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Coradiant sets and ε-efficiency in multiobjective optimization

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Abstract This paper studies ε-efficiency in multiobjective optimization by using the so-called coradiant sets. Motivated by the nonlinear separation property for cones, a similar separation property for coradiant sets is investigated. A new notion, called Bishop–Phelps coradiant set is introduced and some appropriate properties of this set are studied. This paper also introduces the notions of ε-dual and augmented ε-dual for Bishop and Phelps coradiant sets. Using these notions, some scalarization and characterization properties for ε-efficient and proper ε-efficient points are proposed.

Keywords Multiobjective optimization · Efficiency · ε-Efficiency · Bishop and Phelps coradiant set · Scalarization

1 Introduction

Multiobjective optimization has many applications in decision making problems such as those in economic theory, management science, medical science and engineering design. From the

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large amount of relevant publications in multiobjective optimization, we refer reader to books [3,4,19,23].

A large class of methods for solving multiobjective optimization problems is based on scalarization. In scalarization, the multiobjective optimization problem is replaced by some scalar optimization problems including some possible parameters or additional constraints. To solve a scalar optimization problem an iterative algorithm is usually used. Numerical algorithms often cannot attain an exact optimal solution and obtain an approximate solution of scalar optimization problems. On the other hand, the decision maker can be satisfied with a suitable approximation solution. Hence, the study of approximate solution is of interest. The relation between approximate solutions of scalar optimization problem and approximate efficient element of multiobjective optimization problem has been investigated in [5,27]. The first concept of approximate solutions in multiobjective optimization problems was introduced by Kutateladze in [15]. By this concept, necessary and sufficient conditions are obtained for existence of approximate minimal element. The application of this concept to obtain vector variation principle, approximate duality theorems, approximate Kuhn-Tucker type conditions can be seen, for instance [2,6,10,11,16–18,20,24]. After introducing the concept by Kutateladze, some other different definitions of approximate solutions were introduced, for instance [5,9,25–27]. Gutierrez et al. in [7,8], using the concept of coradiant sets, introduced a new concept of $\varepsilon-$efficiency in multiobjective problems. They also showed that many available definitions of approximate solutions can be stated as special cases of their new definitions. In fact, Gutierrez et al. unified almost all available definitions in a new definition.

Kasimbeyli (2010) introduced a nonlinear separation theorem for cones and applied this theorem to investigate a large class of nonconvex vector optimization problems [13]. By using a special class of monotone sublinear functionals in partially ordered linear spaces, he introduced a conic scalarization method [12,13]. It should be noted that the conic scalarization method is applied to characterize efficient solutions of vector optimization problems without the convexity and boundedness assumptions. Motivated by the separation property for cones, a similar separation property for coradiant sets is introduced. This paper also introduces the notions of ε-dual and augmented ε-dual for Bishop and Phelps coradiant sets. Using these notions, some scalarizations and characterizations for ε-efficient and properly ε-efficient points are proposed.

The paper is organized as follows: In Sect. 2, some preliminaries and definitions that are used throughout the paper are given. Bishop–Phelps coradiant sets are given in Sect. 3, where some useful properties of these sets are also introduced. This section investigates ε-dual and augmented ε-dual for Bishop–Phelps coradiant sets. In Sect. 4, scalarization and characterizations for ε-efficient and properly ε-efficient points are proposed. Section 5 provides concluding remarks.

2 Preliminaries

In this paper the following multiobjective optimization problem (MOP) is considered,

$$\min \{f(x) : x \in S\}, \tag{1}$$

where $f: S \to \mathbb{R}^p$ and $S \subseteq \mathbb{R}^n$ is a nonempty set. A nonempty subset $K \subseteq \mathbb{R}^p$ is said to be a cone if $\alpha K \subseteq K$ for all $\alpha > 0$. By $\text{int}(K)$, $cl(K)$, $bd(K)$, $co(K)$ and K^c we denote the interior, the closure, the boundary, convex hull and the complement of the set K. The cone K is said to be convex if $K + K \subseteq K$ and it is called pointed if $K \cap (-K) \subseteq \{0\}$. The optimal elements of an MOP are defined using a partial order which is usually defined by a nontrivial pointed convex cone $K \subset \mathbb{R}^p$. The point $\bar{y} \in A := f(S)$ is said to be an efficient element of A with respect to (w.r.t) the cone K if there does not exist $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + K$; that is

$$(\{\bar{y}\} - K) \cap A \subseteq \{\bar{y}\}. \tag{2}$$

Numerical algorithms often cannot attain an exact optimal solution of optimization problems. On the other hand, the decision maker usually can be satisfied with an appropriate approximate solution. Hence, the study of approximate solution is of great interest. Motivated by this fact, in the sequel, we study the approximate solution of problem (1).

Definition 1 A set $C \subseteq \mathbb{R}^p$ is called a coradiant set if $\alpha C \subseteq C$ for all $\alpha > 1$. In addition, the coradiant C is called a pointed coradiant set if $C \cap (-C) = \emptyset$.

Let C be a convex and pointed coradiant set with nonempty interior. For any $\varepsilon > 0$, the sets $C(\varepsilon)$ and $C(0)$ are defined as follows:

$$C(\varepsilon) := \varepsilon C = \{\varepsilon c \,|c \in C\},$$
$$C(0) := \underset{\varepsilon>0}{\cup} C(\varepsilon).$$

The following lemma (given in [7]) states some properties of convex coradiant sets.

