Quasi-equilibrium problems with non-self constraint map

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Abstract

In 2016 Aussel, Sultana and Vetrivel developed the concept of projected solution for Nash equilibria. The purpose of this work is to study the same concept of solution, but for quasi-equilibrium problems. Our results recover several existence theorems for quasi-equilibrium problems in the literature. Additionally, we show the existence of projected solutions for quasi-optimization problems, quasi-variational inequality problems, and generalized Nash equilibrium problems.

1 Introduction

Given a non-empty subset C of $\mathbb{R}^n$ and a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the *equilibrium problem* (EP) is the following:

$$\text{find } x \in C \text{ such that } f(x, y) \geq 0, \quad \text{for all } y \in C. \tag{EP}$$

The equilibrium problem was introduced in [8] and has been extensively studied in recent years (see e.g. [9,12,19,24] and the references therein). Related to (EP), it is natural to consider the following problem:

$$\text{find } x \in C \text{ such that } f(y, x) \leq 0, \text{ for all } y \in C, \tag{MEP}$$

which was called *Minty equilibrium problem* (MEP) in [9]. This problem corresponds to a particular case of the *Convex Feasibility Problem* [17,26]. It was established in [3], that if f has the upper sign property, then every solution of (MEP) is a solution of (EP). Moreover, both solution sets trivially coincide when f is also pseudomonotone.

The classical example of equilibrium problem is the variational inequality problem (see e.g. [5,25]), which is defined as follows: a Stampacchia variational inequality problem is formulated as

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find $x \in C$ such that there exists $x^* \in T(x)$
with $\langle x^*, y - x \rangle \geq 0$, for all $y \in C$,

where $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. So, if T has compact values, and we define the representative bifunction f_T of T by

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle, \tag{1}$$

it follows that every solution of the equilibrium problem associated to f_T and C is a solution of the variational inequality problem associated to T and C, and conversely.

Given a set-valued map $K : C \rightrightarrows C$, the *quasi-equilibrium problem* (QEP) associated to f and K is the following

find $x \in K(x)$ such that $f(x, y) \geq 0$, for all $y \in K(x)$. (QEP)

The associated *Minty quasi-equilibrium problem* (MQEP), consists of

find $x \in K(x)$ such that $f(x, y) \leq 0$, for all $y \in K(x)$. (MQEP)

A (Minty) quasi-equilibrium problem is an (Minty) equilibrium problem in which the constraint set depends on the optimizing variable. This dependence allows one to model some complex problems such as quasi-optimization problems, quasi-variational inequalities, generalized Nash equilibrium problems, among others. These problems are unified conveniently, and many of the results obtained for one can be extended, with suitable modifications, to general quasi-equilibrium problems, thus obtaining broader applicability.

A quasi-equilibrium problem is defined by a bifunction and a constraint set-valued map. In most of the results on the existence of solutions for quasi-equilibrium problems in the literature, the constraint map is assumed to be a self-map (see for instance [3,10,11,13]). Our aim in this paper is to study quasi-equilibrium problems with non-self constraint maps. This type of problems arises, for example, in the electricity market as in the work of Aussel, Sultana, and Vetrivel in [6]. In this case, there is usually no solution to the quasi-equilibrium problem.

We study the same concept of projected solution introduced in [6], but for quasi-equilibrium problems. In doing so we improve two general results presented in [6]. As a consequence of our results, we obtain applications for quasi-optimization problems, quasi-variational inequalities, and Nash equilibria.

The paper is organized as follows. Notation and basic definitions are given in Sect. 2. In Sect. 3, we show that the notions of generalized monotonicity for bifunctions can be characterized in terms of solution sets of (EP) and (MEP). Moreover, we show that the concept of pseudomonotonicity and upper sign property are related under suitable assumptions. Then, in Sect. 4, we present the projected solution for quasi-equilibrium problems and prove different results on the existence of such solutions and recover several well-know theorems, one of them is due to Fan [18]. Finally, in Sect. 5, we consider three applications: first, we obtain an existence result for quasi-optimization problems with a non-self constraint map; second, an application to quasi-variational inequalities is given; and finally, we show the existence of projected solutions for Nash equilibria.

2 Basic definitions and preliminaries

Let X and Y be Hausdorff topological spaces and let $K : X \rightrightarrows Y$ be a set-valued map. We recall that K is:

– *closed*, when for any net $(x_i, y_i)_{i \in I}$ in the graph of K such that $(x_i, y_i)_{i \in I}$ converges to (x_0, y_0), we have $y_0 \in K(x_0)$;
– *lower semicontinuous by nets*, when for any x_0, and any net $(x_i)_{i \in I}$ converging to x_0 and any $y_0 \in K(x_0)$, there exists a subnet $(x_{\varphi(j)})_{j \in J}$ of (x_i) and a net $(y_j)_{j \in J}$ converging to y_0 such that $y_j \in K(x_{\varphi(j)})$, for all $j \in J$;
– *lower semicontinuous by sets*, when for any x_0 and any neighborhood V of $K(x_0)$, there exists a neighborhood U of x_0 such that for all $x \in U$, the set $K(x) \cap V$ is not empty;
– *upper semicontinuous (usc)*, when for any x_0 and any neighborhood V of $K(x_0)$, there exists a neighborhood U of x_0 such that $K(U) \subset V$.

In [21, Proposition 2.5.6] the authors show that lower semicontinuity by nets is equivalent to lower semicontinuity by sets. Hence, from now on, we can use any of these two definitions interchangeably and refer to them as lsc.

A fixed point of a set-valued map $T : X \rightrightarrows X$ is a point $x \in X$ such that $x \in T(x)$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$.

Our existence result will be obtained as a consequence of Himmelberg's fixed point theorem, which is stated below and it can be found in [23, Theorem 2].

Theorem 1 (Himmelberg) *Let A be a non-empty and convex subset of a Hausdorff, locally convex topological vector space Y, and let $T : A \rightrightarrows A$ be a set-valued map. If T is usc with convex, closed and non-empty values, and $T(A)$ is contained in some compact subset N of A, then $\mathrm{Fix}(T)$ is a non-empty set.*

We will also need the following selection theorem due to Michael which can be found in [27, Theorem 3.1'''].

Theorem 2 (Michael) *Every lower semicontinuous set-valued map Φ from a metric space to $\mathbb{R}^n$ with non-empty and convex values admits a continuous selection. This means that there exists a continuous function h, with the same domain as Φ, such that the graph of h is included in the graph of Φ.*

As a consequence of the two previous theorems, one can deduce the following result in a similar way to a part of the proof of Theorem 2.1 in [15].

Corollary 1 *Given a non-empty, convex and closed subset C of $\mathbb{R}^n$, if $\Phi : C \rightrightarrows C$ is lsc with non-empty, convex values and $\Phi(C)$ is relatively compact, then $\mathrm{Fix}(\Phi)$ is a non-empty set.*

We now recall some different definitions of *generalized monotonicity* (which we use throughout this article).

