\right)\right)\right) < f_k(\bar{x}). \tag{22}$$

To show this, suppose by contradiction that there exists a subsequence of $\{t_n\}_{n\geq\tilde{n}}$, which, for simplicity, we denote by $\{t_n\}_{n\geq\tilde{n}}$ too, such that

$$f_k\left(\bar{x} + t_n v + \frac{t_n^2}{2}\left(w + \gamma_2\left(t_n\right)\right)\right) \geq f_k(\bar{x}),\ \text{ for all } n \geq \bar{n}.$$

From the above inequality, as in case 2), we get $f_k^\circ(\bar{x}; v) \geq 0$, and then, $f_k^\circ(\bar{x}; v) = 0$ as v is a critical direction. This contradicts $k \notin K(\bar{x}; v)$. Thus, inequality (22) holds, which contradicts inequality (21).

(ii) If $k \in K'$ and $k \in K(\bar{x}; v)$, we have $f_k^\circ(\bar{x}; v) = 0$. Inequality (21) implies that the following inequality holds for all $n \geq \bar{n}$

$$\begin{aligned}
&\alpha t_n^2\|v + \frac{t_n}{2}\left(w + \gamma_2\left(t_n\right)\right))\|^2\\
&\leq \frac{t_n^2}{2}\left[\frac{2}{t_n^2}\left(f_k\left(\bar{x} + t_n v + \frac{t_n^2}{2}\left(w + \gamma_2\left(t_n\right)\right)\right) - f_k\left(\bar{x} + t_n v\right)\right)\right]\\
&\quad + \frac{t_n^2}{2}\left[\frac{2}{t_n^2}\left(f_k\left(\bar{x} + t_n v\right) - f_k(\bar{x}) - t_n f_k^\circ(\bar{x}; v)\right)\right].
\end{aligned}$$

Since $v \neq 0$ and $\alpha > 0$, we obtain after dividing the above inequality by $t_n^2/2$ and taking the upper limit as $n \to \infty$, that

$$\begin{aligned}
0 < 2\alpha\|v\|^2 &\leq \limsup_{n\to\infty}\left\{\left[\frac{2}{t_n^2}\left(f_k\left(\bar{x} + t_n v + \frac{t_n^2}{2}\left(w + \gamma_2\left(t_n\right)\right)\right) - f_k\left(\bar{x} + t_n v\right)\right)\right]\right.\\
&\qquad \left. + \left[\frac{2}{t_n^2}\left(f_k(\bar{x} + t_n v) - f_k(\bar{x}) - t_n f_k^\circ(\bar{x}; v)\right)\right]\right\}\\
&\leq \limsup_{n\to\infty}\left[\frac{2}{t_n^2}\left(f_k\left(\bar{x} + t_n v + \frac{t_n^2}{2}\left(w + \gamma_2\left(t_n\right)\right) - f_k\left(\bar{x} + t_n v\right)\right)\right]\\
&\quad + \limsup_{n\to\infty}\left[\frac{2}{t_n^2}\left(f_k\left(\bar{x} + t_n v\right) - f_k(\bar{x}) - t_n f_k^\circ(\bar{x}; v)\right)\right]\\
&\leq f_k^\circ\left(\bar{x}; w\right) + f_k^{\circ\circ}(\bar{x}; v),
\end{aligned}$$

which contradicts the assumption that w is a solution of system (20), i.e., the assumption that $f_k^\circ(\bar{x}; w) + f_k^{\circ\circ}(\bar{x}; v) \le 0$.

Since we arrived at contradictions in both possible cases concerning f_k, $k \in K'$, there is no nonzero critical direction $v \in T_{\bar{x}} D$ for which system (20) has a solution $w \in X$. □

Remark 2 For problems with inequality constraints only ($D = X$), the scalar case of Theorem 9 coincides to Theorem 4, [8]. In view of Remark 5, [8], Theorem 9 improves Theorem 3, [16] for inequality-constrained scalar problems with continuously differentiable data and some second-order directionally differentiable data, and Theorem 6, [17] for inequality-constrained scalar problems with locally Lipschitz, regular and Gâteaux differentiable data and some second-order Hadamard differentiable data.

Next, we apply Theorem 9 to problem (P_1), and we take into account the characterizations of the first and second-order tangent vectors to $D_h = \{z \in X : \ h(z) = 0\}$ at $\bar{x} \in D_h$, given in Theorems 1–4.

