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A subgradient algorithm for a class of nonlinear split feasibility problems: application to jointly constrained Nash equilibrium models

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Abstract

In this paper we propose an algorithm for solving the split feasibility problem $x \in C$, $Ax \in Q$ with C being the solution set of an equilibrium problem and A can be nonlinear. The proposed algorithm is a combination between the projection method for the equilibrium problem and the gradient method for the inclusion $Ax \in Q$. The convergence of the algorithm is investigated. A numerical example for a jointly constrained Nash equilibrium model in electricity production market is provided to demonstrate the behavior of the algorithm.

Keywords Nonlinear split feasibility · Equilibria · Subgradient method · Quasiconvexity · Nash model

1 Introduction

In 1994, Censor and Elfving [12] first introduced the split feasibility problem in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. In this setting the problem is stated as

$$\text{Find } x^* \in C : Ax^* \in Q, \qquad (SFP)$$

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where C, Q are convex subsets in $\mathcal{R}^n$, $\mathcal{R}^m$ respectively, and $A : \mathcal{R}^n \to \mathcal{R}^m$ is a linear continuous mapping on $\mathcal{R}^n$. Problem (SFP) can also be applied to study intensity-modulated radiation therapy and other practical problems see, for instance [8,10,11,13,15] and the references therein. An important special case of Problem (SFP) is obtained when C is the solution set of the equilibrium problem given as

$$\text{Find } x^* \in K : f(x^*, y) \geq 0 \quad \forall y \in K, \qquad (EP)$$

where K is a convex subset in $\mathcal{R}^n$ and f is a monotone bifunction with $f(x, .)$ being convex on K and $f(x, x) = 0$ for every $x \in K$. It is well known that some important problems such as optimization, variational inequality, Kakutani fixed point, saddle point ones and Nash equilibrium model can be formulated equivalently in the form of (EP), see e.g. [7,35] and the survey paper [6].

Some solution methods for solving split feasibility problem (SFP) when C and/or Q are solution sets of some other problems such as fixed point, optimization, variational inequality, equilibrium ones have been developed see [2,9,14,18,23,33,34,38–40] and the survey paper [31]. These methods either use the adjoint operator of A or are based upon a convex mathematical programming formulation, and therefore they fail to apply to the case when A is nonlinear. A nonlinear split feasibility problem with C and Q being the intersection of convex subsets has been considered in [30], where the problem was equivalently formulated as a differentiable mathematical program. Under the assumption that the objective function of the latter program is convex, the problem was further formulated as a cooercive variational inequality that was solved by a splitting projection algorithm.

The purpose of this paper is to propose an algorithm for solving Problem (SFP) in finite dimensional Euclidean spaces where C is the solution set of a paramonotone equilibrium problem and A is a quasilinear operator defined by quasilinear functions. Quasilinear functions play an important role in mathematics and many real-life problems see e.g. [4,19,27,32]. The algorithm that we propose for Problem (SFP) is a combination between the projection method for inclusion $Ax \in Q$ and the gradient projection method for Problem (EP). This algorithm can be considered as an extension to Problem (SFP) of the one by Santos and Scheimberg in [36] for (EP). The main difference between these algorithms is that our algorithm uses an additional projection in order to handle the requirement $Ax \in Q$, where A may not be linear. In the case of equilibrium problem (EP) considered in [36], the set $Q \equiv \mathcal{R}^n$, then our [36] algorithm collapses into the one in [36].

The paper is organized as follows. In the next section we recall some definitions and properties of quasiconvex and quasilinear functions. The algorithm and its convergence analysis are presented in Sect. 3. An application to jointly constraint Nash equilibrium models is discussed and some computational results for a coupled constrained Nash–Cournot equilibrium model in electricity production market are reported in the last section.

2 Preliminaries

The following lemmas will be used for validity and convergence of the algorithm.

Lemma 1 ([5], p. 61) *Let C be a nonempty closed convex subset in a Hilbert space $\mathcal{H}$ and $P_C(x)$ be the metric projection of x onto C. Then*

(i) $\langle x - y, P_C(x) - P_C(y) \rangle \geq \| P_C(x) - P_C(y) \|^2 \quad \forall x, y \in \mathcal{H}$;
(ii) $\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \quad \forall x \in \mathcal{H}, y \in C$.

Lemma 2 ([1]) *Let $\{v_k\}$ and $\{\delta_k\}$ be nonnegative sequences of real numbers satisfying $v_{k+1} \leq v_k + \delta_k$ with $\sum_{k=1}^{\infty} \delta_k < +\infty$. Then the sequence $\{v_k\}$ is convergent.*

The quasiconvex functions were first introduced by De Finetti in [20]. This class of functions are widely used in optimization, game theory, economics and other fields.

Definition 1 ([4]) Let $X \subset \mathbb{R}^n$ be a convex set and $\varphi : X \to R$.

(i) φ is called quasiconvex on X if its sublevel set

$$S_{\varphi,\alpha} = \{x \in X : \quad \varphi(x) \leq \alpha\}$$

   is convex for every $\alpha \in \mathbb{R}$.
(ii) φ is called quasiconcave on X if $-\varphi$ is quasiconvex on X.
(iii) φ is called quasilinear on X if it is both quasiconvex and quasiconcave.

Two important quasilinear functions are $\varphi(x) := \log x$ and $\varphi(x) := \frac{a^T x + b}{c^T x + d}$. The first function is quasilinear on $\mathbb{R}_{++}$, while the second one is quasilinear on the set $\{x : \quad c^T x + d \neq 0\}$.

The following proposition provides a characterization of the quasiconvex functions that will be used in the sequel.

Proposition 1 ([21]) *The following statements are equivalent*

(i) *φ is quasiconvex on X.*
(ii) *For any $x, y \in X$ and $\lambda \in [0, 1]$ one has*

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{\varphi(x_1), \varphi(x_2)\}. \tag{1}$$

Consequently, φ is quasilinear on X if and only if for every $x, y \in X$ and $\lambda \in [0, 1]$ we have

$$\min\{\varphi(x_1), \varphi(x_2)\} \leq \varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{\varphi(x_1), \varphi(x_2)\}. \tag{2}$$

Theorem 1 ([4,21]) *Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is differentiable on an open convex set containing X. Then φ is quasiconvex on X if and only if*

$$x, y \in X, \quad \varphi(y) \leq \varphi(x) \Rightarrow \nabla\varphi(x)^T(y - x) \leq 0. \tag{3}$$

It is easy to see that if φ_i is quasiconvex on X for every $i = 1, 2, \ldots, m$, then $\varphi(x) := \max_{i=1,2,\ldots,m} \varphi_i(x)$ is quasiconvex on X.