Lemma 1 *Let C be a convex and pointed coradiant set with nonempty interior. Then*

a. $C(\varepsilon)$ *is a solid convex coradiant set for all* $\varepsilon > 0$,
b. $C(\varepsilon_2) \subseteq C(\varepsilon_1)$, *for all* $0 < \varepsilon_1 < \varepsilon_2$,
c. $C + C(\alpha) \subseteq C(\alpha)$, *for all* $\alpha > 0$,
e. $C(\varepsilon) + C(\alpha) \subseteq C(\varepsilon)$, *for all* $\varepsilon, \alpha > 0$,
f. $C(\varepsilon) + C(0) \subseteq C(\varepsilon)$, *for all* $\varepsilon > 0$,
g. $C(0)$ *is a solid pointed convex cone.*

Definition 2 [7] Let $\varepsilon > 0$ be a positive number. An element $\bar{y} \in A$ is called an $\varepsilon-$efficient element of A w.r.t coradiant set C if

$$(\bar{y} - C(\varepsilon)) \cap A \setminus \{\bar{y}\} = \emptyset. \tag{3}$$

It is obvious that each cone is a coradiant set. Hence, if in Definition 2 it is assumed that C is a cone then, $C(\varepsilon) = \varepsilon C = C$ and in this special case, ε-efficient elements are efficient elements. The next example illustrates the concept of ε-efficiency.

Example 1 Let $\varepsilon > 0$ and A in Fig. 1 shows the image of the feasible set in the criterion space. Consider the coradiant C defined as follows:

$$C := \{(y_1, y_2) \mid y_1 \geq y_2 \geq 0, y_1 \geq 1\}. \tag{4}$$

By Definition 2, $\bar{y} \in A$ is an $\varepsilon-$efficient element of A w.r.t coradiant set C if

$$(\bar{y} - C(\varepsilon)) \cap A \setminus \{\bar{y}\} = \emptyset. \tag{5}$$

Figure 1 shows that the elements of shaded area satisfy the condition (5). Thus, this area is the $\varepsilon-$efficient set of A w.r.t coradiant set C.

It can be concluded from (4) that $d(0, C) = 1$ and $d(0, C(\varepsilon)) = \varepsilon$.

Throughout this paper it is assumed that C is a closed coradiant set and $0 \notin C$, that is $d(0, C) = \inf\{\|c\| : c \in C\} > 0$.

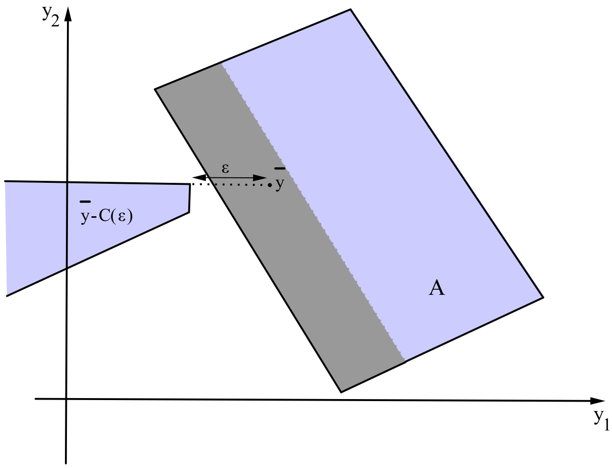

Fig. 1 The *shaded area* shows the set of $\varepsilon-$efficient elements in A

Definition 3 [21] A nonempty subset D of a cone K where $0 \notin D$ is called a base of K if any $y \in K$ has a unique representation as follows,

$$y = \beta d \text{ for some } \beta > 0 \text{ and } d \in D.$$

Definition 4 Let $C \subseteq \mathbb{R}^p$ be a closed coradiant set. Then $B \subseteq C$ is called a base of C if

$$C = corad(B) := \{ty \mid y \in B\ ,\ t \geq 1\}\ .$$

In this case $corad(B)$ is called the coradiant hull of B.

3 Augmented dual coradiant sets

Bishop and Phelps in [1] studied the vector optimization problem by introducing a class of ordering cones with enhanced mathematical structures. They proved some useful properties for these cones. In this section, we investigate $\varepsilon-$efficiency in problem (1) by considering a new class of coradiant sets, similar to BP (Bishop and Phelps) cones.

Remark 1 Let $\ell \in \mathbb{R}^p \backslash \{0\}$ and β be an arbitrary positive number. Then $C(\ell, \beta) := \{y \in \mathbb{R}^p \,|\langle \ell, y\rangle - \|y\| \geq \beta\}$ is a coradiant set.

Proof Let $\alpha > 1$ and $y \in C(\ell, \beta)$.

$$\langle \ell, \alpha y\rangle - \|\alpha y\| = \alpha(\langle \ell, y\rangle - \|y\|) \geq \alpha\beta \geq \beta.$$

Hence $\alpha y \in C(\ell, \beta)$ and $C(\ell, \beta)$ is a coradiant set. □

Definition 5 Let $\beta > 0$ and $\ell \in \mathbb{R}^p \backslash \{0\}$. The set

$$C(\ell, \beta) := \left\{y \in \mathbb{R}^p \,|\langle \ell, y\rangle - \|y\| \geq \beta\right\}$$

is called a Bishop and Phelps (BP) coradiant set.

If $\beta = 0$, then $C(\ell, 0)$ is a BP cone. Thus any BP coradiant set such $C(\ell, \beta)$ is a subset of the BP cone $C(\ell, 0)$.

