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Essential stability in unified vector optimization

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Abstract
The emphasis of the paper is to examine the essential stability of efficient solutions for semicontinuous vector optimization problems, subject to the perturbation of objective function and feasible set. We obtain sufficient conditions for existence and characterization of essential efficient solutions, essential sets and essential components, where the efficient solutions are governed by an arbitrary preference relation in a real normed linear space. Further, we establish the density of the set of stable vector optimization problems in the sense of Baire category. We also exhibit that essential stability is weaker than examining continuity aspects of solution sets.

Keywords Vector optimization · Essential sets · Essential components · Continuity · Semicontinuity

Mathematics Subject Classification 90C29 · 90C31 · 46N10 · 49J45

1 Introduction

In literature various stability aspects, such as continuity, convergence, density and so on have been extensively studied in [1,11,19,23] and references therein. In 1950, Fort [7] proposed another aspect of stability, namely essential stability. In this paper he introduced a notion of essential fixed points for continuous mappings over compact sets. In 1952, Kinoshita [15] considered a notion of essential components of fixed points. Since then, various authors have considered the notions of essential solutions, essential sets and essential components of solution sets for optimization problems; see [17,18,20,24,26–28]. This study has been extended to the study of KKM points, Ky Fan's points, Nash equilibrium points and vector equilibrium problems, for more details see [9,29–31].

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Essential stability ensures some sort of continuity of the solution set mappings. Each perturbed problem which is sufficiently near the original problem, has solutions lying arbitrarily near the solutions of the original problem. There are mainly two aspects widely studied pertaining to essential stability of solution sets, existence results and density of essentially stable problems. In [20], Luo examined necessary and sufficient conditions for the existence of essential solutions and essential components of solution sets for lower semicontinuous scalar optimization problems. Xiang and Zhou [26,27] analyzed these existence results for continuous vector optimization problems. Further, they also examined the relation between essential solutions, essential sets, lower semicontinuity and upper semicontinuity of solution set mappings. Yu [28] established a density of stable problems among a set of all vector optimization problems in the sense of Baire category. Recently, Long et al. [18] proved density results for semi-infinite optimization problems under the perturbation of both constraint and objective mappings.

There are many economic models in infinite-dimensional spaces where preference relations are specified via sets having empty interior; see [14,33] and references therein. This prompts the need for unifying the well-studied convex cone ordering structure to general preference structures. The first attempt in this direction was by Yu [32], followed by other researchers [4–6,13,22,25]. In this direction, we consider a vector optimization problem in real normed linear spaces, where the solution concepts are based on arbitrary preference relations. The available literature on essential stability is mostly confined to continuous vector optimization problems. However, in this paper we examine essential stability for semicontinuous vector optimization problems.

In the above unified setting and under the perturbation of both objective function and feasible set, we study the stability results of [26]. We begin by examining essential sets for (lower) upper semicontinuous efficient solution mappings. Further, we establish the density of stable vector optimization problems in the sense of Baire category. We next establish the existence of essential solution sets, minimal essential sets and essential components under suitable assumptions and exhibit that the essential stability is weaker than the lower semicontinuity and the upper semicontinuity of solution set-valued mappings. Moreover, we also obtain a complete characterization of essential efficient solutions, essential sets and essential components of efficient solution set.

The content of the paper is outlined in 7 sections. Section 2 presents some basic definitions and notations required in the sequel. In Sect. 3, we introduce the notions of essential efficient solutions, essential sets and essential components. In Sect. 4, we examine the relation between continuity of efficient solution mappings and corresponding essential solutions and essential sets, followed by the density aspects of stable problems. Section 5 deals with the existence of minimal essential sets and essential components of efficient solutions, followed by their complete characterizations in Sect. 6. Finally, concluding remarks are highlighted in Sect. 7.

2 Preliminaries

Let $(Y, \|\cdot\|)$ be a real normed linear space. For a nonempty set B in Y, the interior and closure of B are denoted by $\operatorname{int} B$ and $\operatorname{cl} B$, respectively. We denote the open ball of radius $\delta > 0$ centered at 0_Y by $B_\delta(0_Y)$, and the set of nonnegative and positive real numbers by $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively. We consider a preference relation $\preceq_D$ induced by a nonempty proper subset D of Y. For $y,\ y' \in Y$,

$$y \preceq_D y' \quad \Longleftrightarrow \quad y' \in y + D.$$

Clearly, the preference relation $\preceq_D$ is reflexive if $0_Y \in D$, antisymmetric if $D \cap (-D) = \{0_Y\}$ and transitive if $D + D \subseteq D$. Further, if $D + D \subseteq D$ and $\operatorname{int} D \neq \emptyset$, then $D + \operatorname{int} D \subseteq \operatorname{int} D$ and $0_Y \notin \operatorname{int} D$.

Let (X, d) be a metric space, A be a nonempty subset of X and $f : X \to Y$ be a vector-valued mapping. Consider the following vector optimization problem

$$\text{(P)} \quad D\text{-}\min\ f(x) \qquad \text{subject to} \ \ x \in A.$$

A vector $\bar{x} \in A$ is said to be an *efficient solution* of (P), if for each $x \in A$ either $f(x) \npreceq_D f(\bar{x})$ or $f(x) = f(\bar{x})$, that is,

$$(f(A) - f(\bar{x})) \cap (-D) \subseteq \{0_Y\},$$

(see Page 1124 [25]). If $0_Y \notin D$, then the above relation reduces to $(f(A) - f(\bar{x})) \cap (-D) = \emptyset$. Since D is an arbitrary preference relation, the efficiency notion above unifies various existing solution concepts for (P) studied in the literature. Indeed, if $D = K$ ($D = \operatorname{int} K$), where $K \subseteq Y$ is a closed, convex and pointed cone with a nonempty interior, then the above notion reduce to the notions of efficient (weak efficient) solution considered in [12,14,19].

If E denotes the set of efficient solutions of (P) and $M = \{f(x) \mid x \in E\}$, then for each $y \in M$, we have $f^{-1}(y) \subseteq E$ and

$$E = \bigcup_{y \in M} f^{-1}(y),$$

where $f^{-1}(y) := \{x \in A \mid f(x) = y\}$.