Theorem 10 *Let $\bar{x} \in S$ be a strict local minimum of order two of problem (P_1), the functions g_i, $i \notin I(\bar{x})$ be continuous at $\bar{x}$, and the functions f_k, $k \in K$ and g_i, $i \in I(\bar{x})$ be locally Lipschitz on U.*

Suppose in addition that either i) or ii) holds.

i) $h : X \to Y$, Y is a Banach space, h is strictly differentiable at $\bar{x}$ and twice Fréchet differentiable at $\bar{x}$, $h(\bar{x}) = 0$, and $h^{'}(\bar{x})$ is onto.

ii) $h : X \to Y$, Y is a finite dimensional normed space, h is Fréchet differentiable at $\bar{x}$ and continuous near $\bar{x}$, $h(\bar{x}) = 0$, and $h^{'}(\bar{x})$ is onto.

Then, for every critical direction $v \neq 0$ with $h'(\bar{x})(v) = 0$, it follows that there is no $w \in X$ which solves the system

$$\begin{cases} f_k^\circ(\bar{x}; w) + f_k^{\circ\circ}(\bar{x}; v) \le 0, \ k \in K(\bar{x}; v), \\ g_i^\circ(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v) < 0, \ i \in I(\bar{x}; v), \\ h'(\bar{x})(w) + h''(\bar{x})(v)(v) = 0. \end{cases}$$

Theorem 11 *Let $\bar{x} \in S$ be a strict local minimum of order two of problem (P_1), the functions g_i, $i \notin I(\bar{x})$ be continuous at $\bar{x}$, and the functions f_k, $k \in K$ and g_i, $i \in I(\bar{x})$ be locally Lipschitz on U.*

Suppose that $h : X \to Y$, Y is a finite dimensional normed space, h is continuous near $\bar{x} \in D_h = \{z \in X; \ h(z) = 0\}$ and three times Fréchet differentiable at $\bar{x}$, $h'(\bar{x}) = 0$, and h is 2-regular at $\bar{x}$.

Then, for every critical direction $v \neq 0$ with $h''(\bar{x})(v)(v) = 0$, it follows that there is no $w \in X$, which solves the system

$$\begin{cases} f_k^\circ(\bar{x}; w) + f_k^{\circ\circ}(\bar{x}; v) \le 0, \ k \in K(\bar{x}; v), \\ g_i^\circ(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v) < 0, \ i \in I(\bar{x}; v), \\ h'''(\bar{x})(v)(v)(v) + 3h''(\bar{x})(v)(w) = 0. \end{cases}$$

Remark 3 Theorems 9, 10, and 11 extend to problem (P) with inequality constraints and an arbitrary constraint set, and to problem (P_1) with inequality constraints and either regular or degenerate equality constraints, the second-order necessary conditions of Theorem 3.1, [13] for locally Lipschitz multiobjective optimization problems with only inequality constraints. Theorem 3.1, [13] assumes that the Zangwill second-order constraint qualification (ZSCQ) holds, but Theorems 9, 10, 11 do not require any constraint qualification or regularity condition. Theorems 9, 10, 11 can be used to solve problems where Theorem 3.1, [13] is not applicable, as shown by the following two examples.

Example 2 Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f_1(x_1, x_2) = (x_1 - x_2)^4 + x_2^6 + 1$, $f_2(x_1, x_2) = |x_1 + x_2| - x_1 - x_1x_2 + x_2^5$, subject to $x \in C = \{x \in \mathbb{R}^2;\ g_1(x) = 2|x_2| - x_1 - x_2^3 - x_1^2 \leq 0,\ g_2(x) = \sqrt{x_1^2 + x_2^2} - 4 \leq 0\}$, and $x \in D_h = \{(x_1, x_2) \in \mathbb{R}^2;\ h(x) = x_1 + 2x_2 - x_2^2 + x_1^4 = 0\}$.

The point $\bar{x} = (0, 0) \in S = C \cap D_h$ is a strict local minimum of $f = (f_1, f_2)$ on S as $f_1(x) > f_1(\bar{x})$ for all $x \in \mathbb{R}^2, x \neq \bar{x}$.

We have $I(\bar{x}) = \{1\}$. Clearly, f_1 is continuously differentiable and thus, f_1 is locally Lipschitz on $\mathbb{R}^2$, f_2 and g_1 are locally Lipschitz on $\mathbb{R}^2$, but f_2 and g_1 are not differentiable at $\bar{x}$, and g_2 is continuous at $\bar{x}$.