A set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be:

– *pseudomonotone* on a subset C of $\mathbb{R}^n$ if, for all $x, y \in C$ and any $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds

$$\langle x^*, y - x \rangle \geq 0 \implies \langle y^*, y - x \rangle \geq 0,$$

– *quasimonotone* on a subset C of $\mathbb{R}^n$ if, for all $x, y \in C$ and any $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds

$$\langle x^*, y - x \rangle > 0 \implies \langle y^*, y - x \rangle \geq 0,$$

– *properly quasimonotone* on a convex subset C of $\mathbb{R}^n$ if, for all $x_1, \ldots, x_m \in C$ and $x \in \mathrm{co}(\{x_1, \ldots, x_m\})$ (the convex hull), there exists i such that

$$\langle x_i^*, x - x_i \rangle \leq 0, \ \forall x_i^* \in T(x_i).$$

In a similar way, a given bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be:

- *pseudomonotone* on a subset C of $\mathbb{R}^n$ if, for all $x, y \in C$, the following implication holds

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$

- *quasimonotone* on a subset C of $\mathbb{R}^n$ if, for all $x, y \in C$, the following implication holds

$$f(x, y) > 0 \Rightarrow f(y, x) \leq 0;$$

- *properly quasimonotone* on a convex subset C of $\mathbb{R}^n$ if, for all $x_1, \ldots, x_m \in C$, and all $x \in \mathrm{co}(\{x_1, \ldots, x_m\})$, there exists i such that

$$f(x_i, x) \leq 0.$$

When $C = \mathbb{R}^n$, we only say that a set-valued map (or a bifunction) is pseudomonotone (quasimonotone or properly quasimonotone) instead of saying that it is pseudomontone (quasimonotone or properly quasimonotone) on $\mathbb{R}^n$.

In the case of set-valued maps, pseudomonotonicity implies proper quasimonotonicity, which implies quasimonotonicity. However, for bifunctions, pseudomonotonicity implies proper quasimonotonicity, provided that the bifunction is quasiconvex with respect to its second argument (see [7, Proposition 1.1]). Moreover, no relationship exists between quasi-monotonicity and proper quasimonotonicity of bifunctions (see the counter-examples in [7]).

It is very well-known that a set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, with non-empty and compact values, satisfies some generalized monotonicity if and only if, its bifunction f_T, defined as in (1), does too. In a similar spirit, we have the following result which is easy to check.

Proposition 1 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values. If $-T$ is pseudomonotone, then $-f_T$ is too.*

Remark 1 The previous proposition is also true in Banach spaces. In this case we can use weak*-compactness instead of regular compactness.

The converse of the previous result does not hold in general, as the following example shows.

Example 1 Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued map defined by

$$T(x) = \{-1, 1\}, \text{ for all } x \in \mathbb{R}.$$

Clearly, $-T$ is not pseudomonotone but $-f_T$ is pseudomonotone, because $f_T \geq 0$ and it only vanishes on the diagonal of $\mathbb{R} \times \mathbb{R}$.

Another important concept is the upper sign condition, which is given first for set-valued maps and later for bifunctions. Let C be a convex subset of $\mathbb{R}^n$. For a given $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, let $x_t = (1 - t)x + ty$.

- A set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *upper sign-continuous* on C if, for all $x, y \in C$, the following implication holds

$$\left(\forall t \in]0, 1[, \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \right) \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0.$$

- A bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to have the *upper sign property* on C if, for all $x \in C$ and for every $y \in C$, the following implication holds

$$\left(f(x_t, x) \leq 0, \ \forall t \in]0, 1[\right) \Rightarrow f(x, y) \geq 0.$$

Upper sign-continuity ([22]) is a very weak notion of continuity. For instance, any upper semicontinuous set-valued map is upper sign-continuous. Moreover, any positive function on $\mathbb{R}$ is upper sign-continuous. This notion plays an essential role in proving the existence of solutions of variational inequalities and quasi-variational inequalities, see [1,5]. In a similar spirit, the upper sign property plays a vital role to establish the existence of solutions of equilibrium problems and quasi-equilibrium problems, see [3,9].

For the sake of completeness, let us recall also that a function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be:

– *convex* if, for any $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, we have

$$h(x_t) \leq (1 - t)h(x) + th(y);$$

– *quasiconvex* if, for any $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, we have

$$h(x_t) \leq \max\{h(x), h(y)\}.$$

– *semistrictly quasiconvex* if, it is quasiconvex and, for any $x, y \in \mathbb{R}^n$ such that $h(x) \neq h(y)$, the following holds

$$h(x_t) < \max\{h(x), h(y)\}, \text{ for all } t \in]0, 1[.$$

Clearly, every convex function is semistrictly quasiconvex. An equivalent and useful characterization of quasiconvexity is that the function f is quasiconvex if and only if, its sublevel set $S_\lambda = \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ is convex, for all $\lambda \in \mathbb{R}$.

3 Canonical relations

John [25], characterized the proper quasimonotonicity of set-valued maps by the non-emptiness of the solution set of Minty variational inequality problems associated with this set-valued map on compact sets. Bianchi and Pini established a similar result for bifunctions under lower semicontinuity and quasiconvexity, see [7, Theorem 2.1].

In a similar way to [25, Theorem 2 and Corollary of Theorem 1], the next result characterizes quasimonotonicity and pseudomonotonicity. Denote by $EP(f, C)$ and $MEP(f, C)$ the solution sets of the equilibrium problem and Minty equilibrium problem, respectively.

Proposition 2 *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. Then, the following hold*

1. *f is quasimonotone if and only if, $MEP(f, \{x, y\}) \neq \emptyset$, for all $x, y \in \mathbb{R}^n$.*
2. *f is pseudomonotone if and only if, $EP(f, C) \subset MEP(f, C)$, for every subset C of $\mathbb{R}^n$.*
3. *If $-f$ is pseudomonotone, then $MEP(f, C) \subset EP(f, C)$, for every subset C of $\mathbb{R}^n$. The converse holds provided that f vanishes on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof 1. It follows from the fact that f is not quasimonotone if and only if, there exists $x, y \in \mathbb{R}^n$ such that $f(x, y) > 0$ and $f(y, x) > 0$, which is equivalent to $MEP(f, \{x, y\}) = \emptyset$.
2. It is a straightforward adaptation of [25, Theorem 2].
3. Let $x \in MEP(f, C)$, that means $f(y, x) \leq 0$, for all $y \in C$. By pseudomonotonicity of $-f$, we have $f(x, y) \geq 0$. Hence, $x \in EP(f, C)$.
   Conversely, let $x, y \in \mathbb{R}^n$ such that $-f(x, y) \geq 0$. We take $C = \{x, y\}$ and since $f(y, y) = 0$, we have $y \in MEP(f, C)$. Thus, $f(y, x) \geq 0$ or equivalently $-f(y, x) \leq 0$.   □

The following example says that in part 3 of Proposition 2, the reciprocal does not hold in general.

Example 2 The bifunction $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follows

$$f(x, y) = \begin{cases} -1, & \text{if } (x, y) = (0, 1) \\ 1, & \text{if } (x, y) = (0, 0) \\ 0, & \text{otherwise} \end{cases}$$

satisfies that $\text{MEP}(f, C) \subset \text{EP}(f, C)$, for every subset C of $\mathbb{R}$. However, $-f$ is not pseudomonotone.

It was shown in [3, Proposition 3.1] that under the upper sign property, the inclusion in part 3 of Proposition 2 holds. The next two propositions show that pseudomonotonicity and the upper sign property are related under suitable assumptions.

Proposition 3 *Let C be a convex subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction such that one of the following assumptions holds*

1. *$f(\cdot, y)$ is lower semicontinuous, for all $y \in C$;*
2. *$f(x, \cdot)$ is upper semicontinuous, for all $x \in C$;*
3. *f vanishes on the diagonal of C and $-f(\cdot, y)$ is semistrictly quasiconvex, for all $y \in C$;*
4. *f vanishes on the diagonal of C and $f(x, \cdot)$ is semistrictly quasiconvex, for all $x \in C$.*

If $-f$ is pseudomonotone, then f has the upper sign property on C.