We next recall the notions of semicontinuity of vector-valued mappings with respect to the preference relation D from [18, Definition 2.3]. Let $f : X \to Y$ be a vector-valued mapping and $\bar{x} \in X$. Then, f is said to be *D-lower semicontinuous* (resp. *D-upper semicontinuous*) at $\bar{x} \in X$, if for any open neighbourhood V of $f(\bar{x})$ in Y, there exists an open neighbourhood U of $\bar{x}$ in X such that $f(x) \in V + D$ (resp. $f(x) \in V - D$) for all $x \in U$. Moreover, f is said to be *continuous* at $\bar{x}$, if $f(x) \in V$ for all $x \in U$. Further, we say f is D-lower semicontinuous (resp. D-upper semicontinuous, continuous) on a nonempty subset $A \subseteq X$, if f is D-lower semicontinuous (resp. D-upper semicontinuous, continuous) at each $x \in A$.

It can be observed that every continuous mapping is both D-lower semicontinuous and D-upper semicontinuous, if $0_Y \in D$ and the converse holds, if D is normal, that is, for any open neighbourhood V of 0_Y in Y, the set $(V + D) \cap (V - D)$ is bounded (see Theorem 2.2.10 in [10]). Further, from [10, Corollary 2.2.11], it is evident that if $Y = \mathbb{R}^n$, D is closed and $D \cap (-D) = \{0_Y\}$, then D is normal, however the same cannot be claimed if Y is infinite dimensional (see Example 2 below).

Further, we recall the notions of continuity of set-valued mappings from [14, Definition 3.1.1], which we prefer to call as semicontinuity notions. Let $F : A \rightrightarrows Y$ be a set-valued mapping. Then, F is said to be *lower semicontinuous* (resp. *upper semicontinuous*) at $\bar{x} \in A$, if for any open set V in Y with $F(\bar{x}) \cap V \neq \emptyset$ (resp. $F(\bar{x}) \subseteq V$), there exists an open neighbourhood U of $\bar{x}$ such that $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$) for all $x \in U \cap A$. Also, F is said to be *continuous* at $\bar{x}$, if F is both lower semicontinuous and upper semicontinuous at $\bar{x}$. These continuity notions can be extended to the set A, if these notions hold at each $x \in A$.

Proposition 3.1.10.(iv) in [14] states that, if F is upper semicontinuous at $\bar{x} \in A$ and $F(\bar{x})$ is compact, then for any sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ with $x_n \to \bar{x}$ and $y_n \in F(x_n)$, $n \in \mathbb{N}$ with $y_n \to \bar{y}$, we have $\bar{y} \in F(\bar{x})$.

Remark 1 For a vector-valued mapping $f : X \to Y$, if we consider a set-valued mapping $F : X \rightrightarrows Y$ defined as $F(x) := f(x) + D$, where D is preorder relation on Y (that is, reflexive and transitive), then for $\bar{x} \in X$, it is easy to observe that

(a) f is D-upper semicontinuous at $\bar{x}$ if and only if F is lower semicontinuous at $\bar{x}$.
(b) f is D-lower semicontinuous at $\bar{x}$ if F is upper semicontinuous at $\bar{x}$.

The notions of continuity of set-valued mappings with respect to the arbitrary preference relation D on Y can be considered as in [10]. So, for further relations between continuity notions of vector-valued and set-valued mappings we refer to the book by Göpfert et al. [10].

We next propose to provide an existence theorem for (P) with D-lower semicontinuous objective mapping. From [19], we recall that if D is a closed convex cone and $f : X \to Y$ is D-lower semicontinuous on X, then f is level-closed, that is, for each $y \in Y$, the level set L_y defined as

$$L_y := \{x \in X \mid f(x) \preceq_D y\},$$

is closed. Thus, the following lemma is immediate from Corollary 5.10 in [19].

Lemma 1 *Suppose that A is closed, f is D-lower semicontinuous on A, D is closed and $D + D \subseteq D$, then $L_y \cap A$ is closed for all $y \in Y$.*

Flores-Bazán et al. [6] established an existence result for efficient solutions of a vector optimization problem with level-closed objective mapping by considering a general preorder relation. Assuming the conditions $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$, it can be seen that the notion of efficiency considered in [6] coincides with the notion considered in this paper. Hence, the following existence theorem follows from Theorem 5.1 in [6] and Lemma 1.

Theorem 1 *Suppose that A is compact, f is D-lower semicontinuous on A, D is closed, $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$, then there exists an efficient solution of* (P).

Remark 2 Note that there may not exist a solution for (P), if D is a preorder relation with $D \cap (-D) \neq \{0_Y\}$. Consider a problem (P), where $f : \mathbb{R} \to \mathbb{R}^2$ is defined as

$$f(x) = \begin{cases} (0, 0), & \text{if } x = 0, \\ (1, 0), & \text{if } x \neq 0, \end{cases}$$

$D \subseteq \mathbb{R}^2$ be defined as $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \in \mathbb{N} \cup \{0\}\}$ and $A = [0, 1]$. Clearly, f is D-lower semicontinuous on compact set A, however $D \cap (-D) = \{(x, 0) \mid x \in \mathbb{R}\}$ and we have $E = \emptyset$.

3 Essential sets and essential components

An essential solution of a problem is an efficient solution, which has in its neighbourhood, an efficient solution of perturbed problems, lying arbitrarily close to the given problem. Similarly, in the neighbourhood of essential sets there exists efficient solutions of the perturbed problems which are arbitrarily close to the given problem. For problems with compact feasible sets and semicontinuous objective mappings, we now introduce these notions by perturbing both the objective function and the feasible set.

Let $K_0(X)$ be the space of all nonempty compact subsets of X. For $A_1, A_2 \in K_0(X)$, we consider the Hausdorff distance between A_1 and A_2 defined by

$$h(A_1, A_2) := \max\{e(A_1, A_2), e(A_2, A_1)\},$$

where $e(A_1, A_2) = \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} d(a_1, a_2)$ (see Definition 2.3 in [17]). It can be seen that $(K_0(X), h)$ is a metric space with metric h. Further, if (X, d) is a complete metric space, then so is $(K_0(X), h)$ (see Theorem 4.3.9 in [16]). For a sequence $(A_n)_{n\in\mathbb{N}}$ in $K_0(X)$ and $A \in K_0(X)$, we say that $A_n \to A$, if $h(A_n, A) \to 0$. The next lemma follows from [16, Corollary 4.2.3] and [28, Lemma 1].

