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Necessary optimality conditions for a nonsmooth semi-infinite programming problem

Nazih Abderrazzak Gadhi[1]

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Abstract
The nonsmooth semi-infinite programming (*SIP*) is solved in the paper (Mishra et al. in J Glob Optim 53:285–296, 2012) using limiting subdifferentials. The necessary optimality condition obtained by the authors, as well as its proof, is false. Even in the case where the index set is a finite, the result remains false. Two major problems do not allow them to have the expected result; first, the authors were based on Theorem 3.2 (Soleimani-damaneh and Jahanshahloo in J Math Anal Appl 328:281–286, 2007) which is not valid for nonsmooth semi-infinite problems with an infinite index set; second, they would have had to assume a suitable constraint qualification to get the expected necessary optimality conditions. For the convenience of the reader, under a nonsmooth limiting constraint qualification, using techniques from variational analysis, we propose another proof to detect necessary optimality conditions in terms of Karush–Kuhn–Tucker multipliers. The obtained results are formulated using limiting subdifferentials and Fréchet subdifferentials.

Keywords Nonsmooth semi-infinite optimzation · Extremal principle · Fréchet subdifferential · Limiting subdifferential · Fréchet normal cone · Limiting normal cone · Optimality conditions · Constraint qualification

Mathematics Subject Classification Primary 90C29 · 90C26 · 90C70; Secondary 49K99

1 Introduction

Semi-infinite programming problems have been investigated by many authors [2–6,9]. In the paper [6], the authors investigated the following nonsmooth semi-infinite programming problem

$$(SIP): \quad \begin{cases} \text{Minimize} & f(x) \\ \text{s.t.} & g_i(x) \leq 0 \quad \forall i \in I, \end{cases}$$

✉ Nazih Abderrazzak Gadhi
nazih.gadhi@googlemail.com

1 LSO, Department of Mathematics, Sidi Mohamed Ben Abdellah University, Dhar El Mehrez, Fes, Morocco

where I is an index set which is possibly infinite, f and g_i, $i \in I$, are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$.

Using limiting subdifferentials [7], they established necessary and sufficient optimality conditions for (*SIP*) and gave some duality results. The main theorem, Theorem 3.1 [6], where the authors give necessary optimality conditions, is based on Theorem 3.2 [10] (see Theorem 1 below) obtained by Soleimani-damaneh and Jahanshahloo [10] in the case where I is a finite index set. For what follows, let $I = \{1, \ldots, p\}$, $p \in \mathbb{N}^*$, and let (P) be the nonsmooth optimization problem where one minimizes the function f over the set

$$\mathbb{S} = \left\{x \in \mathbb{R}^n : g_i(x) \leq 0, \quad \forall i \in I\right\}.$$

Theorem 1 [10] *Let $\overline{x} \in \mathbb{S}$ be an optimal solution of (P). Suppose that $g_i(x)$ for $i \in I(\overline{x})$ is Lipschitz near $\overline{x}$ and $g_i(x)$ for $i \notin I(\overline{x})$ is continuous at $\overline{x}$. Also suppose that there exists a $d \in \mathbb{R}^n$ such that $\eta^t d < 0$ for all $\eta \in \bigcup_{i \in I(\overline{x})} \partial_L g_i(\overline{x})$. Then*

$$d \in D^{\overline{x}} = \left\{d \in \mathbb{R}^n : d \neq 0 \text{ and } \exists \delta > 0 \text{ such that } \overline{x} + \lambda d \in \mathbb{S},\ \forall \lambda \in]0, \delta[\right\}, \tag{1}$$

where

$$I(\overline{x}) = \{i \in I : g_i(x) = 0\}.$$

Looking closely at the proof of Theorem 3.2 [10], we note that it is neither valid nor usable for the infinite case; one can not guarantee that $\delta \neq 0$ is not zero, and as a result one can not deduces (1). As Theorem 1 is an integral part of the proof of Theorem 3.1 [6], the result obtained by the authors, as well as its proof, is false (setting $f(x) = x$ and $g_n(x) = \exp(-nx) - 1$, $n \in \mathbb{N}$, yields a simple counterexample). Furthermore, since the authors have not assumed any constraint qualification, Theorem 3.1 [6] remains false even in the finite case (Example 4.2.10 [1] yields a simple counterexample). To overcome all those problems, under a nonsmooth limiting constraint qualification and using techniques from variational analysis, we propose another proof to detect necessary optimality conditions of (*SIP*) in terms of Karush–Kuhn–Tucker multipliers. The obtained results are formulated using limiting subdifferentials [7] and Fré chet subdifferentials [7]. Theorem 8 and Theorem 10 are actually two corrected versions of Theorem 3.1 [6].