Lemma 2 *Let $\beta > 0$. The set $C(\ell, \beta)$ is a pointed closed convex coradiant set.*

Proof Let $y_1, y_2 \in C(\ell, \beta)$ and $\alpha \in (0, 1)$. Then $\langle \ell, y_1\rangle - \|y_1\| \geq \beta$ and $\langle \ell, y_2\rangle - \|y_2\| \geq \beta$.

$$\begin{aligned}
&\langle \ell, \alpha y_1 + (1-\alpha)y_2\rangle - \|\alpha y_1 + (1-\alpha)y_2\| \\
&\quad = \alpha\langle \ell, y_1\rangle + (1-\alpha)\langle \ell, y_2\rangle - \|\alpha y_1 + (1-\alpha)y_2\| \\
&\quad \geq \alpha\langle \ell, y_1\rangle + (1-\alpha)\langle \ell, y_2\rangle - \alpha\,\|y_1\| - (1-\alpha)\,\|y_2\| \\
&\quad = \alpha(\langle \ell, y_1\rangle - \|y_1\|) + (1-\alpha)(\langle \ell, y_2\rangle - \|y_2\|) \\
&\quad \geq \alpha\beta + (1-\alpha)\beta = \beta.
\end{aligned}$$

Thus $\alpha y_1 + (1-\alpha)y_2 \in C(\ell, \beta)$ and $C(\ell, \beta)$ is a convex set. For showing pointedness of $C(\ell, \beta)$, by contradiction assume that $y \in C(\ell, \beta) \cap (-C(\ell, \beta))$.

Since $y, -y \in C(\ell, \beta)$,

$$y \in C(\ell, \beta) \Rightarrow \langle \ell, y\rangle - \|y\| \geq \beta, \tag{6}$$

$$-y \in C(\ell, \beta) \Rightarrow -\langle \ell, y\rangle - \|y\| \geq \beta. \tag{7}$$

Adding (6) and (7) leads to $\beta \leq -\|y\| \leq 0$ which is a contradiction. Consequently $C(\ell, \beta)$ is a pointed set. Closeness of $C(\ell, \beta)$ can be proved obviously. □

Let $\varepsilon > 0$ and C be a coradiant set such that $0 \notin C$. Then $\varepsilon-$dual coradiant and its quasi-interior denoted by $C^*(\varepsilon)$ and $C^\#(\varepsilon)$ respectively, are defined as follows:

$$\begin{aligned}
C^*(\varepsilon) &:= \left\{\ell \in \mathbb{R}^p \mid \langle \ell, y\rangle \geq 0 \text{ for all } y \in C(\varepsilon)\right\}, \\
C^\#(\varepsilon) &:= \left\{\ell \in \mathbb{R}^p \mid \langle \ell, y\rangle > 0 \text{ for all } y \in C(\varepsilon)\right\}.
\end{aligned}$$

Assume that $\lambda > 0$ and $C^\#(\varepsilon) \neq \emptyset$. The following coradiant sets $C_\lambda^{a*}(\varepsilon)$ and $C_\lambda^{a\#}(\varepsilon)$ are called augmented $\varepsilon-$dual coradiant set and quasi-interior of $\varepsilon-$dual coradiant set, respectively:

$$\begin{aligned}
C_\lambda^{a*}(\varepsilon) &:= \left\{(\ell, \alpha) \in C^\#(\varepsilon) \times \mathbb{R}_+ \mid \langle \ell, y\rangle - \alpha\,\|y\| \geq \lambda \text{ for all } y \in C(\varepsilon)\right\}, \\
C_\lambda^{a\#}(\varepsilon) &:= \left\{(\ell, \alpha) \in C^\#(\varepsilon) \times \mathbb{R}_+ \mid \langle \ell, y\rangle - \alpha\,\|y\| > \lambda \text{ for all } y \in C(\varepsilon)\right\}.
\end{aligned}$$

The relation between BP coradiant sets and augmented $\varepsilon-$dual coradiant sets is given in the following lemma.

Lemma 3 *Let $\varepsilon > 0$, $\ell \in \mathbb{R}^p\backslash\{0\}$, $\alpha \in [0, 1]$ and let $C := C(\ell, \beta)$ be a BP coradiant set. Then $(\ell, \alpha) \in C_{\varepsilon\beta}^{a*}(\varepsilon)$.*

Proof Since C is a BP coradiant set,

$$\langle \ell, y\rangle - \|y\| \geq \beta \qquad \text{for all } y \in C.$$

Let $d \in C(\varepsilon)$, then $\frac{d}{\varepsilon} \in C$. Thus $\langle \ell, \frac{d}{\varepsilon}\rangle - \left\|\frac{d}{\varepsilon}\right\| \geq \beta$ or equivalently,

$$\langle \ell, d\rangle - \|d\| \geq \varepsilon\beta \qquad \text{for all } d \in C(\varepsilon).$$

Since $\alpha \in [0, 1]$,

$$\langle \ell, d\rangle - \alpha\,\|d\| \geq \langle \ell, d\rangle - \|d\| \geq \varepsilon\beta \qquad \text{for all } d \in C(\varepsilon).$$

This completes the proof. □

Lemma 4 *Assume that $C \subset \mathbb{R}^p$ is a nonempty coradiant set, $\lambda > 0$ and $(\ell, \alpha) \in C_\lambda^{a*}(\varepsilon)$. Then $S_\lambda(\ell, \alpha) := \{y \in \mathbb{R}^p \,|\, \langle \ell, y\rangle + \alpha \|y\| \le -\lambda\}$ is a pointed closed convex coradiant set containing $-C(\varepsilon)$.*

Proof Similar to the proof of Lemma 2 it can be proved that $S_\lambda(\ell, \alpha)$ is a pointed closed convex coradiant set.