Lemma 2 *Let $(A_n)_{n\in\mathbb{N}} \subseteq K_0(X)$ and $A \in K_0(X)$ be such that $A_n \to A$. Then, the following conditions hold:*

(i) If $x_n \in A_n$ for every $n \in \mathbb{N}$ and $x_n \to x$, then $x \in A$.
(ii) If $x \in A$ and U is any neighbourhood of x in X, then $A_n \cap U \neq \emptyset$ for sufficiently large n.
(iii) $\bigcup_{n=1}^{\infty} A_n \cup A \in K_0(X)$.

In the above lemma, it is evident from the condition (ii), that if $A_n \to A$, then for any $x \in A$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$, where $x_n \in A_n$ for each $n \in \mathbb{N}$. Thus from (i) and (ii), it follows that $A_n \to A$ in the sense of Kuratowski–Painlevé set convergence (see Definition 5.2.3 in [2]).

Further, let $\mathcal{LU}(X, Y)$ be the space of all vector-valued mappings from X to Y that are both D-lower semicontinuous and D-upper semicontinuous on X. For $f_1,\ f_2 \in \mathcal{LU}(X, Y)$, we consider a metric ρ on $\mathcal{LU}(X, Y)$ defined as

$$\rho(f_1, f_2) = \min\left\{ \sup_{x\in X} \| f_1(x) - f_2(x)\|,\ \frac{1}{2} \right\}.$$

It is stated fact that $(\mathcal{LU}(X, Y), \rho)$ is a metric space (see Page 56 in [21]).

Next, we consider the space S defined as

$$S := K_0(X) \times \mathcal{LU}(X, Y).$$

It can be seen that (S, ϱ) is a metric space, where for $s_i = (A_i, f_i) \in S$, $i = 1, 2$, the metric ϱ is defined as

$$\varrho(s_1, s_2) = h(A_1, A_2) + \rho(f_1, f_2).$$

For each $s = (A, f) \in S$, we consider the following vector optimization problem

$$\begin{aligned} (\mathrm{P})_s \quad & D\text{-}\min\ f(x) \\ & \text{subject to}\ \ x \in A. \end{aligned}$$

We denote the set of efficient solutions of $(\mathrm{P})_s$ by $E(s)$. The corresponding mapping $E : S \rightrightarrows X$ is referred to as *efficient solution mapping*. Further, the mapping $M : S \rightrightarrows Y$, where $M(s) := \{f(x) \mid x \in E(s)\}$ is referred to as *minimal solution mapping*.

Throughout this section, we assume that the set $E(s) \neq \emptyset$ for any $s = (A, f) \in S$.

We next define notions of essential solutions and essential solution sets similar to Definitions 2 and 3 considered by Xiang and Zhou in [26].

Definition 1 Let $s = (A, f) \in S$. An element $\bar{x} \in E(s)$ is said to be an *essential efficient solution* of $(\mathrm{P})_s$, if for every open neighbourhood U of $\bar{x}$ in X, there exists an open neighbourhood O of s in S such that $E(s') \cap U \neq \emptyset$ for all $s' \in O$.

The problem $(\mathrm{P})_s$ is said to be *stable*, if every $\bar{x} \in E(s)$ is an essential efficient solution of $(\mathrm{P})_s$.

Definition 2 Let $s = (A, f) \in S$. A nonempty set $e(s) \subseteq E(s)$ is said to be an *essential set* of $E(s)$, if

(a) $e(s)$ is closed in X;
(b) for any open set $U \supseteq e(s)$, there exists an open neighbourhood O of s in S such that $E(s') \cap U \neq \emptyset$ for all $s' \in O$.

Moreover, $e(s)$ is said to be a *minimal essential set*, if it is minimal among all the essential sets of $E(s)$ in terms of set inclusion.

Remark 3 If $e(s)$ is an essential set of $E(s)$ and B is a closed set in X such that $e(s) \subseteq B \subseteq E(s)$, then B is an essential set of $E(s)$.

Remark 4 Any closed subset of $E(s)$ containing an essential efficient solution of $(\mathrm{P})_s$ is an essential set of $E(s)$. However, there may exist an essential set having no essential efficient solution (see Remark 6 below).

Next, we define the notion of an essential component of the efficient solution set similar to the notion considered by Yu et al. [30] (see Page e2416).

Definition 3 Let $s = (A, f) \in S$. A maximal connected subset of $E(s)$ is said to be a *component* of $E(s)$, where maximality is in terms of set inclusion among all connected subsets of $E(s)$. A component $c(s)$ of $E(s)$ containing an essential set of $E(s)$ is said to be an *essential component* of $E(s)$.

4 Essential stability and continuity

In the sequel, we characterize essential sets of lower semicontinuous and upper semicontinuous efficient solution mappings. Moreover, the density of stable problems is obtained in the sense of Baire category.

The following characterization of essential efficient solution and essential set is obvious for a lower semicontinuous efficient solution mapping.

Theorem 2 *Let $s = (A, f) \in S$. Then, every $x \in E(s)$ is an essential efficient solution of $(\mathrm{P})_s$ if and only if E is lower semicontinuous at s. Moreover, the set $e(s) = \{x\}$ is an essential set for every $x \in E(s)$ if and only if E is lower semicontinuous at s.*

To characterize essential sets of the upper semicontinuous efficient solution mappings, we require $\operatorname{int} D \neq \emptyset$. In the next theorem, we establish that strict domination property is a necessary and sufficient condition for the upper semicontinuity of the efficient solution mapping. We next recall that, *strict domination property* holds for $s = (A, f) \in S$, if for each $\bar{x} \in A$ either $\bar{x} \in E(s)$ or there exists $x' \in A$ such that $f(x') \preceq_{\operatorname{int} D} f(\bar{x})$ (see Definition 2.1 in [13]). Further, for $q \in Y \backslash \{0_Y\}$, we denote by $\mathbb{R}_{++}q := \{\lambda q \mid \lambda > 0\}$.