3 Algorithm and its convergence analysis

In this section we describe a projection method for solving the following split feasibility problem

$$\text{Find } z \in K \text{ such that } f(z, u) \geq 0 \quad \forall u \in K \text{ and } F(z) \in Q, \qquad (NSEP)$$

where $\emptyset \neq K \subseteq \mathbb{R}^n$ is convex, $f : K \times K \to \mathbb{R}$, $\emptyset \neq Q \subseteq \mathbb{R}^m$ and F is a map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

In what follows we suppose that Problem (NSEP) admits a solution and that

(A1) $Q = Q_1 \times Q_2 \times \cdots \times Q_m$ where Q_i is a closed convex subset of $\mathcal{R}$ for every $i = 1, 2, \ldots, m$;

(A2) $F = (F_1, F_2, \ldots, F_m)$ where $F_i : \mathcal{R}^n \to \mathcal{R}$ is quasilinear, i.e., both quasiconvex and quasiconcave, and continuously differentiable on an open set containing K.

We recall some well known definitions on monotonicity of a bifunction see e.g. [24,26]

Definition 2 Let K be a convex set and $S \subseteq K$. A bifunction $f : K \times K \to \mathcal{R}$ is said to be

(a) monotone on K with respect to S if

$$f(x, y) + f(y, x) \leq 0 \quad \forall x \in S, y \in K;$$

(b) pseudomonotone on K with respect to S if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x \in S, y \in K;$$

(c) paramonotone on K with respect to S if

$$x \in S, y \in K : f(x, y) = f(y, x) = 0 \Rightarrow y \in S.$$

Paramonotone bifunctions have been introduced and studied in [24] and used in some papers [36,40]. Clearly, $f(x, y) := \varphi(y) - \varphi(x)$ is paramonotone for every function φ. Another example for paramonotone bifunction is $f(x, y) := \langle Ax + b, y - x \rangle$ where A is a $(n \times n)$-nonsymmetric matrix such that $\hat{A} = \frac{1}{2}(A + A^T)$ is positive semidefinite and $\ker(\hat{A}) \subset \ker(A)$. In [24] (Prop. 3.2), Iusem proved that $f(x, y) := \langle Ax + b, y - x \rangle$ is paramonotone if and only if $\hat{A}$ is positive semidefinite and $\ker(\hat{A}) \subset \ker(A)$. Note that since A is nonsymmetric, $Ax + b$ can not be expressed as the gradient of any function.

Let $Sol(EP)$ denote the solution set of the equilibrium problem

$$\text{Find } z \in K \text{ such that } f(z, u) \geq 0 \quad \forall u \in K. \qquad (EP)$$

Then it is not hard to see that under Assumptions $(A1)$, $(A2)$, Problem (NSEP) can take the form

$$\min_{x \in C} \max_{i=1,2,\ldots,m} |(I - P_{Q_i})(F_i(x))|^2, \qquad (OP)$$

with C being the solution set of (EP). Moreover, the function $p_i(x) := |(I - P_{Q_i})(F_i(x))|^2$ is quasiconvex and hence, $p(x) := \max_{i=1,2,\ldots,m} p_i(x)$ is quasiconvex, too.

The use of the function p is motivated mainly by two facts. The first one is that the split convex feasibility problem

$$\text{find } x \in C : Fx \in Q, \qquad (P)$$

introduced by Censor et al where C and Q are closed convex sets, has been considered in a lot papers when F is a linear bounded mapping. In this case, there are two approaches to solve the problem. The first one uses the adjoint operator of the linear mapping, while the second one formulates the problem equivalently as a convex programming problem. However both these approaches fail to apply to the case F is not linear. The second fact is that the quasiconvex function and its properties were well studied in some references [19–21,32], and some algorithms were developed to find a local or a global minimizer of a quasiconvex function over a convex set, see [19,21,28,32]. Note that in Problem (OP), the feasible domain C is not given explicitly, but the solution set of an equilibrium problem.

For each $x \in K$, let

$$I(x) := \{i : p_i(x) = p(x)\},$$

then we have the following lemma:

Lemma 3 *Under the assumptions (A1) and (A2), it holds that*

(i) *The function p_i is quasiconvex and differentiable on K;*
(ii) *The function p is quasiconvex on K.*

Proof (i) Since F_i is differentiable on K, then it is easy to see that $p_i(x) = |(I - P_{Q_i})(F_i(x))|^2$ is also differentiable on K and

$$\nabla p_i(x) = 2(I - P_{Q_i})(F_i(x))\nabla F_i(x).$$

Since F_i is quasilinear on K for every i, then for any $x_1, x_2 \in K$ and $\lambda \in [0, 1]$ we have

$$\min\{F_i(x_1), F_i(x_2)\} \leq F_i(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{F_i(x_1), F_i(x_2)\}.$$

Hence, there exists $\alpha \in [0, 1]$ such that

$$F_i(\lambda x_1 + (1 - \lambda)x_2) = \alpha F_i(x_1) + (1 - \alpha)F_i(x_2).$$

It is well-known that the function $|(I - P_{Q_i})(.)|^2$ is convex. Then one has

$$
\begin{aligned}
p_i(\lambda x_1 &+ (1 - \lambda)x_2) \\
&= |(I - P_{Q_i})(F_i(\lambda x_1 + (1 - \lambda)x_2))|^2 \\
&= |(I - P_{Q_i})[\alpha F_i(x_1) + (1 - \alpha)F_i(x_2)]|^2 \\
&\leq \alpha|(I - P_{Q_i})(F_i(x_1))|^2 + (1 - \alpha)|(I - P_{Q_i})(F_i(x_2))|^2 \\
&= \alpha p_i(x_1) + (1 - \alpha)p_i(x_2) \\
&\leq \max\{p_i(x_1), p_i(x_2)\} \quad \forall x_1, x_2 \in K,
\end{aligned}
$$

which implies that p_i is quasiconvex on K.

(ii) This assertion comes directly from (i) and the definition of p. □

We need the following widely used assumptions for bifunction f.

(A3) For every $x \in K$, the function $f(x, .)$ is convex, subdifferentiable, $f(., x)$ is upper semicontinuous on an open convex set containing K and $f(x, x) = 0$ for every $x \in K$.

(A4) The bifunction f is pseudomonotone on K with respect to the solution set S of Problem (EP), that is

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \quad \forall x \in S, y \in K.$$

For a fixed $\epsilon \geq 0$, let $\partial_2^\epsilon f(x, x)$ denote the ϵ-subdifferentiable of the convex function $f(x, .)$ at x, that is

$$\partial_2^\epsilon f(x, x) := \{g : \langle g, y - x \rangle \leq f(x, y) - f(x, x) + \epsilon \quad \forall y\}.$$

Note that since the function $f(x, .)$ is convex and subdifferentiable on an open convex set containing K, the ϵ-subdifferential mapping $\partial_2^\epsilon f(x, .)$ maps a bounded set in K to a bounded set.

The algorithm then can be described as follows:

Algorithm 1.

