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Sequential characterizations of approximate solutions in convex vector optimization problems with set-valued maps

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Abstract
This paper deals with a convex vector optimization problem with set-valued maps. In the absence of constraint qualifications, it provides, by the scalarization theorem, sequential Lagrange multiplier conditions characterizing approximate weak Pareto optimal solutions for the problem in terms of the approximate subdifferentials of the marginal function associated with corresponding set-valued maps. The paper shows also that this result yields the approximate Lagrange multiplier condition for the problem under a new constraint qualification which is weaker than the Slater-type constraint qualification. Illustrative examples are also provided to discuss the significance of the sequential conditions.

Keywords Convex vector optimization problems with set-valued maps · Sequential Lagrange multiplier conditions · Constraint qualifications · Scalarizations · Approximate weak Pareto optimal solutions

Mathematics Subject Classification 90C26 · 90C29 · 90C46 · 90C48

1 Introduction

Vector optimization problems involving set-valued maps have received increasing attention in the optimization community as the value of a given function can be made to vary in a specified set because of forecasting errors or lack of complete information. Over the years, there has been a growing interest in establishing Lagrangian-type optimality conditions for

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several kinds of solutions of vector optimization problems with set-valued maps in a general setting by many scholars; see, e.g., [4,12,14,15,17,18,21,42] and the references therein.

On the other hand, vector optimization problems with set-valued maps do not necessarily have the exact solutions, in which the global optimality is guaranteed. Moreover, a lot of solution methods produce approximations to the theoretical solutions. Therefore, from both the theoretical and the practical points of view, it is meaningful to consider various concepts of approximate solutions instead to optimization problems. Rong and Wu [36] initially introduced an approximate weak Pareto optimal solution of vector optimization problems with set-valued maps. Since the appearance of that paper, Lagrangian-type conditions for several notions of approximate solutions of vector optimizations with set-valued maps have been given in the literature; see, e.g., [9,28–30,38,40,43] and the references therein.

It is worth noting that in order to investigate optimality conditions for vector optimization problems with set-valued maps we often formulate a corresponding scalar optimization problem with set-valued maps. As a consequence, by employing the marginal function, one can characterize inevitable solutions of the scalar optimization problem with set-valued maps as a solution of a scalar optimization problem with single-valued maps. Nevertheless, by following this approach, the scalar optimization problem may not satisfy any constraint qualifications. Besides, most iterative algorithms or heuristic algorithms do not try out constraint qualifications at all, even though (approximate)-type Lagrangian conditions are always evaluated. This manner of facts leads one to modify Lagrange multiplier conditions without constraint qualifications; see, e.g., [6,24,25,39] and the references therein. Recently, sequential optimality conditions not only have been admitted to be valuable in designing algorithms for finding approximate optimal solutions of nonlinear programming problems [1,2,32] but also have been used as a termination condition to optimization algorithms [10,16,41] and the references therein. Some modified Lagrangian optimality conditions have also been shown to establish sequential Lagrange multipliers for a weak Pareto optimal solution as well as a proper Pareto optimal solution of convex vector optimization problems with set-valued maps as indicated in [15]. It is noteworthy, however, that the convexity of a set-valued map considered in [15] is based on prescription by graph of the corresponding set-valued map so that this notion is stronger than the cone-convexity notions, see Remark 1.

Motivated and inspired by the works in the literature, the main purposes of this work is to establish sequential optimality conditions for approximate weak Pareto optimal solutions in convex vector optimization problem with set-valued maps that do not require any constraint qualifications by using the cone-convexity notions and the scalarization theorems. We employ the relation between the conjugate function of the scalar set-valued map and of the marginal function associated with corresponding the scalar set-valued map to examine an approximate optimal solution of the obtained scalarization problem. We then obtain a new sequential form of Lagrange multiplier condition for the approximate optimality in terms of the approximate subdifferentials. This is achieved by employing the description of the epigraph of a conjugate function written in terms of the approximate subdifferential. Our result, in the case of exact solutions, differs from another sequential optimality result [15] established in the literature, where the scalarization function the so-called oriented distance functions and coderivatives are used. The significance of this result is that it yields the standard (approximate) Lagrange multiplier rule condition for the vector optimization problem with set-valued maps under a new constraint qualification which is guaranteed by the Slater-type constraint qualification. For other results concerning on coderivatives and suboptimality of convex problems as well as nonconvex problems, we refer the readers to [33] and the references therein. The interested reader is referred to [34] for more information on basic properties of the marginal function.

The layout of the paper is as follows. In Sect. 2, some basic definitions, notations and several auxiliary results that will be used later in the paper are presented. In Sect. 3, without any constraint qualifications, we obtain some sequential optimality conditions for both an approximate weak Pareto optimal solution and a weak Pareto optimal solution in a convex vector optimization problem with set-valued maps in terms of the subdifferentials of the marginal function associated with scalar set-valued maps. Section 4 describes a new constraint qualification which guarantees that the approximate Lagrange multiplier condition holds for the approximate weak Pareto optimal solution. Section 5 summarizes the obtained results.

2 Preliminaries

In this section, we recall some notations, basic definitions, and preliminary results which will be used in succeeding sections. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm $\| \cdot \|$. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geqq 0,\ i = 1, \ldots, n\}$. A nonempty subset S of $\mathbb{R}^n$ is said to be a *cone* if $tS \subseteq S$ for all $t \geqq 0$. The *dual* (*positive polar*) *cone* of S is denoted by $S^+ := \{v \in \mathbb{R}^n : \langle v, x \rangle \geqq 0 \text{ for all } x \in S\}$. For a nonempty set E in $\mathbb{R}^n$, by $\operatorname{int}(E)$ (resp. $\operatorname{ri}(E), \operatorname{cl}(E)$) we will denote the *interior* (resp. *relative interior* (see, e.g., [35]), *closure*) of the set E. The *support function* σ_E is defined by $\sigma_E(v) := \sup_{x \in E} \langle v, x \rangle$, and the indicator function δ_E respect to a set E is defined as $\delta_E(x) := 0$ if $x \in E$ and $\delta_E(x) := +\infty$ else. We say that E is *convex* whenever $tx_1 + (1-t)x_2 \in E$ for all $t \in [0, 1]$, $x_1, x_2 \in E$.

For an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, the *effective domain* and the *epigraph* are respectively defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leqq \alpha\}$. We say that f is *proper* if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\operatorname{dom} f \neq \emptyset$. A function f is said to be *convex* if $f(tx_1 + (1-t)x_2) \leqq tf(x_1) + (1-t)f(x_2)$ for all $t \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}^n$ with the conventions: $(+\infty) + (-\infty) = (-\infty) + (+\infty) = 0 \cdot (+\infty) = +\infty$, $0 \cdot (-\infty) = 0$. The *conjugate function* of f, $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, is defined by $f^*(v) = \sup\{\langle v, x \rangle - f(x) : x \in \mathbb{R}^n\}$ for any $v \in \mathbb{R}^n$. Let $\epsilon \geqq 0$ and $\bar{x} \in \mathbb{R}^n$ be such that $f(\bar{x}) \in \mathbb{R}$. The ϵ-*subdifferential* of f at $\bar{x}$ [7] is the set $\partial_\epsilon f(\bar{x}) := \{v \in \mathbb{R}^n : f(x) \geqq f(\bar{x}) + \langle v, x - \bar{x} \rangle - \epsilon,\ \forall x \in \mathbb{R}^n\}$. The ϵ-*normal set* of E at $\bar{x} \in E$ is given by $N^\epsilon_E(\bar{x}) := \partial_\epsilon \delta_E$. When $\epsilon = 0$, we denote the *subdifferential* of f at $\bar{x}$ and the *normal cone* of E at $\bar{x} \in E$, respectively, by $\partial f(\bar{x})$ and $N_E(\bar{x})$. For a proper lower semicontinuous convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and $\bar{x} \in \operatorname{dom} f$, the epigraph of f^* can be represented as follows (see, e.g., [22]):

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geqq 0} \{(u, \langle u, \bar{x} \rangle + \epsilon - f(\bar{x})) : u \in \partial_\epsilon f(\bar{x})\}. \tag{1}$$

The following lemma is needed for our study.