The origin verifies our first-order necessary conditions for a weak local minimum of Theorem 2, [12] because the system $f_k^\circ(\bar{x}; v) < 0$, $k = 1, 2$, and $g_i^\circ(\bar{x}; v) < 0$, $\forall i \in I(\bar{x})$ and $v \in T_{\bar{x}}D_h$, cannot have as a solution any critical direction v as $f_1^\circ(\bar{x}; v) = 0 \not< 0$, for any $v \in \mathbb{R}^2$.

Since $h'(\bar{x}) = (1, 2)$ is onto, by Theorem 3, i) (Pavel-Ursescu, Corollary 3.1, [31]), we get $T_{(0,0)}D_h = \{v = (v_1, v_2) \in \mathbb{R}^2;\ v_1 + 2v_2 = 0\}$, and also by Theorem 3, ii) (Pavel-Ursescu, Corollary 3.2, [31]), we get $w \in T_{\bar{x}}^2 D_h$ with associated vector $v \in T_{\bar{x}}D_h$, if and only if $w_1 + 2w_2 - 2v_2^2 = 0$.

A vector $v = (v_1, v_2)$ is a critical direction, if and only if it solves the system $|v_1 + v_2| - v_1 \leq 0$, $2|v_2| - v_1 \leq 0$, and thus, $-\dfrac{v_1}{2} \leq v_2 \leq 0$. It follows that a critical direction v belongs to $T_{(0,0)}D_h$, if and only if $v_1 = -2v_2 \geq 0$. For any critical direction v, we have $I(\bar{x}; v) = \{1\}$, $K(\bar{x}; v) = \{1\}$ if $v \neq 0$, and $K(\bar{x}; v) = \{1, 2\}$ if $v = 0$.

Our second-order necessary conditions for a weak local minimum of Theorem 3, [12] and Luu's second-order necessary conditions for a weak local minimum (Corollary 3.2, [25]) are satisfied at $\bar{x}$. Indeed, for a critical direction $v \in T_{\bar{x}}D_h$, the systems of those results have no solution $w \in T_{\bar{x}}^2 D_h$ with associated vector v as they contain the inequality $f_1^{\circ\circ}(\bar{x}; v) + f_1^\circ(\bar{x}; w) < 0$, which is not verified for any vector w as $f_1^{\circ\circ}(\bar{x}; v) = f_1^\circ(\bar{x}; w) = 0$, for any $v,\ w \in \mathbb{R}^2$.

For a nonzero critical direction $v \in T_{\bar{x}}D_h$, i.e., for v with $v_1 = -2v_2 > 0$, we can find a $w \in \mathbb{R}^2$, which is a solution of the system of Theorem 10: $f_1^\circ(\bar{x}; w) + f_1^{\circ\circ}(\bar{x}; v) = 0 \leq 0$, $g_1^\circ(\bar{x}; w) + g_1^{\circ\circ}(\bar{x}; v) = 2|w_2| - w_1 - 2v_1^2 < 0$, $h'(\bar{x})(w) + h''(\bar{x})(v)(v) = w_1 + 2w_2 - 2v_2^2 = 0$. Any w with $w_1 + 2w_2 - 2v_2^2 = 0$ and $w_1 \geq 0$, $w_2 \geq 0$ is a solution because for such w, we have $g_1^\circ(\bar{x}; w) + g_1^{\circ\circ}(\bar{x}; v) = -2w_1 - 6v_2^2 < 0$ as $v \neq 0$ implies $v_2 \neq 0$. Thus, Theorem 10 is not satisfied, and $\bar{x}$ is not a strict local minimum of order two. Theorem 3.1, [13] cannot be used here because an equality constraint is present besides the inequality constraints.

Next we give an example with only inequality constraints, that can be analyzed with the aid of the present results, but to which the results from [13] are not applicable.

Example 3 Let us consider the function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}$ subject to $x \in D = \{x \in \mathbb{R}^2 :\ g_1(x) = |x_1| - x_2 + x_1^5 \leq 0\}$, where $f_1(x) = |x_2 - x_1|$ and $f_2(x) = x_1^2 + x_2^2$.

Clearly, $\bar{x} = (0, 0)$ is a strict local minimum of order two of f_2 on $\mathbb{R}^2$, and thus, on D. We have $I(\bar{x}) = \{1\}$. The functions f_1, f_2, and g_1 are locally Lipschitz near $\bar{x}$, and f_1 and g_1 are not differentiable at $\bar{x}$.