For all the sequel, unless otherwise stated, $\mathbb{B}_{\mathbb{R}^n}$ denotes the closed unit ball of $\mathbb{R}^n$ and $\|(x, y)\| := \|x\| + \|y\|$ is the l_1-norm of (x, y). For a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the expressions

$$\limsup_{x \to \overline{x}} F(x) := \left\{x^* \in \mathbb{R}^n \mid \exists x_k \to \overline{x},\ \exists x_k^* \to x^* : x_k^* \in F(x_k)\ \forall k \in \mathbb{N}\right\}$$

and

$$\liminf_{x \to \overline{x}} F(x) := \left\{x^* \in \mathbb{R}^n \mid \forall x_k \to \overline{x},\ \exists x_k^* \to x^* : x_k^* \in F(x_k)\ \forall k \in \mathbb{N}\right\}$$

signify, respectively, the *sequential* Painlevé-Kuratowski upper/outer and lower/inner limits in the norm topology in $\mathbb{R}^n$.

The rest of the paper is organized in this way: Sect. 2 contains basic definitions and preliminary material from nonsmooth variational analysis. Section 3 addresses main results (optimality conditions).

2 Preliminaries

For a subset $D \subseteq \mathbb{R}^n$, $cl\ D$, $co\ D$ and $cone\ D$ stand for the closure, the convex hull and the convex cone generated by D, repectively. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the graph of f is the set of points in $\mathbb{R}^{n+1}$ defined by

$$grf = \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y = f(x)\big\}.$$

The following definitions are crucials for our investigation.

Definition 2 [7] Let Ω_1 and Ω_2 be nonempty closed subsets of $\mathbb{R}^n$. We say that $\{\Omega_1, \Omega_2\}$ is an extremal system in $\mathbb{R}^n$ if these sets have at least one (locally) *extremal point* $\bar{x} \in \Omega_1 \cap \Omega_2$; that is, there exists a neighborhood U of $\bar{x}$ such that for every $\varepsilon > 0$ there is a vector $a \in \varepsilon\mathbb{B}_{\mathbb{R}^n}$ with

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset.$$

In this case, $\{\Omega_1, \Omega_2, \bar{x}\}$ is said to be an extremal system in $\mathbb{R}^n$.

Definition 3 [7] Let $\Omega \subset \mathbb{R}^n$ be locally closed around $\bar{x} \in \Omega$. Then the *Fréchet normal cone* $\widehat{N}(\bar{x}; \Omega)$ and the *Mordukhovich normal cone (limiting normal cone)* $N(\bar{x}; \Omega)$ to Ω at $\bar{x}$ are defined by

$$\widehat{N}(\bar{x}; \Omega) := \left\{ x^* \in \mathbb{R}^n : \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0 \right\}, \tag{2}$$

$$N(\bar{x}; \Omega) := \limsup_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x; \Omega), \tag{3}$$

where $x \xrightarrow{\Omega} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$.

Definition 4 [7] Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous around $\bar{x}$.

1. The *Fréchet subdifferential* of φ at $\bar{x}$ is

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

2. The *Mordukhovich (limiting) subdifferential* of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) := \limsup_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x), \tag{4}$$

where $x \xrightarrow{\varphi} \bar{x}$ means that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$.

One clearly has

$$\widehat{N}(\bar{x}; \Omega) = \widehat{\partial}\delta(\bar{x}; \Omega), \quad N(\bar{x}; \Omega) = \partial\delta(\bar{x}; \Omega),$$

where $\delta(\cdot; \Omega)$ is the indicator function of Ω.