Take a positive number δ and real sequences $\{\delta_k\}, \{\beta_k\}, \{\epsilon_k\}$ satisfying the conditions

$$\delta_k > \delta > 0, \quad \beta_k > 0, \quad \epsilon_k \geq 0, \quad \forall k \in \mathbb{N}; \tag{4}$$

$$\sum_{k=1}^{\infty} \epsilon_k < +\infty, \quad \sum_{k=1}^{\infty} \frac{\beta_k \epsilon_k}{\delta_k} < +\infty; \tag{5}$$

$$\sum_{k=1}^{\infty} \frac{\beta_k}{\delta_k} = +\infty, \quad \sum_{k=1}^{\infty} \beta_k^2 < +\infty; \tag{6}$$

Step 0: Seek $x_1 \in K$ and let $k := 1$.
Iteration k: Given $x_k \in K$. Take $g_k \in \partial_2^{\epsilon_k} f(x_k, x_k)$ and define

$$\alpha_k = \frac{\beta_k}{\gamma_k} \quad \text{where } \gamma_k = \max\{\delta_k, \|g_k\|\}.$$

Compute $y_k = P_K(x_k - \alpha_k g_k)$.
If $\nabla p_i(y_k) = 0 \quad \forall i \in I(y_k)$ then take $\widehat{h}_k = 0$;
Otherwise, take $0 \neq h_k \in \text{co}\{\nabla p_i(y_k), \quad i \in I(y_k)\}$, where co stands for the convex hull notion. Set

$$\widehat{h}_k = \frac{h_k}{\|h_k\|}$$

and compute

$$x_{k+1} = P_K\left(y_k - \alpha_k \widehat{h}_k\right),$$

then increase k by one and go to iteration k.

Remark 1 (i) If $\epsilon_k = 0$, $x_k = y_k$ and $p(x_k) = 0$, then x_k is an exact solution. So, we can call x_k is an ϵ-solution if $\epsilon_k \leq \epsilon$, $\|x_k - y_k\| \leq \epsilon$ and $p(x_k) \leq \epsilon$.

(ii) If $Q_i \equiv \mathcal{R}$ for every i, then $p_i(x) = |(I - P_{Q_i})F_i(x)| = 0$ for every i. Thus Problem (*NSEP*) becomes the equilibrium problem (*EP*). In this case Algorithm 1 becomes the one in [36] with exact projection. In fact, since $\widehat{h}_k = 0$ for every k, the iteration k of Algorithm 1 reads

Iteration k: Given $x_k \in K$. Take $g_k \in \partial_2^{\epsilon_k} f(x_k, x_k)$ and define

$$\alpha_k = \frac{\beta_k}{\gamma_k} \quad \text{where } \gamma_k = \max\{\delta_k, \|g_k\|\}.$$

Compute $x_{k+1} = P_K(x_k - \alpha_k g_k)$.

The proof of the lemma below can be done by a similar way as the one in [36].

Lemma 4 ([36]) *For every k, the following inequalities hold*

(i) $\alpha_k \|g_k\| \leq \beta_k$;
(ii) $\|y_k - x_k\| \leq \beta_k$.

The following lemmas will be used in the proof of the convergence for the proposed algorithm.

Lemma 5 *It holds that*

$$\|x_{k+1} - z\|^2 \leq \|y_k - z\|^2 - 2\alpha_k \langle \widehat{h}_k, y_k - z \rangle + \alpha_k^2, \quad \forall z \in K.$$

Proof By nonexpansiveness of P_K, one has

$$\|x_{k+1} - z\|^2 = \|P_K\left(y_k - \alpha_k \widehat{h}_k\right) - z\|^2$$
$$\leq \|y_k - \alpha_k \widehat{h}_k - z\|^2 = \|y_k - z\|^2 - 2\alpha_k \langle \widehat{h}_k, y_k - z \rangle + \alpha_k^2 \quad \forall z \in K.$$

Lemma 6 (i) *Under Assumptions (A1), (A2), (A3), for $z \in K$, it holds that*

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + 2\alpha_k f(x_k, z) - 2\alpha_k \langle \widehat{h}_k, y_k - z \rangle + A_k, \tag{7}$$

where $A_k = 2(\alpha_k \epsilon_k + \beta_k^2) + \alpha_k^2$.
(ii) *If there exist $z \in K$, $\epsilon > 0$ and $\delta > 0$ such that*

$$p(y) < p(y_k) - \delta \quad \forall y \in B(z, \epsilon),$$

then

$$\langle \widehat{h}_k, y_k - z \rangle \geq \epsilon \quad \forall k$$

whenever $\widehat{h}_k \neq 0$.

Proof (i) It is clear that,

$$\begin{aligned}
\|y_k - z\|^2 &= \|z - x_k + x_k - y_k\|^2 \\
&= \|x_k - z\|^2 - \|x_k - y_k\|^2 + 2\langle x_k - y_k, z - y_k \rangle \\
&\leq \|x_k - z\|^2 + 2\langle x_k - y_k, z - y_k \rangle.
\end{aligned}$$

Since $y_k = P_K(x_k - \alpha_k g_k)$, we have

$$\begin{aligned}
&\langle y_k - x_k + \alpha_k g_k, z - y_k \rangle \geq 0 \\
&\Leftrightarrow \langle \alpha_k g_k, z - y_k \rangle \geq \langle x_k - y_k, z - y_k \rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y_k - z\|^2 &\leq \|x_k - z\|^2 + 2\langle \alpha_k g_k, z - y_k \rangle \\
&= \|x_k - z\|^2 + 2\langle \alpha_k g_k, z - x_k \rangle + 2\langle \alpha_k g_k, x_k - y_k \rangle. \tag{8}
\end{aligned}$$

From $g_k \in \partial_2^{\epsilon_k} f(x_k, x_k)$ it follows that

$$\begin{aligned}
&f(x_k, z) - f(x_k, x_k) \geq \langle g_k, z - x_k \rangle - \epsilon_k \\
&\Leftrightarrow f(x_k, z) + \epsilon_k \geq \langle g_k, z - x_k \rangle. \tag{9}
\end{aligned}$$

On the other hand, by Lemma 4, it holds that

$$\langle \alpha_k g_k, x_k - y_k \rangle \leq \alpha_k \|g_k\| \|x_k - y_k\| \leq \beta_k^2. \tag{10}$$

From (8), (9) and $\alpha_k > 0$, it follows that

$$\|y_k - z\|^2 \leq \|x_k - z\|^2 + 2\alpha_k f(x_k, z) + 2\alpha_k \epsilon_k + 2\beta_k^2, \tag{11}$$

which, by Lemma 5, implies

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + 2\alpha_k f(x_k, z) - 2\alpha_k \langle \widehat{h}_k, y_k - z \rangle + A_k, \tag{12}$$

where $A_k = 2(\alpha_k \epsilon_k + \beta_k^2) + \alpha_k^2$.
(ii) Since $z + \epsilon \widehat{h}_k \in B(z, \epsilon)$, by the assumption, we have

$$p(z + \epsilon \widehat{h}_k) < p(y_k).$$

Then from the definitions of p and $I(y_k)$, we obtain

$$p_i(z + \epsilon \widehat{h}_k) < p_i(y_k) \quad \forall i \in I(y_k).$$

Applying Theorem 1 with $\varphi = p_i$, $y = z + \epsilon \widehat{h}_k$, and $x = y_k$ we have

$$\langle \nabla p_i(y_k), z + \epsilon \widehat{h}_k - y_k \rangle \leq 0, \quad \forall i \in I(y_k).$$