Lemma 1 [5,11] *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be proper convex functions such that* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$.

(i) *If f_1 and f_2 are lower semicontinuous, then,*

$$\operatorname{epi}(f_1 + f_2)^* = \operatorname{cl}(\operatorname{epi} f_1^* + \operatorname{epi} f_2^*).$$

(ii) *If one of f_1 and f_2 is continuous at some $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then,*

$$\operatorname{epi}(f_1 + f_2)^* = \operatorname{epi} f_1^* + \operatorname{epi} f_2^*.$$

Let $K \subseteq \mathbb{R}^p$ be a nonempty convex cone. For a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, the domain and graph of F are, respectively, defined by $\text{dom}(F) := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$, $\text{gph}(F) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in \text{dom}(F),\ y \in F(x)\}$. F is called *proper* if $\text{dom}(F) \neq \emptyset$. For a nonempty set E in $\mathbb{R}^n$, the indicator function $\widetilde{\delta}_E$ of E in the set-valued version is defined as $\widetilde{\delta}_E(x) := \{0\}$ if $x \in E$ and $\widetilde{\delta}_E(x) := \emptyset$ else. Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. We say that F is *K-convex on C* whenever $tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + K$ for any $t \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}^n$ with the conventions: $E + \emptyset = \emptyset$ for any subset E in $\mathbb{R}^p$, and $t \cdot \emptyset = \emptyset$ for any real numbers t. We also say that F is *K-convex* if $\text{dom}(F)$ is a convex set and F is K-convex on $\text{dom}(F)$. It should be noted that if F is K-convex, then $F(x) + K$ is convex for any $x \in \mathbb{R}^n$. In addition, for any proper set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ we can associate F with a linear scalarization set-valued map with respect to some $\lambda \in K^+$ defined by $(\lambda \circ F)(x) := \{\langle \lambda, y\rangle : y \in F(x)\}$ if $x \in \text{dom}(F)$ and $(\lambda \circ F)(x) := \emptyset$ else. Clearly, $\text{dom}(F) = \text{dom}(\lambda \circ F)$ for every $\lambda \in K^+$.

Remark 1 Let us now recall that a proper set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is *convex* if $\text{gph}(F)$ is convex. Due to the following characterization: F is K-convex if and only if $\text{gph}(F)+(\{0\}\times K)$ is convex, see, e.g., [27, Proposition 3.3], we can verify that if F is convex then it is K-convex. In general, K-convexity of F needs not imply convexity of F. For a simple example, it can be observed that a proper set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$, defined by $F(x) := \{y \in \mathbb{R} : x^3 \leqq y \leqq x^2\}$ if $x \in [0, 1]$ and $F(x) := \emptyset$ else, is not convex but it is $\mathbb{R}_+$-convex.

Unless otherwise stated, let A be a nonempty closed convex set in $\mathbb{R}^n$, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a proper set-valued mapping and $K \subseteq \mathbb{R}^p$ be a nonempty pointed ($K \cap (-K) = \{0\}$) closed convex cone with nonempty interior. In this paper, we consider the following vector optimization problem with set-valued map:

$$\min_K F(x) \text{ subject to } x \in A. \tag{P}$$

Let $\theta \in K$ be given. A point $(\bar{x}, \bar{y}) \in \text{gph}(F)$ with $\bar{x} \in A$ is said to be an *θ-weak Pareto optimal solution* with respect to K of the problem (P) if

$$(F(\Omega) - \bar{y} + \theta) \cap (-\text{int}(K)) = \emptyset,$$

where $\Omega := A \cap \text{dom}(F)$ and $F(\Omega) := \bigcup_{x\in\Omega} F(x)$. When $\theta := 0$, θ-weak Pareto optimal solution deduces to be a *weak Pareto optimal solution* (if exists) of (P).

The next theorem yields a characterization of an θ-weak Pareto optimal solution of (P) in terms of approximate solutions of the associated scalar set-valued optimization problem. In the following, let us recall the scalar set-valued optimization problem (SP):

$$\min H(x) \text{ subject to } x \in A, \tag{SP}$$

where A is a nonempty closed convex set in $\mathbb{R}^n$ and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a proper set-valued map.

Given $\epsilon \geqq 0$, let us recall also that a point $(\bar{x}, \bar{y}) \in \text{gph}(H)$ with $\bar{x} \in A$ is said to be an *ϵ-optimal solution* of (SP) if for any $x \in A \cap \text{dom}(H)$ and any $y \in H(x)$,

$$\bar{y} - \epsilon \leqq y.$$

When $\epsilon := 0$, ϵ-optimal solution deduces to be an *optimal solution* of (SP).

Theorem 1 [36, Theorem 2.1](see also [38, Theorem 3.2]) *Let $\theta \in K$, F be a proper set-valued map from $\mathbb{R}^n$ into $\mathbb{R}^p$ and $(\bar{x}, \bar{y}) \in \text{gph}(F)$ with $\bar{x} \in A$. Assume that $F(A\cap\text{dom}(F))+ \text{int}K$ is convex. Then $(\bar{x}, \bar{y})$ is an θ-weak Pareto optimal solution of* (P) *if and only if there exists $\lambda \in K^+\backslash\{0\}$ such that $(\bar{x}, \langle\lambda, \bar{y}\rangle)$ is an $\langle\lambda, \theta\rangle$-optimal solution of the problem* (SP) *with $H(\cdot) := (\lambda \circ F)(\cdot)$.*

Remark 2 If a proper set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is K-convex, then F is K-convex on $A \cap \mathrm{dom}(F)$, and so, $F(A \cap \mathrm{dom}(F)) + K$ is a convex set. Consequently, $F(A \cap \mathrm{dom}(F)) + \mathrm{int}K = (F(A \cap \mathrm{dom}(F)) + K) + \mathrm{int}K$ is convex.

Interestingly, we point out that the scalar set-valued map can be employed by the marginal function, which is used to establish a characterization of weak Pareto optimal solutions in [18, Theorem 4.2]. In what follows, given a proper set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}$, by $\varphi_H : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ we denote the marginal function of H, i.e.,

$$\varphi_H(x) := \begin{cases} \inf\{y \in \mathbb{R} : y \in H(x)\}, & \text{if } x \in \mathrm{dom}(H), \\ +\infty, & \text{ohterwise.} \end{cases}$$

It is clear that $\mathrm{dom}(H) = \mathrm{dom}(\varphi_H)$. Recall that the conjugate function of the proper set valued map (see e.g., [28]) H, $H^* : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, defined by for any $v \in \mathbb{R}^n$,

$$H^*(v) := \sup\{\langle v, x\rangle - y : x \in \mathrm{dom}(H),\ y \in H(x)\}.$$

For a nonempty set E in $\mathbb{R}^n$, it could be convenient to observe that $(\widetilde{\delta}_E)^* = \sigma_E = (\delta_E)^*$.

We close the section by the following results that justify why we are allowed to characterize an θ-weak Pareto optimal solution of (P) by using conjugate function of the scalar set valued map.

Proposition 1 [28, Proposition 4.1] *Let $H_1,\ H_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be proper set-valued maps such that $\mathrm{dom}(H_1) \cap \mathrm{dom}(H_2) \neq \emptyset$. Suppose that for any $x \in \mathrm{dom}(H_1) \cap \mathrm{dom}(H_2)$, $\varphi_{H_1}(x) > -\infty$ and $\varphi_{H_2}(x) > -\infty$. Then,*

$$(H_1 + H_2)^* = (\varphi_{H_1} + \varphi_{H_2})^*.$$

In particular, for any proper set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}$ such that $\varphi_H(x) > -\infty$ for all $x \in \mathrm{dom}(H)$, the identity $H^* = (\varphi_H)^*$ holds.

