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Stability of a convex feasibility problem

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Abstract
The 2-sets convex feasibility problem aims at finding a point in the intersection of two closed convex sets A and B in a normed space X. More generally, we can consider the problem of finding (if possible) two points in A and B, respectively, which minimize the distance between the sets. In the present paper, we study some stability properties for the convex feasibility problem: we consider two sequences of sets, each of them converging, with respect to a suitable notion of set convergence, respectively, to A and B. Under appropriate assumptions on the original problem, we ensure that the solutions of the perturbed problems converge to a solution of the original problem. We consider both the finite-dimensional and the infinite-dimensional case. Moreover, we provide several examples that point out the role of our assumptions in the obtained results.

Keywords Convex feasibility problem · Stability · Set-convergence

Mathematics Subject Classification Primary 90C25; Secondary 90C31 · 49J53

1 Introduction

The convex feasibility problem is the classical problem of finding a point in the intersection of a finite collection of closed and convex sets (see [5, Sect. 4.5] for the main results on this subject). Many concrete problems in applications can be formulated as a convex feasibility problem. As typical examples, we mention solution of convex inequalities, partial differential equations, minimization of convex nonsmooth functions, medical imaging, computerized tomography and image reconstruction. For some details and other applications see, e.g. [2,7]

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and the references therein. Moreover, it is worth to mention the recent annotated bibliography [6], about projection methods, containing several references to the convex feasibility problem and its applications.

Many efforts have been devoted to the study of algorithmic procedures to solve convex feasibility problems, both from a theoretical and from a computational point of view (see, e.g. [2–4,9] and the references therein).

Often in concrete applications data are affected by some uncertainties. Hence stability of solutions with respect to data perturbations is a desirable property, also in view of the development of a computational approach to solve the convex feasibility problem. Our paper is devoted to investigate some stability properties of the 2-sets convex feasibility problem by using set convergence notions. We will also consider the case of a pair of closed and convex sets with empty intersection: in this case a solution of the problem is a pair of minimal distance elements of the two sets.

In this paper, we investigate a sequence of perturbed convex feasibility problems whose data are obtained by considering two sequences of closed and convex sets $\{A_n\}$ and $\{B_n\}$ converging respectively to the sets A and B. If the intersection of A_n and B_n is empty, we consider, as a solution of the n-th perturbed problem, the pair of elements $a_n \in A_n$ and $b_n \in B_n$ such that the distance between A_n and B_n is $\|a_n - b_n\|$.

Our aim is to find some conditions that guarantee the convergence of the solutions of the perturbed convex feasibility problems to a solution of the original convex feasibility problem.

We obtain some stability results both in the finite-dimensional and in the infinite-dimensional framework, using the Kuratowski–Painlevé convergence notion in the finite-dimensional case and the Attouch–Wets convergence in the infinite-dimensional setting. Moreover, we give some examples showing that the assumptions that we use to guarantee the stability features of a given convex feasibility problem cannot be avoided, both in the finite and in the infinite-dimensional case.

The paper is organized as follows. Section 2 is devoted to definitions and preliminary results, mainly concerning the various notions of set-convergence. Section 3 presents a stability result for the convex feasibility problem when A and B are contained in a finite-dimensional normed vector space and the sequences of closed and convex sets $\{A_n\}$ and $\{B_n\}$ converge in the Kuratowski–Painlevé sense respectively to A and B. Section 4 is devoted to study the stability properties of a convex feasibility problem in an infinite-dimensional setting. Here, we use the Attouch–Wets convergence, that is stronger than the Kuratowski–Painlevé convergence, even if they coincide in the finite-dimensional setting. Moreover, it is worth to be noticed that we obtain results concerning both weak and norm convergence of the solutions of perturbed problems to a solution of the original problem. In order to obtain the norm convergence of a sequence of solutions of perturbed problems, we strengthen the convexity assumptions by assuming that A has nonempty interior and it is locally uniformly rotund (LUR) at a given solution a. Finally, in Sect. 5, we provide some rather involved examples in ℓ_2 that point out the role of our assumptions even in a Hilbert space framework.

2 Notations and preliminaries

Throughout all this paper, X denotes a real normed space with the topological dual X^*. We denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. For $x, y \in X$, $[x, y]$ denotes the closed segment in X with endpoints x and y, and $(x, y) = [x, y] \setminus \{x, y\}$ is the corresponding "open" segment. For a subset K of X, $\alpha > 0$, and a functional $x^* \in S_{X^*}$

bounded on K, let

$$S(x^*, \alpha, K) = \{x \in K;\ x^*(x) \geq \sup x^*(K) - \alpha\}$$

be the closed slice of K given by α and x^*.

For a subset A of X, we denote by int (A), conv (A) and $\overline{\text{conv}}\,(A)$ the interior, the convex hull and the closed convex hull of A, respectively. Moreover,

$$\overline{\text{cone}}(A) = \overline{\text{conv}}\,\big([0, \infty) \cdot A\big)$$

is the closed convex cone generated by the set A. We denote by

$$\text{diam}(A) = \sup_{x,y \in A} \|x - y\|,$$

the (possibly infinite) diameter of A. For $x \in X$, let

$$\text{dist}(x, A) = \inf_{a \in A} \|a - x\|.$$

Moreover, given A, B nonempty subsets of X, we denote by $\text{dist}(A, B)$ the usual "distance" between A and B, that is,

$$\text{dist}(A, B) = \inf_{a \in A} \text{dist}(a, B).$$

2.1 Convergence of sets

By $\text{c}(X)$ we denote the family of all nonempty closed subsets of X. Let $\{A_n\}$ be a sequence in $\text{c}(X)$ and let us consider the following sets:

$$\text{Li } A_n = \{x \in X;\ x = \lim_n x_n, x_n \in A_n\}$$

and

$$\text{Ls } A_n = \{x = \lim_k x_k \in X;\ x_k \in A_{n_k}, \{n_k\} \text{ is a subsequence of the integers}\}.$$

Definition 2.1 Let $\{A_n\}$ be a sequence in $\text{c}(X)$ and $A \in \text{c}(X)$.

(i) $\{A_n\}$ converges to A for the *lower Kuratowski–Painlevé convergence* if $A \subset \text{Li}A_n$.
(ii) $\{A_n\}$ converges to A for the *upper Kuratowski–Painlevé convergence* if $A \supset \text{Ls}A_n$.

Moreover, we say that $\{A_n\}$ converges to A for the *Kuratowski–Painlevé convergence* ($A_n \xrightarrow{K} A$) if $\{A_n\}$ converges to A for the upper and the lower Kuratowski–Painlevé convergence.