Proof Let x and y be two elements of C such that

$$f(x_t, x) \leq 0, \text{ for all } t \in]0, 1[. \tag{2}$$

1. If $f(\cdot, y)$ is lower semicontinuous, then $f(y, x) \leq 0$. Thus, the result follows from the pseudomonotonicity of $-f$.
2. Since $-f$ is pseudomonotone, condition (2) implies that $f(x, x_t) \geq 0$, for any $t \in]0, 1[$. By upper semicontinuity of $f(x, \cdot)$ we deduce that $f(x, y) \geq 0$.
3. If $f(x, y) < 0$, then $f(y, x) > 0$ which in turn implies $f(x_t, x) > 0$ for all $t \in]0, 1[$, due to semistric quasiconvexity of $-f(\cdot, x)$. However, this fact is a contradiction with (2). Hence, $f(x, y) \geq 0$.
4. Suppose that f does not have the upper sign property on C. Thus, there exist x and y in C such that $f(x, y) < 0$ and (2) holds. Since $f(x, x) = 0$ and $f(x, \cdot)$ is semistrictly quasicovex, we have $f(x, x_t) < 0$, for all $t \in]0, 1[$. Now, by pseudomonotonicity of $-f$, we obtain $f(x_t, x) > 0$, for all $t \in]0, 1[$, which is a contradiction. $\square$

An important consequence of the previous result is the following corollary.

Corollary 2 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact and non-empty values. If $-T$ is pseudomonotone, then it is upper sign-continuous.*

Proof It is enough to show that f_T, defined in (1), has the upper sign property because from the definition of f_T, we have that T is upper sign-continuous if and only if, f_T has the upper sign property. So, by Proposition 1, we have that $-f_T$ is pseudomonotone. Thus, f_T has the upper sign property due to part 4 of Proposition 3. $\square$

Remark 2 The previous corollary is also true in Banach spaces. In that case, we can use weak*-compactness instead of regular compactness.

Proposition 4 *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction such that the following assumptions hold*

1. f vanishes on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ and
2. $f(\cdot, y)$ is quasiconvex for all $y \in \mathbb{R}^n$.

If f has the upper sign property on $\mathbb{R}^n$, then $-f$ is pseudomonotone.

Proof Let x and y be two elements of $\mathbb{R}^n$ such that

$$f(x, y) \leq 0 \text{ and } f(y, x) < 0. \tag{3}$$

By quasiconvexity we obtain $f(x_t, y) \leq 0$, for all $t \in]0, 1[$. We now apply the upper sign property of f and deduce that $f(y, x) \geq 0$, which is a contradiction with (3). □

Remark 3 A few remarks about the previous results are given below.

– A bifunction satisfying condition 3 in Proposition 3 is actually properly quasimonotone, due to [7, Proposition 1.1].
– In general, the upper sign property of f and pseudomonotonicity of $-f$ are independents. Consider for instance the bifunctions $f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x, y) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \ y \notin \mathbb{Q} \text{ or } x \notin \mathbb{Q}, \ y \in \mathbb{Q} \\ 0, & \text{if } x = y \\ -1, & \text{otherwise} \end{cases}$$

and

$$f_2(x, y) = \begin{cases} -x, & \text{if } y = 0, x > 0 \\ y, & \text{if } x = 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f_1 has the upper sign property and $-f_2$ is pseudomonotone, but neither $-f_1$ is pseudomonotone nor f_2 has the upper sign property.
– The bifunction f_2 also shows that the quasiconvexity assumption in Proposition 4 can not be dropped.

4 Main results

Any solution of (QEP) (or (MQEP)) will be called a "*classical solution*". From now on, we denote the Euclidean norm of $\mathbb{R}^n$ by $\| \cdot \|$.

Definition 1 Given a non-empty subset C of $\mathbb{R}^n$, a set-valued map $K : C \rightrightarrows \mathbb{R}^n$ and a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, a point x_0 of C is called a *projected solution* of:

– the QEP if, there exists $z_0 \in \mathrm{EP}(f, K(x_0))$ such that $x_0 \in P_C(z_0)$,
– the MQEP if, there exists $z_0 \in \mathrm{MEP}(f, K(x_0))$ such that $x_0 \in P_C(z_0)$,

where P_C denotes the projection onto C, that means

$$P_C(z) = \{x \in C : \|z - x\| \leq \|z - w\| \text{ for all } w \in C\}.$$

Remark 4 It is clear that every classical solution is a projected solution. If additionally, K is defined from C to C, then the set of classical solutions is equal to the set of projected solutions.

On the other hand, we can see the projection as a set-valued map in general. Furthermore, if C is a convex, closed and non-empty subset of $\mathbb{R}^n$, then P_C is a continuous function.

In a similar way to [11,15], we now show the existence of projected solutions for quasi-equilibrium problems without upper semicontinuity of the constraint map by using Corollary 1. But before that, we need to introduce a few definitions.

Consider a non-empty set $C \subset \mathbb{R}^n$, a set-valued map $K : C \rightrightarrows \mathbb{R}^n$, and a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Let $Q : C \times \mathbb{R}^n \rightrightarrows C \times \mathbb{R}^n$ be the set-valued map defined by

$$Q(x, z) = P_C(z) \times K(x),$$

$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be defined by

$$F(z) = \{y \in \mathbb{R}^n : \ f(z, y) < 0\},$$

and $R : C \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be defined by

$$R(x, z) = F(z) \cap K(x).$$

The following lemma is not difficult to check and it establishes a characterization of projected solutions in terms of properties of the set-valued maps Q and R.

Lemma 1 *Let C be a non-empty subset of $\mathbb{R}^n$, $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map with non-empty values, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction, and $x \in C$. Then, x is a projected solution of* (QEP) *if and only if, there exists $z \in \mathbb{R}^n$ such that $(x, z) \in \mathrm{Fix}(Q)$ and $R(x, z) = \emptyset$.*

We are now ready for our first existence result.

Theorem 3 *Let C be a non-empty, compact and convex subset of $\mathbb{R}^n$, $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. Assume that*

1. *Q is lsc with non-empty convex values;*
2. *$Q(C \times \mathbb{R}^n)$ is relatively compact;*
3. *$\mathrm{Fix}(Q)$ is closed;*
4. *R is lsc with convex values on $\mathrm{Fix}(Q)$;*
5. *$f(z, z) \geq 0$, for every $z \in M$, where*

$$M = \{w \in K(C) : \ \text{there exists } u \in C \text{ such that } (u, w) \in \mathrm{Fix}(Q)\}.$$

Then, there exists a projected solution of (QEP).

In order to prove the previous result we need the following lemma, which can be found in [28, Lemma 2.3].

Lemma 2 *Let X, Y two topological spaces and A a closed subset of X. Consider two lsc set-valued maps $F : X \rightrightarrows Y$, $\Phi : A \rightrightarrows Y$ such that, for every $x \in A$, one has $\Phi(x) \subset F(x)$. Let $G : X \rightrightarrows Y$ be defined as*

$$G(x) = \begin{cases} F(x), & \text{if } x \in X \backslash A \\ \Phi(x), & \text{if } x \in A. \end{cases}$$

Then, the set-valued map G is lsc.