Theorem 3 *Suppose that $s \in S$, D is closed, $\operatorname{int} D \neq \emptyset$, $D + D \subseteq D$ and there exists $q \in Y \backslash \{0_Y\}$ such that $\mathbb{R}_{++}q \subseteq \operatorname{int} D$. Then, E is upper semicontinuous at s if and only if the strict domination property holds for s.*

Proof Let E be upper semicontinuous at $s = (A, f)$ and $\bar{x} \in A$ be such that $\bar{x} \notin E(s)$. Define $\alpha : X \to \mathbb{R}_{++}$ as

$$\alpha(x) = \frac{1}{1 + d(x, \bar{x})}.$$

Clearly, α is continuous on X, $\alpha(\bar{x}) = 1$ and $\alpha(x) < 1$ for all $x \neq \bar{x}$. For $n \in \mathbb{N}$, define $A_n = A$ and $f_n : X \to Y$ as

$$f_n(x) = f(x) - \frac{\alpha(x)}{n} q.$$

It can be seen that $f_n \in \mathcal{LU}(X, Y)$, hence $s_n = (A_n, f_n) \in S$ for all $n \in \mathbb{N}$ and $s_n \to s$. Since $\bar{x} \notin E(s)$, there exists an open set $U \supseteq E(s)$ such that $\bar{x} \notin U$. Since E is upper semicontinuous at s, there exists $\bar{n} \in \mathbb{N}$ such that $E(s_n) \subseteq U$ for all $n \geq \bar{n}$, which implies that $\bar{x} \notin E(s_{\bar{n}})$. Hence, there exists $x' \in A_{\bar{n}} = A$ such that $f_{\bar{n}}(x') \preceq_D f_{\bar{n}}(\bar{x})$ and $f_{\bar{n}}(x') \neq f_{\bar{n}}(\bar{x})$. Thus, we have

$$f(x') - \frac{\alpha(x')}{\bar{n}} q \preceq_D f(\bar{x}) - \frac{\alpha(\bar{x})}{\bar{n}} q.$$

Since $\alpha(\bar{x}) = 1$ and $\alpha(x') < 1$, we have

$$f(x') \preceq_{\operatorname{int} D} f(x') + \left(\frac{1 - \alpha(x')}{\bar{n}}\right) q \preceq_D f(\bar{x}).$$

As $D + \operatorname{int} D \subseteq \operatorname{int} D$, it follows that $f(x') \preceq_{\operatorname{int} D} f(\bar{x})$.

Conversely, let the strict domination property hold for $s = (A, f)$ and E be not upper semicontinuous at s. Then, there exist an open set $U \supseteq E(s)$, a sequence $(s_n)_{n\in\mathbb{N}} \subseteq S$, where $s_n = (A_n, f_n)$ with $s_n \to s$ and $x_n \in E(s_n)$ such that $x_n \notin U$ for all $n \in \mathbb{N}$. Since $x_n \in A_n \subseteq \bigcup_{n=1}^{\infty} A_n \cup A \in K_0(X)$, by Lemma 2$(iii)$, without loss of generality, we assume that $x_n \to \bar{x}$. By Lemma 2(i), it follows that $\bar{x} \in A$. Clearly, $\bar{x} \notin U$ and hence $\bar{x} \notin E(s)$. By the strict domination property for s, there exists $x' \in A$ such that $f(x') \preceq_{\operatorname{int} D} f(\bar{x})$, which implies that there exists $\lambda \in \mathbb{R}_{++}$ such that $f(\bar{x}) - f(x') - 2\lambda q \in \operatorname{int} D$. Let $d = \lambda q$, then $d \in \operatorname{int} D$ and

$$f(x') + d \preceq_{\operatorname{int} D} f(\bar{x}) - d. \tag{1}$$

Let $\delta > 0$ be such that $d + B_{2\delta}(0_Y) \subseteq \operatorname{int} D$. Since the sequence $(f_n)_{n\in\mathbb{N}}$ uniformly converges to f, there exists $n_1 \in \mathbb{N}$ such that $f_n(x) \in f(x) + B_\delta(0_Y)$ for all $x \in X$ and $n \geq n_1$. Further, since f is D-lower semicontinuous at $\bar{x}$, there exists an open neighbourhood U_1 of $\bar{x}$ in X such that $f(u) \in f(\bar{x}) + B_\delta(0_Y) + D$ for all $u \in U_1$. Thus, for all $u \in U_1$ and $n \geq n_1$, it follows that $f_n(u) \in f(\bar{x}) + B_{2\delta}(0_Y) + D$, which further implies that

$$f(\bar{x}) - d \preceq_{\operatorname{int} D} f_n(u) \quad \forall\, u \in U_1,\ n \geq n_1. \tag{2}$$

Similarly, as f is D-upper semicontinuous at x', there exists an open neighbourhood U_2 of x' in X and $n_2 \in \mathbb{N}$ such that

$$f_n(u) \preceq_{\operatorname{int} D} f(x') + d \quad \forall\, u \in U_2,\ n \geq n_2. \tag{3}$$

Since $x_n \to \bar{x}$ and $x' \in A$, by Lemma 2(ii), there exists $n_3 \in \mathbb{N}$ such that $x_n \in U_1$ and $A_n \cap U_2 \neq \emptyset$ for all $n \geq n_3$. Let $\bar{n} = \max\{n_1, n_2, n_3\}$. For each $n \geq \bar{n}$, let $x'_n \in A_n \cap U_2$. Thus, from (1)-(3) and the fact that $D + \operatorname{int} D \subseteq \operatorname{int} D$, it follows that $f_n(x'_n) \preceq_{\operatorname{int} D} f_n(x_n)$ for all $n \geq \bar{n}$, which contradicts that $x_n \in E(s_n)$ for all $n \in \mathbb{N}$. □

In the next theorem, we identify an essential set when the efficient solution set mapping is upper semicontinuous.

Theorem 4 *Suppose that* $s = (A, f) \in S$, *D is closed,* $\operatorname{int} D \neq \emptyset$, $D + D \subseteq D$ *and there exists* $q \in Y \backslash \{0_Y\}$ *such that* $\mathbb{R}_{++} q \subseteq \operatorname{int} D$. *If E is upper semicontinuous at s, then $E(s)$ is an essential set of itself.*

Proof Since E is upper semicontinuous at s, it is enough to prove that $E(s)$ is a closed set. On the contrary, let there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E(s)$ with $x_n \to \bar{x} \in A$ such that $\bar{x} \notin E(s)$. By Theorem 3, the strict domination property holds for s, hence there exists $x' \in A$ such that $f(x') \preceq_{\text{int}\,D} f(\bar{x})$. Let $d = f(\bar{x}) - f(x') \in \text{int}\,D$ and $\delta > 0$ be such that $d + B_\delta(0_Y) \subseteq \text{int}\,D$. Since f is D-lower semicontinuous at $\bar{x}$ and $x_n \to \bar{x}$, it follows that $f(x_n) \in f(\bar{x}) + B_\delta(0_Y) + D$ for sufficiently large n, which further implies that $f(\bar{x}) - d \preceq_{\text{int}\,D} f(x_n)$ for sufficiently large n. Hence, $f(x') \preceq_{\text{int}\,D} f(x_n)$ for sufficiently large n, which is a contradiction. □

In view of preceding results, we next aim to show that for any $s \in S$, $(\mathrm{P})_s$ can be approximated arbitrarily by stable problems emanating from S, provided the efficient solution set mapping is upper semicontinuous for each $s \in S$. For this purpose, in the remaining section we assume (X, d) is a complete metric space and $(Y, \|\cdot\|)$ is a Banach space and we recall the following result. Also by Baire category theorem [3], it is known that a complete metric space is a Baire Space.