Now, let us introduce the (extended) Hausdorff metric h on $\text{c}(X)$. For $A, B \in \text{c}(X)$, we define the excess of A over B as

$$e(A, B) = \sup_{a \in A} d(a, B).$$

Moreover, if $A \neq \emptyset$ and $B = \emptyset$ we put $e(A, B) = \infty$, if $A = \emptyset$ we put $e(A, B) = 0$. We define

$$h(A, B) = \max\big\{e(A, B), e(B, A)\big\}.$$

Definition 2.2 A sequence $\{A_j\}$ in $\text{c}(X)$ is said to *Hausdorff converge* to $A \in \text{c}(X)$ if

$$\lim_j h(A_j, A) = 0.$$

Finally, we introduce the so called Attouch–Wets convergence (see, e.g. [11, Definition 8.2.13]), which can be seen as a localization of the Hausdorff convergence. If $N \in \mathbb{N}$ and $A, B \in \mathrm{c}(X)$, define

$$e_N(A, B) = e(A \cap NB_X, B) \in [0, \infty),$$
$$h_N(A, B) = \max\{e_N(A, B), e_N(B, A)\}.$$

Definition 2.3 A sequence $\{A_j\}$ in $\mathrm{c}(X)$ is said to *Attouch–Wets converge* to $A \in \mathrm{c}(X)$ if, for each $N \in \mathbb{N}$,

$$\lim_j h_N(A_j, A) = 0.$$

Several times without mentioning it, we shall use the following theorem in proving the results contained in Sects. 3 and 4.

Theorem 2.4 (See, e.g. [11, Theorem 8.2.14]) *The sequence of sets $\{A_n\}$ Attouch–Wets converges to A iff*

$$\sup_{\|x\| \le N} |d(x, A_n) - d(x, A)| \to 0 \quad (n \to \infty),$$

whenever $N \in \mathbb{N}$.

We recall that in the finite-dimensional case the Attouch–Wets convergence and the Kuratowski–Painlevé convergence coincide (see, e.g. [11, Sect. 8.2]).

In the sequel, we shall use the following easy-to-prove fact. For the convenience of the reader we provide a proof.

Fact 2.5 *Let A and B be two closed and convex subsets of a normed space X. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the lower Kuratowski–Painlevé convergence. Then*

$$\limsup_n \mathrm{dist}(A_n, B_n) \le \mathrm{dist}(A, B).$$

In particular, if $A \cap B \ne \emptyset$ we have $\lim_n \mathrm{dist}(A_n, B_n) = 0$.

Proof Let $\epsilon > 0$, then there exist $x \in A$ and $y \in B$ such that $\|x - y\| \le \mathrm{dist}(A, B) + \epsilon$. Since $A_n \to A$ and $B_n \to B$ for the lower Kuratowski–Painlevé convergence, there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$, $y_n \to y$ and, for each $n \in \mathbb{N}$, $x_n \in A_n$, $y_n \in B_n$. In particular, it eventually holds $\|x_n - x\| \le \epsilon$ and $\|y_n - y\| \le \epsilon$. Hence, the following inequalities eventually hold:

$$\mathrm{dist}(A_n, B_n) \le \|x_n - y_n\| \le \mathrm{dist}(A, B) + 3\epsilon.$$

By the arbitrariness of $\epsilon > 0$, we have the thesis. □

3 Convergence of minimal distance points of a pair of convex sets: the finite-dimensional case

In this section, we denote by X a finite-dimensional normed space.

Definition 3.1 Let A, B be nonempty closed convex set in X. Let

$$\mathrm{m}(A, B) = \{a \in A;\ \mathrm{dist}(a, B) = \mathrm{dist}(A, B)\}.$$

It is easy to see that $\mathrm{m}(A, B)$ is a closed convex set.

Definition 3.2 Let C be a non empty closed convex subset of X and $x \in C$. Let us define

$$D(x) = \{d \in X;\ x + td \in C,\ \forall t > 0\}.$$

Remark 3.3 By [1, Proposition 2.1.5], if $x, y \in C$ then $D(x) = D(y)$. That is, the set $D(x)$ does not depend on $x \in C$. We denote this set, called the *asymptotic cone* of C, by C_∞.

We prove the following lemma that will be useful in the sequel (it can be seen as a slight generalization of [1, Proposition 2.1.9]).

Lemma 3.4 *Let A and B be nonempty closed convex sets in X such that* $\mathrm{m}(A, B)$ *is nonempty. Then*

$$A_\infty \cap B_\infty = [\mathrm{m}(A, B)]_\infty.$$

Proof Let $a \in A$ and $b \in B$ be such that $\|b - a\| = \mathrm{dist}(A, B)$.

Let us prove that $A_\infty \cap B_\infty \subset [\mathrm{m}(A, B)]_\infty$. Let $d \in A_\infty \cap B_\infty = D(a) \cap D(b)$. Since, for each $t > 0$,

$$\|a - b\| = \|a + td - (b + td)\| = \mathrm{dist}(A, B),$$

we have that $a + td \in \mathrm{m}(A, B)$, whenever $t > 0$. Hence $d \in [\mathrm{m}(A, B)]_\infty$.

For the reverse inclusion, suppose that $a + td \in \mathrm{m}(A, B)$, whenever $t > 0$. Clearly, $d \in A_\infty$. Now, we prove that $d \in B_\infty$. Let us fix $t > 0$ and $n \in \mathbb{N}$, and let us observe that

$$\|b + ntd - (a + ntd)\| = \|b - a\| = \mathrm{dist}(A, B).$$

Hence, there exists $d_n \in B$ such that

$$\|b + ntd - d_n\| \le 2\,\mathrm{dist}(A, B).$$

Then,

$$\|b + td - (b + \tfrac{d_n - b}{n})\| \le \tfrac{2}{n}\,\mathrm{dist}(A, B).$$

By the arbitrariness of $n \in \mathbb{N}$, since $b + \frac{d_n - b}{n} \in B$, and since B is closed, it holds that $b + td \in B$. By the arbitrariness of $t > 0$, the thesis is proved. □

The following theorem is the main result of this section. It proves that, under mild assumption, the 2-sets convex feasibility problem has a considerable degree of stability.

Theorem 3.5 *Let* $\{A_n\}$ *and* $\{B_n\}$ *be two sequences of nonempty closed convex sets in X, A and B two nonempty closed convex subsets of X such that*

$$A_n \to A \quad \text{and} \quad B_n \to B,$$

for the Kuratowski–Painlevé convergence. Suppose that $\mathrm{m}(A, B)$ *is a nonempty bounded set. Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences such that* $a_n \in A_n$, $b_n \in B_n$ ($n \in \mathbb{N}$) *and*

$$dist(A_n, B_n) = \|a_n - b_n\|.$$

Then there exists a subsequence $\{a_{n_k}\}$ *such that*

$$\lim_{k\to\infty} a_{n_k} = c \in \mathrm{m}(A\,, B).$$

Moreover, if $\mathrm{m}(A, B) = \{a\}$ *then* $a_n \to a$.

Proof Let us prove the first part of the theorem. By Fact 2.5, it holds

$$\limsup_n \|a_n - b_n\| \le \operatorname{dist}(A, B). \tag{1}$$

We claim that $\{a_n\}$ and $\{b_n\}$ are bounded.