Thus, since $\widehat{h}_k \neq 0$, $\widehat{h}_k = \frac{h_k}{\|h_k\|}$ with $h_k \in co\{\nabla p_i(y_k), \quad i \in I(y_k)\}$. So,

$$\langle \widehat{h}_k, z + \epsilon \widehat{h}_k - y_k \rangle \leq 0.$$

Hence,

$$\langle \widehat{h}_k, y_k - z \rangle \geq \epsilon. \qquad \qquad \square$$

Before stating the convergence result of the algorithm we recall [16] that the directional derivative of a local Lipschitz function $\varphi : \mathcal{R}^n \to \mathcal{R}$ at x in direction d is defined by

$$\varphi^0(x, d) := \limsup_{y \to x, t \searrow 0} \frac{\varphi(y + td) - \varphi(y)}{t}.$$

The subdifferential (in Clarke's sense) of f at x then is defined as

$$\partial^0 \varphi(x) := \{\xi \in \mathcal{R}^n : \varphi^0(x, d) \geq \langle \xi, d \rangle \quad \forall d \in \mathcal{R}^n\}.$$

Note that, since $p(x) := \max\{p_i(x) : i \in I(x)\}$, and p_i is locally Lipschitz, differentiable, by Proposition 2.3.12 [16], we have $\partial^0 p(x) = co\{\nabla p_i(x) : i \in I(x)\}$, where co stands for the convex hull notion. As usual, we say that $x^* \in C$ is a stationary point of the problem $\min_{x \in C} \varphi(x)$ (or x^* is a stationary point of φ over C), if $0 \in \partial^0 \varphi(x^*) + N_C(x^*)$, in particular, $0 \in \partial^0 \varphi(x^*)$.

Let S denote the solution set of Problem (NSEP), then we have the following convergence result:

Theorem 2 *Under the assumptions (A1)–(A4) we suppose further that f is paramonotone with respect to the solution set $Sol(EP)$ and that the sequence $\{g_k\}$ is bounded. Then the sequence $\{x_k\}$ converges to a solution of Problem (NSEP) or to a solution of equilibrium problem (EP) which is also a stationary point of problem $\min\{p(x) : x \in K\}$.*

More precisely, let

$$J = \left\{ k \mid \widehat{h}_k \neq 0 \right\}, \tag{13}$$

then

(i) *If $\sum_{k \in J} \alpha_k = +\infty$, the sequence $\{x_k\}$ converges to a solution of (NSEP).*
(ii) *If $\sum_{k \in J} \alpha_k < +\infty$, the sequence $\{x_k\}$ converges to a solution x^* of the equilibrium problem (EP), which is also a stationary point of problem $\min\{p(x) : x \in K\}$.*

Remark 2 The condition for which the sequence $\{g_k\}$ is bounded has been given in [36]. Note that in the variational inequality case where $f(x, y) := \langle \phi(x), y - x \rangle$, the sequence $\{g_k\}$ is bounded whenever ϕ is continuous. The condition $\sum_{k \in J} \alpha_k = +\infty$ is satisfied if there exists an integer number k_0 such that $\widehat{h}_k \neq 0$ for $k \geq k_0$.

Proof (of the theorem) We divide the proof into several steps.

Step 1: First we prove that, in any case, the sequence $\{\|x_k - z\|^2\}$ is convergent for all $z \in S$, and hence $\{x_k\}$ and $\{y_k\}$ are both bounded.

In fact, since f is pseudomonotone on K with respect to the solution set of (EP), one has

$$f(y_k, z) \leq 0 \quad \forall z \in Sol(EP).$$

On the other hand, since z is a minimizer of p, we have $p(y_k) \geq p(z)$. Hence, from the definitions of p and p_i, it follows that

$$\langle \nabla p_i(y_k), z - y_k \rangle \leq 0 \quad \forall k.$$

Since $h_k \in co\{\nabla p_i(y_k), \quad i \in I(y_k)\}$ and $\widehat{h}_k = 0$ or $\widehat{h}_k = \frac{h_k}{\|h_k\|}$, one has

$$\langle \widehat{h}_k, z - y_k \rangle \leq 0 \quad \forall k.$$

Then, by virtue of Lemma 6 (i), we obtain

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + A_k \quad \forall z \in S, \tag{14}$$

where $A_k = 2(\alpha_k \epsilon_k + \beta_k^2) + \alpha_k^2$. Since $\alpha_k = \frac{\beta_k}{\gamma_k}$ with $\gamma_k = \max\{\delta_k, \|g_k\|\}$,

$$\sum_{k=1}^{+\infty} \alpha_k \epsilon_k = \sum_{k=1}^{+\infty} \frac{\beta_k}{\gamma_k} \epsilon_k \leq \sum_{k=1}^{+\infty} \frac{\beta_k}{\delta_k} \epsilon_k < +\infty.$$

Since $\delta_k > \delta > 0$,

$$\sum_{k=1}^{+\infty} \alpha_k^2 = \sum_{k=1}^{+\infty} \frac{\beta_k^2}{\gamma_k^2} \leq \sum_{k=1}^{+\infty} \frac{\beta_k^2}{\delta_k^2} < \sum_{k=1}^{+\infty} \frac{\beta_k^2}{\delta^2},$$

which together with $\sum_{k=1}^{+\infty} \beta_k^2 < +\infty$ implies

$$\sum_{k=1}^{+\infty} A_k = \sum_{k=1}^{+\infty} \left[2(\alpha_k \epsilon_k + \beta_k^2) + \alpha_k^2\right] < +\infty.$$

Using Lemma 2, we see that $\{\|x_k - z\|^2\}$ is convergent for every $z \in S$. Hence, $\{x_k\}$ is bounded. Then, since $y_k = P_K(x_k - \alpha_k g_k)$, by Lemma 1, the sequence $\{y_k\}$ is bounded too.

Now we consider two distinct cases.

(i) *Case 1.* $\sum_{k \in J} \alpha_k = +\infty$.