Let $y \in -C(\varepsilon)$, then $-y \in C(\varepsilon)$. Since $(\ell, \alpha) \in C_\lambda^{a*}(\varepsilon)$, $\langle \ell, -y\rangle - \alpha \|-y\| \ge \lambda$ or equivalently $\langle \ell, y\rangle + \alpha \|y\| \le -\lambda$. Therefore, $y \in S_\lambda(\ell, \alpha)$, thus $-C(\varepsilon) \subseteq S_\lambda(\ell, \alpha)$. □

It is seen that a coradiant set has many bases, trivially C is a base of C. Naturally, we are interested in determining the smallest base for a given coradiant set. To this aim, the minimal base is defined in the sequel.

Definition 6 Let C be a closed coradiant set. Base $B \subseteq C$ is called a minimal base of C if there is no exists base $\widehat{B} \subseteq C$ such that $\widehat{B} \underset{\neq}{\subset} B$.

Proposition 1 *If C is a closed coradiant set with a bounded base $B \subset C$, then there exists number $M > 0$ such that $C_t := \{y \in C \mid \|y\| = t\}$ is a norm base of $cone(C)$ for all $t \ge M$.*

Proof Let $y \in cone(C)$. Then there are $c \in C$ and $\lambda > 0$ such that $y = \lambda c$. Since $c \in C$ and B is a base of coradiant set C, there are $\bar{\lambda} > 0$ and $b \in B$ such that $c = \bar{\lambda} b$. Therefore $y = \lambda\bar{\lambda} b$.

Since B is bounded, there is $M > 0$ such that $\|b\| \le M$ for all $b \in B$. Assume that $t \in \mathbb{R}$ and $t \ge M$, then $\frac{t}{\|b\|} \ge 1$. On the other hand C is a coradiant set and $b \in B \subset C$, then $\frac{t}{\|b\|} b \in C$. Since $\left\|\frac{t}{\|b\|} b\right\| = t$, $\bar{b} := \frac{t}{\|b\|} b \in C_t$. Set $\bar{\bar{\lambda}} := \frac{\lambda\bar{\lambda}\|b\|}{t} > 0$, then $y = \lambda\bar{\lambda} b = \bar{\bar{\lambda}}\bar{b}$, that is C_t is a base of $cone(C)$. □

The following lemma proves that every coradiant set with bounded closed base, has a norm base. This property is trivial for the cones, but not for the coradiant sets. The existence of a norm base for coradiant sets, is an important condition which will be used to prove characterization theorems and to analyse the separation property for coradiant sets.

Lemma 5 *If C is a closed coradiant set with a bounded closed base $B \subset C$, then there exists a number $t > 0$ such that $C_t := \{y \in C : \|y\| = t\}$ is a norm base of C.*

Proof Let $y \in C$, then there exists $b \in B$ and $\lambda \ge 1$ such that $y = \lambda b$. Since B is a closed bounded set, there exists $t > 0$ with $t = max\{\|b\| : b \in B\}$. Let $C_t = \{y \in C : \|y\| = t\}$. Then, there exists $\gamma \ge 1$ with $\gamma b = c_t \in C_t$. Then we have $b = (1/\gamma)c_t$ and thus $y = (\lambda/\gamma)c_t$, which proves the lemma. □

Kasimbeyli in [13] introduced a separation property for two cones. In this paper this property is extended to coradiant sets.

Definition 7 [22] Let $C, K \subset \mathbb{R}^p$ be two closed coradiant sets with some bounded bases and let $C_{t_1} := \{y \in C \mid \|y\| = t_1\}$ and $K_{t_2} := \{y \in K \mid \|y\| = t_2\}$ be norm bases of coradiant sets C and K, respectively. Set $t := max\{t_1, t_2\}$, $K_t^\partial := K_t \cap bd(K)$ and denote by $\tilde{C}$ and $\tilde{K}^\partial$ the closure of sets $co(C_t)$ and $co(K_t^\partial \cup \{0\})$, respectively. Then coradiant sets K and C are said to satisfy the separation property if

$$\tilde{C} \cap \tilde{K}^\partial = \emptyset.$$

Theorem 1 [22]

(a) Assume that C and K are two closed coradiant sets with some bounded bases and $-C$ and K satisfy the separation property; that is $(-\tilde{C}) \cap \tilde{K}^{\partial} = \emptyset$. Then there exists $\lambda > 0$ and $(\ell, \alpha) \in C_{\lambda}^{a\#}$ such that

$$\langle \ell, y \rangle + \alpha \|y\| \leq -\lambda \quad \textit{for all } y \in -C, \tag{8}$$
$$\langle \ell, z \rangle + \alpha \|z\| \geq \lambda \quad \textit{for all } z \in bd(K). \tag{9}$$

(b) Assume that C and K are two closed convex coradiant sets with some bounded bases, $(\ell, \alpha) \in C_{\lambda}^{a\#}$ and

$$\langle \ell, y \rangle + \alpha \|y\| < 0 \leq \langle \ell, z \rangle + \alpha \|z\| \quad \textit{for all } y \in -C, \ z \in bd(K). \tag{10}$$

Then $-C$ and K satisfy the separation property.