Suppose that this is not the case and let $a \in A$ and $b \in B$ be such that $\|a-b\| = \operatorname{dist}(A, B)$. Without loss of generality, we can suppose that $\|a_n\|, \|b_n\| \to \infty$. By the lower part of the convergence of $\{A_n\}$ there exists a sequence $\{a'_n\}$ such that $a'_n \in A_n$ and $a'_n \to a$. Since A_n is a convex set, for any $\alpha \in [0, 1]$ it holds:

$$\alpha a_n + (1-\alpha)a'_n = a'_n + \alpha(a_n - a'_n) \in A_n.$$

The sequence

$$\left\{ \frac{a_n - a'_n}{\|a_n - a'_n\|} \right\}$$

has a subsequence converging to $d \neq 0$. There is no loss of generality in assuming

$$\lim_{n\to+\infty} \frac{a_n - a'_n}{\|a_n - a'_n\|} = d.$$

Therefore, it holds

$$a + \beta d = \lim_{n\to+\infty} \left(a'_n + \frac{\beta}{\|a_n - a'_n\|}(a_n - a'_n) \right).$$

Since for every $\beta > 0$ there exists $n_2(\beta) \in \mathbb{N}$ such that $\frac{\beta}{\|a_n - a'_n\|} \in [0, 1]$, whenever $n > n_2(\beta)$, it holds

$$a + \beta d \in A,$$

for every $\beta > 0$. Hence, $d \in A_\infty$.

Analogously, we may prove that

$$\lim_{n\to+\infty} \frac{b_n - b'_n}{\|b_n - b'_n\|} = d' \in B_\infty,$$

where $\{b'_n\}$ is a sequence such that $b'_n \in B_n$ and $b'_n \to b$.

Let us observe that $\{a'_n\}$, $\{b'_n\}$ and $\{a_n - b_n\}$ are bounded sequences in X. Since $\|a_n\|, \|b_n\| \to \infty$, we have $\|a_n - a'_n\| \sim \|b_n - b'_n\|$ and hence

$$d = \lim_{n\to+\infty} \frac{a_n - a'_n}{\|a_n - a'_n\|} = \lim_{n\to+\infty} \frac{b_n - b'_n}{\|b_n - b'_n\|} = d',$$

Therefore we have

$$0 \neq d \in A_\infty \cap B_\infty.$$

By Lemma 3.4, we have

$$A_\infty \cap B_\infty = [\mathrm{m}(A, B)]_\infty.$$

Then $\mathrm{m}(A, B)$ is not a bounded set, a contradiction.

By the claim and compactness, there exist two subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$, respectively of $\{a_n\}$ and of $\{b_n\}$, such that

$$\lim_{k\to+\infty} a_{n_k} = u, \quad \lim_{k\to+\infty} b_{n_k} = v,$$

where $u \in A$ and $v \in B$. By Fact 2.5, $\|u - v\| = \mathrm{dist}(A, B)$ and the thesis is proved.

The second part of the theorem follows easily by the first part. □

Remark 3.6 (i) The above theorem can be proved in an alternative way, by using known results concerning stability theory for convex optimization problem. However, we preferred to present a direct and more geometrical proof. We give a sketch of the alternative proof below. (See, e.g. [11] for definitions and main results about convergence of functions and well-posed problems).

Let $f, f_n : X \times X \to (\infty, \infty]$ $(n \in \mathbb{N})$ be the convex lower semicontinuous functions defined as follows. For each $(x_1, x_2) \in X \times X$ and $n \in \mathbb{N}$, put

$$f(x_1, x_2) = \begin{cases} \|x_1 - x_2\| & \text{if } x_1 \in A \text{ and } x_2 \in B; \\ \infty & \text{otherwise;} \end{cases}$$

and

$$f_n(x_1, x_2) = \begin{cases} \|x_1 - x_2\| & \text{if } x_1 \in A_n \text{ and } x_2 \in B_n; \\ \infty & \text{otherwise.} \end{cases}$$

Since $A_n \to A$ and $B_n \to B$ for the Kuratowski–Painlevé convergence (equivalently, for the Attouch–Wets convergence), we have that $f_n \to f$ for the Kuratowski–Painlevé convergence. Moreover, proceeding as in the proof of Theorem 3.5, we may prove that f is Tykhonov well-posed in the generalized sense. Hence, we can apply [11, Theorem 10.2.24] to obtain the thesis.

(ii) It is interesting to observe that, under the hypothesis of the above theorem, we have $\|a_n - b_n\| \to \mathrm{dist}(A, B)$ $(n \to \infty)$. Indeed, if $\{n_k\}$ is a subsequence of the integers, by the proof of Theorem 3.5 there exists $\{n_{k_h}\}$, a subsequence of $\{n_k\}$, such that $\lim_h a_{n_{k_h}} = u \in A$, $\lim_h b_{n_{k_h}} = v \in B$, and $\|u - v\| = \mathrm{dist}(A, B)$. Then $\lim_h \|a_{n_{k_h}} - b_{n_{k_h}}\| = \mathrm{dist}(A, B)$ and the proof is complete.

Whenever the two limit sets are such that $A \cap B \neq \emptyset$, we have the following corollary.

Corollary 3.7 *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of nonempty closed convex sets in X, A and B two nonempty closed convex subsets of X such that*

$$A_n \to A \quad \text{and} \quad B_n \to B,$$

for the Kuratowski–Painlevé convergence. Suppose that $A \cap B$ is a nonempty bounded set. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $a_n \in A_n$, $b_n \in B_n$ $(n \in \mathbb{N})$ and

$$dist(A_n, B_n) = \|a_n - b_n\|.$$

Then there exist two subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ such that

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} b_{n_k} = c \in A \cap B.$$

Moreover, if $A \cap B = \{c\}$ then $a_n, b_n \to c$.

The following examples show that both the assumptions in Theorem 3.5 play an independent role and each of them cannot be deleted. The first one focuses on the role of convexity assumptions.