Step 2(i): We show that, for $z \in S$,

$$\liminf_{\substack{k \in J \\ k \to \infty}} \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z)\right] = 0.$$

Indeed, by Lemma 6(i), for every k, we have

$$2\alpha_k \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z)\right] \leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + A_k. \tag{15}$$

Summing up we obtain

$$\sum_{k=1}^{\infty} 2\alpha_k \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z)\right] < +\infty. \tag{16}$$

Since z is a solution of Problem (NSEP), by pseudomonotonicity of f, we have $f(x_k, z) \leq 0$ and $\langle \widehat{h}_k, y_k - z \rangle \geq 0$. Hence,

$$\sum_{k \in J} \alpha_k \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z)\right] < +\infty. \tag{17}$$

But from $\sum_{k \in J} \alpha_k = +\infty$, it holds that

$$\liminf_{\substack{k \in J \\ k \to \infty}} \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z)\right] = 0.$$

Step 3(i): We show that if $z \in S$ and $\{x_{k_j}\}$ is a subsequence of $\{x_k\}$ such that

$$\widehat{h}_{k_j} \neq 0 \quad \forall j,$$

$$\liminf_{\substack{k \in J \\ k \to \infty}} \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \right] = \lim_{j \to \infty} \left[\langle \widehat{h}_{k_j}, y_{k_j} - z \rangle - f(x_{k_j}, z) \right], \tag{18}$$

and x^* is a cluster point of $\{x_{k_j}\}$, then x^* belongs to S.
Combining (18) with $f(x_{k_j}, z) \leq 0$ and $\langle \widehat{h}_{k_j}, y_{k_j} - z \rangle \geq 0$, we obtain

$$\lim_{j \to \infty} \langle \widehat{h}_{k_j}, y_{k_j} - z \rangle = \lim_{j \to \infty} f(x_{k_j}, z) = 0. \tag{19}$$

Without loss of generality, we can assume that x_{k_j} converges to x^* as $j \to \infty$. Since $f(., z)$ is upper semicontinuous,

$$f(x^*, z) \geq \lim_{j \to \infty} f(x_{k_j}, z) = 0.$$

By assumption (A4), f is pseudomonotone on K with respect to S, we have $f(x^*, z) \leq 0$ if $z \in S$. Thus, $f(x^*, z) = 0$. Again, by the pseudomonotonicity of f, $f(z, x^*) \leq 0$. Hence, $f(z, x^*) = f(x^*, z) = 0$. Then using the paramonotonicity of f, we can conclude that x^* is a solution of (EP).
Now, it remains to prove that $p(x^*) = 0$. Indeed, otherwise, there exists $\alpha > 0$ such that

$$p(x^*) > \alpha.$$

Since $\lim_{j \to \infty} x_{k_j} = x^*$ and $\lim_{k \to \infty} \|x_k - y_k\| \leq \lim_{k \to \infty} \beta_k = 0$, we have $\lim_{j \to \infty} y_{k_j} = x^*$. Thus,

$$\lim_{j \to \infty} p(y_{k_j}) = p(x^*) > \alpha > 0,$$

and therefore there exists $j_0 \in \mathbb{N}$ such that

$$p(y_{k_j}) > \frac{\alpha}{2} \quad \forall j \geq j_0.$$

Note that $0 = p(z) < \frac{\alpha}{2}$ for every $z \in S$. Since p is continuous, there exists $\epsilon > 0$ such that

$$p(y) < \frac{\alpha}{2} \quad \forall y \in B(z, \epsilon).$$

Then $p(y) < p(y_{k_j})$ for all $j \geq j_0$ and $y \in B(z, \epsilon)$. Since $\widehat{h}_{k_j} \neq 0$ for every j, by applying Lemma 6(ii), we obtain

$$\langle \widehat{h}_{k_j}, y_{k_j} - z \rangle \geq \epsilon \quad \forall j \geq j_0,$$

which contradicts with (19). Thus $p(x^*) = 0$, which means that $x^* \in S$.
Step 4(i): Now we prove that $\{x_k\}$ converges to a solution of (NSEP).
In Step 1(i), we have seen that the sequence $\{\|x_k - z\|\}$ converges for every $z \in S$. Combining this fact with Step 3(i) to obtain

$$\lim_{k \to \infty} \|x_k - x^*\| = \lim_{j \to \infty} \|x_{k_j} - x^*\| = 0.$$

(ii) *Case 2.* $\sum_{k \in J} \alpha_k < +\infty$

Step 2(ii): We show that, for $z \in S$,

$$\liminf_{k \to \infty} \left[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \right] = 0.$$

Indeed, by Lemma 6, for every k, we have

$$2\alpha_k \Big[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \Big] \leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + A_k. \qquad (20)$$

Summing up we obtain

$$\sum_{k=1}^{\infty} 2\alpha_k \Big[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \Big] < +\infty. \qquad (21)$$

By the assumption that $\{\|g_k\|\}$ is bounded, there exists $L > \delta$ such that $\|g_k\| \leq L$ for every k. Thus,

$$\frac{\gamma_k}{\delta_k} = \max \left\{ 1, \frac{\|g_k\|}{\delta_k} \right\} \leq \frac{L}{\delta}.$$

Hence,

$$\alpha_k = \frac{\beta_k}{\gamma_k} \geq \frac{\delta}{L} \frac{\beta_k}{\delta_k}.$$

Since z is a solution of Problem (NSEP), by pseudomonotonicity of f, we have $f(x_k, z) \leq 0$ and $\langle \widehat{h}_k, y_k - z \rangle \geq 0$. The it follows from (21) that

$$\sum_{k=1}^{\infty} \frac{\beta_k}{\delta_k} \Big[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \Big] < +\infty. \qquad (22)$$

But from $\sum_{k=1}^{\infty} \frac{\beta_k}{\delta_k} = +\infty$, it holds that

$$\liminf_{k \to \infty} \Big[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \Big] = 0.$$

Step 3(ii): We show that if $z \in S$ and $\{x_{k_j}\}$ is a subsequence of $\{x_k\}$ such that

$$\liminf_{k \to \infty} \Big[\langle \widehat{h}_k, y_k - z \rangle - f(x_k, z) \Big] = \lim_{j \to \infty} \Big[\langle \widehat{h}_{k_j}, y_{k_j} - z \rangle - f(x_{k_j}, z) \Big],$$

and x^* is a cluster point of $\{x_{k_j}\}$, then x^* belongs to $Sol(EP)$.
Indeed, without loss of generality, we can assume that x_{k_j} converges to x^* as $j \to \infty$. We have

$$\lim_{j \to \infty} \Big[\langle \widehat{h}_{k_j}, y_{k_j} - z \rangle - f(x_{k_j}, z) \Big] = 0.$$

Combining this with $f(x_{k_j}, z) \leq 0$ and $\langle \widehat{h}_{k_j}, y_{k_j} - z \rangle \geq 0$, we obtain

$$\lim_{j \to \infty} \langle \widehat{h}_{k_j}, y_{k_j} - z \rangle = \lim_{j \to \infty} f(x_{k_j}, z) = 0. \qquad (23)$$

By the same argument as in Step 3(i), we can show that x^* is a solution of (EP).
Now, if there exists a subsequence of $\{k_j\}$ such that $\widehat{h}_{k_j} \neq 0$, then by the same argument as in Step 3(i), we can conclude that x^* belongs to S. Otherwise, there exist j_1 such that for all $j \geq j_1$,

$$\widehat{h}_{k_j} = 0,$$

which implies that $\nabla p_i(y_{k_j}) = 0$ for all $i \in I(y_{k_j})$. Note that $\{y_{k_j}\}$ also converges to x^*.

Step 4(ii): We prove that the sequence $\{\|x_k - x^*\|\}$ converges, and hence the whole sequence $\{x_k\}$ converges to x^*.