Remark 5 [8]

1. For any closed set $\Omega \subset \mathbb{R}^n$ and $\overline{x} \in \Omega$ one has
$$N_c(\bar{x}; \Omega) = cl\ coN(\bar{x}; \Omega) \tag{5}$$
and for any Lipschitz continuous function $\varphi \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ around $\bar{x}$, one has
$$\partial_c \varphi(\bar{x}) = cl\ co\partial\varphi(\bar{x}) \tag{6}$$
where $N_c(\bar{x}; \Omega)$ and $\partial_c\varphi(\bar{x})$ denote respectively the Clarke's normal cone and the Clarke's subdifferential.
2. The *Fréchet normal cone* $\widehat{N}(\bar{x}; \Omega)$ is always convex while the *Mordukhovich normal cone* $N(\bar{x}; \Omega)$ is nonconvex in general.

Definition 6 [7] Let $\{\Omega_1, \Omega_2, \bar{x}\}$ be an extremal system in $\mathbb{R}^n$. $\{\Omega_1, \Omega_2, \bar{x}\}$ satisfies the approximate extremal principle if for every $\varepsilon > 0$ there are $x_1 \in \Omega_1 \cap (\overline{x} + \varepsilon\mathbb{B}_{\mathbb{R}^n})$, $x_2 \in \Omega_2 \cap (\overline{x} + \varepsilon\mathbb{B}_{\mathbb{R}^n})$ and $x^* \in \mathbb{R}^n$ such that $\|x^*\| = 1$ and
$$x^* \in \left(\widehat{N}(x_1; \Omega_1) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}\right) \cap \left(-\widehat{N}(x_2; \Omega_2) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}\right).$$

Remark 7 A common point $\overline{x}$ of sets is locally extremal if these sets can be locally pushed apart by a (linear) *small translation* in such a way that the resulting sets have empty intersection in some neighborhood of $\overline{x}$.

3 Necessary optimality conditions

Let S be the feasible set of (*SIP*) defined by
$$S = \left\{x \in \mathbb{R}^n : g_i(x) \leq 0,\ \forall i \in I\right\}.$$

Theorem 8 *Assume that f is locally Lipschitz with constant k around the local optimal point $\overline{u} \in S$. Then, for any $\varepsilon > 0$, there exist $u_1, u_2 \in \overline{u} + \varepsilon\mathbb{B}_{\mathbb{R}^n},\ v_1, v_2 \in (f(\overline{u}) - \varepsilon,\ f(\overline{u}) + \varepsilon)$ and $\beta^*_\varepsilon \in \mathbb{R}_+ \setminus \{0\}$ such that $u_1 \in S,\ v_1 \leq f(\overline{u}),\ v_2 = f(u_2)$ and*
$$0 \in \widehat{\partial}\left(\beta^*_\varepsilon f\right)(u_2) + \widehat{N}(u_1; S) + \varepsilon\mathbb{B}_{\mathbb{R}^n}.$$

Proof Since $\overline{u}$ is a local optimal solution of (*SIP*), there exists a neighborhood V of $\overline{u}$ such that for all $u \in V \cap S$
$$f(u) \geq f(\overline{u}).$$

- Take
$$\Omega_1 := S \times (-\infty, f(\overline{u})] \text{ and } \Omega_2 := grf.$$
Then, it is easy to show that $(\overline{u}, f(\overline{u}))$ is an extremal point of the system (Ω_1, Ω_2). Indeed, suppose that is not the case, i.e., for any neighborhood U of $(\overline{u}, f(\overline{u}))$ there is $\varepsilon > 0$ such that for all $a \in \varepsilon B_{\mathbb{R}^n \times \mathbb{R}}$ one has
$$(\Omega_1 + a) \cap \Omega_2 \cap U \neq \emptyset.$$
Let $a = \left(0, -\frac{\varepsilon}{2}\right)$ and $(u, v) \in (\Omega_1 + a) \cap \Omega_2 \cap U$. Thus,
$$u \in S \text{ and } f(u) \in \left]-\infty, f(\overline{u}) - \frac{\varepsilon}{2}\right].$$

Hence $f(u) < f(\overline{u})$, which contradicts the fact that $\overline{u}$ is a local optimal solution of (SIP).