Example 3.8 Let us consider the sets ($n \geq 2$):

$$A_n = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq -\frac{1}{n},\ n\frac{x_1+1}{n-1} \leq x_2 \leq 2 + n\frac{x_1+1}{1-n}\right\} \cup \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq -\tfrac{1}{2n},\ \tfrac{2}{x_1^2} \leq x_2 \leq 8n^2\right\}$$

and

$$B_n = \left\{(x_1, x_2) \in \mathbb{R}^2 : \frac{1}{n} \leq x_1 \leq 1,\ n\frac{x_1-1}{1-n} \leq x_2 \leq 2 + n\frac{x_1-1}{n-1}\right\} \cup \left\{(x_1, x_2) \in \mathbb{R}^2 : \frac{1}{2n} \leq x_1 \leq 1,\ \frac{2}{x_1^2} \leq x_2 \leq 8n^2\right\}.$$

The sequences $\{A_n\}$ and $\{B_n\}$ converge respectively to

$$A = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 0,\ x_1 + 1 \leq x_2 \leq 1 - x_1\right\} \cup \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 < 0,\ x_2 \geq \frac{2}{x_1^2}\right\}$$

and

$$B = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 1 - x_1 \leq x_2 \leq 1 + x_1\right\} \cup \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ x_2 \geq \frac{2}{x_1^2}\right\}.$$

It is easy to see that $A \cap B = \{(0, 1)\}$ and

$$A_n \xrightarrow{K} A, \quad B_n \xrightarrow{K} B.$$

All the assumptions of Theorem 3.5 are satisfied except for the convexity of A_n and B_n. The minimal distance between the sets A_n and B_n is achieved only at the pair of points

$$a_n = \left(-\frac{1}{2n}, 8n^2\right) \in A_n \text{ and } b_n = \left(\frac{1}{2n}, 8n^2\right) \in B_n.$$

It is clear that the sequences $\{a_n\}$ and $\{b_n\}$ are not bounded. Hence, the thesis of Theorem 3.5 does not hold.

The second example proves that the boundedness assumption on the set $\mathrm{m}(A, B)$ cannot be dropped.

Example 3.9 Let A_n and B_n be defined as in Example 3.8. Let us consider the sets

$$C_n = \mathrm{conv}(A_n) \quad \text{and} \quad D_n = \mathrm{conv}(B_n).$$

It is easy to see that

$$C_n \xrightarrow{K} C, \quad D_n \xrightarrow{K} D,$$

where

$$C = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \le x_1 \le 0,\ x_1 + 1 \le x_2\right\}$$

and

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1,\ 1 - x_1 \le x_2\right\}.$$

Moreover, we have $C \cap D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0,\ x_2 \ge 1\right\}$.

All the assumptions of Theorem 3.5 are satisfied except for the boundedness of the set $C \cap D$. The minimal distance between the sets C_n and D_n is achieved only at the same pair of points $a_n \in C_n$ and $b_n \in D_n$ as in Example 3.8. Of course, as in the previous example both the sequences $\{a_n\}$ and $\{b_n\}$ have no convergent subsequences. Therefore the thesis of Theorem 3.5 does not hold.

4 Convergence of minimal distance points of a pair of convex sets: the infinite-dimensional case

In an infinite-dimensional setting, we need some strengthenings of the assumptions to obtain stability results for our problems. Indeed, Example 5.2, in Sect. 5, shows that an analogue of Theorem 3.5 does not hold, even if we assume that the sequences of sets converge for the Hausdorff convergence and that the space X is a Hilbert space. In this section, we prove that an additional geometric condition on the limit sets ensures the stability result (see Theorem 4.5 below). Moreover, we use the Attouch–Wets convergence of sets instead of the Kuratowski–Painlevé convergence (cf. Example 5.6).

We start with some definitions and preliminary results. Let us recall that a *body* in X is a closed convex set in X with nonempty interior.

Definition 4.1 (*See, e.g.* [8, Definition 7.10]) Let A be a nonempty subset of a normed space X. A point $a \in A$ is called a *strongly exposed point* of A if there exists a support functional $f \in X^*\setminus\{0\}$ for A in a (i.e., $f(a) = \sup f(A)$), such that $x_n \to a$ for all sequences $\{x_n\}$ in A such that $\lim f(x_n) = \sup f(A)$. In this case, we say that f strongly exposes A at a.

Let us observe that $f \in S_{X^*}$ strongly exposes A at a iff $f(a) = \sup f(A)$ and

$$\operatorname{diam}\big(S(f, \alpha, A)\big) \to 0 \text{ as } \alpha \to 0.$$

Definition 4.2 (*See, e.g.* [10, Definition 1.3]) Let $A \subset X$ be a body. We say that $x \in \partial A$ is an *LUR (locally uniformly rotund) point* of A if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in A$ and $\operatorname{dist}(\partial A, (x + y)/2) < \delta$ then $\|x - y\| < \epsilon$.

If $A = B_X$, the previous definition coincides with the standard definition of local uniform rotundity of the norm at x. We say that A is an *LUR body* if each point in ∂A is an LUR point of A.

Lemma 4.3 *Let A be a body in X and suppose that $a \in \partial A$ is an LUR point of A. Then, if $f \in S_{X^*}$ is a support functional for A in a, f strongly exposes A at a. Moreover, every slice S of the form $S = S(f, \alpha, A)$ is a bounded set.*

The first part of the lemma is well-known in the case the body is a ball (see, e.g. [8, Exercise 8.27]) and in the general case the proof is similar. However, for the convenience of the reader we include a proof.

Proof Without loss of generality, we can suppose that $a = 0$. Fix $w \in \operatorname{int} A$ and observe that $f(w) < 0$.

Let us prove the first part of the lemma. Let $\alpha > 0$, $z \in S = S(f, \alpha, A)$ and

$$z' = z - \frac{f(z)}{f(w)} w.$$

Since $\frac{z}{2} \in A$ and $f(z') = 0$, we have that $[\frac{z}{2}, \frac{z'}{2}] \cap \partial A \neq \emptyset$. Hence

$$\operatorname{dist}(\partial A, \tfrac{z}{2}) \leq \tfrac{1}{2}\|z' - z\| \leq \tfrac{1}{2}\tfrac{\|w\|}{|f(w)|}\alpha.$$

Since $a = 0$ is an LUR point of A, if $\alpha \to 0$ then $\operatorname{diam}(S) \to 0$ and the proof is concluded.

Now, the second part of the lemma follows easily. Suppose on the contrary that there exists $\alpha > 0$ such that $S = S(f, \alpha, A)$ is unbounded. Then there exists a sequence $\{y_n\}$ in $S \backslash \{0\}$ such that $\|y_n\| \to \infty$. Put $z_n = \frac{y_n}{\|y_n\|}$ and observe that $\|z_n\| = 1$ and $z_n \in S(f, \alpha/\|y_n\|, A)$, a contradiction by the first part of the lemma. □