In the sequel, it is shown that there exists a large class of coradiant sets which satisfy the separation property introduced in Definition 7. In particular, it is also shown that two disjoint BP coradiant sets satisfy the separation property.

Theorem 2 *Let C be a BP coradiant set:*

$$C = \{y \in \mathbb{R}^p : \langle l, y \rangle - \alpha \|y\| \geq \beta.\} \tag{11}$$

Then

$$cl(co(C_t)) = \{y \in \mathbb{B}_t : \langle l, y \rangle \geq \alpha t + \beta\} \tag{12}$$

where C_t is a norm base of C for some $t > 0$.

Proof Let

$$\widetilde{C} = \text{cl}(\text{co}(C_t)). \tag{13}$$

It is clear that the norm base C_t can be represented as

$$C_t = \{y \in U_t : \langle l, y \rangle - \alpha \|y\| \geq \beta\} = \{y \in U_t : \langle l, y \rangle - \alpha t \geq \beta\}, \tag{14}$$

where U_t denotes the circle with radius t, centered at the origin. By Lemma 2, the set C is a convex closed pointed coradiant set. Let

$$D := \{y \in \mathbb{B}_t : \ \langle l, y \rangle \geq \alpha t + \beta\}. \tag{15}$$

We now show that

$$\text{co}(C_t) = D. \tag{16}$$

Let $y \in \text{co}(C_t)$. Then, by definition of the convex hull, there exists a set of nonnegative numbers β_i, $i \in I$ such that, y can be represented as

$$y = \sum_{i \in I} \beta_i y_i, \ \text{ where } y_i \in C_t \text{ and } \sum_{i \in I} \beta_i = 1.$$

Clearly $y \in B_t$. On the other hand

$$\langle l, y \rangle = \sum_{i \in I} \beta_i \langle l, y_i \rangle \geq \sum_{i \in I} \beta_i (\beta + \alpha t) = \beta + \alpha t.$$

Then, from (15) we have $y \in D$; that is, $\text{co}(C_t) \subset D$.

Now, let $y \in D$. We will show that $y \in \text{co}(C_t)$.
If $\|y\| = t$ then $y \in \mathbb{U}_t$ and the inclusion $y \in C_t \subset \text{co}(C_t)$ follows from (14).

Consider the case $\|y\| < t$, that is $y \in \mathrm{int}(\mathbb{B}_t)$. Denote $\nu = \langle l, y \rangle$. Clearly $\nu \geq \alpha t + \beta$. Take any non-zero vector $b \in \mathbb{R}^n$ with $\langle l, b \rangle = 0$. Consider

$$y_\lambda = y + \lambda b, \quad \lambda \in (-\infty, \infty).$$

We have

$$\langle l, y_\lambda \rangle = \langle l, y \rangle + \lambda \langle l, b \rangle = \nu \geq \alpha t + \beta. \tag{17}$$

As $b \neq \mathbf{0}$, we have $\|y_\lambda\| \to \infty$ if $|\lambda| \to \infty$ which means that $y \notin \mathbb{B}_t$ for sufficiently large values of λ. On the other hand, since $y \in \mathrm{int}(\mathbb{B})$, the inclusion $y_\lambda \in \mathrm{int}(\mathbb{B})$ holds for sufficiently small in absolute value numbers $\lambda > 0$ and $\lambda < 0$. Then, since $\|y_\lambda\|$ is a semicontinuous function of λ, and $\mathbb{B}_t$ is compact, there exist numbers $\lambda_1 > 0$ and $\lambda_2 < 0$ such that the corresponding points $y_1 \doteq y_{\lambda_1}$ and $y_2 \doteq y_{\lambda_2}$ belong to the boundary of $\mathbb{B}_t$ (as maximum values of $\|y_\lambda\|$ w.r.t. $\lambda > 0$ and $\lambda < 0$ respectively). That is,

$$y_i \in \mathbb{U}_t, \quad i = 1, 2.$$

These inclusions together with (17) and (14) imply that $y_i \in C_t, \quad i = 1, 2.$

Finally, denoting $\lambda' = \lambda_1/(\lambda_1 - \lambda_2)$, it is not difficult to check that,

$$\lambda' \in (0, 1) \text{ and } y = (1 - \lambda')y_1 + \lambda' y_2.$$

Therefore, $y \in \mathrm{co(C_t)}$, which means that $D \subset \mathrm{co(C_t)}$.

Thus, we have shown that the relation (16) is true. From this relation, we have

$$\widetilde{C} = \{y \in \mathbb{B}_t : \langle l, y \rangle \geq \alpha t + \beta\},$$

and the proof of the theorem is completed. □

Theorem 3 *Let C be a closed convex pointed coradiant set. Assume that there exists a triple of positive numbers (α, β, t) such that the condition (12) is satisfied, where $t > 0$ defines the norm base C_t for C. Then, for the part $C_{\geq t}$ of C defined by*

$$C_{\geq t} = \{y \in C : \|y\| \geq t\}, \tag{18}$$

we have:

$$co(C_{\geq t}) \subseteq C(l, \alpha, \beta) := \{y \in \mathbb{R}^p : \langle l, y \rangle - \alpha \|y\| \geq \beta\}. \tag{19}$$

By Theorem 2, the BP coradiant set $C(l, \alpha, \beta)$ satisfies condition (12) for some $\bar{t}$. If additionally, $t = \bar{t}$, then we have also $C(l, \alpha, \beta) = co(C_{\geq t})$.