Lemma 3 [8, Theorem 2] *Suppose that Λ is a Baire space, Z is a metric space and $F : \Lambda \rightrightarrows Z$ is a upper semicontinuous mapping with compact values. Then, there exists a dense residual subset G of Λ (that is, G contains a countable intersection of open dense subsets of Λ) such that F is lower semicontinuous at each $\lambda \in G$.*

Next, we prove the following lemma.

Lemma 4 *(S, ϱ) is a complete metric space.*

Proof Let $(s_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in S, where $s_n = (A_n, f_n)$ for all $n \in \mathbb{N}$. Thus for any $0 < \delta < \dfrac{1}{2}$, there exists $n_0 \in \mathbb{N}$ such that $\varrho(s_n, s_m) < \frac{\delta}{4}$ for all $n,\ m \geq n_0$, that is,

$$h(A_n, A_m) < \frac{\delta}{4} \quad \text{and} \quad \|f_n(x) - f_m(x)\| < \frac{\delta}{4}, \quad \forall x \in X,\ n, m \geq n_0. \tag{4}$$

Hence, $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(K_0(X), h)$ and for any fixed $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Y, \|\cdot\|)$. Since $K_0(X)$ is a complete metric space and Y is a Banach space, there exist $A \in K_0(X)$ and $f(x) \in Y$ such that $A_n \to A$ and $f_n(x) \to f(x)$. From (4), letting $m \to +\infty$, we have

$$h(A_n, A) \leq \frac{\delta}{4} \quad \text{and} \quad \|f_n(x) - f(x)\| \leq \frac{\delta}{4}, \quad \forall x \in X,\ n \geq n_0, \tag{5}$$

which further implies that $\varrho(s_n, s) \leq \frac{\delta}{2} < \delta$, where $s = (A, f)$. We next prove that $f \in \mathcal{LU}(X, Y)$. Let $x \in X$ be fixed. Since f_{n_0} is D-lower semicontinuous at x, there exists an open neighbourhood U of x in X such that

$$f_{n_0}(x') \in f_{n_0}(x) + B_{\delta/4}(0_Y) + D, \quad \forall x' \in U. \tag{6}$$

From (5) and (6), for all $x' \in U$ it can be seen that $f(x') \in f(x) + B_\delta(0_Y) + D$. Similarly, we can show that f is D-upper semicontinuous on X. Hence, for $(s_n)_{n\in\mathbb{N}} \subseteq S$, we have $s \in S$ such that $\varrho(s_n, s) \to 0$, which concludes that (S, ϱ) is a complete metric space. □

In view of preceding lemmas and Theorem 4, we now establish existence of a dense subset in S which comprise of stable problems.

Corollary 1 *Suppose that D is closed,* $\text{int}\,D \neq \emptyset$, *$D + D \subseteq D$ and there exists $q \in Y \backslash \{0_Y\}$ such that $\mathbb{R}_{++}q \subseteq$* $\text{int}\,D$. *If E is upper semicontinuous for each $s \in S$, then there exists a dense residual subset G of S such that* $(\mathrm{P})_s$ *is stable for all $s \in G$.*

Proof From Lemma 4, clearly S is a Baire Space. By Theorem 4 and compactness of set A, it follows that $E : S \rightrightarrows X$ is upper semicontinuous on S with compact values. Further, from Lemma 3, we have there exists a dense residual subset G of S such that E is lower semicontinuous at each $s \in G$. Hence, the result follows from Theorem 2. □

5 Existence of essential set and essential component

In the sequel, we provide sufficient conditions for the existence of essential sets, minimal essential sets and essential components of the efficient solution set. Moreover, we assert that existence of essential set is strictly weaker than lower semicontinuity or upper semicontinuity of efficient solution set mapping.

To begin with, we prove that $f^{-1}(\bar{y})$ for any $\bar{y} \in M(s)$ is an essential set of $E(s)$.

Theorem 5 *Suppose that $s = (A, f) \in S$, D is closed, $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$. If $\bar{y} \in M(s)$, then $f^{-1}(\bar{y})$ is an essential set of $E(s)$.*

Proof Let $\bar{x} \in E(s)$ be such that $f(\bar{x}) = \bar{y}$ and $e(s) = f^{-1}(\bar{y})$. We first show that $e(s)$ is a closed set. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $e(s)$ such that $x_n \to x$. Let $\delta > 0$. As f is D-lower semicontinuous at x, there exists $n_0 \in \mathbb{N}$ such that $f(x) + B_\delta(0_Y) \preceq_D f(x_n)$ for all $n \geq n_0$. Since $f(x_n) = f(\bar{x})$ for each n, it follows that $f(x) + B_\delta(0_Y) \preceq_D f(\bar{x})$ for each $\delta > 0$. As D is closed, it follows that $f(x) \preceq_D f(\bar{x})$, which implies that $f(x) = \bar{y}$ and hence, $x \in e(s)$.

Let U be an open set such that $e(s) \subseteq U$. We next need to show that there exists a neighbourhood of s such that $E(s') \cap U \neq \emptyset$ for every s' in that neighbourhood. On the contrary, assume that there exists a sequence $(s_n)_{n\in\mathbb{N}} \subseteq S$ with $s_n \to s$ such that $E(s_n)\cap U=\emptyset$ for all n. Clearly, $e(s)$ is a compact set being a closed subset of the compact set A. Hence, there exists an open set W in X such that $e(s) \subseteq W \subseteq \overline{W} \subseteq U$.