Lemma 4.4 *Let X be a normed space. There exists a constant $\Gamma > 0$ such that if $R > 1$, if $x, y, a, b \in X$ are such that $\|x\|, \|y\| < R$ and $\|a\|, \|b\| > 2R$, then, if $[x, a] \cap RS_X = \{a'\}$ and $[y, b] \cap RS_X = \{b'\}$, it holds*

$$\|b' - a'\| \leq \Gamma \max\{\|x\|, \|y\|, \|a - b\|\}.$$

Proof Let $\lambda, \mu \in (0, 1)$ be such that $a' = \lambda a + (1 - \lambda)x$ and $b' = \mu b + (1 - \mu)y$. By the triangle inequality, it follows easily that

$$\frac{R - \|y\|}{\|b\| - \|y\|} \leq \mu.$$

Moreover, since

$$R = \|\lambda a + (1 - \lambda)x\| \geq \lambda\|a\| - (1 - \lambda)\|x\|,$$

we have

$$\lambda \leq \frac{R + \|x\|}{\|a\| + \|x\|}.$$

Without loss of generality, we can assume that $\lambda \geq \mu$. If we denote

$$d = \max\{\|x\|, \|y\|, \|a - b\|\},$$

we have

$$\begin{aligned}
\|b' - a'\| &\leq \lambda\|a - b\| + (1 - \lambda)\|x - y\| + |\lambda - \mu|(\|y\| + \|b\|) \\
&\leq 3d + \Big(\frac{R + \|x\|}{\|a\| + \|x\|} - \frac{R - \|y\|}{\|b\| - \|y\|}\Big)(\|y\| + \|b\|) \\
&\leq 3d + \frac{R(\|b\| - \|a\| - \|y\| - \|x\|) + \|a\|\,\|y\| + \|b\|\,\|x\|}{(\|a\| + \|x\|)(\|b\| - \|y\|)}(\|y\| + \|b\|) \\
&\leq 3d + \frac{R(\|b\| - \|a\| - \|y\| - \|x\|) + \|a\|\,\|y\| + \|b\|\,\|x\|}{(\|a\| + \|x\|)(\|b\|/2)}(2\|b\|) \\
&\leq 3d + 4\frac{R(\|b\| - \|a\| - \|y\| - \|x\|)}{2R} + 4\frac{\|a\|\,\|y\| + \|b\|\,\|x\|}{\|a\| + \|x\|} \\
&\leq 3d + 2\big(\big|\|b\| - \|a\|\big|\big) + \frac{4\|a\|\,\|y\|}{\|a\| + \|x\|} + \frac{4\|b\|\,\|x\|}{\|a\| + \|x\|} \\
&\leq 5d + \frac{4\|a\|}{\|a\| + \|x\|}d + \frac{4\|b - a + a\|\,\|x\|}{\|a\| + \|x\|} \\
&\leq 5d + 4d + \frac{4\|x\|}{\|a\| + \|x\|}\|b - a\| + \frac{4\|a\|}{\|a\| + \|x\|}\|x\| \leq 17d
\end{aligned}$$

The proof is concluded if we set $\Gamma = 17$ □

The following theorem is the main result of this section.

Theorem 4.5 *Let X be a normed space, B a nonempty closed convex subset of X, A a body in X and $a \in \partial A$ an LUR point of A. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the Attouch–Wets convergence. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $a_n \in A_n$, $b_n \in B_n$ ($n \in \mathbb{N}$) and*

$$\operatorname{dist}(A_n, B_n) = \|a_n - b_n\|.$$

Suppose that at least one of the following conditions holds.

1. $A \cap B = \{a\}$.
2. $A \cap B = \emptyset$ *and there exists $b \in B$ such that* $\operatorname{dist}(A, B) = \|a - b\|$.

Then $a_n \to a$ in the $\|\cdot\|$-topology.

Proof There is no loss of generality in assuming $a = 0$. Let us assume that (1) holds.

Since $\operatorname{int}(A) \cap B = \emptyset$, by the Hahn–Banach theorem there exists $f \in S_{X^*}$ such that

$$\sup f(A) = 0 = \inf f(B).$$

In particular, f is a support functional for A in 0. Let $\alpha > 0$ and observe that, by Lemma 4.3, there exists $r > 1$ such that $S = S(f, 3\alpha, A) \subset rB_X$. Put $R = r + \alpha$.

We claim that $\{a_n\}$ and $\{b_n\}$ are eventually contained in $2RB_X$. Suppose that this is not the case and let $\{a_{n_k}\}$ and $\{b_{n_k}\}$ be two subsequences such that $\|a_{n_k}\| > 2R$ and $\|b_{n_k}\| > 2R$ whenever $k \in \mathbb{N}$. Now, let $x_{n_k} \in A_{n_k}$ and $y_{n_k} \in B_{n_k}$ be such that $\|x_{n_k}\| \to 0$ and $\|y_{n_k}\| \to 0$ as $k \to \infty$. Let $[x_{n_k}, a_{n_k}] \cap RS_X = \{a'_{n_k}\}$ and $[y_{n_k}, b_{n_k}] \cap RS_X = \{b'_{n_k}\}$, and observe that, by Lemma 4.4, it holds $\|b'_{n_k} - a'_{n_k}\| \to 0$ as $n \to \infty$.

Since $A_n \to A$ for the Attouch–Wets convergence, $a'_{n_k} \in A_{n_k} \cap RS_X$ and

$$A = S(f, 3\alpha, A) \cup [A \cap \{x \in X;\ f(x) \le -3\alpha\}] \subset rB_X \cup \{x \in X;\ f(x) \le -3\alpha\},$$

it eventually holds $a'_{n_k} \in \{x \in X;\ f(x) \le -2\alpha\}$.

Analogously, since $B_n \to B$ for the Attouch–Wets convergence, $b'_{n_k} \in B_{n_k} \cap RS_X$ and

$$B \subset \{x \in X;\ f(x) \ge 0\},$$

it eventually holds $b'_{n_k} \in \{x \in X;\ f(x) \ge -\alpha\}$.

In particular, it eventually holds $\|b'_{n_k} - a'_{n_k}\| \ge f(b'_{n_k} - a'_{n_k}) \ge \alpha$, a contradiction. Therefore our claim is proved.

Now, since $\{a_n\}$ and $\{b_n\}$ are bounded, there exist sequences $\{w_n\} \subset A$ and $\{z_n\} \subset B$ such that $\|w_n - a_n\| \to 0$ and $\|z_n - b_n\| \to 0$. Since clearly $\lim_n \|z_n - w_n\| = 0$, it holds

$$0 \le \liminf_n [f(z_n) - \|w_n - z_n\|] \le \liminf_n f(w_n) \le \limsup_n f(w_n) \le 0,$$

and hence that $f(w_n) \to 0$ as $n \to \infty$. Since, by Lemma 4.3, f strongly exposes 0, we have that $w_n \to 0$ and hence that $a_n \to 0$ in the $\|\cdot\|$-topology. This concludes the proof in case (1).

If assumption (2) holds, the proof is similar, but some additional efforts are needed. Let $d = \operatorname{dist}(A, B)$ and observe that:

(i) $\operatorname{int}(A) \cap (B + dB_X) = \emptyset$;
(ii) $0 \in B + dB_X$;
(iii) $\limsup_n \|a_n - b_n\| \le d$

Then there exists $f \in S_{X^*}$ such that

$$\sup f(A) = 0 = \inf f(B + dB_X).$$

In particular, f is a support functional for A in 0 and $\inf f(B) = d$. Let Γ be the constant given by Lemma 4.4 and let us consider $S = S(f, (\Gamma + 2)d, A)$ and observe that, by Lemma 4.3, there exists $r > 1$ such that $S \subset rB_X$. Let $R = r + d$.