Proof Clearly $C_{\geq t}$ is a coradiant set. Let $y \in C_{\geq t}$. Then, there exists a number $\gamma \in (0, 1)$ such that $\gamma y \in C_t$. By condition (12), we have

$$\langle l, \gamma y \rangle \geq \alpha t + \beta.$$

Since $\gamma y \in C_t$, we have $\alpha \|\gamma y\| = \alpha t$, and hence

$$\langle l, \gamma y \rangle \geq \alpha \|\gamma y\| + \beta.$$

This means that $\gamma y \in C(l, \alpha, \beta)$, and since $C(l, \alpha, \beta)$ is a coradiant set, and $\gamma \in (0, 1)$, we obtain that $1/\gamma > 1$ and hence $(1/\gamma)\gamma y = y \in C(l, \alpha, \beta)$. Now, since $C(l, \alpha, \beta)$ is convex set, the inclusion (19) is established.

Finally, assume that C is a coradiant set, which satisfies condition (12) for some triple of positive numbers (α, β, t). Assume also that the BP coradiant set $C(l, \alpha, \beta)$ satisfies condition (12) for the same number t. Then it obviously follows from this condition that both of sets $C(l, \alpha, \beta)$ and $co(C_{\geq t})$ are the coradiant hulls of the same set, which proves the theorem. □

Theorem 4 *Let $C \subset \mathbb{R}^p$ be a BP coradiant set and $K \subset \mathbb{R}^p$ be a closed coradiant set. Assume that C and K have norm bases C_t and K_t respectively, for the same $t > 0$. If $C \cap K = \emptyset$ then C and K satisfy the separation property given in Definition 7.*

Proof Since C is a BP coradiant set, there are $\ell \in \mathbb{R}^p$ and positive numbers α and β such that

$$C = \left\{ y \in \mathbb{R}^p \mid \langle \ell, y \rangle - \alpha \|y\| \geq \beta \right\}.$$

Let $C_t \subseteq C$ be a norm base, then by Theorem 2

$$cl(co(C_t)) = \{y \in B_t : \langle l, y \rangle \geq \alpha t + \beta\}. \tag{20}$$

Since $C \cap K = \emptyset$, it can be concluded that $\langle \ell, y \rangle < \alpha t + \beta$ for all $y \in K_t$. We can assume that there exists positive number $\beta' < \beta$ such that $\langle \ell, y \rangle \leq \alpha t + \beta'$ for all $y \in K_t$, since K_t is compact. This inequality holds also for $y = 0$. Therefore, we conclude that

$$\langle \ell, y \rangle \leq \alpha t + \beta' \qquad \text{for all} \quad y \in (co(K_t \cup \{0\}),$$

and Consequently,

$$\langle \ell, y \rangle \leq \alpha t + \beta' \qquad \text{for all} \quad y \in cl((co(K_t^{\partial} \cup \{0\})). \tag{21}$$

Setting $\widetilde{C} := cl(co(C_t))$ and $\widetilde{K}^{\partial} := cl(co(K_t^{\partial} \cup \{0\}))$, from (20) and (21) it follows that $\widetilde{C} \cap \widetilde{K}^{\partial} = \emptyset$. □

4 Scalarization

An important class of methods for solving MOP problems is based on scalarization. In scalarization, the MOP problem is replaced by some scalar optimization problems involving possibly some parameters or additional constraints. In this section, we introduce a new scalarization approach by using the coradiant sets. This approach uses the idea of the conic scalarization method proposed in [12,13]. The new scalarization approach is used to characterize $\varepsilon-$efficient (in the sense of Definition 2) and $\varepsilon-$properly efficient elements of a MOP.

Theorem 5 *Let ε and λ be positive real numbers and let C be a coradiant set. Assume that $(\ell, \alpha) \in C_{\lambda}^{a*}(\varepsilon)$. Set*

$$\begin{aligned} T(y) &:= \langle \ell, y \rangle + \alpha \|y\|, \\ r &\in \arg\min\{\|c\| : c \in C(\varepsilon)\}. \end{aligned} \tag{22}$$

If $\bar{y} \in A$ is an optimal solution of the problem $\min\{T(y) : y \in A\}$, *then any element of the set $\left\{ \bar{y} + \theta r \,:\, 0 \leq \theta < \frac{\lambda}{T(r)} \right\} \cap A$ is an $\varepsilon-$efficient element of A w.r.t. coradiant set C.*

Proof Since $r \in C(\varepsilon)$ and $(\ell, \alpha) \in C_{\lambda}^{a*}(\varepsilon)$, $\langle \ell, r \rangle - \alpha \|r\| \geq \lambda$. Consequently, $T(r) \geq \langle \ell, r \rangle - \alpha \|r\| \geq \lambda > 0$.

First we show that for some $y_1, y_2 \in A$, $y_1 - y_2 \in C(\varepsilon)$ implies $T(y_1) - T(y_2) \geq \lambda$.