Proof of Theorem 3 First notice that the non-emptyness of $\mathrm{Fix}(Q)$ is guaranteed by Corollary 1. Let $S : C \times \mathbb{R}^n \rightrightarrows C \times \mathbb{R}^n$ be the set-valued map defined by

$$S(x, z) = \begin{cases} Q(x, z), & \text{if } (x, z) \in C \times \mathbb{R}^m \backslash \mathrm{Fix}(Q) \\ P_C(z) \times R(x, z), & \text{if } (x, z) \in \mathrm{Fix}(Q). \end{cases}$$

Since $S(x, z) \subset Q(x, z)$ for each $(x, z) \in C \times \mathbb{R}^n$, the lower semicontinuity of S follows from Lemma 2 and the fact that Q is lsc. Moreover, S is convex valued. Since $Q(C \times \mathbb{R}^n)$ is relatively compact, so it is $S(C \times \mathbb{R}^n)$. Assuming that S has non-empty values, Corollary 1 implies that there exists $(x_0, z_0) \in C \times \mathbb{R}^n$ such that $(x_0, z_0) \in S(x_0, z_0)$. This in turn implies $(x_0, z_0) \in P_C(z_0) \times R(x_0, z_0)$, that means $z_0 \in R(x_0, z_0)$, but this a contradiction because $z_0 \in M$. Thus, there exists some $(x_0, z_0) \in C \times \mathbb{R}^n$ such that $S(x_0, z_0) = \emptyset$, which means that $(x_0, z_0) \in \text{Fix}(Q)$ and $R(x_0, z_0) = \emptyset$. The result follows then from Lemma 1.   □

Remark 5 Since P_C is a continuous function, Q is lsc with non-empty and convex values, provided that the set-valued map K is lsc with non-empty convex values. On the other hand, if the bifunction f is quasiconvex with respect to its second argument, then F and R are convex valued. Moreover, if f is continuous, then F has an open graph.

In order to show the lower semicontinuity of R we give sufficient conditions in the following result, which is inspired by [31, Lemma 4.2].

Proposition 5 *Let X, Y, Z be topological spaces, and $T_1 : X \rightrightarrows Y$, $T_2 : Z \rightrightarrows Y$ be set-valued maps such that T_1 has open graph and T_2 is lsc. Then, the set-valued map $T : X \times Z \rightrightarrows Y$, defined by*

$$T(x, z) = T_1(x) \cap T_2(z)$$

is lsc.

Proof Let V be an open subset of Y and (x_0, z_0) be an element of $X \times Z$ such that $T(x_0, z_0) \cap V \neq \emptyset$. For $y_0 \in T(x_0, z_0) \cap V$, since T_1 has open graph, we deduce that there exist V_{x_0} and V_{y_0}, open subsets of X and Y respectively, such that $(x_0, y_0) \in V_{x_0} \times V_{y_0}$, where $V_{x_0} \times V_{y_0}$ is a subset of the graph of T_1. By the lower semicontinuity of T_2, there exists V_{z_0}, an open subset of Z, such that $z_0 \in V_{z_0}$ and $T_2(z') \cap V_{y_0} \neq \emptyset$. Thus, taking $V_{x_0} \times V_{z_0}$ we have that

$$T(x', z') \cap V = T_1(x') \cap T_2(z') \cap V \neq \emptyset.$$ □

Thanks to Theorem 3, Remark 5 and Proposition 5 we have the following corollary.

Corollary 3 *Assume that C is compact, convex and non-empty. If the following hold*

1. *K is closed and lsc with convex values;*
2. *$K(C)$ is a compact subset of $\mathbb{R}^n$;*
3. *f is continuous and quasiconvex with respect to its second argument;*
4. *f vanishes on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$;*

then, there exists a projected solution of (QEP).

We now present an alternative proof of the previous result, which does not follow from Theorem 3 but it also works in Banach spaces. However, we need to introduce first a few definitions.

The set-valued map $S : C \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by

$$S(x, z) = \arg \min_{y \in K(x)} f(z, y)$$

and $T : C \times \mathbb{R}^n \rightrightarrows C \times \mathbb{R}^n$ is defined by

$$T(x, z) = P_C(z) \times S(x, z).$$

It is clear that $T(C \times \mathbb{R}^n) \subset C \times \mathbb{R}^n$.

We also need the following two lemmas to establish the proof of Corollary 3. The first one characterizes projected solutions of (QEP) as fixed points of T.

Lemma 3 *Let $x_0 \in C$ and assume that f vanishes on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. Then, x_0 is a projected solution of* (QEP) *if and only if, there exists $z_0 \in X$ such that (x_0, z_0) is a fixed point of T.*

Proof Since $f(x, x) = 0$ for all $x \in \mathbb{R}^n$, the result follows from the fact that for any $(x, z) \in C \times \mathbb{R}^n$, $z \in \mathrm{EP}(f, K(x))$ if and only if, $z \in S(x, z)$. $\qquad\square$

Remark 6 The equivalence in Lemma 3 does not hold if we assume f to be positive on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. Consider for instance the bifunction $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the constraint set-valued map $K : [0, 1] \rightrightarrows \mathbb{R}$, both defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases} \quad \text{and} \quad K(x) = [0, 1 + x].$$

Clearly K is closed and lsc. Also, for each $z \in \mathbb{R}$ we have that $S(x, z) = K(x)\backslash\{z\}$, for all $x \in [0, 1]$ and $z \in \mathbb{R}$. So, T does not have fixed points. However, 0 is a projected solution of (QEP) associated to f and K.

The second lemma is the following.

Lemma 4 *Let C be a closed and non-empty subset of $\mathbb{R}^n$ and f be a continuous bifunction. If K is closed and lsc, then the set-valued map S is closed.*

Proof Let $(x_\alpha, z_\alpha, w_\alpha)_{\alpha \in A}$ be a net in the graph of S converging to (x_0, z_0, w_0). The closeness of K implies $w_0 \in K(x_0)$. Since K is lsc, for each $u \in K(x_0)$, there exists a subnet $(x_{\varphi(\beta)})_{\beta \in B}$ and a net $(u_\beta)_{\beta \in A}$ converging to u such that $u_\beta \in K(x_{\varphi(\beta)})$ and $f(z_{\varphi(\beta)}, w_{\varphi(\beta)}) \leq f(z_{\varphi(\beta)}, u_\beta)$, for all $\beta \in B$. By continuity of f, we obtain $f(z_0, w_0) \leq f(z_0, u)$. Therefore, $w_0 \in S(x_0, z_0)$. $\qquad\square$

Proof of Corollary 3 By Lemma 4 and the quasiconvexity in the second argument of f, we deduce that S is closed with compact and convex values. Moreover, since $K(C)$ is compact, S must be usc. Since P_C is continuous, we deduce that T is usc with compact, convex and non-empty values. Hence, by Theorem 1, there exists a fixed point of T. The result follows from Lemma 3. $\qquad\square$

Remark 7 In Banach spaces, the projection P_C is always usc with convex, compact and non-empty values, provided that C is convex, compact and non-empty; and it is enough to obtain the upper semicontinuity of T in the previous proof.