We claim that $\{a_n\}$ and $\{b_n\}$ are eventually contained in $2RB_X$. Suppose that this is not the case and let $\{a_{n_k}\}$ and $\{b_{n_k}\}$ be two subsequences such that $\|a_{n_k}\| > 2R$ and $\|b_{n_k}\| > 2R$ whenever $k \in \mathbb{N}$. Now, let $x_{n_k} \in A_{n_k}$ $y_{n_k} \in B_{n_k}$ be such that $x_{n_k} \to a$ and $y_{n_k} \to b$ as $k \to \infty$. Let $[x_{n_k}, a_{n_k}] \cap RS_X = \{a'_{n_k}\}$ and $[y_{n_k}, b_{n_k}] \cap RS_X = \{b'_{n_k}\}$, and observe that, by Lemma 4.4, it eventually holds $\|b'_{n_k} - a'_{n_k}\| < (\Gamma + 1)d$.

Since $A_n \to A$ for the Attouch–Wets convergence, $a'_{n_k} \in A_n \cap RS_X$ and

$$\begin{aligned} A &= S(f, (\Gamma + 2)d, A) \cup [A \cap \{x \in X;\ f(x) \le -(\Gamma + 2)d\}] \\ &\subset rB_X \cup \{x \in X;\ f(x) \le -(\Gamma + 2)d\}, \end{aligned}$$

it eventually holds $a'_{n_k} \in \{x \in X;\ f(x) \le -(\Gamma + 1)d\}$.

Analogously, since $B_n \to B$ for the Attouch–Wets convergence, $b'_{n_k} \in B_n \cap RS_X$ and

$$B \subset \{x \in X;\ f(x) \ge d\},$$

it eventually holds $b'_{n_k} \in \{x \in X;\ f(x) \ge 0\}$.

In particular, it eventually holds $\|b'_{n_k} - a'_{n_k}\| \ge f(b'_{n_k} - a'_{n_k}) \ge (\Gamma + 1)d$, a contradiction and our claim is proved.

Now, since $\{a_n\}$ and $\{b_n\}$ are bounded, there exist sequences $\{w_n\} \subset A$ and $\{z_n\} \subset B$ such that $\|w_n - a_n\| \to 0$ and $\|z_n - b_n\| \to 0$. Let us observe that, by Fact 2.5,

$$d \le \liminf \|z_n - w_n\| \le \limsup_n \|z_n - w_n\| = \limsup_n \|a_n - b_n\| \le d$$

and

$$0 \le \liminf_n [f(z_n) - \|w_n - z_n\|] \le \liminf_n f(w_n) \le \limsup_n f(w_n) \le 0.$$

Hence, we obtain $f(w_n) \to 0$ as $n \to \infty$. Since, by Lemma 4.3, f strongly exposes 0, we have that $w_n \to 0$ and hence that $a_n \to 0$ in the $\|\cdot\|$-topology. □

Remark 4.6 (i) As in the finite-dimensional case (see Remark 3.6), the theorem above can be proved in an alternative way, by using known results concerning stability theory for convex optimization problem. However, the well-posedness of the involved problems requires a proof with techniques similar to those used in Theorem 4.5. As in the finite-dimensional case, we preferred to present a direct and more geometrical proof.

(ii) Similarly to the finite-dimensional case, under the hypothesis of the above theorem, we have $\|a_n - b_n\| \to \text{dist}(A, B)$. Indeed, by the proof of Theorem 4.5, $\{a_n\}$ and $\{b_n\}$ are bounded, and hence there exist sequences $\{w_n\} \subset A$ and $\{z_n\} \subset B$ such that $\|w_n - a_n\| \to 0$ and $\|z_n - b_n\| \to 0$. In particular, $\liminf_n \|a_n - b_n\| = \liminf_n \|w_n - z_n\| \ge \text{dist}(A, B)$. Since, by Fact 2.5, $\text{dist}(A, B) \ge \limsup_n \|a_n - b_n\|$, the proof is complete.

(iii) It is not difficult to see that, in general, condition (2) in Theorem 4.5 does not ensure that the sequence $\{b_n\}$ is convergent. However, under the additional requirement that $\frac{b-a}{\|b-a\|}$ is an LUR point of the unit ball of X, it is easy to prove (proceeding as in the proof of the above theorem) that $b_n \to b$.

If the limit sets A and B satisfy a strong condition about non-separation, we obtain a result similar to Corollary 3.7.

Proposition 4.7 *Let A and B be two closed convex subsets of a reflexive Banach space X such that $A \cap B$ is bounded and such that $(\text{int}\, A) \cap B \neq \emptyset$. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the Attouch–Wets convergence. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $a_n \in A_n,\ b_n \in B_n$ ($n \in \mathbb{N}$) and*

$$\text{dist}(A_n, B_n) = \|a_n - b_n\|.$$

Then there exist two subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ that weakly converge to a point of $A \cap B$.

Proof By [11, Corollary 9.2.8], the sequence $\{A_n \cap B_n\}$ converges to $A \cap B$ for the Attouch–Wets convergence. In particular, the sets $A_n \cap B_n$ ($n \in \mathbb{N}$) are eventually nonempty and hence a_n and b_n eventually coincide. Since $A \cap B$ is bounded and X is reflexive, the thesis holds. □

By combining the above proposition with Theorem 4.5, we obtain the following corollary.

Corollary 4.8 *Let X be a reflexive Banach space. Let A be an LUR body of X and B a closed convex subset of X such that $A \cap B$ is nonempty and bounded. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of closed convex sets such that $A_n \to A$ and $B_n \to B$ for the Attouch–Wets convergence. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $a_n \in A_n,\ b_n \in B_n$ ($n \in \mathbb{N}$) and*

$$\text{dist}(A_n, B_n) = \|a_n - b_n\|.$$

Then there exist subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ that weakly converge to a point $c \in A \cap B$. Moreover, if $(\text{int}\, A) \cap B = \emptyset$ then $a_n, b_n \to c$ with respect to the norm convergence.

5 Examples and final remarks

In this section we provide three examples to illustrate the role of the assumptions in the infinite-dimensional case. We point out that all of them are in ℓ_2, therefore the assumptions used in Sect. 4 cannot be avoided even in the "simplest" infinite-dimensional space.

The following example shows that an analogous of Theorem 3.5 does not hold in the infinite-dimensional setting.