Since $x^* \in K$, by Lemma 6(i), one has

$$\|x_{k+1} - x^*\|^2 \le \|x_k - x^*\|^2 + 2\alpha_k f(x_k, x^*) - 2\alpha_k \langle \widehat{h}_k, y_k - x^* \rangle + A_k.$$

In Step 3(ii), we have shown that x^* belongs to $Sol(EP)$, and, by pseudomonotonicity of f,

$$f(x_k, x^*) \ge 0 \quad \forall k.$$

So,

$$\|x_{k+1} - x^*\|^2 \le \|x_k - x^*\|^2 - 2\alpha_k \langle \widehat{h}_k, y_k - x^* \rangle + A_k. \tag{24}$$

Note that if $k \notin J$, then $\widehat{h}_k = 0$ and thus,

$$\|x_{k+1} - x^*\|^2 \le \|x_k - x^*\|^2 + A_k. \tag{25}$$

If $k \in J$, then $\widehat{h}_k \ne 0$ and $\|\widehat{h}_k\| = 1$, hence

$$\langle \widehat{h}_k, y_k - x^* \rangle \ge -\|y_k - x^*\|.$$

Note that $\{y_k\}$ is bounded, so there exists $M > 0$ large enough such that

$$\|y_k - x^*\| < M \quad \forall k.$$

Then,

$$\langle \widehat{h}_k, y_k - x^* \rangle \ge -M.$$

The it follows from (24) that

$$\|x_{k+1} - x^*\|^2 \le \|x_k - x^*\|^2 + 2M\alpha_k + A_k \quad \forall k \in J. \tag{26}$$

From (25) and (26), we have

$$\|x_{k+1} - x^*\|^2 \le \|x_k - x^*\|^2 + B_k,$$

where

$$B_k = \begin{cases} A_k + 2M\alpha_k & \text{if } k \in J, \\ A_k & \text{otherwise.} \end{cases}$$

Since $\sum_{k=1}^{\infty} A_k < +\infty$ and $\sum_{k \in J} \alpha_k < +\infty$, we have

$$\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} A_k + \sum_{k \in J} 2M\alpha_k < +\infty.$$

Thanks to Lemma 2, we see that $\{\|x_k - x^*\|^2\}$ is convergent, and hence $\{x_k\}$ converges to x^*, as, by Step 3(ii), it has a subsequence converging to x^*. More precisely, if there exists N_0 such that $\widehat{h}_k \ne 0$ for every $k \ge N_0$, then by the same argument as in Case 1, we can show that x^* is a solution of Problem (NSFP). If $\widehat{h}_k = 0$ for infinitely many k, then there exists an infinite subsequence $\{y_{k_j}\}$ such that $\nabla p_i(y_{k_j}) = 0$ for every j and $i \in I(y_{k_j})$. Let

$$I_0 := \{i \mid i \in I(y_{k_j}), \ \nabla p_i(y_{k_j}) = 0 \text{ for infinitely many j}\}.$$

Since $\nabla p_i(y_{k_j}) = 0$ for infinitely many j, and $I(y_{k_j}) \subseteq \{1, \dots, m\}$, we have $I_0 \ne \emptyset$. Letting $j \to \infty$ we obtain $\nabla p_i(x^*) = 0$ for every $i \in I_0$. On the other hand, for $i \in I_0$,

$$p_i(x^*) = \lim_{i \to \infty} p_i(y_{k_j}) = \lim_{i \to \infty} p(y_{k_j}) = p(x^*),$$

which implies $I_0 \subseteq I(x^*)$. Thus $0 \in co(\{\nabla p_i(x^*) = 0 \ i \in I_0\}) \subseteq co(\{\nabla p_i(x^*) = 0 \ i \in I(x^*)\})$ that means that x^* is a stationary point of p over C. Since both sequences $\{x_k\}, \{y_k\}$ converging to x^*, the proof is complete.   □

4 Numerical experiments

In this section, we test our algorithm, denoted by NSEP and compare it with the MACEP algorithm proposed in [37] for solving constrained equilibrium problems on 3 examples. The first one is a differentiated jointly constrained Nash equilibrium model. The second is an example taken from [37] for a Nash–Cournot model. In the last example, we consider a NSEP problem where each function $F_i(x_i) = \log(a_i x_i + b_i)$. The algorithms have been code in Matlab 7.8 on a 8Gb RAM core i7.

Example 1 (**Application to Jointly Constrained Nash Equilibrium Models**)

Suppose that there are $i = 1, \dots, n$ players participating in a game. Each player can take an individual action, which is represented by a vector x_i in $\mathcal{R}$. All players together can take a collective action $x \in \mathcal{R}^n$. Each player i uses a payoff function f_i which depends on actions of other players. Then the Nikaido–Isoda function of the game is defined as

$$f(x, y) := \sum_{i=1}^{n} \Big(f_i(x) - f_i(x[y_i]) \Big), \tag{27}$$

where the vector $x[y_i]$ is obtained from x by replacing component x_i by y_i. Let $K_i \subset \mathcal{R}$ be the strategy of player i. Then the strategy set of the game is $K := K_1 \times \cdots \times K_n$. We recall that a point $x^* \in K$ is a Nash equilibrium point of the game if

$$f_i(x^*) = \max_{y_i \in K_i} f_i(x^*[y_i]) \quad \forall y_i \in K_i, \quad \forall i.$$

It is well known that x^* is an equilibrium point if and only if $f(x^*, y) \geq 0 \ \ \forall y \in K$. In some practical models an equilibrium point is required to satisfy certain additional constraints defined as $F(x) \in Q$, where F is a mapping from $\mathcal{R}^n$ to Q with Q being a convex subset in another space $\mathcal{R}^m$. Clearly, the problem of finding such an equilibrium point can take the form of Problem (NSEP).

To illustrate let us consider a model in electricity production where it is assumed that a company possesses n-plants, each of them producing a type of electricity, for instance, nuclear, solar, wind, hydro and thermoelectricity.

We suppose that the price for producing at plant i is an affine function given by $p_i(x_1, \dots, x_n) := \alpha - \sum_{k=1}^{n} \tau_{ik} x_k$ for every i, where $\alpha > 0$ (in general large), $\tau_{ik} > 0$. This price function arises in differentiated good models [22], where the users prefer the commodity produced by one plant to the other ones, for example, many users prefer solar and wind electricity to thermoelectricity or nuclear ones. Note that when $t_{ik} = \tau$ for every i and k, the price function becomes the usual one. The profit made by plant i takes the form

$$f_i(x) := p_i(x_1, \dots, x_n)x_i - c_i(x_i), \tag{28}$$

where $c_i(x_i)$ is the cost (including the fee for environmental pollution) for producing x_i by plant i. In general c_i is an increasingly convex function. The convexity means that the cost for producing a unit increases as the amount of the production gets larger.