Let $y_1 - y_2 \in C(\varepsilon)$, since $(\ell, \alpha) \in C_{\lambda}^{a*}(\varepsilon)$,

$$\langle \ell, y_1 - y_2 \rangle - \alpha \|y_1 - y_2\| \geq \lambda. \tag{23}$$

On the other hand, since $| \, \|y_1\| - \|y_2\| \, | \leq \|y_1 - y_2\|$, we have:

$$\begin{aligned} T(y_1) - T(y_2) &= \langle \ell, y_1 \rangle + \alpha \|y_1\| - \langle \ell, y_2 \rangle - \alpha \|y_2\| \\ &= \langle \ell, y_1 - y_2 \rangle + \alpha \|y_1\| - \alpha \|y_2\| \\ &\geq \langle \ell, y_1 - y_2 \rangle - \alpha \|y_1 - y_2\| \geq \lambda, \end{aligned}$$

that is $y_1 - y_2 \in C(\varepsilon)$, and consequently $T(y_1) - T(y_2) \geq \lambda$.

Now let $\bar{y} + \theta r \in \left\{ \bar{y} + \theta r \,:\, 0 \leq \theta < \frac{\lambda}{T(r)} \right\} \cap A$. On the contrary, assume that $\bar{y} + \theta r$ is not an $\varepsilon-$efficient element of A w.r.t. C. This means that $((\bar{y} + \theta r) - C(\varepsilon)) \cap A \neq \emptyset$. Hence there exists $\hat{y} \in A$ such that $\hat{y} \in (\bar{y} + \theta r) - C(\varepsilon)$ or equivalently $\hat{y} - (\bar{y} + \theta r) \in -C(\varepsilon)$. That is $(\bar{y} + \theta r) - \hat{y} \in C(\varepsilon)$ and $T(\bar{y} + \theta r) - T(\hat{y}) \geq \lambda$.

On the other hand, since $\|\bar{y} + \theta r\| \leq \|\bar{y}\| + \theta \|r\|$,

$$T(\bar{y}) + \theta T(r) - T(\hat{y}) \geq T(\bar{y} + \theta r) - T(\hat{y}) \geq \lambda,$$

thus,

$$T(\bar{y}) - T(\hat{y}) \geq \lambda - \theta T(r). \tag{24}$$

By assuming $0 \leq \theta < \frac{\lambda}{T(r)}$, we obtain $\theta T(r) < \lambda$. Therefore, (24) implies $T(\bar{y}) - T(\hat{y}) > 0$. But this contradicts the optimality of $\bar{y}$ for the problem $\min \{\langle \ell, y \rangle + \alpha \|y\| \,:\, y \in A\}$ and therefore completes the proof of the theorem. □

Application of Theorem 5 for obtaining $\varepsilon-$efficient elements of set A is illustrated in details in Example 2.

Example 2 Let $A \subset \mathbb{R}^p$ and C be a given coradiant set as following

$$\begin{aligned} A &:= \{(y_1, y_2) \mid y_1 + y_2 \geq 10, \quad 0 \leq y_1, y_2 \leq 10\}, \\ C &:= \left\{ (y_1, y_2) \;\middle|\; y_1 + y_2 \geq \frac{1}{10}, \quad 0 \leq y_1 \leq y_2, \quad y_2 \leq 2y_1 \right\}. \end{aligned}$$

Choosing $\varepsilon = 1$ implies $r = \arg\min\{\|c\| : c \in C(\varepsilon)\} = (\frac{1}{20}, \frac{1}{20})$ and $\|r\| = 0.0707$.

Considering $\lambda = 0.025$ it easily can be seen that if $(\ell, \alpha) \in \{(1, 1, 1), (\frac{3}{2}, 1, 1)\}$ then $(\ell, \alpha) \in C_{\lambda}^{a*}(\varepsilon)$.

Choosing $(\ell, \alpha) = (1, 1, 1)$, the element $\bar{y} = (5, 5)$ becomes an optimal solution of the problem $\min\{T(y) \,:\, y \in A\}$ and $\lambda / T(r) = 0.1465$. The line segment L_1 is a subset of $\varepsilon-$efficient elements of the set A w.r.t C, where

$$L_1 := \left\{ \bar{y} + \theta r \,:\, 0 \leq \theta < \frac{\lambda}{T(r)} \right\} = \left\{ (5, 5) + \left(\frac{1}{20}, \frac{1}{20} \right) \theta \,:\, 0 \leq \theta < 0.1465 \right\}.$$

Choosing $(\ell, \alpha) = (\frac{3}{2}, 1, 1)$, the element $\bar{y} = (3.11, 6.89)$ is an optimal solution of problem $\min\{T(y) \,:\, y \in A\}$ and $\lambda / T(r) = 0.1277$. The line segment L_2 is a subset of $\varepsilon-$efficient elements of the set A w.r.t C, where

$$L_2 := \left\{ (3.11, 6.89) + \left(\frac{1}{20}, \frac{1}{20} \right) \theta \,:\, 0 \leq \theta < 0.1277 \right\}.$$

In Fig. 2 the sets L_1 and L_2 are shown.

Definition 8 An element $\bar{y} \in A$ is said to be an $\varepsilon-$properly efficient element of set A w.r.t coradiant set C if there exists a coradiant set K such that $C \subset \mathrm{int}(K)$ and $(\bar{y} - K(\varepsilon)) \cap A = \emptyset$.