Lemma 2 *Let $\epsilon \geqq 0$ be given and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be a proper set-valued map. If $\mathrm{dom}(H) \cap A \neq \emptyset$, then the point $(\bar{x}, \bar{y})$ with $\bar{x} \in A$ is an ϵ-optimal solution of (SP) if and only if $(0, -\bar{y} + \epsilon) \in \mathrm{epi}(H + \widetilde{\delta}_A)^*$.*

Proof A point $(\bar{x}, \bar{y})$ with $\bar{x} \in A$ is an ϵ-optimal solution of (SP) if and only if for any $x \in A \cap \mathrm{dom}(H) = \mathrm{dom}(H + \widetilde{\delta}_A)$ and any $y \in H(x) = (H + \widetilde{\delta}_A)(x)$, $-y \leqq -\bar{y} + \epsilon$, or equivalently, $(H + \widetilde{\delta}_A)^*(0) \leqq -\bar{y} + \epsilon \Leftrightarrow (0, -\bar{y} + \epsilon) \in \mathrm{epi}(H + \widetilde{\delta}_A)^*$. □

Remark 3 It is worth noting that from Lemma 2 and Proposition 1 we can obtain a result in the line of [3, Lemma 3.1] by considering $H(x) := \{f(x)\}$ if $f(x) \in \mathbb{R}$ and $H(x) := \emptyset$ else, where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function. In this case, for $\bar{x} \in \mathrm{dom} f \cap A$, $\bar{y} = f(\bar{x})$, $\varphi_H = f$ and $\varphi_{\widetilde{\delta}_A} = \delta_A$.

3 Sequential Lagrange multiplier conditions for θ-weak Pareto solutions

In this section, we consider the vector optimization problem with set-valued map (P). Here, the feasible set A of the problem (P) is given by

$$A := \{x \in C : G(x) \cap -S \neq \emptyset\},$$

where C is a nonempty closed convex subset of $\mathbb{R}^n$, S is a nonempty closed convex cone of $\mathbb{R}^m$ which does not necessarily have a nonempty interior, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a proper set-valued map. The feasible set A has a quite general formulation, which provides a unified framework for examining various feasible sets for scalar/vector optimization problems. For example, if $G(x) := \{g(x)\}$, where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function, the set A reduces to $\{x \in C : g(x) \in -S\}$, which can be expressed of the form $\{x \in C : \langle \mu, g(x)\rangle \leqq 0, \ \forall \mu \in S^+\}$. Unfortunately, in the set-valued setting, the equality

$$\widetilde{A} := \left\{x \in \mathbb{R}^n : G(x) \cap -S \neq \emptyset\right\} = \left\{x \in \mathbb{R}^n : \varphi_{\mu \circ G}(x) \leqq 0, \ \forall \mu \in S^+\right\} \tag{2}$$

may fail to be true in general. In fact, by taking $G(x) := \{z := (z_1, z_2) \in \mathbb{R}^n : \exp(z_1) - x \leqq z_2\}$ for all $x \in \mathbb{R}$ and $S := \mathbb{R} \times \mathbb{R}_+$. We see that $\mathrm{dom}(G) = \mathbb{R}$ and $0 \notin \widetilde{A}$. On the one hand, for each $\mu := (\mu_1, \mu_2) \in S^+ = \{0\} \times \mathbb{R}_+$, we have

$$0 \leqq \mu_2 \exp(z_1) \leqq \mu_2 z_2 = \langle \mu, z\rangle, \ \forall z \in G(0).$$

This shows that $\varphi_{\mu \circ G}(0) = 0$ for all $\mu \in S^+$, and so, $0 \in \{x \in \mathbb{R}^n : \varphi_{\mu \circ G}(x) \leqq 0, \ \forall \mu \in S^+\}$.

Recently, an additional condition on the set-valued map G has been shown to guarantee the equality (2) (see, e.g., [37]). In what follows, one says that **Assumption (A)** holds if one of the following conditions holds:

- **(A1)** For each $x \in \mathrm{dom}(G)$, there exists $z \in G(x)$ such that $\varphi_{\mu \circ G}(x) = \langle \mu, z\rangle$ for every $\mu \in S^+$.
- **(A2)** For each $x \in \mathrm{dom}(G)$, $G(x)$ is a compact subset of $\mathbb{R}^m$.

It is worth noting that there are no implication relations between **(A1)** and **(A2)**, see, e.g., [37]. Note also that under the validity of assumption **(A)**, the closedness of the set $\widetilde{A}$ can be guaranteed by the lower semicontinuity of $\varphi_{\mu \circ G}$ for all $\mu \in S^+$. In the sequel, let us recall that G is *S-proper* (resp. *nonnegatively S-lsc*) if $\varphi_{\mu \circ G}$ is proper (resp. lower semicontinuous) for all $\mu \in S^+$. Hereafter, for the problem (P), we always assume in the rest of this paper that the proper set-valued mapping F is K-proper, K-convex and nonnegatively K-lsc, and the proper set-valued mapping G is S-proper, S-convex and nonnegatively S-lsc satisfying **(A)**.

Remark 4 The following points are taken from [19, Remark 3.1], [26, Theorem 4.1] and [37], respectively. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a proper set-valued map. One has:

- (i) $\varphi_{\lambda \circ F}$ is convex for each $\lambda \in K^+$ if and only if F is K-convex.
- (ii) $\varphi_{\lambda \circ F}$ is lower semicontinuous for each $\lambda \in K^+$ provided that F is *upper K-continuous* on $\mathrm{dom}(F)$ (see, e.g., [13, Definition 2.5.16] and [31, Definition 7.1]), i.e., for any $x \in \mathrm{dom}(F)$ and any open set $V \supseteq F(x)$, there exists a neighborhood U of x such that $F(u) \subseteq V + K$ for all $u \in U$.
- (iii) $\varphi_{\lambda \circ F}$ is proper for each $\lambda \in K^+$ if either F has compact values on $\mathrm{dom}(F)$ or F is *K-bounded from below on* $\mathrm{dom}(F)$ in the sense that there exists $a \in \mathbb{R}^p$ such that $F(x) \subseteq a + K$ for all $x \in \mathrm{dom}(F)$.

Next, we will obtain some sequential characterizations of θ-weak Pareto optimal solution for the problem (P) in terms of the approximate subdifferentials of the marginal functions associated with F and G. We begin with the following two lemmas.

Lemma 3 [37, Proposition 2] *Let a proper set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be S-proper, S-convex and nonnegatively S-lsc satisfying* **(A)**. *If $\widetilde{A} := \{x \in \mathbb{R}^n : G(x) \cap -S \neq \emptyset\} \neq \emptyset$, then*

$$\mathrm{epi}\sigma_{\widetilde{A}} = \mathrm{cl}\left(\cup_{\mu \in S^+} \mathrm{epi}(\varphi_{\mu \circ G})^*\right).$$

Remark 5 We point out that the set $\cup_{\mu\in S^+}\text{epi}(\varphi_{\mu\circ G})^*$ is a convex cone due to [37, Proposition 1]. Consequently, since $\text{epi}\delta_C^*$ is a convex cone, the set $\cup_{\mu\in S^+}\text{epi}(\varphi_{\mu\circ G})^* + \text{epi}\delta_C^*$ is also a convex cone.