- Since f is locally Lipschitz, one has

$$|f(u) - f(\widehat{u})| \leq k\|u - \widehat{u}\| \qquad \text{for } u, \widehat{u} \text{ sufficiently close to } \overline{u}.$$

Let $1/4 > \varepsilon > 0$. Since $\{\Omega_1, \Omega_2, (\overline{u}, f(\overline{u}))\}$ is an extremal system, due to [7, Theorem 2.10], the approximate extremal principle holds at $(\overline{u}, \overline{v})$. Choosing $\theta = \dfrac{\varepsilon}{4(k+1)}$, there exist $u_1, u_2 \in \overline{u} + \theta\mathbb{B}_{\mathbb{R}^n},\ v_1, v_2 \in (f(\overline{u}) - \theta, f(\overline{u}) + \theta)$ and $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $u_1 \in S,\ v_1 \in (-\infty, f(\overline{u})],\ v_2 = f(u_2),\ \|(x^*, y^*)\| = 1$ and

$$\left(x^*, y^*\right) \in \left[\widehat{N}((u_1, v_1); \Omega_1) + \theta\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}}\right] \cap \left[-\widehat{N}((u_2, v_2); \Omega_2) + \theta\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}}\right]. \quad (7)$$

Hence we can find $(u^*, v^*) \in \widehat{N}((u_2, v_2); \Omega_2)$ and $\left(a_i^*, b_i^*\right) \in \mathbb{B}_{\mathbb{R}^n\times\mathbb{R}},\ i = 1,\ 2$, and $\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right) \in \widehat{N}((u_1, v_1); \Omega_1)$ such that

$$\left(u^*, v^*\right) = -\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right) + \theta\left(a_2^*, b_2^*\right) - \theta\left(a_1^*, b_1^*\right),$$

and

$$\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right) + \theta\left(a_1^*, b_1^*\right) = \left(x^*, y^*\right). \quad (8)$$

- Since $(u^*, v^*) \in \widehat{N}((u_2, v_2); \Omega_2)$, one has for all $(u, v) \in grf$ sufficiently close to (u_2, v_2) :

$$\left\langle u^*, u - u_2\right\rangle + \left\langle v^*, v - v_2\right\rangle - 2\theta\|(u - u_2, v - v_2)\| \leq 0.$$

Since the definition of Fréchet normals (2) implies that

$$\inf_{\delta>0} \sup_{(u,v)\in\mathbb{B}_\delta(u_2,v_2)\cap\Omega_2} \frac{\langle u^*, u - u_2\rangle + \langle v^*, v - v_2\rangle}{\|(u - u_2, v - v_2)\|} \leq 0,$$

thus there exists $\delta > 0$ such that for all $(u, v) \in \mathbb{B}_\delta(u_2, v_2) \cap \Omega_2$,

$$\frac{\langle u^*, u - u_2\rangle + \langle v^*, v - v_2\rangle}{\|(u - u_2, v - v_2)\|} \leq 2\theta,$$

or

$$\left\langle u^*, u - u_2\right\rangle + \left\langle v^*, v - v_2\right\rangle - 2\theta\|(u - u_2, v - v_2)\| \leq 0.$$

Consequently,

$$\begin{aligned} 0 \geq\ & \langle -\alpha_\varepsilon^*, u - u_2\rangle + \langle -\beta_\varepsilon^*, f(u) - f(u_2)\rangle \\ & + \theta\left\langle a_2^* - a_1^*, u - u_2\right\rangle + \theta\left(b_2^* - b_1^*\right)(f(u) - f(u_2)) \\ & - 2\theta\|(u - u_2, f(u) - f(u_2))\| \end{aligned}$$

for $(u, v) \in grf$ sufficiently close to (u_2, v_2).

- The locally Lipschitz property of f together with the fact that $\left(a_i^*, b_i^*\right) \in \mathbb{B}_{\mathbb{R}^n\times\mathbb{R}}$ for $i = 1,\ 2$, gives us for each u sufficiently close to u_2,

$$\begin{aligned} \langle\alpha_\varepsilon^*, u - u_2\rangle + \beta_\varepsilon^*(f(u) - f(u_2)) &\geq -4\theta\left[\|u - u_2\| + |f(u) - f(u_2)|\right] \\ &\geq -4\theta(k+1)\|u - u_2\| \\ &\geq -\varepsilon\|u - u_2\|. \end{aligned}$$

for $\theta = \dfrac{\varepsilon}{4(k+1)}$ and u sufficiently close to u_2. Then, u_2 minimizes locally the function

$$\Psi(u) := \langle \alpha_\varepsilon^*, u - u_2 \rangle + \beta_\varepsilon^* (f(u) - f(u_2)) + \varepsilon \|u - u_2\|.$$