Since $\bar{x} \in A$ and $A_n \to A$, there exists $x_n \in A_n$ for all $n \in \mathbb{N}$ such that $x_n \to \bar{x}$. As $\bar{x} \in e(s) \subseteq W$, so $x_n \in W$ for sufficiently large n. Without loss of generality we assume that $x_n \in A_n \cap W$ for all $n \in \mathbb{N}$. Let $\delta > 0$. Since f is D-upper semicontinuous at $\bar{x}$ and $f_n \to f$ uniformly, there exists $n_1 \in \mathbb{N}$ such that

$$f_n(x_n) \preceq_D f(\bar{x}) - B_\delta(0_Y) \quad \forall\, n \geq n_1. \tag{7}$$

We next claim that for sufficiently large n, there exists $x'_n \in A_n\backslash\overline{W}$ such that $f_n(x'_n) \preceq_D f_n(x_n)$. On the contrary, assume that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $f_{n_k}(x) \npreceq_D f_{n_k}(x_{n_k})$ for all $x \in A_{n_k}\backslash\overline{W}$. For each $k \in \mathbb{N}$, define

$$T_{n_k} := \{w \in A_{n_k} \cap \overline{W} \mid f_{n_k}(w) \preceq_D f_{n_k}(x_{n_k})\}.$$

As $0_Y \in D$, it is clear that $x_{n_k} \in T_{n_k}$ and hence $T_{n_k} \neq \emptyset$ for each $k \in \mathbb{N}$. Proceeding as in the proof of Lemma 1, it can be seen that T_{n_k} is closed subset of A_{n_k} and hence a compact set. Thus, from Theorem 1, we have $E(T_{n_k}, f_{n_k}) \neq \emptyset$. Let $w_{n_k} \in E(T_{n_k}, f_{n_k})$. Also, for each $k \in \mathbb{N}$ we have $f_{n_k}(x) \npreceq_D f_{n_k}(x_{n_k})$ for all $x \in A_{n_k}\backslash T_{n_k}$. As $f_{n_k}(w_{n_k}) \preceq_D f_{n_k}(x_{n_k})$, it follows that $f_{n_k}(x) \npreceq_D f_{n_k}(w_{n_k})$ for all $x \in A_{n_k}\backslash T_{n_k}$. Hence, it follows that $w_{n_k} \in E(s_{n_k})$. Since $w_{n_k} \in \overline{W} \subseteq U$, it follows that $E(s_{n_k})\cap U \neq \emptyset$ for each $k \in \mathbb{N}$, which is a contradiction. Thus, there exists $x'_n \in A_n\backslash\overline{W}$ and $n_2 \in \mathbb{N}$ such that

$$f_n(x'_n) \preceq_D f_n(x_n), \quad \forall\, n \geq n_2. \tag{8}$$

The sequence $(x'_n)_{n\in\mathbb{N}}$ has a convergent subsequence, as $\bigcup_{n=1}^{\infty} A_n \cup A$ is a compact set. Without loss of generality, let $x'_n \to x'$. As f is D-lower semicontinuous at x', $f_n \to f$ and $\delta > 0$, there exists $n_3 \in \mathbb{N}$ such that

$$f(x') + B_\delta(0_Y) \preceq_D f_n(x'_n) \quad \forall n \geq n_3. \tag{9}$$

By choosing $\bar{n} = \max\{n_1, n_2, n_3\}$, it is clear from (7)-(9) that

$$f(x') + B_\delta(0_Y) \preceq_D f(\bar{x}) - B_\delta(0_Y),$$

for each $\delta > 0$, which implies that $f(x') \preceq_D f(\bar{x})$, as D is a closed set. Since $\bar{x} \in E(s)$, it follows that $x' \in e(s) \subseteq W$, which contradicts the fact that $x'_n \in A_n \backslash \overline{W}$, for n sufficiently large. □

The following example illustrates that if D-upper semicontinuity of f is relaxed while defining the set of vector-valued functions $\mathcal{LU}(X, Y)$ and the parameter space S, then above theorem may fail to hold.

Example 1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and the mapping $f : X \to Y$ be defined as

$$f(x) = \begin{cases} (0,0), & \text{if } x \leq 0, \\ (0,1), & \text{if } x > 0. \end{cases}$$

Let $D \subseteq \mathbb{R}^2$ be defined as $D = (0, x_2) \in R^2 | x_2 \geq 0$ and $A = [0, 1]$. Clearly, D is a set satisfying all the assumptions of Theorem 5, A is compact, f is D-lower semicontinuous on X but fails to be D-upper semicontinuous at $x = 0$. For $s = (A, f)$, we have $(0, 0) \in M(s)$, however we show that $E(s) = f^{-1}(0, 0) = \{0\}$ is not an essential set of itself. Consider a neighbourhood $U \supseteq \{0\}$ such that $1 \notin U$ and a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = (A_n, f_n)$, where $A_n = \left[\frac{1}{n}, 1\right]$ and

$$f_n(x) = \begin{cases} (0,0), & \text{if } x \leq 0, \\ (0,1), & \text{if } x \in (0,1), \\ \left(0, 1 - \dfrac{1}{n}\right), & \text{if } x \geq 1, \end{cases}$$

for all $n \in \mathbb{N}$. Clearly, $s_n \to s$ and $E(s_n) = \{1\}$ for each $n \in \mathbb{N}$, but $E(s_n) \cap U = \emptyset$.

Remark 5 We observe that, if we perturb only the objective function, then we can relax D-upper semicontinuity of f in Theorem 5. In other words, the theorem can be proved on similar lines, if $S := A \times \mathcal{L}(X, Y)$, where A is assumed to be a nonempty compact subset of X and $\mathcal{L}(X, Y) := \{f \mid f : X \to Y \text{ is } D\text{-lower semicontinuous on } X\}$. Thus, Theorem 5 improves and generalize Lemma 1 of Xian and Zhou in [26], in the following aspects:

- (i) the continuity of objective function is relaxed to D-lower semicontinuity.
- (ii) the framework of Euclidean space endowed with ordering cone $\mathbb{R}^n_+$ is unified to normed space endowed with any partial order relation.
- (iii) not only objective function is perturbed but the perturbations with respect to the feasible set is also taken into the consideration.

In view of Theorem 1, 5 and Corollary 3.2 in [24], we next obtain sufficient conditions for the existence of minimal essential sets.