Lemma 4 *If $A \neq \emptyset$, then*

$$\text{epi}(\widetilde{\delta}_A)^* = \text{cl}\left(\cup_{\mu\in S^+}\text{epi}(\varphi_{\mu\circ G})^* + \text{epi}\delta_C^*\right).$$

Proof Note by the definition of the indicator function in the set-valued version that $\widetilde{\delta}_A = \widetilde{\delta}_{\widetilde{A}} + \widetilde{\delta}_C$. So, invoking Proposition 1, Lemmas 1(i) and 3, we have

$$\begin{aligned}\text{epi}\left(\widetilde{\delta}_A\right)^* &= \text{epi}\left(\varphi_{\widetilde{\delta}_{\widetilde{A}}} + \varphi_{\widetilde{\delta}_C}\right)^* \\ &= \text{cl}\left(\text{epi}\left(\varphi_{\widetilde{\delta}_{\widetilde{A}}}\right)^* + \text{epi}\left(\varphi_{\widetilde{\delta}_C}\right)^*\right) \\ &= \text{cl}\left(\text{epi}\left(\widetilde{\delta}_{\widetilde{A}}\right)^* + \text{epi}\left(\widetilde{\delta}_C\right)^*\right) \\ &= \text{cl}\left(\text{epi}\sigma_{\widetilde{A}} + \text{epi}\sigma_C\right) \\ &= \text{cl}\left(\text{cl}\left(\cup_{\mu\in S^+}\text{epi}\left(\varphi_{\mu\circ G}\right)^*\right) + \text{epi}\delta_C^*\right) \\ &= \text{cl}\left(\cup_{\mu\in S^+}\text{epi}\left(\varphi_{\mu\circ G}\right)^* + \text{epi}\delta_C^*\right),\end{aligned}$$

and the proof is complete. □

Theorem 2 *For the problem* (P)*, if $\Omega \neq \emptyset$, then $(\bar{x}, \bar{y}) \in \text{gph}(F)$ with $\bar{x} \in A$ is an θ-weak Pareto optimal solution of* (P) *if and only if there exist $\lambda \in K^+\backslash\{0\}$, $\{\mu_l\} \subset S^+$, $\{\epsilon_l\}$, $\{\eta_l\}$, $\{\zeta_l\} \subset \mathbb{R}_+$, $u_l \in \partial_{\epsilon_l}\varphi_{\lambda\circ F}(\bar{x})$, $v_l \in \partial_{\eta_l}\varphi_{\mu_l\circ G}(\bar{x})$, $w_l \in N_C^{\zeta_l}(\bar{x})$ such that*

$$u_l + v_l + w_l \to 0, \ \text{as } l \to +\infty \tag{3}$$

and

$$\lim_{l\to+\infty}(\epsilon_l + \eta_l + \zeta_l - \varphi_{\mu_l\circ G}(\bar{x})) + \langle\lambda, \bar{y}\rangle - \varphi_{\lambda\circ F}(\bar{x}) = \langle\lambda, \theta\rangle. \tag{4}$$

Proof Assume that a pair $(\bar{x}, \bar{y}) \in \text{gph}(F)$ with $\bar{x} \in A$ is an θ-weak Pareto optimal solution of (P). On account of the K-convexity of F, by Remark 2, we apply Theorem 1 to assert that there exists $\lambda \in K^+\backslash\{0\}$ such that $(\bar{x}, \langle\lambda, \bar{y}\rangle)$ is a $\langle\lambda, \theta\rangle$-optimal solution of (SP) where $H(x) := (\lambda \circ F)(x)$. So, Lemma 2 together with Proposition 1 yields

$$(0, -\langle\lambda, \bar{y}\rangle + \langle\lambda, \theta\rangle) \in \text{epi}(\lambda \circ F + \widetilde{\delta}_A)^* = \text{epi}(\varphi_{\lambda\circ F} + \varphi_{\widetilde{\delta}_A})^*. \tag{5}$$

In addition, Lemma 4 gives us that

$$\begin{aligned}\text{epi}\left(\varphi_{\lambda\circ F} + \varphi_{\widetilde{\delta}_A}\right)^* &= \text{cl}\left(\text{epi}\left(\varphi_{\lambda\circ F}\right)^* + \text{epi}\left(\varphi_{\widetilde{\delta}_A}\right)^*\right) \\ &= \text{cl}\left(\text{epi}\left(\varphi_{\lambda\circ F}\right)^* + \text{cl}\left(\cup_{\mu\in S^+}\text{epi}\left(\varphi_{\mu\circ G}\right)^* + \text{epi}\delta_C^*\right)\right) \\ &= \text{cl}\left(\text{epi}\left(\varphi_{\lambda\circ F}\right)^* + \cup_{\mu\in S^+}\text{epi}\left(\varphi_{\mu\circ G}\right)^* + \text{epi}\delta_C^*\right).\end{aligned}$$

Taking (5) into account, we assert that there exist sequences $\{\mu_l\} \subset S^+$, $\{(u_l, \alpha_l)\} \subset \text{epi}(\varphi_{\lambda\circ F})^*$, $\{(v_l, \beta_l)\} \subset \text{epi}(\varphi_{\mu_l\circ G})^*$ and $\{(w_l, \gamma_l)\} \subset \text{epi}\delta_C^*$ such that $u_l + v_l + w_l \to 0$ and $\alpha_l + \beta_l + \gamma_l \to -\langle\lambda, \bar{y}\rangle + \langle\lambda, \theta\rangle$ as $l \to +\infty$. In view of (1), there exist sequences $\{\epsilon_l\}$, $\{\eta_l\}$, $\{\zeta_l\} \subset \mathbb{R}_+$ such that

$$\begin{cases} u_l \in \partial_{\epsilon_l}\varphi_{\lambda\circ F}(\bar{x}), \ \alpha_l = \langle u_l, \bar{x}\rangle + \epsilon_l - \varphi_{\lambda\circ F}(\bar{x}), \\ v_l \in \partial_{\eta_l}\varphi_{\mu_l\circ G}(\bar{x}), \ \beta_l = \langle v_l, \bar{x}\rangle + \eta_l - \varphi_{\mu_l\circ G}(\bar{x}), \\ w_l \in \partial_{\zeta_l}\delta_C(\bar{x}), \ \gamma_l = \langle w_l, \bar{x}\rangle + \zeta_l, \ \forall l \in \mathbb{N}. \end{cases}$$

It follows that $\alpha_l + \beta_l + \gamma_l = \langle u_l + v_l + w_l, \bar{x}\rangle - \varphi_{\lambda\circ F}(\bar{x}) - \varphi_{\mu_l\circ G}(\bar{x}) + (\epsilon_l + \eta_l + \zeta_l)$ for each $l \in \mathbb{N}$. Now, passing to the limit as $l \to +\infty$, we get

$$-\langle\lambda, \bar{y}\rangle + \langle\lambda, \theta\rangle = \lim_{l\to+\infty}(\alpha_l + \beta_l + \gamma_l) = -\varphi_{\lambda\circ F}(\bar{x}) + \lim_{l\to+\infty}(\epsilon_l + \eta_l + \zeta_l - \varphi_{\mu_l\circ G}(\bar{x})),$$

and so, (3) and (4) have been justified.