Remark 8 We note that

$$\mathrm{Fix}(Q) = \{(x, x) \in C \times C : x \in \mathrm{Fix}(K)\},$$

when the constraint map is a self-map. Moreover, the set M in Theorem 3 coincides with $\mathrm{Fix}(K)$, and the set-valued map R restricted to $\mathrm{Fix}(Q)$ is

$$R(x, x) = \{y \in K(x) : f(x, y) < 0\}.$$

Moreover, if $K(x) = C$, for all $x \in C$, and f is upper semicontinuous with respect to its first argument, then by Proposition 5, R is lsc.

As a corollary of Theorem 3, Proposition 5 and Remark 4, we recover the following existence result due to Cubiotti [15] by considering K to be a self-map.

Corollary 4 [15, Theorem 2.1] *Let C be a non-empty compact convex subset of $\mathbb{R}^n$, $K : C \rightrightarrows C$ be a set-valued map, and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that*

1. *K is lsc with non-empty convex values;*
2. *Fix(K) is closed;*
3. *the set $\{(x, y) \in C \times C : f(x, y) \geq 0\}$ is closed;*
4. *for each $x \in C$, $f(x, \cdot)$ is quasiconvex on $K(x)$;*
5. *for each $x \in$ Fix(K), $f(x, x) \geq 0$.*

Then, there exists a classical solution of (QEP).

As another corollary of Theorem 3, we recover the famous minimax inequality due to Ky Fan on finite dimensional spaces, which can be found in [18].

Corollary 5 (Ky Fan) *Let C be a non-empty compact and convex subset of $\mathbb{R}^n$ and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that*

1. *f is upper semicontinuous with respect to its first argument,*
2. *f is quasiconvex with respect to its second argument, and*
3. *f is not negative on the diagonal of $C \times C$.*

Then, there exists a solution of (EP).

Now, we will establish the existence of projected solutions for Minty quasi-equilibrium problems.

Theorem 4 *Let $C \subset \mathbb{R}^n$ be a non-empty, compact and convex set, $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. If the following assumptions hold*

1. *Q is lsc with non-empty convex values;*
2. *$Q(C \times \mathbb{R}^n)$ is relatively compact;*
3. *Fix(Q) is closed;*
4. *the set-valued map $G :$ Fix$(Q) \rightrightarrows \mathbb{R}^n$, defined by*

$$G(x, z) = \{y \in K(x) : f(y, z) > 0\}$$

 is lsc;
5. *f is properly quasimonotone on co$(K(C))$;*

then, there exists a projected solution of (MQEP). *Moreover, the set of projected solutions of* (QEP) *is non-empty whether f has the upper sign property or $-f$ is pseudomonotone.*

In order to proof the previous result we need the following lemma, which can be found in [30, Theorem 5.9].

Lemma 5 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. If T is lsc, then so is the set-valued map* co$(T) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *defined by*

$$\mathrm{co}(T)(x) = \mathrm{co}(T(x)).$$

Proof of Theorem 4 Due to Lemma 5, the set-valued map co$(G) :$ Fix$(Q) \rightrightarrows \mathbb{R}^n$, which is defined as

$$\mathrm{co}(G)(x, z) = \mathrm{co}(G(x, z)),$$

is lsc too. Thanks to Lemma 2, the set-valued map $\Phi : C \times \mathbb{R}^n \rightrightarrows C \times \mathbb{R}^n$, defined as

$$\Phi(x, z) = \begin{cases} Q(x, z), & \text{if } (x, z) \in C \times \mathbb{R}^n \setminus \text{Fix}(Q) \\ P_C(z) \times \text{co}(G)(x, z), & \text{if } (x, z) \in \text{Fix}(Q) \end{cases}$$

is lsc. If G is non-empty valued then, by Corollary 1, there exists $(x, z) \in \text{Fix}(Q)$ such that $x \in P_C(z)$ and $z \in \text{co}(G)(x, z)$. That means that there exist $z_1, z_2, \ldots, z_n$ belonging to $G(x, z)$ such that $z \in \text{co}(\{z_1, z_2, \ldots, z_n\})$. However, we get a contradiction with the proper quasimonotonicity of f, because $f(z_i, z) > 0$ for all i. Hence, there exists $(x, z) \in C \times \mathbb{R}^m$ such that $(x, z) \in \text{Fix}(Q)$ and $G(x, z) = \emptyset$. Therefore, x is a projected solution of (MQEP).

Finally, the existence of projected solutions for (QEP) is due to [3, Proposition 3.1] when f has the upper sign property and part 3 of Proposition 2 if $-f$ is pseudomonotone. □

As a consequence of the previous result and Proposition 5 we obtain the following corollary.

Corollary 6 *Assume that $C \subset \mathbb{R}^n$ is compact, convex and non-empty set. If the following hold*

1. *K is closed and lsc with convex values;*
2. *$K(C)$ is a compact subset of $\mathbb{R}^n$;*
3. *f is properly quasimonotone;*
4. *f is quasiconvex with respect to its second argument;*
5. *$\{(x, y) \in K(C) \times K(C) : f(x, y) \leq 0\}$ is closed;*

then, there exists a projected solution of (MQEP)*. Moreover, the set of projected solutions of* (QEP) *is non-empty, whether f has the upper sign property or $-f$ is pseudomonotone.*

As in Corollary 3, we will present an alternative proof of the previous corollary, which also works in Banach spaces. First, we define the following set-valued maps. Let $M : C \rightrightarrows \mathbb{R}^n$ be defined by

$$M(x) = \text{MEP}(f, K(x))$$

and let $T' : C \times X \rightrightarrows C \times X$ be defined by

$$T'(x, z) = P_C(z) \times M(x).$$

In a similar way to Lemma 1, we characterize projected solutions of (MQEP) as fixed points of T'.

Lemma 6 *A point $x_0 \in C$ is a projected solution of* (MQEP) *if and only if, there exists a $z_0 \in X$ such that $(x_0, z_0) \in \text{Fix}(T')$.*

Finally, we need the following lemma which is a very straightforward adaptation of [13, Proposition 2.2].

Lemma 7 *If K is a closed and lower semicontinuous set-valued map, and the set $\{(x, y) \in K(C) \times K(C) : f(x, y) \leq 0\}$ is closed; then, the set-valued map M is closed.*

Proof of Corollary 6 By [3, Proposition 2.4], M is non-empty valued. Moreover, by Lemma 7 and the fact that $M(C)$ is relatively compact, we deduce the upper semicontinuity of M. It is also clear that T' is usc with convex and closed values. As $T'(C \times X) \subset C \times K(C)$, by Theorem 1, T' admits at least one fixed point. Therefore, x_0 is a projected solution of (MQEP) by Lemma 6.

Finally, the existence of projected solution for (QEP) is due to [3, Proposition 3.1] when f has the upper sign property and part 3 of Proposition 2 if $-f$ is pseudomonotone. □

As a consequence of Theorem 4 we recover the following result.

Corollary 7 [3, Theorem 4.5] *Let* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a bifunction, C be a convex, compact and non-empty subset of* $\mathbb{R}^n$, *and* $K : C \rightrightarrows C$ *be a set-valued map. Suppose that the following properties hold*

1. *the map K is closed and lsc with convex values, and* $\text{int}(K(x)) \neq \emptyset$, *for all* $x \in C$;
2. f *is properly quasimonotone;*
3. f *is semistrictly quasiconvex and lower semicontinuous with respect to its second argument;*
4. *for all* $x, y \in \mathbb{R}^n$ *and all sequence* $(y_k)_k \subset \mathbb{R}^n$ *converging to y, the following implication holds*

$$\liminf_{k \to +\infty} f(y_k, x) \leq 0 \implies f(y, x) \leq 0,$$

5. f *has the upper sign property.*

Then, the quasi-equilibrium problem admits a classical solution.