Using [7, Proposition 1.107] together with the fuzzy sum rule [7, Theorem 2.33], we can find a point $\widetilde{u}_2 \in u_2 + \frac{\varepsilon}{2}\mathbb{B}_{\mathbb{R}^n}$ such that

$$0 \in \alpha_\varepsilon^* + \widehat{\partial}\left(\beta_\varepsilon^* f + \varepsilon \|. - u_2\|\right)(u_2) \tag{9}$$
$$\subseteq \alpha_\varepsilon^* + \widehat{\partial}\left(\beta_\varepsilon^* f\right)(\widetilde{u}_2) + \varepsilon \mathbb{B}_{\mathbb{R}^n}. \tag{10}$$

- $\beta_\varepsilon^* \neq 0$. Indeed, from (9), there exist $s^* \in \widehat{\partial}\left(\beta_\varepsilon^* f\right)(\widetilde{u}_2)$ and $e^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$0 = \alpha_\varepsilon^* + s^* + \varepsilon e^*.$$

Then,

$$\left\|\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right)\right\| = \left\|\alpha_\varepsilon^*\right\| + \left\|\beta_\varepsilon^*\right\| = \left\|s^* + \varepsilon e^*\right\| + \left\|\beta_\varepsilon^*\right\| \leq \left\|s^*\right\| + \varepsilon + \beta_\varepsilon^*.$$

Moreover,

$$\left\|\left(x^*, y^*\right)\right\| = \left\|\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right) + \theta\left(a_1^*, b_1^*\right)\right\| \leq \left\|\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right)\right\| + \theta\left\|\left(a_1^*, b_1^*\right)\right\|.$$

Since $\left(a_1^*, b_1^*\right) \in \mathbb{B}_{\mathbb{R}^{n+1}}$, $\|(x^*, y^*)\| = 1$ and $\theta = \dfrac{\varepsilon}{4(k+1)} < \varepsilon < 1/4$, one gets

$$\frac{3}{4} \leq 1 - \theta \leq \left\|\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right)\right\|.$$

Thus,

$$\frac{3}{4} \leq \left\|s^*\right\| + \varepsilon + \beta_\varepsilon^*.$$

Using the Lipschitz property of f, one deduces that

$$\frac{3}{4} \leq \beta_\varepsilon^* k + \varepsilon + \beta_\varepsilon^*.$$

Consequently,

$$0 < \frac{\frac{3}{4} - \varepsilon}{k+1} \leq \beta_\varepsilon^*.$$

Hence, $\beta_\varepsilon^* > 0$. This implies part 1 by (9). □

The following constraint qualification will be used to get necessary optimality conditions in terms of limiting subdifferentials and Karush–Kuhn–Tucker multipliers.

Definition 9 We say that the nonsmooth limiting constraint qualification holds at $\overline{u} \in S$ if

$$N(\overline{u}, S) \subseteq cl\left(\sum_{i \in I(\overline{u})} cone\, \partial g_i(\overline{u})\right),$$

where

$$I(\overline{u}) = \{i \in I : g_i(\overline{u}) = 0\}.$$

Theorem 10 gives exact optimality conditions for our nonsmooth semi-infinite programming problem.

Theorem 10 *Assume that f is locally Lipschitz at $\overline{u} \in S$ with Lipschitz constant k and that $\overline{u}$ is a local optimal solution of (SIP). Suppose that the nonsmooth limiting constraint qualification holds at $\overline{u}$. Then,*

$$0 \in \partial f(\overline{u}) + cl\left(\sum_{i \in I(\overline{u})} cone\, \partial g_i(\overline{u})\right).$$

Proof Fix arbitrary $\varepsilon > 0$. Since $\overline{u}$ is a local optimal solution of (*SIP*), there exist $u_1, u_2 \in \overline{u} + \varepsilon\mathbb{B}_{R^n}$, $v_1, v_2 \in (f(\overline{u}) - \varepsilon, f(\overline{u}) + \varepsilon)$ and $\beta_\varepsilon^* \in \mathbb{R}_+ \setminus \{0\}$ such that $u_1 \in S$, $v_1 \leq f(\overline{u})$, $v_2 = f(u_2)$ and

$$0 \in \widehat{\partial}\left(\beta_\varepsilon^* f\right)(u_2) + \widehat{N}(u_1; S) + \varepsilon\mathbb{B}_{\mathbb{R}^n}.$$