Example 5.1 Let $X = \ell_2$ and let $\{e_n\}_n$ be its standard basis. We denote by $\{e_n^*\}_n$ the dual basis. Let $A, B, A_n, B_n \subset X$ ($n \in \mathbb{N}, n \geq 2$) be defined as follows:

$$\begin{aligned} A &= \overline{\text{cone}}\,(\{e_k + \tfrac{1}{k}e_1;\ k \in \mathbb{N}\}); \\ B &= \{x \in X;\ e_1^*(x) = 0\}; \\ A_n &= \overline{\text{conv}}\,\big(\{\ln n\, e_n + \tfrac{1}{n}e_1\} \cup (\tfrac{1}{n}e_1 + A)\big); \\ B_n &= B. \end{aligned}$$

Let $a_n = \ln n\, e_n + \frac{1}{n}e_1 \in A_n$ and $b_n = \ln n\, e_n \in B_n$. Then:

(i) $A \cap B = \{0\}$;
(ii) $A_n \to A$ and $B_n \to B$ for the Hausdorff convergence (and, hence, for the Attouch–Wets convergence);
(iii) $\text{dist}(A_n, B_n) = \|a_n - b_n\|$;

(iv) $\|a_n\|, \|b_n\| \to \infty$.

Proof We just have to prove (i) and (ii), since the proofs of (iii) and (iv) are straightforward.

(i) For $n \in \mathbb{N}\setminus\{1\}$, let $f_n = ne_1^* - e_n^*$ and $g_n = e_n^*$ and observe that

$$\{e_k + \tfrac{1}{k}e_1;\ k \in \mathbb{N}\} \subset \{x \in X;\ f_n(x) \geq 0,\ g_n(x) \geq 0\}.$$

Then $A \subset \bigcap_{n=1}^{\infty}\{x \in X;\ f_n(x) \geq 0,\ g_n(x) \geq 0\}$. Now, if $x \in A \cap B$, it holds $e_1^*(x) = 0$, $f_n(x) = -e_n^*(x) \geq 0$ and $g_n(x) = e_n^*(x) \geq 0$. Then $x = 0$.

(ii) We just have to prove that $A_n \to A$ for the Hausdorff convergence. Let us observe that

$$\operatorname{dist}(a_n, \tfrac{1}{n}e_1 + A) \leq \|\tfrac{1}{n}e_1 + \ln n(e_n + \tfrac{1}{n}e_1) - a_n\| = \tfrac{\ln n}{n}.$$

Hence, it holds

$$h(A_n, A) \leq h(A_n, \tfrac{1}{n}e_1 + A) + h(\tfrac{1}{n}e_1 + A, A) \leq \tfrac{\ln n}{n} + \tfrac{1}{n},$$

and the proof is concluded.

□

Given two sets $A, B \subset X$, we say that A and B are *separated* iff there exists $x^* \in X^*\setminus\{0\}$ such that

$$\sup x^*(A) \leq \inf x^*(B).$$

The following example shows that, in Proposition 4.7, the condition

$$(\operatorname{int} A) \cap B \neq \emptyset$$

cannot be replaced with the weaker condition "A and B are not separated".

Example 5.2 Let $X = \ell_2$ and for $n \in \mathbb{N}$ let us consider the following subsets of X:

$$\begin{aligned}
C_n &= \overline{\operatorname{cone}}\left(\{e_{2n-1}, e_{2n} + \tfrac{1}{n}e_{2n-1}\}\right) - \tfrac{1}{n}e_{2n-1};\\
D_n &= \overline{\operatorname{cone}}\left(\{-e_{2n-1}, e_{2n} - \tfrac{1}{n}e_{2n-1}\}\right) + \tfrac{1}{n}e_{2n-1};\\
C'_n &= C_n - \tfrac{1}{\ln n}e_{2n-1};\\
D'_n &= D_n - \tfrac{1}{\ln n}e_{2n-1};\\
A &= \overline{\operatorname{conv}}\left(\textstyle\bigcup_{n\in\mathbb{N}} C_n\right);\\
B &= \overline{\operatorname{conv}}\left(\textstyle\bigcup_{n\in\mathbb{N}} D_n\right);\\
A_n &= \overline{\operatorname{conv}}\left(\textstyle\bigcup_{k\in\mathbb{N}\setminus\{n\}} C_k \cup C'_n\right);\\
B_n &= \overline{\operatorname{conv}}\left(\textstyle\bigcup_{k\in\mathbb{N}\setminus\{n\}} D_k \cup D'_n\right).
\end{aligned}$$

Then:

(i) A and B are not separated;
(ii) $A \cap B$ is bounded;
(iii) $A_n \to A$ and $B_n \to B$ for the Hausdorff convergence (and, hence, for the Attouch–Wets convergence);
(iv) let $a_n = b_n = (1 + \frac{n}{\ln n})e_{2n} \in A_n \cap B_n$ then $\operatorname{dist}(A_n, B_n) = \|a_n - b_n\| = 0$ and $\|a_n\| = \|b_n\| = 1 + \frac{n}{\ln n} \to \infty$.

In order to prove Example 5.2 we need some preliminary work. Let us define $X_n = \text{span}\,\{e_{2n-1}, e_{2n}\}$, $B_{X_n} = B_X \cap X_n$ and $Y_N = \text{span}\,(\bigcup_{n=1}^N X_n)$. Observe that

$$Y_N = X_1 \oplus_2 \cdots \oplus_2 X_N,$$

where we denote by $X_1 \oplus_2 \cdots \oplus_2 X_N$ the direct sum $X_1 \oplus \cdots \oplus X_N$ endowed with the norm

$$\|(x_1, \ldots, x_N)\| = (\|x_1\|^2 + \cdots + \|x_N\|^2)^{\frac{1}{2}}.$$

The easy proof of the following lemma is left to the reader.

Lemma 5.3 *Let $C_n, D_n \subset X_n$ be defined as above, then the following inclusion holds:*

$$(C_n + \tfrac{1}{\sqrt{n^2+1}} B_{X_n}) \cap (D_n + \tfrac{1}{\sqrt{n^2+1}} B_{X_n}) \subset 2B_{X_n}.$$

Lemma 5.4 *Let W_n be convex subsets of X_n containing the origin ($n = 1, \ldots, N$) and let $\epsilon > 0$, then the following inclusion holds:*

$$\text{conv}\left(\bigcup_{n=1}^N W_n + \epsilon B_{Y_N}\right) \subset 2\,\text{conv}\left(\bigcup_{n=1}^N [W_n + \sqrt{N}\epsilon B_{X_n}]\right).$$

Proof Since $Y_N = X_1 \oplus_2 \cdots \oplus_2 X_N$, it is not difficult to prove that

$$B_{Y_N} \subset \sqrt{N}\text{conv}\left(\bigcup_{n=1}^N B_{X_n}\right),$$

hence the following inclusions hold:

$$\text{conv}\left(\bigcup_{n=1}^N W_n + \epsilon B_{Y_N}\right) \subset \text{conv}\left(\bigcup_{n=1}^N W_n + \epsilon\sqrt{N}\text{conv}\left(\bigcup_{n=1}^N B_{X_n}\right)\right)$$
$$\subset 2\,\text{conv}\left(\bigcup_{n=1}^N [W_n + \sqrt{N}\epsilon B_{X_n}]\right).$$