Conversely, suppose that there exist $\lambda \in K^+\backslash\{0\}, \{\epsilon_l\}, \{\eta_l\}, \{\zeta_l\} \subset \mathbb{R}_+, u_l \in \partial_{\epsilon_l}\varphi_{\lambda\circ F}(\bar{x})$, $v_l \in \partial_{\eta_l}\varphi_{\mu_l\circ G}(\bar{x}), w_l \in N_C^{\zeta_l}(\bar{x})$ such that (3) and (4) hold. Let $x \in A\cap\mathrm{dom}(F)$ and $y \in F(x)$ be arbitrary. Then $x \in C$ and there exists $z \in G(x)$ such that $z \in -S$. By taking into account the definitions of $\varphi_{\lambda\circ F}$, $\varphi_{\mu_l\circ G}$ and $N_C^{\zeta_l}(\bar{x})$, we have, for each positive integer l,

$$\begin{aligned}\langle\lambda, y\rangle - \varphi_{\lambda\circ F}(\bar{x}) &\geqq \langle u_l, x - \bar{x}\rangle - \epsilon_l,\\ \langle\mu_l, z\rangle - \varphi_{\mu_l\circ G}(\bar{x}) &\geqq \langle v_l, x - \bar{x}\rangle - \eta_l,\\ 0 &\geqq \langle w_l, x - \bar{x}\rangle - \zeta_l.\end{aligned}$$

Adding these inequalities, it holds that

$$\begin{aligned}\langle\lambda, y\rangle - \varphi_{\lambda\circ F}(\bar{x}) &\geqq \langle\lambda, y\rangle - \varphi_{\lambda\circ F}(\bar{x}) + \langle\mu_l, z\rangle - \varphi_{\mu_l\circ G}(\bar{x}) + \varphi_{\mu_l\circ G}(\bar{x})\\ &\geqq \langle u_l + v_l + w_l, x - \bar{x}\rangle - (\epsilon_l + \eta_l + \zeta_l - \varphi_{\mu_l\circ G}(\bar{x})).\end{aligned}$$

Passing to the limit as $l \to +\infty$, we obtain that $\langle\lambda, y\rangle \geqq \langle\lambda, \bar{y}\rangle - \langle\lambda, \theta\rangle$, showing that $(\bar{x}, \langle\lambda, \bar{y}\rangle)$ is a $\langle\lambda, \theta\rangle$-optimal solution of (SP). Therefore, in view of Theorem 1, $(\bar{x}, \bar{y})$ is an θ-weak Pareto optimal solution of (P) as desired. □

Next let us provide an example illustrating Theorem 2 where the Slater-type constraint qualification fails. Here, the set $A := \{x \in C : G(x) \cap -S \neq \emptyset\}$ is said to satisfy the *Slater-type constraint qualification* if $\mathrm{int}S \neq \emptyset$ and there exists $\hat{x} \in \mathrm{ri}C$ such that $G(\hat{x})\cap-\mathrm{int}S \neq \emptyset$.

Example 1 Let $K = S := \mathbb{R}^2_+$, $C := [-1, 1]$, $\theta := (0.1, 0.1)$, $(\bar{x}, \bar{y}) := (0, (0.1, 0.1))$ and let F and G be defined by $F(x) := (x, -\frac{1}{2}\sqrt{x}) + \mathbb{R}^2_+$ if $x \in [0, +\infty[$ and $F(x) := \emptyset$ else, $G(x) := \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x)^2 + (z_2 - 1)^2 \leqq 1\}$ for every $x \in \mathbb{R}$. It can be easily checked that $K^+ = S^+ = \mathbb{R}^2_+$, $A = [-1, 0]$, $(\bar{x}, \bar{y})$ is an θ-weak Pareto optimal solution of (P) and that the Slater-type constraint qualification fails. Now, it can also be verified that for each $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$, $\varphi_{\lambda\circ F}(x) = \lambda_1 x - \frac{\lambda_2}{2}\sqrt{x}$ for all $x \geqq 0$; otherwise $\varphi_{\lambda\circ F}(x) = +\infty$, and for each $(\mu_1, \mu_2) \in \mathbb{R}^2_+$, $\varphi_{\mu\circ G}(x) = \mu_1 x + \mu_2 - \sqrt{\mu_1^2 + \mu_2^2}$ for all $x \in \mathbb{R}$. Putting $\bar{\lambda} := (0, 1)$, $\epsilon_l = \eta_l := \frac{1}{\sqrt{l}}$, $\zeta_l := \frac{1}{2\sqrt{l}}$, $\mu_l := (\mu_1^l, \mu_2^l) := (\frac{\sqrt{l}}{2}, \frac{\sqrt{l}}{4}(\frac{l^2-1}{l}))$ for each $l \in \mathbb{N}$. Then, $-\frac{\sqrt{l}}{2} \in \partial_{\epsilon_l}\varphi_{\bar{\lambda}\circ F}(\bar{x})$ and $\frac{\sqrt{l}}{2} \in \partial_{\eta_l}\varphi_{\mu_l\circ G}(\bar{x})$. Indeed, a direct calculation shows that for any $x \in \mathbb{R}^n$, $-\frac{\sqrt{l}}{2}x \leqq \varphi_{\bar{\lambda}\circ F}(x) + \epsilon_l$, since

$$-\frac{\sqrt{l}}{2}x \begin{cases} < +\infty = \varphi_{\bar{\lambda}\circ F}(x), & \text{if } x < 0;\\ \leqq 0 \leqq -\frac{1}{2}\sqrt{x} + \epsilon_l = \varphi_{\bar{\lambda}\circ F}(x) + \epsilon_l, & \text{if } 0 \leqq x \leqq \frac{1}{\sqrt{l}};\\ < -\frac{1}{2} < -\frac{1}{2}\sqrt{x} < \varphi_{\bar{\lambda}\circ F}(x) + \epsilon_l, & \text{if } \frac{1}{\sqrt{l}} < x < 1;\\ \leqq -\frac{1}{2}x \leqq -\frac{1}{2}\sqrt{x} < \varphi_{\bar{\lambda}\circ F}(x) + \epsilon_l, & \text{if } x \geqq 1,\end{cases}$$

and $\frac{\sqrt{l}}{2}x \leqq \frac{\sqrt{l}}{2}x + \eta_l = \varphi_{\mu_l\circ G}(x) - \varphi_{\mu_l\circ G}(\bar{x}) + \eta_l$. In addition, it can be verified that

$$\varphi_{\mu_l\circ G}(\bar{x}) = \frac{\sqrt{l}}{4}\left(\frac{l^2-1}{l}\right) - \sqrt{\frac{l}{4} + \frac{(l^2-1)^2}{16l}} = \frac{1}{4\sqrt{l}}((l^2 - 1) - (l^2 + 1)) = -\frac{1}{2\sqrt{l}},$$

$\langle \bar{\lambda}, \theta \rangle - \langle \bar{\lambda}, \bar{y} \rangle + \varphi_{\bar{\lambda} \circ F}(\bar{x}) = 0$, and $N_C^{\zeta_l}(\bar{x}) = [-\zeta_l, \zeta_l]$. Letting $u_l := -\frac{\sqrt{l}}{2}$, $v_l := \frac{\sqrt{l}}{2}$ and $w_l := \zeta_l$ for each $l \in \mathbb{N}$, we see that $u_l + v_l + w_l = \zeta_l \to 0$ and $\epsilon_l + \eta_l + \zeta_l - \varphi_{\mu_l \circ G}(\bar{x}) = \frac{3}{\sqrt{l}} \to 0$ as $l \to +\infty$, showing that the sequential conditions of Theorem 2 hold. □

The special case of $\theta := 0$ in the preceding theorem gives us a new characterization of weak Pareto optimal solutions of (P) as follows.

Corollary 1 *For the problem* (P)*, if* $\Omega \neq \emptyset$*, then* $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ *with* $\bar{x} \in A$ *is a weak Pareto optimal solution of* (P) *if and only if there exist* $\lambda \in K^+ \backslash \{0\}$*,* $\{\mu_l\} \subset S^+$*,* $\{\epsilon_l\} \subset \mathbb{R}_+$*,* $u_l \in \partial_{\epsilon_l} \varphi_{\lambda \circ F}(\bar{x})$*,* $v_l \in \partial_{\epsilon_l} \varphi_{\mu_l \circ G}(\bar{x})$*,* $w_l \in N_C^{\epsilon_l}(\bar{x})$ *such that* $\langle \lambda, \bar{y} \rangle = \varphi_{\lambda \circ F}(\bar{x})$*,*

$$u_l + v_l + w_l \to 0, \ \epsilon_l \to 0 \textit{ and } \varphi_{\mu_l \circ G}(\bar{x}) \to 0 \textit{ as } l \to +\infty.$$