Proof Let the set-valued map $G : \text{Fix}(K) \rightrightarrows \mathbb{R}^n$ be defined by

$$G(x) = \{y \in K(x) : f(y, x) > 0\}.$$

It is enough to show that G is lower semicontinuous. Let V be an open subset such that $G(x) \cap V \neq \emptyset$ and let $y \in G(x) \cap V$. Since $\text{int}(K(x)) \neq \emptyset$, there exists $w \in \text{int}(K(x)) \cap V$ such that $f(w, x) > 0$, due to assumption 4. Now, by the lower semicontinuity of f in its second variable, there exists U, an open subset, with $x \in U$ such that $f(w, x') > 0$, for any $x' \in U$, that means $V \cap G(x') \neq \emptyset$. □

As another direct consequence of Theorem 4, we have an existence result for Minty equilibrium problems.

Corollary 8 *Let* $C \subset \mathbb{R}^n$ *be a convex, compact and non-empty subset. Let also* $f : C \times C \to \mathbb{R}$ *be a properly quasimonotone bifunction. If the set-valued map* $F : C \rightrightarrows C$, *defined as*

$$F(y) = \{x \in C : f(x, y) > 0\}$$

is lsc, then there exists a solution of (MEP).

We finish this section with Theorem 5, which is inspired by [17, Lemma 1].

Let C be a non-empty and convex subset of $\mathbb{R}^n$. A set-valued map $T : C \rightrightarrows C$ is said to be a *KKM map* if, for any $x_1, x_2, \ldots, x_m \in C$ the following holds

$$\text{co}(\{x_1, x_2, \ldots, x_m\}) \subset \bigcup_{i=1}^{m} T(x_i).$$

Given a set-valued map $T : C \rightrightarrows C$, we define the set-valued map $S : C \rightrightarrows C$ as

$$S(y) = \{x \in C : y \notin T(x)\}.$$

Clearly, it satisfies $\cap_{x \in C} T(x) = \{y \in C : S(y) = \emptyset\}$.

Theorem 5 *Let* C *be a non-empty and convex subset of* $\mathbb{R}^n$ *and* $T : C \rightrightarrows C$ *be a KKM map such that S is lsc. If there exists K, a non-empty, convex and compact subset of C, such that* $S(C) \subset K$; *then*

$$\bigcap_{x \in C} T(x) \neq \emptyset.$$

Proof Due to Lemma 5, the set-valued map $\mathrm{co}(S) : C \rightrightarrows C$, which is defined as

$$\mathrm{co}(S)(x) = \mathrm{co}(S(x))$$

is lsc too. If S has non-empty values then, by Corollary 1, there exists $x \in C$ such that $x \in \mathrm{co}(S)(x)$. But this contradicts the fact that T is a KKM map. Hence, there exists $y \in C$ such that $S(y) = \emptyset$. □

Notice that Theorem 5 cannot be deduced from [17, Lemma 1]. On the other hand, we can see that Corollary 8 is also a consequence of Theorem 5.

Remark 9 It is important to notice that we can generalize the concept of projected solution if we introduce a set-valued map P from $\mathbb{R}^n$ to C with similar properties of the projection.

5 Applications

In this section, as was mentioned earlier, we consider applications to the study of solutions of three particular problems. The first one is a special optimization problem known as a quasi-optimization problem, the second one is a quasi-variational inequality problem, and finally, the third one is a generalized Nash equilibrium problem. We will establish sufficient conditions to guarantee the existence of projected solutions for these problems, which were introduced by Aussel et al. [6].

5.1 Quasi-optimization

Given a real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ and a set-valued map $K : C \rightrightarrows C$, where C is a subset of $\mathbb{R}^n$, the *quasi-optimization problem* (QOpt) is described as

$$\text{find } x_0 \in K(x_0) \text{ such that } \quad x_0 \in \arg\min_{z \in K(x_0)} h(z). \qquad \text{(QOpt)}$$

The terminology of quasi-optimization problem comes from [20] (see formula (8.3) and Proposition 12) and has been recently used in [1,13,16]. It emphasizes the fact that it is not a standard optimization problem since the constraint set depends on the solution, and it also highlights the parallelism to quasi-equilibrium problems.

As in [6], a point $x_0 \in C$ is said to be a projected solution of the QOpt if there exists $z_0 \in \arg\min_{z \in K(x_0)} h(z)$ such that $x_0 \in P_C(z_0)$.

Using a reformulation as quasi-equilibrium problem, similar to the one in [13], we will characterize the projected solutions of (QOpt). In that sense, associated to h, let us define the bifunction $f^h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$f^h(x, y) = h(y) - h(x).$$

Clearly, f^h vanishes on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. The following lemma follows from the definition of f^h.

Lemma 8 *Let $x_0 \in C$. Then, x_0 is a projected solution of* (QOpt) *if and only if, x_0 is a projected solution of* (QEP) *associated to f^h and K.*

We now show the existence of projected solutions for (QOpt) without the continuity nor the quasiconvexity of h, which is a consequence of Theorem 4 and it generalizes [6, Theorem 4.1].

Theorem 6 *Let C be a closed, convex and non-empty subset of $\mathbb{R}^n$, let $h : \mathbb{R}^n \to \mathbb{R}$ be a function and let $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map such that $K(C)$ is relatively compact. If the following assumptions hold*

1. *K is lsc with convex values;*
2. *the set $D = \{(x, z) \in C \times \mathbb{R}^n : z \in K(x) \text{ and } x \in P_C(z)\}$ is closed;*
3. *the set-valued map $H : D \rightrightarrows \mathbb{R}^n$ defined as*

$$H(x, z) = \{y \in K(x) : h(z) > h(y)\}$$

   is lsc with convex values;

then, there exists a projected solution of (QOpt).

Notice that under continuity of h and lower semicontinuity of K, the set-valued map H is lsc too. However, the converse is not true in general as we can see in the following example.

Example 3 Let $h : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$h(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ x + 2, & x > 1 \end{cases}$$

which is clearly quasiconvex but not continuous. Now, consider the constraint map $K : C \rightrightarrows C$ defined as $K(x) = C = [0, 2]$. It is clear that K is lsc with convex values. Moreover, it is not difficult to verify that the set-valued map $H : C \rightrightarrows C$, defined by

$$H(x) = \{y \in C : h(x) > h(y)\} = [0, x]$$

is lsc.

We can see that under quasiconvexity of h, if K is convex valued, then H is convex valued too. However, the following example shows us that the converse does not hold in general.

Example 4 Let $h : \mathbb{R} \to \mathbb{R}$ be a function and let $K : [0, 2] \rightrightarrows [0, 2]$ be a set-valued map both defined by

$$h(x) = \begin{cases} |x - \frac{1}{2}|, & x \le 1 \\ |x - \frac{3}{2}|, & 1 < x \end{cases} \quad \text{and} \quad K(x) = \begin{cases} [1/2, 1] \cup \{x\}, & 0 \le x < 1/2 \\ [x, 1], & 1/2 \le x \le 1 \\ [1, x], & 1 < x \le 2. \end{cases}$$

Figure 1 shows the graphs of h and K. It is clear that the map $H : [0, 2] \rightrightarrows [0, 2]$, defined as

$$H(x) = \{y \in K(x) : h(y) < h(x)\}$$

has convex values. However, neither h is quasiconvex nor K is convex valued.