Using the Lipschitz property of f, from (9), there exist $s_\varepsilon^* \in \widehat{\partial}\left(\beta_\varepsilon^* f\right)(u_2)$, $\alpha_\varepsilon^* \in \widehat{N}(u_1; S)$ and $e_\varepsilon^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$0 = \alpha_\varepsilon^* + s_\varepsilon^* + \varepsilon e_\varepsilon^*,$$
$$\left\| s_\varepsilon^* \right\| \leq \beta_\varepsilon^* k,$$

and

$$\left(s_\varepsilon^*, -\beta_\varepsilon^*\right) \in \widehat{N}\left((u_2, f(u_2)), grf\right).$$

Since $\|(x^*, y^*)\| = 1$, $\left\|\left(a_1^*, b_1^*\right)\right\| \leq 1$ and

$$\begin{aligned} \left\|\alpha_\varepsilon^*\right\| + \left\|\beta_\varepsilon^*\right\| = \left\|\left(\alpha_\varepsilon^*, \beta_\varepsilon^*\right)\right\| &= \left\|\left(x^*, y^*\right) - \frac{\varepsilon}{4(k+1)}\left(a_1^*, b_1^*\right)\right\| \\ &\leq \left\|\left(x^*, y^*\right)\right\| + \frac{\varepsilon}{4(k+1)}\left\|\left(a_1^*, b_1^*\right)\right\| \end{aligned}$$

one gets

$$0 < \beta_\varepsilon^* \leq 1 + \frac{\varepsilon}{4(k+1)}.$$

Letting $\varepsilon \to 0$, $u_2 \to \overline{u}$ and $f(u_2) \to f(\overline{u})$, there exist $s^* \in -N(\overline{u}; S)$ and $0 < \beta^* \leq 1$ such that

$$\left(s^*, -\beta^*\right) \in N\left((\overline{u}, f(\overline{u})), grf\right).$$

Thus,

$$\begin{cases} s^* \in \partial\left(\beta^* F\right)(\overline{u}), \\ -s^* \in N(\overline{u}, S). \end{cases}$$

Then,

$$0 \in \partial\left(\beta^* f\right)(\overline{u}) + N(\overline{u}, S) = \beta^* \partial f(\overline{u}) + N(\overline{u}, S).$$

Consequently,

$$0 \in \partial f(\overline{u}) + N(\overline{u}, S).$$

The nonsmooth limiting constraint qualification implies that

$$0 \in \partial f\left(\overline{u}\right) + cl\left(\sum_{i \in I(\overline{u})} cone\ \partial g_i\left(\overline{u}\right)\right).$$

□

Example 11 Consider the following optimization problem :

$$\left(SIP^*\right): \quad \begin{cases} \text{Minimize} & f\left(x, y\right) = -3x + 2\left|y\right| \\ \text{s.t.} & g_i\left(x, y\right) = x + e^{-i}y \leq 0, \quad \forall i \in \mathbb{N} \cup \{0\}. \end{cases}$$

We remark that $\overline{u} = (0, 0) \in S$ is an optimal solution of (SIP^*) with

$$I\left(\overline{u}\right) = \mathbb{N} \cup \{0\} \text{ and } S = \left\{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } x + y \leq 0\right\}.$$

The nonsmooth limiting constraint qualification holds at $\overline{u}$. It is easy to show that

$$N\left(\overline{u}, S\right) = \left\{(d_1, d_2) \in \mathbb{R}^2 : 0 \leq d_2 \leq d_1\right\} = cl\left(\sum_{i \in I(\overline{u})} cone\ \partial g_i(\overline{u})\right)$$

On the other hand, $\partial f\left(\overline{u}\right) = \{-3\} \times [-2,\ 2]$, hence we get

$$(-3, -1) \in \partial f\left(\overline{u}\right) \cap \left(-cl\left(\sum_{i \in I(\overline{u})} cone\ \partial g_i(\overline{u})\right)\right).$$

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