Proof By taking $\theta := 0$ in Theorem 2, we know that $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ with $\bar{x} \in A$ is a weak Pareto optimal solution of (P) if and only if there exist $\lambda \in K^+ \backslash \{0\}$, $\{\mu_l\} \subset S^+$, $\{\widetilde{\epsilon}_l\}$, $\{\widetilde{\eta}_l\}$, $\{\widetilde{\zeta}_l\} \subset \mathbb{R}_+$, $u_l \in \partial_{\widetilde{\epsilon}_l} \varphi_{\lambda \circ F}(\bar{x})$, $v_l \in \partial_{\widetilde{\eta}_l} \varphi_{\mu_l \circ G}(\bar{x})$, $w_l \in N_C^{\widetilde{\zeta}_l}(\bar{x})$ such that

$$u_l + v_l + w_l \to 0, \text{ as } l \to +\infty$$

and

$$\lim_{l \to +\infty} (\widetilde{\epsilon}_l + \widetilde{\eta}_l + \widetilde{\zeta}_l - \varphi_{\mu_l \circ G}(\bar{x})) + \langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) = 0.$$

Due to the feasibility of $\bar{x}$, there exits $\bar{z} \in G(\bar{x})$ such that $\bar{z} \in -S$. It follows that for each $l \in \mathbb{N}$, $\varphi_{\mu_l \circ G}(\bar{x}) \leqq \langle \mu_l, \bar{z} \rangle \leqq 0$. By $\{\widetilde{\epsilon}_l\}$, $\{\widetilde{\eta}_l\}$, $\{\widetilde{\zeta}_l\} \subset \mathbb{R}_+$, $-\varphi_{\mu_l \circ G}(\bar{x}) \geqq 0$ and $\langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) \geqq 0$, we can obtain that $\langle \lambda, \bar{y} \rangle = \varphi_{\lambda \circ F}(\bar{x})$ and $\lim_{l \to +\infty} \widetilde{\epsilon}_l = \lim_{l \to +\infty} \widetilde{\eta}_l = \lim_{l \to +\infty} \widetilde{\zeta}_l = \lim_{l \to +\infty} \varphi_{\mu_l \circ G}(\bar{x}) = 0$.

Letting $\epsilon_l := \max\{\widetilde{\epsilon}_l, \widetilde{\eta}_l, \widetilde{\zeta}_l\}$. Then, we have that $\epsilon_l \to 0$ as $l \to +\infty$ and $u_l \in \partial_{\epsilon_l} \varphi_{\lambda \circ F}(\bar{x})$, $v_l \in \partial_{\epsilon_l} \varphi_{\mu_l \circ G}(\bar{x})$, $w_l \in N_C^{\epsilon_l}(\bar{x})$. So, we obtain the desired result. □

4 A new constrained qualification for θ-weak Pareto optimal solution

In this section, we give a new constrained qualification for θ-weak Pareto optimal solution of problem (P). From now on, we say that the set $A := \{x \in C : G(x) \cap -S \neq \emptyset\}$ is said to satisfy the *closed cone constraint qualification* when the set

$$\cup_{\mu \in S^+} \operatorname{epi}(\varphi_{\mu \circ G})^* + \operatorname{epi}\delta_C^* \qquad \text{(CCCQ)}$$

is closed.

Remark 6 In the case of G is a single-valued map, namely, $G(x) := \{g(x)\}$ for every $x \in \mathbb{R}^n$ where $g : \mathbb{R}^n \to \mathbb{R}^m$, we get $A = \{x \in C : g(x) \in -S\}$ and $\varphi_{\mu \circ G} = \mu \circ g$ for each $\mu \in S^+$. Then (CCCQ) collapses to the usual closed cone constraint qualification which was proposed in [23] and was used in [3,11,24,25] and the references therein to establish optimality conditions for convex (infinite) programming problems.

The following theorem establishes necessary/sufficient optimality criteria for θ-weak Pareto optimal solution of problem (P) in terms of approximate subdifferentials under the condition (CCCQ).

Theorem 3 *For the problem* (P)*, suppose that* $\mathrm{dom}(F) = \mathbb{R}^n$*. If* $A \neq \emptyset$ *and* (CCCQ) *is fulfilled, then* $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ *with* $\bar{x} \in A$ *is an* θ*-weak Pareto optimal solution of* (P) *if and only if there exist* $\lambda \in K^+ \backslash \{0\}$*,* $\mu \in S^+$*,* $\epsilon, \eta, \zeta \in \mathbb{R}_+$ *such that*

$$0 \in \partial_\epsilon \varphi_{\lambda \circ F}(\bar{x}) + \partial_\eta \varphi_{\mu \circ G}(\bar{x}) + N_C^\zeta(\bar{x}) \tag{6}$$

and

$$\epsilon + \eta + \zeta - \varphi_{\mu \circ G}(\bar{x}) + \langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) = \langle \lambda, \theta \rangle. \tag{7}$$

Proof As seen before, we know that $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ with $\bar{x} \in A$ is an θ-weak Pareto optimal solution of (P) if and only if there exists $\lambda \in K^+ \backslash \{0\}$ such that (5) holds. Note that $\varphi_{\lambda \circ F}$ is a convex function for each $\lambda \in K^+$ due to K-convexity of F, and so, it is continuous on $\mathrm{ri}(\mathrm{dom}(\varphi_{\lambda \circ F}))$. As $\mathrm{dom}(\varphi_{\lambda \circ F}) = \mathrm{dom}(\lambda \circ F) = \mathrm{dom}(F) = \mathbb{R}^n$, $\varphi_{\lambda \circ F}$ is a continuous function on $\mathbb{R}^n$. By Lemma 1(ii), one has

$$\mathrm{epi}(\varphi_{\lambda \circ F} + \varphi_{\widetilde{\delta}_A})^* = \mathrm{epi}(\varphi_{\lambda \circ F})^* + \mathrm{epi}(\widetilde{\delta}_A)^*.$$

Since $\cup_{\mu \in S^+} \mathrm{epi}(\varphi_{\mu \circ G})^* + \mathrm{epi}\delta_C^*$ is closed, the above equality, by Lemma 4, becomes

$$\mathrm{epi}(\varphi_{\lambda \circ F} + \widetilde{\delta}_A)^* = \mathrm{epi}(\varphi_{\lambda \circ F})^* + \cup_{\mu \in S^+} \mathrm{epi}(\varphi_{\mu \circ G})^* + \mathrm{epi}\delta_C^*.$$

This together with (5) in turn implies that there exist $\mu \in S^+$, $(u, \alpha) \in \mathrm{epi}(\varphi_{\lambda \circ F})^*$, $(v, \beta) \in \mathrm{epi}(\varphi_{\mu \circ G})^*$ and $(w, \gamma) \in \mathrm{epi}\delta_C^*$ such that $0 = u + v + w$ and $\alpha + \beta + \gamma = -\langle \lambda, \bar{y} \rangle + \langle \lambda, \theta \rangle$. In view of (1), there exist $\epsilon, \eta, \zeta \in \mathbb{R}_+$ such that

$$\begin{cases} u \in \partial_\epsilon \varphi_{\lambda \circ F}(\bar{x}), \ \alpha = \langle u, \bar{x} \rangle + \epsilon - \varphi_{\lambda \circ F}(\bar{x}), \\ v \in \partial_\eta \varphi_{\mu \circ G}(\bar{x}), \ \beta = \langle v, \bar{x} \rangle + \eta - \varphi_{\mu \circ G}(\bar{x}), \\ w \in \partial_\zeta \delta_C(\bar{x}), \ \gamma = \langle w, \bar{x} \rangle + \zeta. \end{cases}$$

Therefore,

$$\begin{aligned} \langle \lambda, \theta \rangle &= \alpha + \beta + \gamma + \langle \lambda, \bar{y} \rangle \\ &= \langle u + v + w, \bar{x} \rangle + \langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) - \varphi_{\mu \circ G}(\bar{x}) + \epsilon + \eta + \zeta \\ &= \langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) - \varphi_{\mu \circ G}(\bar{x}) + \epsilon + \eta + \zeta. \end{aligned}$$

The converse conclusion can also be obtained easily by Theorem 2 and will be omitted. □

The next example demonstrates that the approximate subgradient conditions (6) and/or (7) in Theorem 3 may fail for an θ-weak Pareto optimal solution of (P) under the violation of the condition (CCCQ).