Theorem 6 *Suppose that D is closed, $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$. Then, for each $s \in S$, there is at least one minimal essential set of $E(s)$.*

Proof Let $s \in S$, then Theorem 1 implies $E(s) \neq \emptyset$. Let $\bar{x} \in E(s)$, then by Theorem 5, it follows that $e(s) = f^{-1}(f(\bar{x})) = \{x \in A \mid f(x) = f(\bar{x})\}$ is an essential set of $E(s)$.

For every essential set $e'(s) \subseteq E(s)$, if $e'(s) \nsubseteq e(s)$, then $e(s)$ is a minimal essential set of $E(s)$. Otherwise, let Q be the family of all essential subsets of $e(s)$ indexed by set inclusion. Since $e(s)$ is compact, by Cantor's intersection theorem, every decreasing chain in Q has a nonempty intersection in Q, which is precisely a lower bound of this chain. Hence, by Zorn's lemma, there is an essential set $\bar{e}(s) \subseteq e(s)$, which is a minimum element of Q. We next claim $\bar{e}(s)$ is an minimal essential set of $E(s)$. Let $e'(s)$ be an essential set of $E(s)$ such that $e'(s) \neq \bar{e}(s)$. If $e'(s) \nsubseteq e(s)$, then clearly $e'(s) \nsubseteq \bar{e}(s)$, and if $e'(s) \subseteq e(s)$, the minimality of $\bar{e}(s)$ implies that $e'(s) \nsubseteq \bar{e}(s)$. Hence, either $e(s)$ is a minimal essential set or it contains a minimal essential set of $E(s)$. □

The following example shows that for $\bar{x} \in E(s)$, the set $f^{-1}f(\bar{x})$ is not necessarily a minimal essential set of $E(s)$.

Example 2 Let $X = [-1\,0] \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup [1, 2]$ be a metric space with a usual topology on $\mathbb{R}$, $Y = l^\infty$ and the mapping $f : X \to Y$ be defined as

$$f(x) = \begin{cases} (x, 3x, 0, \dots, 0, \dots), & \text{if } x \in [-1, 0], \\ -e_n, & \text{if } x \in \left\{\dfrac{1}{n} \mid n \in \mathbb{N}\backslash\{1\}\right\}, \\ (0, 1-x, 0, \dots, 0, \dots), & \text{if } x \in [1, 2], \end{cases}$$

where $e_n = (0, \dots, 0, 1, 0, \dots)$, 1 is at the nth position. Let $D \subseteq Y$ be defined as

$$D = \left\{(x_1, x_2, \dots, x_i, \dots) \in l^\infty \mid x_1 \geq \frac{x_i}{i},\ x_1 \geq 0\right\} \cup \left\{(x_1, x_2, \dots, x_i, \dots) \in l^\infty \mid x_1 \geq 1\right\}.$$

Observe that for any $\delta > 0$ and $i \geq 1$, we have

$$\left(0, \dots, 0, \frac{i\delta}{2}, 0, \dots\right) \in (\mathrm{B}_\delta(0_Y) + D) \cap (\mathrm{B}_\delta(0_Y) - D),$$

and hence $(\mathrm{B}_\delta(0_Y) + D) \cap (\mathrm{B}_\delta(0_Y) - D)$ is unbounded for each $\delta > 0$. Clearly D is a set satisfying all the assumptions of Theorem 6. Also, it can be seen that f is not continuous at $0 \in X$. However, it is evident for any $\delta > 0$ and $\frac{1}{n} < \delta$, that $f\left(\frac{1}{n}\right) = -e_n \in (B_\delta(0_Y) + D) \cap (B_\delta(0_Y) - D)$. Hence, f is D-lower semicontinuous and D-upper semicontinuous on X. Let $A = [-\frac{1}{2}, 0] \cup \{1\}$, then for $s = (A, f) \in S$, clearly $E(s) = A$. By Theorem 5 for $0_Y \in M(s)$, the set $f^{-1}(0_Y) = \{0, 1\}$ is an essential set of $E(s)$, however it is evident that $\{0\}$ is a minimal essential set of $E(s)$ (refer Theorem 7, where $q = (2, 0, \dots, 0, \dots) \in Y$).

Remark 6 In Example 2, if we restrict $X = \{0\} \cup [1, 2]$ and consider $A = X$, then $E(s) = \{0, 1\}$ is an essential set of itself. However, E is neither lower semicontinuous nor upper semicontinuous at s. It is evident for a sequence $(s_n)_{n \in \mathbb{N}}$ with $s_n = (A_n, f_n)$, where $A_n = A$ and

$$f_n(x) = \begin{cases} -\dfrac{e_n}{n}, & \text{if } x = 0 \\ \left(-\dfrac{x}{n}, 1-x, 0, \dots, 0, \dots\right), & \text{if } x \in [1, 2], \end{cases}$$

for all $n \in \mathbb{N}$, we have $s_n \to s$ and $E(s_n) = [1, 2]$ for all $n \geq 3$. Hence, the existence of essential sets is strictly weaker than both the continuity properties of efficient solution set mapping.

The preceding argument also depicts that $0 \in E(s)$ is not an essential solution, since for a neighbourhood U of 0 with $[1, 2] \nsubseteq U$, the sequence $s_n \to s$ with $E(s_n) \cap U = \emptyset, n \in \mathbb{N}$. Further, for the sequence $(s_n)_{n\in\mathbb{N}}$, if we redefine

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in X\backslash\{0\}, \\ -\dfrac{e_1}{n}, & \text{if } x = 0, \end{cases}$$

for all $n \in \mathbb{N}$, we have $E(s_n) = \{0\}$ and it follows that $1 \in E(s)$ is not an essential solution of $E(s)$. Hence, we conclude $E(s)$ has no essential solution and is the minimal essential set which is not connected.

We next establish the existence of essential components under an additional assumption that $f^{-1}f(\bar{x}) = \{\bar{x}\}$ for some $\bar{x} \in E(s)$. Further, the assumption cannot be relaxed is evident from Remark 6.

Corollary 2 *Suppose that $s \in S$, D is closed, $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$. If $f^{-1}f(\bar{x}) = \{\bar{x}\}$ for some $\bar{x} \in E(s)$, then there is at least one essential component of $E(s)$.*

6 Characterizations of essential set and essential component

In this section, we obtain characterizations of essential efficient solutions, essential sets and essential components.

The following theorem presents necessary and sufficient conditions for a nonempty closed subset of efficient solution set for $(\mathrm{P})_s$ to be an essential set provided there exists $q \in Y\backslash\{0_Y\}$ such that $\mathbb{R}_+q \subseteq D$.