5.2 Quasi-variational inequalities

Let C be a non-empty subset of $\mathbb{R}^n$ and T, K two set-valued maps $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$. As in [6], a point $x_0 \in C$ is said to be a *projected solution* of the quasi-variational inequality if, there exists $z_0 \in X$ such that

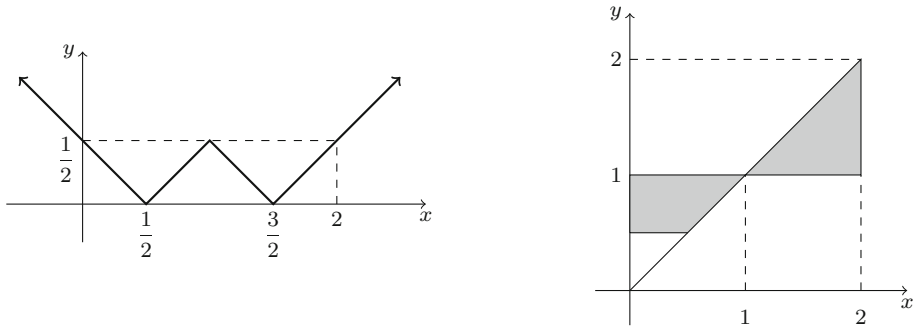1. x_0 is a projection of z_0 onto C, $x_0 \in P_C(z_0)$;

Fig. 1 Graphs of h and K

2. z_0 is a solution of the Stampacchia variational inequality associated to T and $K(x_0)$, that is, $z_0 \in K(x_0)$ and

$$\text{there exists } z_0^* \in T(z_0) \text{ such that } \langle z_0^*, y - z_0 \rangle \geq 0, \text{ for all } y \in K(x_0).$$

Also, $x_0 \in C$ is said to be a *projected solution* of the Minty quasi-variational inequality if, there exists $z_0 \in X$ such that

1. x_0 is the projection of z_0 onto C, $x_0 \in P_C(z_0)$;
2. z_0 is a solution of the Minty variational inequality associated to T and $K(x_0)$, that is, $z_0 \in K(x_0)$ and

$$\text{for all } y \in K(x_0) \text{ and for all } y^* \in T(y), \ \langle y^*, y - z_0 \rangle \geq 0.$$

It is clear that under pseudomonotonicity every projected solution of the Stampacchia variational inequality is a projected solution of the Minty variational inequality. The converse holds if we assume the upper sign-continuity, see for instance [2,5].

The following lemma establishes the relationship between the projected solution of a (Minty) quasi-variational inequality and the projected solution of a (Minty) quasi-equilibrium problem. The proof is a direct consequence of the definitions.

Lemma 9 *Let C be a subset of $\mathbb{R}^n$, $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values and $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map. Any projected solution of the (Minty) quasi-variational inequality associated to T and K is a projected solution of the (Minty) quasi-equilibrium problem associated to f_T and K, where f_T is defined as (1). And conversely.*

In [6], the authors established two results on the existence of projected solutions for quasi-variational inequality problems on finite dimensional spaces. They used generalized monotonicity, specifically pseudomonotonicity and quasimonotonicity, for their proof. In a similar spirit, we present another existence result using proper quasimonotonicity.

Theorem 7 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values, C be a compact, convex and non-empty subset of $\mathbb{R}^n$, and $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map. If the following assumptions hold*

1. *K is closed and lsc with non-empty and convex values;*
2. *$K(C)$ is compact;*
3. *T is properly quasimonotone;*
4. *$\{(x, y) \in K(C) \times K(C) : \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0\}$ is closed;*

then, the Minty quasi-variational inequality admits at least one projected solution. Moreover, if T is upper sign-continuous, then the Stampacchia quasi-variational inequality has a projected solution.

Proof Clearly, f_T (defined as (1)) and K satisfy the conditions of Theorem 4. The result follows from Lemma 9. □

As a direct consequence of the previous theorem, we have the following result on the existence of solutions to the quasi-variational inequality.

Corollary 9 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values, C be a compact, convex and non-empty subset of $\mathbb{R}^n$, and $K : C \rightrightarrows C$ be a set-valued map. If the following assumptions hold*

1. *K is closed and lower semicontinuous with convex values;*
2. *T is properly quasimonotone;*
3. *T is upper sign-continuous;*
4. $\{(x, y) \in C \times C : \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0\}$ *is closed;*

then, the Stampacchia quasi-variational inequality admits at least one solution.

Remark 10 Here a few remarks are needed.

1. Theorem 7 is also a consequence of Corollary 6, and hence, it works in Banach spaces, where T could be considered with weak*-compact values.
2. An analogous result to the previous corollary was proved in [1, Proposition 3.5], where they did not require our assumption 4. Instead, they assumed the following technical condition: for all $x_\alpha \to x$ and all $y_\alpha \to y$

$$\liminf_{x_\alpha^* \in T(x_\alpha)} \sup \langle x_\alpha^*, y_\alpha - x_\alpha \rangle \leq 0 \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0. \qquad (4)$$

This condition implies assumption 4 in Corollary 9, but it is stronger. Indeed, consider for instance the set-valued map $T : \mathbb{R} \rightrightarrows \mathbb{R}$, defined by

$$T(x) = \begin{cases} \{x\}, & x \neq 0 \\ \{1\}, & x = 0 \end{cases},$$

which satisfies assumption 4 in Corollary 9. However, for $x_n = 1/n$ and $y_n = 1$, for all $n \in \mathbb{N}$, we have

$$\liminf \langle x_n, y_n - x_n \rangle = 0 \text{ and } \langle 1, 1 - 0 \rangle > 0.$$

Thus, it fails to satisfy implication (4).

Another result on the existence of projected solutions for quasi-variational inequality problems without generalized monotonicity is given below.

Theorem 8 *Let C be a convex, compact and non-empty subset of $\mathbb{R}^n$ and consider the set-valued maps $K : C \rightrightarrows \mathbb{R}^n$ and $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Assume that*

1. *K is lsc with non-empty convex values;*
2. *K(C) is compact;*
3. *the set $\{(x, z) \in C \times \mathbb{R}^n : z \in K(x) \text{ and } x \in P_C(z)\}$ is closed;*
4. *the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0\}$ is closed.*

Then, the Stampacchia quasi-variational inequality has a projected solution.

Proof Clearly, f_T (defined as (1)) and K satisfy the conditions of Theorem 3. The result follows from Lemma 9.   □

The following examples show that Theorems 7 and 8 are independent.

Example 5 Let $K : [-1, 1] \rightrightarrows \mathbb{R}$ be a set-valued map and $T : \mathbb{R} \to \mathbb{R}$ be a single-valued map defined as

$$K(x) = [-3, 0] \quad \text{and} \quad T(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}.$$

Clearly, K is closed and lsc with convex values, and T is properly quasimonotone. It is not difficult to see that the set

$$\{(x, y) \in [-3, 0] \times [-3, 0] : \langle T(x), y - x \rangle \leq 0\}$$

is closed. However the following set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : \langle T(x), y - x \rangle \geq 0\}$$

is not closed. Hence, we can use Theorem 7 in order to guarantee the existence of projected solutions.