Actually, the company seeks to maximize its profit by choosing a corresponding production level at each plant under the presumption that the production at other plants are parametric

	n	m	Iter	CPU-times (s)
Table 1 Algorithm 1 with $\beta_k = \dfrac{7}{2(k+1)}$	10	5	248.11	7.2859
	20	5	674.82	24.9143
	30	5	1224.2	40.4671
	40	5	1670	58.5344
	50	5	2259.7	84.3867

input. Let K_i be the strategy set for plant i, that is the the production level x_i must be chosen in K_i. A commonly used approach to this model is based upon the famous Nash equilibrium concept by using the Nikaido–Isoda function defined in (27). This function has been used to models in electricity markets and others [17,25].

In practice the level of the production at each plant should satisfy a certain ratio, for example, the ratio of hydroelectricity x_1 and the total production $\sum_{j \neq 1} x_j$ of all other electricity types should be restricted in a given percent, that can be written as $l_1 \leq \frac{x_1}{\sum_{j\neq 1} x_j} \leq u_1$, where l_1 and u_1 are given constants. In this case, the problem of finding an equilibrium point that satisfies joint constraints leads to a nonlinear split feasibility problem of the form (NSEP). Coupled constraint models in electricity market and in the River Basin Pollution game was considered in some papers, see e.g. [3,29,40] and the references therein.

We tested the proposed algorithm with the cost function given by

$$c_j(x_j) := \frac{1}{2} r_j x_j^2 + q_j x_j, \ r_j > 0$$

and the price functions

$$p_i\left(\sum_{j=1}^n x_j\right) := 30 - \sum_{j=1}^n \tau_{ij} x_j,$$

where r_j, q_j and each τ_{ij} are randomly generated in the interval $[0, 20]$, $[0, 3]$ and in $[0, 1/n]$ respectively. For these cost and price functions, by using Proposition 3.2 in [24], one can check that the bifunction f defined by (27) with f_i being given by (28) is paramonotone.

We took the strategy set $K_i := [0, 6]$ for every i and we require that the ratio of each type of electricity and the sum of all electricity production is less than fifty percent, which is represented by the constraints $0 \leq F_i(x) \leq 0.5$ for every $i = 1, \ldots, m$.

We choose the sequences of the parameters as

$$\epsilon_k = 0, \delta_k = 3, \quad \forall k.$$

and we computed the model with $m = 5$ and different number n from 10 to 50 and

$$F_i(x) = \frac{\sum_{k \in I_i} x_k}{\sum_{j=1}^n x_j}, \ i = 1, \ldots, m,$$

where I_i is the set of plants that produce type i of electricity.

The computational results are shown in Table 1 with different sizes, one hundred problems have been tested for each size. We stopped the computation at iteration k if $\|x_k - y_k\| \leq 10^{-4}$ and $p(x_k) \leq 10^{-4}$ or the number of iterations exceeds 20,000.

Figure 1 shows that the errors $p(x_k)$ and $\|x_k - y_k\|$ both go to 0 as the number of iterations k goes to $+\infty$. We can see that in this example, both of the two errors decrease quickly as
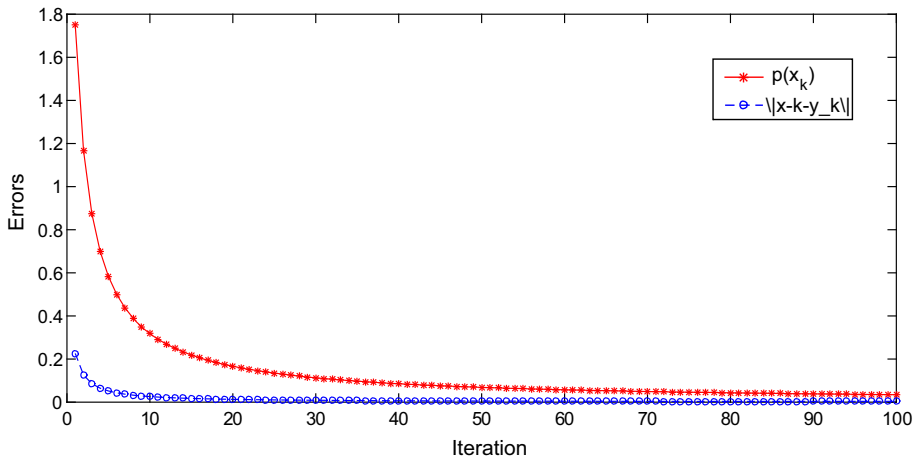
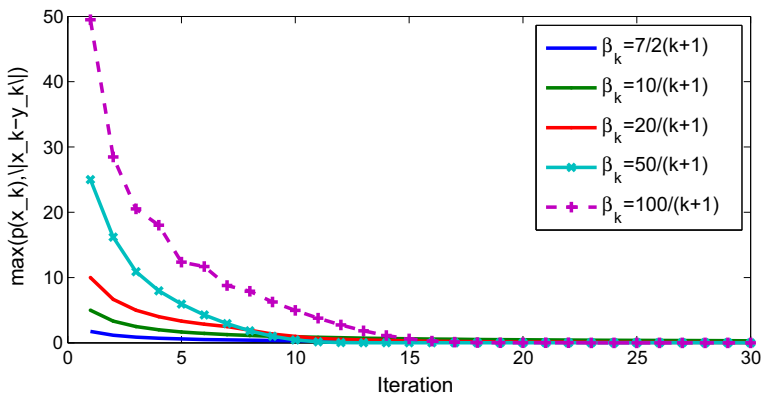**Fig. 1** Behavior of the values $p(x_k)$ and $\|x_k - y_k\|$

Fig. 2 Behavior of the error with different choice of parameter β_k

the number of iteration k gets larger. However, in some cases, the value of function $p(x_k)$ may not be monotone decreasingly. Nevertheless, for all tested problems, the value of the function p at the stoping iterate is very small.

We also tested the algorithm with different values of β_k: $\frac{7}{2(k+1)}$, $\frac{10}{(k+1)}$, $\frac{20}{(k+1)}$, $\frac{50}{k+1}$, $\frac{100}{k+1}$. We observe that the choice of the parameter β_k plays a crucial role for efficiency of the algorithm, in general, the difference $\|x_k - y_k\|$ goes to 0 quite slow, when this parameter is too small or too big, see Fig. 2.

Recently, the authors in [37] considered the equilibrium problem

$$\text{Find } x \in C \cap D : f(x, y) \geq 0 \quad \forall y \in C, \qquad \textit{EP(f, C,D)}$$

where C, D are convex subsets in $\mathcal{R}^n$ and f is a finite bifunction defined on an open set containing C and D.