Theorem 7 *Suppose that D is closed, $D + D \subseteq D$, $D \cap (-D) = \{0_Y\}$ and there exists $q \in Y\backslash\{0_Y\}$ such that $\mathbb{R}_+q \subseteq D$. Let $s \in S$ and $e(s) \subseteq E(s)$ be a nonempty closed subset of X. Then, $e(s)$ is an essential set of $E(s)$ if and only if for any open set U in X with $U \supseteq e(s)$, there exists $y \in M(s)$ such that $f^{-1}(y) \subseteq U$.*

Proof Let $e(s)$ be an essential set of $E(s)$. On the contrary, let there exists an open set U in X with $e(s) \subseteq U$ and $f^{-1}(y) \nsubseteq U$ for every $y \in M(s)$. Since $e(s)$ is a closed subset of A, it is a compact set. Hence, there exists an open set W in X such that $e(s) \subseteq W \subseteq \overline{W} \subseteq U$. Define $\alpha : X \to \mathbb{R}_{++}$ as

$$\alpha(x) = \frac{1}{1 + d(x, A\backslash\overline{W})}.$$

Clearly, α is continuous on X, $\alpha(x) = 1$ for all $x \in A\backslash\overline{W}$ and $\alpha(x) < 1$ for all $x \in A \cap W$. For $n \in \mathbb{N}$, define $A_n = A$ and $f_n : X \to Y$ as

$$f_n(x) = f(x) - \frac{\alpha(x)}{n}q.$$

It can be seen that $f_n \in \mathcal{LU}(X, Y)$, hence $s_n = (A_n, f_n) \in S$ for all $n \in \mathbb{N}$ and $s_n \to s$.

We next claim that $E(s_n) \cap W = \emptyset$ for each $n \in \mathbb{N}$. Let $x \in A \cap W$. Clearly, $x \notin E(s)$. Otherwise, $y = f(x) \in M(s)$ and by the assumption we have $f^{-1}(y) \nsubseteq U$ and hence $x \notin W$. Define a set

$$T := \{w \in A \cap \overline{W} \mid f(w) \preceq_D f(x)\}.$$

Clearly, T is a nonempty set as $x \in T$. By Lemma 1, it follows that T is closed subset of A and hence a compact set. Thus, from Theorem 1, we have $E(T, f) \neq \emptyset$. Let $\bar{w} \in E(T, f)$. As observed earlier, it is clear that $\bar{w} \notin E(s)$. Hence, there exists $x' \in A$ with $f(x') \neq f(\bar{w})$ such that $f(x') \preceq_D f(\bar{w})$. As $\bar{w} \in E(T, f)$, we have $x' \notin \overline{W}$. Further, from $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$, it follows that $f(x') \preceq_D f(x)$ and $f(x') \neq f(x)$. Since $\alpha(x') = 1$ and $\alpha(x) < 1$, we have $-\frac{\alpha(x')}{n} q \preceq_D -\frac{\alpha(x)}{n} q$. Also, as $D + D \subseteq D$ and $D \cap (-D) = \{0_Y\}$, it follows that $f_n(x') \preceq_D f_n(x)$ and $f_n(x') \neq f_n(x)$ for each $n \in \mathbb{N}$. Thus $x \notin E(s_n)$, that is, $E(s_n) \cap W = \emptyset$ for all $n \in \mathbb{N}$, which contradicts the fact $e(s)$ is an essential set of $E(s)$.

Conversely, let U be an open set in X such that $e(s) \subseteq U$. By the assumption, there exists $y \in M(s)$ such that $f^{-1}(y) \subseteq U$. By Theorem 5, $e_1(s) = f^{-1}(y)$ is an essential set of $E(s)$ and hence there exists an open neighbourhood O of s in S such that $E(s') \cap U \neq \emptyset$ for all $s' \in O$, which implies that $e(s)$ is an essential set of $E(s)$. □

In view of above theorem, we obtain the following characterization of essential efficient solutions and essential components. A similar characterization was obtained by Xian and Zhou [26] assuming f to be continuous, $A = X$, $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$.

Theorem 8 *Suppose that $s \in S$, D is closed, $D + D \subseteq D$, $D \cap (-D) = \{0_Y\}$ and there exists $q \in Y \backslash \{0_Y\}$ such that $\mathbb{R}_+ q \subseteq D$. Then, $\bar{x} \in E(s)$ is an essential efficient solution if and only if for any open set U of $\bar{x}$ in X, there exists $y \in M(s)$ such that $f^{-1}(y) \subseteq U$.*

Corollary 3 *Suppose that $s \in S$, D is closed, $D + D \subseteq D$, $D \cap (-D) = \{0_Y\}$. If $f^{-1}(f(\bar{x})) = \{\bar{x}\}$ for all $\bar{x} \in E(s)$, then $\bar{x}$ is essential efficient solution for any $\bar{x} \in E(s)$ and thus E is lower semicontinuous at $s \in S$.*

Theorem 9 *Suppose that $s \in S$, D is closed, $D + D \subseteq D$, $D \cap (-D) = \{0_Y\}$ and there exists $q \in Y \backslash \{0_Y\}$ such that $\mathbb{R}_+ q \subseteq D$. Then, a component $c(s)$ of $E(s)$ is essential if and only if for any open set in X with $U \supseteq c(s)$, there exists $y \in M(s)$ such that $f^{-1}(y) \subseteq U$.*

Corollary 4 *Suppose that $s \in S$, D is closed, $D + D \subseteq D$, $D \cap (-D) = \{0_Y\}$. If there exists $y \in M(s)$ such that $f^{-1}(y)$ is contained in some component of $E(s)$, then there is at least one essential component of $E(s)$.*

7 Conclusion

Since each $s \in S$, comprises of a pair $(A, f) \in K_0(X) \times \mathcal{LU}(X, Y)$, in the present paper we have examined the essential stability of problem $(\mathrm{P})_s$ under the perturbation of both objective function and feasible set. Existence, characterization and density results we established improves and generalize essential stability results established by various authors [20,26,28]. In fact, we relaxed continuity of the objective mapping by D-lower semicontinuity and D-upper semicontinuity assumptions. By Remark 5, the results are further generalized for the problem with only D-lower semicontinuous objective mappings. Moreover, we have considered the unified setting of normed space endowed with a general preference relation having possibly empty interior.

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