The critical directions at $\bar{x}$ are the vectors $v = (v_1, v_2) \in \mathbb{R}^2$ with $v_1 = v_2 \geq 0$. For a critical direction v, we get $K(\bar{x}, v) = \{1,\ 2\}$, and $I(\bar{x}, v) = \{1\}$.

Theorem 9 is applicable with $D = \mathbb{R}^2$. For any nonzero critical direction v, the system (20), formed by the equations $f_1^\circ(\bar{x}; w) + f_1^{\circ\circ}(\bar{x}; v) = |w_2 - w_1| \leq 0$, $f_2^\circ(\bar{x}; w) + f_2^{\circ\circ}(\bar{x}; v) = 2v_1^2 + 2v_2^2 \leq 0$, and $g_1^\circ(\bar{x}; w) + g_1^{\circ\circ}(\bar{x}; v) = |w_1| - w_2 < 0$, has no solution $w = (w_1, w_2) \in \mathbb{R}^2$. Thus, the second-order necessary conditions of Theorem 9 are verified at $\bar{x}$.

The second-order necessary conditions of [13], for locally Lipschitz multiobjective problems with only inequality constraints cannot be used because (ZSCQ) does not hold at $\bar{x}$ in any nonzero critical direction. Let v be any nonzero critical direction at $\bar{x}$, that is, $v_1 = v_2 > 0$. We will show that $B(\bar{x}, v) \nsubseteq cl\ A(\bar{x}, v)$. The set $B(\bar{x}, v) := \{w \in \mathbb{R}^2 : g_i^{\circ}(\bar{x}; w) + g_i^{\circ\circ}(\bar{x}; v) \leq 0, \ \forall i \in I(\bar{x}; v)\}$ becomes $B(\bar{x}, v) = \{w \in \mathbb{R}^2 : |w_1| - w_2 \leq 0\}$, and the set $A(\bar{x}, v) := \{w \in X : \forall i \in I(\bar{x}; v) \ \exists \epsilon_i > 0 \ g_i(\bar{x} + tv + \frac{1}{2}t^2 w) \leq 0, \forall t \in (0, \epsilon_i)\}$ becomes $A(\bar{x}, v) = \{w \in \mathbb{R}^2 : \exists \epsilon > 0 \ g_1(\bar{x} + tv + \frac{1}{2}t^2 w) \leq 0, \forall t \in (0, \epsilon)\}$. Let $\bar{w} = (0, 0) \in B(\bar{x}, v)$. We prove that $\bar{w} \notin cl\ A(\bar{x}, v)$. Suppose by contradiction that there exists $w_n = (w_{n1}, w_{n2}) \in A(\bar{x}, v)$ such that $\lim_{n \to \infty} w_n = \bar{w}$. We have that there exists $\bar{\epsilon} > 0$ such that $|v_1 + \frac{1}{2}tw_{1n}| - (v_2 + \frac{1}{2}tw_{2n}) + t^4(v_1 + \frac{1}{2}tw_{1n})^5 \leq 0, \forall t \in (0, \bar{\epsilon})$. Letting $n \to \infty$, we obtain $|v_1| - v_2 + t^4 v_1^5 \leq 0, \forall t \in (0, \bar{\epsilon})$, and we arrived at a contradiction as $v_1 = v_2 > 0$. So $\bar{w} \notin$ cl $A(\bar{x}, v)$. Therefore, $B(\bar{x}, v) \nsubseteq$ cl $A(\bar{x}, v)$, i.e., (ZSCQ) does not hold at $\bar{x}$ in the direction v. Thus, the second-order necessary conditions for a weak local minimum and for a strict local minimum of order two given in [13] are not applicable here.

5 Conclusions

In this paper, we considered the same problems (P) with inequality constraints and an arbitrary constraint set and (P_1) with inequality and equality constraints as in [12]. We proved that our dual necessary conditions for a weak local minimum of (P_1) with degenerate equality constraints become of Kuhn-Tucker type when either the new constraint qualification (C_1) or the new regularity condition (R) are assumed. Then, we gave necessary conditions for a strict local minimum two for those problems under the same hypotheses as in [12]: locally Lipschitz objective and active inequality constraint functions, and either regular or degenerate equality constraint functions. Thus, we extended results from [8,12,13,16,17,25].

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