Example 2 Let K, S, C, and G be defined as in Example 1. Let $(\bar{x}, \bar{y}) := (0, (1, 0.1))$, $\theta := (0, 0.1)$, and let F be defined by $F(x) := (x, -x) + \mathbb{R}^2_+$ for every $x \in \mathbb{R}$. Then we have already seen that $K^+ = S^+ = \mathbb{R}^2_+$, and for each $(\mu_1, \mu_2) \in \mathbb{R}^2_+$, $\varphi_{\mu \circ G}(x) = \mu_1 x + \mu_2 - \sqrt{\mu_1^2 + \mu_2^2}$ for all $x \in \mathbb{R}$. It can be easily checked that $(\bar{x}, \bar{y})$ is an θ-weak Pareto optimal solution of (P) and that for each $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$, $\varphi_{\lambda \circ F}(x) = (\lambda_1 - \lambda_2)x$ for all $x \in \mathbb{R}$. We assert that the conditions (6) and (7) in Theorem 3 do not hold for this setting. Otherwise, there exist $\lambda := (\lambda_1, \lambda_2) \in K^+ \backslash \{0\}$, $\mu := (\mu_1, \mu_2) \in S^+$, $\epsilon, \eta, \zeta \in \mathbb{R}_+$ such that

$$0 \in \partial_\epsilon \varphi_{\lambda \circ F}(\bar{x}) + \partial_\eta \varphi_{\mu \circ G}(\bar{x}) + N_C^\zeta(\bar{x}) = \{\lambda_1 - \lambda_2\} + \{\mu_1\} + [-\zeta, \zeta]$$

and

$$0 = \epsilon + \eta + \zeta - \varphi_{\mu \circ G}(\bar{x}) + \langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) - \langle \lambda, \theta \rangle$$
$$= \epsilon + \eta + \zeta - \mu_2 + \sqrt{\mu_1^2 + \mu_2^2} + \lambda_1.$$

We obtain that $\epsilon = \eta = \zeta = \lambda_1 = -\mu_2 + \sqrt{\mu_1^2 + \mu_2^2} = 0$ due to $\epsilon, \ \eta, \ \zeta, \ \lambda_1, \ -\mu_2 + \sqrt{\mu_1^2 + \mu_2^2} \geqq 0$. It follows that $\mu_1 = 0$, which result in $\lambda_2 = 0$, and therefore, we arrive at a contradiction that $\lambda \neq 0$. Consequently, the conclusion of Theorem 3 fails to hold. The reason is that the cone $\cup_{\mu \in S^+} \text{epi}(\varphi_{\mu \circ G})^* + \text{epi}\delta_C^*$ is not closed. To see this, for each $\mu \in S^+$,

$$(\varphi_{\mu \circ G})^*(\xi) = \begin{cases} \sqrt{\mu_1^2 + \mu_2^2} - \mu_2, & \text{if } \xi = \mu_1; \\ +\infty, & \text{otherwise,} \end{cases}$$

and $\delta_C^*(\cdot) = |\cdot|$. So,

$$\cup_{\mu \in S^+} \text{epi}(\varphi_{\mu \circ G})^* = \cup_{\mu_1, \mu_2 \geqq 0} \{\mu_1\} \times [\sqrt{\mu_1^2 + \mu_2^2} - \mu_2, +\infty[$$
$$= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geqq 0, \ x_2 > 0\},$$

and hence,

$$\cup_{\mu \in S^+} \text{epi}(\varphi_{\mu \circ G})^* + \text{epi}\delta_C^* = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leqq 0, \ x_1 + x_2 \geqq 0\} \cup \text{int}\mathbb{R}^2_+,$$

which is not closed.

On the other hand, letting $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2) := (0, 1)$, $\epsilon_l = \eta_l = \zeta_l := \frac{1}{2l}$, $\mu_l := (\mu_1^l, \mu_2^l) := (1 - \frac{1}{2l}, l - 1)$, $u_l := -1$, $v_l := \mu_1^l$, and $w_l := \zeta_l$ for each $l \in \mathbb{N}$. Then, elementary calculations give us

$$\varphi_{\mu_l \circ G}(\bar{x}) = l - 1 - \sqrt{\left(1 - \frac{1}{2l}\right)^2 + (l-1)^2} = l - 1 - \sqrt{\left(l - 1 + \frac{1}{2l}\right)^2} = -\frac{1}{2l},$$

$\langle \bar{\lambda}, \bar{y} \rangle - \varphi_{\bar{\lambda} \circ F}(\bar{x}) - \langle \bar{\lambda}, \theta \rangle = 0$, $u_l + v_l + w_l = 0$ and $\epsilon_l + \eta_l + \zeta_l - \varphi_{\mu_l \circ G}(\bar{x}) = \frac{2}{l} \to 0$ as $l \to +\infty$. So, the sequential conditions of Theorem 2 hold. □

In the following corollary we derive an optimality condition for a weak Pareto optimal solution of (P) under the condition (CCCQ).

Corollary 2 *For the problem* (P)*, suppose that* $\text{dom}(F) = \mathbb{R}^n$. *If* $A \neq \emptyset$ *and* (CCCQ) *is fulfilled, then* $(\bar{x}, \bar{y}) \in \text{gph}(F)$ *with* $\bar{x} \in A$ *is a weak Pareto optimal solution of* (P) *if and only if there exist* $\lambda \in K^+ \backslash \{0\}$ *and* $\mu \in S^+$ *such that* $\langle \lambda, \bar{y} \rangle = \varphi_{\lambda \circ F}(\bar{x})$,

$$0 \in \partial \varphi_{\lambda \circ F}(\bar{x}) + \partial \varphi_{\mu \circ G}(\bar{x}) + N_C(\bar{x}) \text{ and } \varphi_{\mu \circ G}(\bar{x}) = 0.$$

Proof We apply Theorem 3 with $\theta := 0$ to assert that $(\bar{x}, \bar{y}) \in \text{gph}(F)$ with $\bar{x} \in A$ is a weak Pareto optimal solution of (P) if and only if there exist $\lambda \in K^+ \backslash \{0\}$, $\mu \in S^+$, $\epsilon, \eta, \zeta \in \mathbb{R}_+$ such that

$$0 \in \partial_\epsilon \varphi_{\lambda \circ F}(\bar{x}) + \partial_\eta \varphi_{\mu \circ G}(\bar{x}) + N_C^\zeta(\bar{x})$$

and

$$\epsilon + \eta + \zeta - \varphi_{\mu \circ G}(\bar{x}) + \langle \lambda, \bar{y} \rangle - \varphi_{\lambda \circ F}(\bar{x}) = 0. \tag{8}$$

Due to the feasibility of $\bar{x}$, there exists $\bar{z} \in G(\bar{x})$ such that $\bar{z} \in -S$. This leads us to $\varphi_{\mu\circ G}(\bar{x}) \leqq \langle \mu, \bar{z}\rangle \leqq 0$. By $\epsilon,\ \eta,\ \zeta \in \mathbb{R}_+$, $-\varphi_{\mu\circ G}(\bar{x}) \geqq 0$ and $\langle \lambda, \bar{y}\rangle - \varphi_{\lambda\circ F}(\bar{x}) \geqq 0$, it entails especially by (8) that

$$\epsilon = \eta = \zeta = \varphi_{\mu\circ G}(\bar{x}) = \langle \lambda, \bar{y}\rangle - \varphi_{\lambda\circ F}(\bar{x}) = 0.$$

The rest of the proof for the converse conclusion follows by using Theorem 3 and so is omitted here. □

Remark 7 In view of [28, Proposition 2.2.(ii)], we can also formulate a version of Theorem 2 and Corollary 1 (resp. Theorem 3 and Corollary 2) in terms of the *radial ϵ-subdifferentials* (see e.g. [28, Definition 1.2]) (resp. radial subdifferentials) by assuming that $\bar{x} \in \operatorname{int}(\operatorname{dom}(F))$.