Example 6 Let $K : [0, 1] \rightrightarrows \mathbb{R}$ and $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be set-valued maps defined by

$$K(x) = \begin{cases}](x + 1)/2, 2], & x \in [0, 1[\\ [1, 2], & x = 1 \end{cases}$$

and $T(x) = \{-1, 1\}$, for all $x \in \mathbb{R}$. Clearly, T is not properly quasimonotone and K is lsc with convex values, but it is not closed. On the other hand,

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0\} = \mathbb{R} \times \mathbb{R}$$

and, as the projection onto $C = [0, 1]$ is single valued and continuous, we have

$$\{(x, z) \in [0, 1] \times \mathbb{R} : z \in K(x) \text{ and } x \in P_C(z)\} = \{1\} \times [1, 2].$$

Therefore, the existence of projected solutions is due to Theorem 8 and not Theorem 7.

5.3 GNEPs

A generalized Nash equilibrium problem (GNEP) consists of p players. Each player ν controls the decision variable $x^\nu \in C_\nu$, where C_ν is a non-empty convex and closed subset of $\mathbb{R}^{n_\nu}$. We denote by $x = (x^1, \ldots, x^p) \in \prod_{\nu=1}^p C_\nu = C$ the vector formed by all these decision variables and by $x^{-\nu}$, we denote the strategy vector of all the players different from player ν. The set of all such vectors will be denoted by $C^{-\nu}$. We sometimes write $(x^\nu, x^{-\nu})$ instead of x in order to emphasize the ν-th player's variables within x. Note that this is still the vector $x = (x^1, \ldots, x^\nu, \ldots, x^p)$, and the notation $(x^\nu, x^{-\nu})$ does not mean that the block components of x are reordered in such a way that x^ν becomes the first block. Each player ν has an objective function $\theta_\nu : C \to \mathbb{R}$ that depends on all player's strategies. Each player's strategy must belong to a set identified by the set-valued map $K_\nu : C^{-\nu} \rightrightarrows C_\nu$ in the sense that the strategy space of player ν is $K_\nu(x^{-\nu})$, which depends on the rival player's strategies

$x^{-\nu}$. Given the strategy $x^{-\nu}$, player ν chooses a strategy x^ν such that it solves the following optimization problem

$$\min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), \text{ subject to } x^\nu \in K_\nu(x^{-\nu}), \qquad (5)$$

for any given strategy vector $x^{-\nu}$ of the rival players. The solution set of problem (5) is denoted by $\mathrm{Sol}_\nu(x^{-\nu})$. Thus, a *generalized Nash equilibrium* is a vector $\hat{x}$ such that $\hat{x}^\nu \in \mathrm{Sol}_\nu(\hat{x}^{-\nu})$, for any ν.

Suppose that n and n_ν are natural numbers that satisfy $n = \sum_{\nu=1}^p n_\nu$. For any $\nu \in \{1, 2, \ldots, p\}$, let C_ν be a non-empty subset of $\mathbb{R}^{n_\nu}$, and let $K_\nu : C^{-\nu} \rightrightarrows \mathbb{R}^{n_\nu}$ and $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$, be set-valued maps. As in [6], a vector $\hat{x}$ of C is said to be a *projected solution* of the generalized Nash equilibrium problem if, there exists $\hat{z} \in \mathbb{R}^n$ such that:

1. $\hat{x}$ is a projection of $\hat{z}$ onto C;
2. $\hat{z}$ is a solution of the Nash equilibrium problem defined by all functions θ_ν, where $\nu \in \{1, 2, \ldots, p\}$, and the constraint sets $K_\nu(\hat{x})$, $\nu \in \{1, 2, \ldots, p\}$, that is, for any ν, $\hat{z}^\nu \in K_\nu(\hat{x}^{-\nu})$ is a solution of the following optimization problem

$$\min_{z^\nu} \theta_\nu(z^\nu, \hat{z}^{-\nu}), \text{ subject to } z^\nu \in K_\nu(\hat{x}^{-\nu}). \qquad (6)$$

Associated to a GNEP, there is a bifunction $f^{NI} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$f^{NI}(x, y) = \sum_{\nu=1}^p \{\theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu})\},$$

which is called Nikaidô–Isoda bifunction and was introduced in [29]. This bifunction has a simple interpretation. Suppose that x and y are two feasible points for the GNEP. Each summand in the definition represents the improvement in the objective function of player ν when he changes his action from x^ν to y^ν, while all the other players stick to the choice $x^{-\nu}$.

Additionally, we define the set-valued map $K : C \rightrightarrows \mathbb{R}^n$ as

$$K(x) = \prod_{\nu=1}^p K_\nu(x^{-\nu}).$$

Now, we can characterize all projected solution of a GNEP in the following result.

Lemma 10 *A vector $\hat{x}$ is a projected solution of a GNEP if and only if, it is a projected solution of* (QEP) *associated to f^{NI} and K.*

Proof Clearly, every projected solution of a GNEP is a projected solution of (QEP) associated to f^{NI} and K. Conversely, if $\hat{x} \in C$ and there exists $\hat{z} \in K(x)$ such that $\hat{x} \in P_C(\hat{z})$, and

$$f^{NI}(\hat{z}, y) \geq 0, \text{ for all } y \in K(\hat{x}), \qquad (7)$$

then, for all $z^\nu \in K_\nu(\hat{x}^{-\nu})$, the vector $(z^\nu, \hat{z}^{-\nu}) \in K(\hat{x})$ and the inequality in (7) becomes

$$\theta_\nu(\hat{z}^\nu, \hat{z}^{-\nu}) \leq \theta_\nu(z^\nu, \hat{z}^{-\nu}),$$

that is, $\hat{z}^\nu$ is a solution of (6). □

Thanks to Lemma 10, we can deduce from Theorem 3 the following result on the existence of projected solutions of a GNEP, which generalizes [6, Theorem 4.2].

Theorem 9 *For any $\nu \in \{1, 2, \dots, p\}$, let C_ν be a non-empty, closed and convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ be a function and $K_\nu : C^{-\nu} \rightrightarrows \mathbb{R}^{n-n_\nu}$ be a set-valued map. Then, the GNEP admits a projected solution if*

1. *for each ν, θ_ν is continuous and convex with respect to the x^ν variable;*
2. *for each ν, the map K_ν is lsc with non-empty and convex values, and $K_\nu(C)$ is relatively compact;*
3. *the set $D = \{(x, z) \in C \times \mathbb{R}^n : x \in P_C(z) \text{ and } z \in K(x)\}$ is closed.*

Proof It is enough to see that the set-valued map $R : D \rightrightarrows \mathbb{R}^n$, defined as

$$R(x, z) = \{y \in K(x) : f^{NI}(z, y) < 0\}$$

is lsc. The result follows from Theorem 3.   □

Remark 11 If in the previous theorem we add the closeness of the graph of each constraint map, then assumption 3 holds. Moreover, it is true in Banach spaces. The existence results for generalized Nash equilibrium problems for infinite dimensional spaces has been recently studied, see for instance [4,10,14]

We finish this subsection with the following result, which is a consequence of Theorem 9.

Corollary 10 [15, Theorem 2.2] *For any $\nu \in \{1, 2, \dots, p\}$, let C_ν be a non-empty, compact and convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ be a function and $K_\nu : C^{-\nu} \rightrightarrows C_\nu$ be a set-valued map. Then, the GNEP admits a solution if*

1. *for each ν, θ_ν is continuous and convex with respect to the x^ν variable;*
2. *for each ν, the map K_ν is lsc with non-empty and convex values;*
3. Fix(K) *is closed.*

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