In order to compare our algorithm, denoted by NSEP, with the one, denoted by MACEP, in [37] on the above example, we formulate Problem (NSEP) in the form of Problem EP(C, D, f) by taking $C := K$, $D := \{x : F(x) \in Q\}$. and apply the algorithm MACEP in

Table 2 MACEP versus NSFP (n=10,k=5)

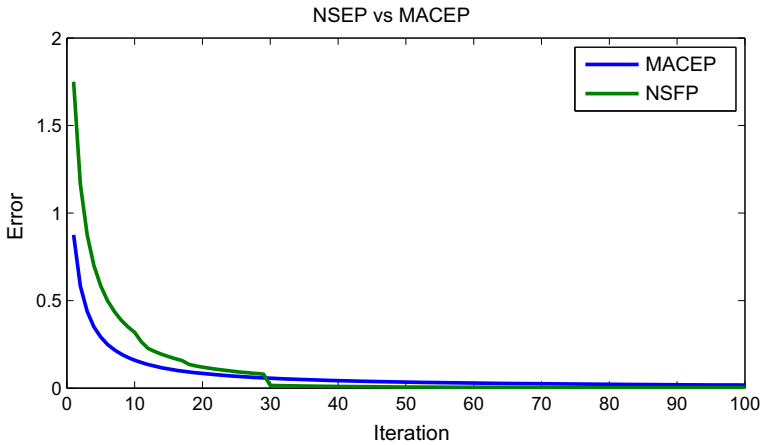|	Iter.	Cpu (s)	Err.
MACEP	150	2.3349	0.0116
NSFP	150	4.2481	0.0012

Fig. 3 Behavior of the MACEP and NSFP algorithms in error and iteration

[37] to solve the model. We try to use the same stopping criteria $\max\{\|x_k - y_k\|, p(x_k)\} \leq 10^{-4}$, and see that the algorithm MACEP takes a lot of time even in the small dimension $n = 10, k = 5$.

Table 2 reports the computed results in average on the first 150-iterations for these two algorithms.

From the table, one can see that, for this example, the computational CPU time of MACEP algorithm is less than that of NSEP algorithm in the first 150-iterations, but the error obtained by NSFP goes to zero more quickly than MACEP.

The plot about error of two algorithms can be seen in Fig 3:

Figure 3 shows that the errors of MACEP at the first 30 iterations decrease more quickly, but from iteration 30, the NSFP is convergent to zero more quickly.

Example 2 ([37])

We test our algorithm on Example 4.3 in [37], where the bifunction is given by

$$f(x, y) = \langle Ax + \chi^5(y + x) + \mu - \alpha, y - x \rangle,$$

with

$$A = \begin{pmatrix} 0 & \chi & \chi & \cdots & \chi \\ \chi & 0 & \chi & \cdots & \chi \\ \chi & \chi & 0 & \cdots & \chi \\ . & . & . & \cdots & . \\ \chi & \chi & . & \cdots & 0 \end{pmatrix}_{10 \times 10}, \chi = 3, \alpha = (2, 2, \ldots, 2)^T, \mu = (3, 4, 5, 6, 7)^T.$$

Table 3 Example 2

	Iter.	Cpu (s)	Err.	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k
MACEP	46	1.4078	8.8654×10^{-5}	0.9466	0.9449	0.9425	0.9404	0.4510
NSFP	21	0.1551	4.8698×10^{-6}	0.9467	0.9447	0.9426	0.9405	0.4509

As in Example 4.3 ([37]),

$$C = D = \begin{cases} x \in \mathbb{R}^5, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 \leq 10, \\ 2x_1 + x_2 - x_3 + x_4 + 3x_5 \leq 15, \\ x_1 + x_2 + x_3 + x_4 + 0.5x_5 \geq 4. \end{cases}$$

By setting

$$Q = (-\infty; 10] \times (-\infty; 15] \times (4, +\infty]$$

and $F(x) = Mx$ where

$$M = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 & 0.5 \end{pmatrix},$$

we can easily see that $D = \{x \mid F(x) \in Q\}$.

We choose $x^0 = (1; 2; 1; 1; 1)$ and $\lambda_k = k/(k + 1)$, $\rho_k = 1$, $\beta_k = 40/k$, $x^0 = (1; 2; 1; 1; 1)$. The computed results are reported in table 3:

Example 3 We consider the (NSEP) problem where

$$K = \underbrace{[0, 6] \times [0, 6] \times \cdots \times [0, 6]}_{10},$$

$$F(x) = (F_1(x_1), \ldots, F_5(x_5))$$

with $F_i(x_i) = \log(a_i x_i + b_i)$, $a = (1, 2, 3, 4, 5)$, $b_i = 14$ for all $i = 1, \ldots, 5$, and

$$Q = \underbrace{[0, 3] \times [0, 3] \times \cdots \times [0, 3]}_{5}.$$

The bifuction f is defined by

$$f(x, y) = \langle Px + q, y - x \rangle,$$

where $q = (-28, -29, -27, -29, -29, -29, -28, -29, -28, -29)$,

$$P = \begin{pmatrix} 23 & 10 & 3 & 7 & 6 & 4 & 7 & 10 & 2 & 2 \\ 1 & 23 & 8 & 2 & 9 & 10 & 1 & 9 & 9 & 5 \\ 4 & 3 & 14 & 1 & 10 & 9 & 5 & 4 & 5 & 3 \\ 1 & 6 & 7 & 21 & 6 & 9 & 7 & 5 & 3 & 8 \\ 2 & 10 & 4 & 4 & 20 & 4 & 10 & 3 & 9 & 3 \\ 8 & 8 & 2 & 10 & 6 & 15 & 9 & 8 & 8 & 9 \\ 8 & 9 & 5 & 10 & 1 & 9 & 14 & 9 & 9 & 9 \\ 9 & 5 & 4 & 3 & 5 & 10 & 8 & 24 & 3 & 4 \\ 6 & 5 & 8 & 9 & 7 & 7 & 5 & 6 & 20 & 5 \\ 1 & 6 & 8 & 9 & 6 & 3 & 10 & 6 & 7 & 20 \end{pmatrix}.$$

Table 4 Example 3

	Iter.	Cpu (s)	Err.
MACEP	4415	70.4365	10^{-4}
NSFP	737	15.4585	10^{-4}

Table 5 Example 3 (cont.)

	Approximate solution
MACEP	(0.3402, 0.2121, 0.9073, 0.4221, 0.6421, 0.3109, 0.2946, 0.3426, 0.1756, 0.1518)
NSFP	(0.3466, 0.2203, 0.8904, 0.4399, 0.6305, 0.2709, 0.2648, 0.3585, 0.2040, 0.1703)

Note that this bifunction f satisfies all assumptions for ensuring convergence of both algorithms. We take $\rho_k = 3$, $\beta_k = 7/2(k+1)$, $x^0 = (3; 3; 3; 3; 3; 0; 0; 0; 0; 0)$ and test MACEP and our algorithm. Computed results are reported in Tables 4 and 5.

5 Conclusion

We have proposed an algorithm for solving the split feasibility problem of finding a solution x of a paramonotone equilibrium problem such that $F(x)$ belongs to a convex set in another space, where F can be a quasilinear mapping. The proposed algorithm is a combination between the projection one for both the equilibrium and inclusion problems. Applications to jointly constrained Nash equilibrium problem have been discussed and some preliminary computational results for coupled constraint Nash–Cournot models in electricity production have been reported.

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