□

Lemma 5.5 *For $n = 1, \ldots, N$, let W_n and Z_n be convex subsets of X_n containing the origin. Then the following inclusion holds:*

$$\text{conv}\left(\bigcup_{n=1}^N W_n\right) \cap \text{conv}\left(\bigcup_{n=1}^N Z_n\right) \subset 2\text{conv}\left(\bigcup_{n=1}^N W_n \cap Z_n\right)$$

Proof Let $x \in \text{conv}\left(\bigcup_{n=1}^N W_n\right) \cap \text{conv}\left(\bigcup_{n=1}^N Z_n\right)$, then there exist $\alpha_n, \beta_n \in [0, 1]$, $w_n \in W_n$ and $z_n \in Z_n$ ($n = 1, \ldots, N$) such that

$$x = \sum_{i=1}^N \alpha_n w_n = \sum_{i=1}^N \beta_n z_n.$$

Since $Y_N = X_1 \oplus \cdots \oplus X_N$, it holds $\alpha_n w_n = \beta_n z_n$ ($n = 1, \ldots, N$). Now suppose that $\alpha_n \geq \beta_n > 0$, then $w_n = \frac{\beta_n}{\alpha_n} z_n \in W_n \cap Z_n$. Analogously, if $0 < \alpha_n \leq \beta_n$, then $z_n = \frac{\alpha_n}{\beta_n} w_n \in W_n \cap Z_n$. Hence

$$x \in (\alpha_1 + \beta_1)(W_1 \cap Z_1) + \cdots + (\alpha_N + \beta_N)(W_N \cap Z_N) \subset 2\text{conv}\left[\bigcup_{n=1}^N (W_n \cap Z_n)\right].$$

□

Proof of Example 5.2 Let us prove assertions (i), (ii) and (iii); the proof of (iv) is obvious.

(i) Let us observe that, for each $n \in \mathbb{N}$, the segments $[-\frac{1}{n}e_{2n-1}, \frac{1}{n}e_{2n-1}]$ and $[0, e_{2n}]$ are contained in $A \cap B$. Now, suppose that there exists $f \in X^*$ such that $\sup f(A) \leq \inf f(B)$, then f is constant on $A \cap B$ and, by the above remark, it holds $f(e_n) = 0$ whenever $n \in \mathbb{N}$. Hence $f = 0$.

(ii) Let us prove that $A \cap B$ is bounded. For $k \in \mathbb{N}$, let us denote by P_k the canonical projection on the first k coordinates. Let $x \in A \cap B$ and let $N \in \mathbb{N}$ be such that $\|x - P_{2N}x\| \leq 1$. To conclude the proof it suffices to show that $\|P_{2N}x\| \leq 8$
We claim that $P_{2N}x \in \overline{\mathrm{conv}}\left(\bigcup_{n=1}^N C_n\right)$. Indeed, since $x \in A$, there exists a sequence $\{y_k\}$, converging in norm to x, such that $y_k \in \mathrm{conv}\left(\bigcup_{n=1}^k C_n\right)$. Then the sequence $\{P_{2N}y_k\} \subset \overline{\mathrm{conv}}\left(\bigcup_{n=1}^N C_n\right)$ converges in norm to $P_{2N}x$ and the claim is proved.
Analogously, it holds $P_{2N}x \in \overline{\mathrm{conv}}\left(\bigcup_{n=1}^N D_n\right)$ and hence,

$$\begin{aligned} P_{2N}x &\in [\mathrm{conv}\left(\textstyle\bigcup_{n=1}^N C_n\right) + \tfrac{1}{\sqrt{N^3+N}} B_{Y_N}] \cap [\mathrm{conv}\left(\textstyle\bigcup_{n=1}^N D_n\right) + \tfrac{1}{\sqrt{N^3+N)}} B_{Y_N}] \\ &\subset 2\,\mathrm{conv}\left(\textstyle\bigcup_{n=1}^N [C_n + \tfrac{1}{\sqrt{N^2+1}} B_{X_n}]\right) \cap 2\,\mathrm{conv}\left(\textstyle\bigcup_{n=1}^N [D_n + \tfrac{1}{\sqrt{N^2+1}} B_{X_n}]\right) \\ &\subset 4\,\mathrm{conv}\left(\textstyle\bigcup_{n=1}^N [C_n + \tfrac{1}{\sqrt{N^2+1}} B_{X_n}] \cap [D_n + \tfrac{1}{\sqrt{N^2+1}} B_{X_n}]\right) \\ &\subset 4\,\mathrm{conv}\left(\textstyle\bigcup_{n=1}^N 2B_{X_n}\right) \subset 8B_X, \end{aligned}$$

where the above inclusions hold by Lemmas 5.3, 5.4 and 5.5, respectively.

(iii) Let us prove that $A_n \to A$ for the Hausdorff convergence, the proof that $B_n \to B$ for the Hausdorff convergence is similar. Let us observe that $h(C_n, C'_n) = \frac{1}{\ln n}$, hence we have:

$$h(A, A_n) = h(\overline{\mathrm{conv}}\left(\textstyle\bigcup_{k\in\mathbb{N}\setminus\{n\}} C_k \cup C'_n\right), \overline{\mathrm{conv}}\left(\textstyle\bigcup_{k\in\mathbb{N}\setminus\{n\}} C_k \cup C_n\right)) \leq \tfrac{1}{\ln n},$$

and the proof is concluded.

□

The following example shows that in Theorem 4.5 it is not possible to replace the Attouch–Wets convergence with the Kuratowski–Painlevé convergence.

Example 5.6 Let $X = \ell_2$ and for $n \in \mathbb{N}$ let us consider the following closed convex subsets of X:

$$\begin{aligned} A &= e_1 + B_X; \\ B &= \{x \in X;\ e_1^*(x) = 0\}; \\ A_n &= \mathrm{conv}\left(A \cup \{e_n\}\right); \\ B_n &= B. \end{aligned}$$

Then:

(i) $A \cap B = \{0\}$ and 0 is an LUR point of A;
(ii) $A_n \to A$ and $B_n \to B$ for the Kuratowski–Painlevé convergence;
(iii) $e_n \in A_n \cap B_n$;
(iv) $\{e_n\}$ does not converge in norm.

Proof (iii) and (iv) are trivial. (i) follows by the well-known fact that the unit ball of ℓ_2 is uniformly rotund and hence LUR. It remains to prove that $A_n \to A$ for the Kuratowski–Painlevé convergence. Since $A \subset A_n$ ($n \in \mathbb{N}$), it is clear that $A \subset \mathrm{Li}A_n$. On the other hand, if $\{n_k\}$ is a subsequence of the integers and $x = \lim_k x_k \in X$ with $x_k \in A_{n_k}$, it is easy to see that $x \in A$ (observe that $x_k = \lambda_k a_k + (1 - \lambda_k)e_{n_k}$ for some $\lambda_k \in [-1, 1]$ and $a_k \in A$, since $e_{n_k} \to 0$ in the weak topology and A is weakly closed we have $x \in A$). Then $A \supset \mathrm{Ls} A_n$ and the proof is concluded. □

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