Remark 8 It is interesting to note here that we can obtain in a similar manner the sequential characterizations of an approximate Pareto optimal solution as well as an approximate Benson proper Pareto optimal solution of (P) by using the scalarization theorems that given in [40, Theorem 4.1 and Theorem 4.4].

Next, under an additional condition, we will see that the Slater-type constraint qualification guarantees the validity of (CCCQ). To this aim, we need the following lemma.

Lemma 5 *Suppose that* $\operatorname{int}(S) \neq \emptyset$ *and there exists* $\hat{x} \in \mathbb{R}^n$ *such that* $G(\hat{x}) \cap -\operatorname{int}(S) \neq \emptyset$. *Then the set* $\cup_{\mu\in S^+}\operatorname{epi}(\varphi_{\mu\circ G})^*$ *is closed.*

Proof Let $(v_l, \alpha_l) \in \cup_{\mu\in S^+}\operatorname{epi}(\varphi_{\mu\circ G})^*$ be such that $(v_l, \alpha_l) \to (v, \alpha)$ as $l \to +\infty$. We only need to show that

$$(v, \alpha) \in \cup_{\mu\in S^+}\operatorname{epi}(\varphi_{\mu\circ G})^*.$$

To see this, as $\operatorname{int}(S) \neq \emptyset$, there is a compact convex set $\mathcal{B} \subseteq S^+$ with $0 \notin \mathcal{B}$ and $S^+ = \operatorname{cone}(\mathcal{B})$ (see, e.g., [20, Lemma 1.28(a)]). Then, there exist $\{\mu_l\} \subset S^+$, $\{t_l\} \subset \mathbb{R}_+$ and $\{b_l\} \subset \mathcal{B}$ such that $\mu_l = t_l b_l$, $(x_l, \alpha_l) \in \operatorname{epi}(\varphi_{\mu_l\circ G})^* = \operatorname{epi}(\mu_l \circ G)^*$ and $b_l \to b \in \mathcal{B}$ as $l \to +\infty$.

We first show that $\{t_l\}$ is bounded. Otherwise, we may assume that $t_l \to +\infty$ as $l \to +\infty$. Then, for each $l \in \mathbb{N}$,

$$\langle v_l, x\rangle - \langle \mu_l, y\rangle \leqq (\mu_l \circ G)^*(v_l) \leqq \alpha_l,\ \ \forall x \in \operatorname{dom}(G),\ \ y \in G(x).$$

In particular, due to $\hat{x} \in \mathbb{R}^n$ satisfying $G(\hat{x}) \cap -\operatorname{int}(S) \neq \emptyset$, there is $\hat{y} \in G(\hat{x})$ such that $-\hat{y} \in \operatorname{int}(S)$. So, $\langle v_l, \hat{x}\rangle - \langle \mu_l, \hat{y}\rangle \leqq \alpha_l$, and consequently,

$$\frac{1}{t_l}\langle v_l, \hat{x}\rangle - \langle b_l, \hat{y}\rangle \leqq \frac{\alpha_l}{t_l}.$$

Passing to limit, we see that $\langle b, \hat{y}\rangle \geqq 0$. On the one hand, as $b \in S^+\backslash\{0\}$ and $-\hat{y} \in \operatorname{int}(S)$, we have $\langle b, \hat{y}\rangle < 0$, a contradiction.

Now, since $\{t_l\}$ is bounded, we may assume that $t_l \to t$ as $l \to +\infty$ for some $t \in \mathbb{R}_+$. Thus, for each $x \in \operatorname{dom}(G)$ and $y \in G(x)$, one has $\langle v, x\rangle - \langle tb, y\rangle \leqq \alpha$, and consequently, $(\bar{\mu} \circ G)^*(v) \leqq \alpha$ where $\bar{\mu} := tb \in S^+$. Therefore,

$$(v, \alpha) \in \operatorname{epi}(\bar{\mu} \circ G)^* \subseteq \cup_{\mu\in S^+}\operatorname{epi}(\mu \circ G)^* = \cup_{\mu\in S^+}\operatorname{epi}(\varphi_{\mu\circ G})^*,$$

and the proof is complete. □

Theorem 4 *Suppose that* $\text{int}(S) \neq \emptyset$ *and there exists* $\hat{x} \in \text{ri}(C)$ *such that* $G(\hat{x}) \cap -\text{int}(S) \neq \emptyset$. *If* G *is* S*-upper semicontinuous [13, Definition 2.5.21], i.e.,* $\{x \in \mathbb{R}^n : G(x) \cap (y - \text{int}(S)) \neq \emptyset\}$ *is open for every* $y \in \mathbb{R}^m$*, then* (CCCQ) *holds.*

Proof As G is S-upper semicontinuous, the set $\{x \in \mathbb{R}^n : G(x) \cap (-\text{int}(S)) \neq \emptyset\}$ is open, and so, $\hat{x} \in \{x \in \mathbb{R}^n : G(x) \cap (-\text{int}(S)) \neq \emptyset\} \subseteq \text{int}(\widetilde{A}) \subseteq \text{ri}(\widetilde{A})$. Since C and $\widetilde{A}$ are closed convex sets, and $\text{ri}(C) \cap \text{ri}(\widetilde{A}) \neq \emptyset$, we have by [8, Proposition 3.2] that the set $\text{epi}\delta^*_{\widetilde{A}} + \text{epi}\delta^*_C$ is closed. According to Lemma 3 and 5, we obtain that $\cup_{\mu \in S^+} \text{epi}(\varphi_{\mu \circ G})^* + \text{epi}\delta^*_C$ is closed. □

To this end, we give the following example that illustrates the case where (CCCQ) holds, whereas the Slater-type constraint qualification fails.

Example 3 Let $C := [-1, 1]$, $S := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geqq 0\}$ and $G(x) := \{(x, r) \in \mathbb{R}^2 : r \geqq \max\{x, 0\}\}$ for every $x \in \mathbb{R}$. Then, $\text{int}(S) \neq \emptyset$ and $G(x) \cap -\text{int}(S) = \emptyset$ for all $x \in \mathbb{R}$. On the one hand, direct calculations show that $\delta^*_C(\cdot) = |\cdot|$, $S^+ = \{(\mu_1, \mu_2) \in \mathbb{R}^2 : \mu_1 = 0,\ \mu_2 \geqq 0\}$, and for each $\mu \in S^+$, $\varphi_{\mu \circ G}(x) = \mu_2 \max\{x, 0\}$,

$$(\varphi_{\mu \circ G})^*(u) = \begin{cases} 0, & \text{if } u \in [0, \mu_2]; \\ +\infty, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \cup_{\mu \in S^+} \text{epi}(\varphi_{\mu \circ G})^* + \text{epi}\delta^*_C &= \cup_{\mu_2 \geqq 0} [0, \mu_2] \times \mathbb{R}_+ + \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leqq x_2\} \\ &= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leqq 0,\ x_1 + x_2 \geqq 0\right\} \cup \mathbb{R}^2_+, \end{aligned}$$

which is a closed set.

5 Conclusions

In this paper, we have employed the scalarization theorem and the relation between the conjugate function of the scalar set-valued map and of the marginal function associated with corresponding the scalar set-valued map to provide sequential Lagrange multiplier conditions characterizing approximate weak Pareto optimal solutions for the convex vector optimization problem with set-valued maps. These conditions are expressed in terms of the approximate subdifferentials of the related marginal functions. Moreover, the approximate Lagrange multiplier conditions for the problem have also been provided by using a new proposed constraint qualification, which is implied by the Slater-type condition.

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