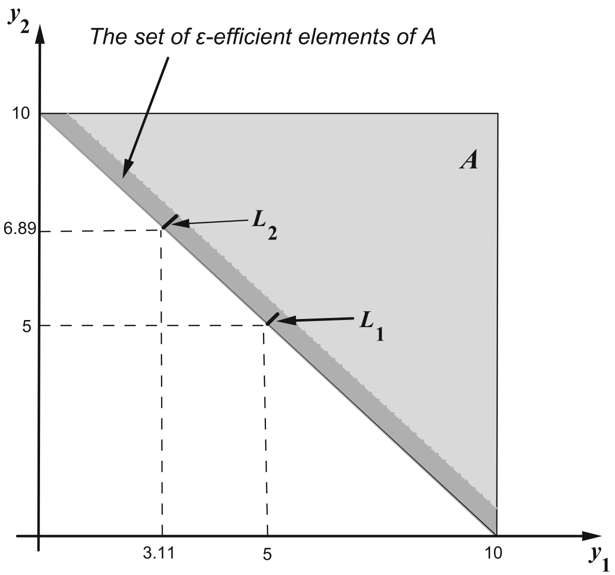

Fig. 2 L_1 and L_2 are subsets of $\varepsilon-$efficient elements in A

Definition 9 Let C be coradiant set with a bounded minimal base $\widehat{B}$, $\theta \in (0, 1)$ and D_θ is the coradiant hull of $\widehat{B} + \theta\mathbb{B}$. Then D_θ is called a θ-coradiant neighborhood of C.

The following theorem provides a necessary condition for ε-properly efficient elements.

Theorem 6 *Let $\varepsilon, \lambda > 0$ and $C \subseteq \mathbb{R}^p$ be a closed convex coradiant set having a compact base. Assume that C and the θ-coradiant neighborhood of C satisfy the separation property for every $\theta \in (0, 1)$. If $\bar{y}$ is an $\varepsilon-$properly efficient element of set $A \subset \mathbb{R}^p$ w.r.t C then there exists $(\ell, \alpha) \in C_\lambda^{a\#}(\varepsilon)$ such that*

$$\min_{y \in A} \{\langle \ell, y - \bar{y} \rangle + \alpha \|y - \bar{y}\|\} > -\lambda.$$

Proof Since $\bar{y}$ is an $\varepsilon-$properly efficient element of A w.r.t coradiant set C, there exists a coradiant set K such that $C \subset \text{int}(K)$ and $(\bar{y} - K(\varepsilon)) \cap A = \emptyset$ or equivalently,

$$-K(\varepsilon) \cap (A - \{\bar{y}\}) = \emptyset. \tag{25}$$

Let $\widehat{B}$ be a compact base of C. Since $C \subset \text{int}(K)$ and C is a closed set, there is some $\theta \in (0, 1)$ such that $D_\theta := corad(\widehat{B} + \theta\mathbb{B}) \subset K$.

Since D_θ and C satisfy the separation property (by the hypothesis), by Theorem 1 there are $\lambda > 0$ and $(\ell, \alpha) \in C_\lambda^{a\#}(\varepsilon)$ such that:

$$\langle \ell, y \rangle + \alpha \|y\| \le -\lambda < 0 \le \langle \ell, z \rangle + \alpha \|z\| \text{ for all } y \in -C(\varepsilon) \text{ and } z \in bd(-D_\theta(\varepsilon)). \tag{26}$$

Using the left part of inequality (26), it is concluded that $-C(\varepsilon) \subseteq S_\lambda(\ell, \alpha) := \{y \in \mathbb{R}^p \,|\langle \ell, y \rangle + \alpha \|y\| \le -\lambda\}$.

Now we show that $S_\lambda(\ell, \alpha) \subseteq -D_\theta(\varepsilon)$. Assume to the contrary $\hat{y} \in S_\lambda(\ell, \alpha)$ but $\hat{y} \notin -D_\theta(\varepsilon)$. Let $T(y) = \langle \ell, y \rangle + \alpha \|y\|$. Then

$$T(\hat{y}) = \langle \ell, \hat{y} \rangle + \alpha \|\hat{y}\| \le -\lambda. \tag{27}$$

Let $y_1 \in -C(\varepsilon)$. Then applying the left part of inequality (26) we obtain $T(y_1) \leq -\lambda$. Since $C(\varepsilon) \subset D_\theta(\varepsilon),\ y_1 \in -D_\theta(\varepsilon)$.

Since $y_1 \in -D_\theta(\varepsilon)$ and $\hat{y} \notin -D_\theta(\varepsilon)$ there exists $t \in (0,1)$ such that $\tilde{y} = t\hat{y} + (1-t)y_1$ belongs to the boundary of $-D_\theta(\varepsilon)$. Hence by the right part of inequality (26),

$$T(\tilde{y}) = \langle \ell, \tilde{y} \rangle + \alpha \|\tilde{y}\| \geq 0. \tag{28}$$

On the other hand, by (28) we have

$$\begin{aligned} T(\tilde{y}) = T(t\hat{y} + (1-t)y_1) &= \langle \ell, t\hat{y} + (1-t)y_1 \rangle + \alpha \left\| t\hat{y} + (1-t)y_1 \right\| \\ &= t\langle \ell, \hat{y} \rangle + (1-t)\langle \ell, y_1 \rangle + \alpha \left\| t\hat{y} + (1-t)y_1 \right\| \\ &\leq t\langle \ell, \hat{y} \rangle + \alpha t \left\| \hat{y} \right\| + (1-t)\langle \ell, y_1 \rangle + \alpha(1-t) \|y_1\| \\ &= tT(\hat{y}) + (1-t)T(y_1) \leq -t\lambda - (1-t)\lambda = -\lambda < 0. \end{aligned}$$

Thus $T(\tilde{y}) < 0$. This contradicts (28). Therefore $\hat{y} \in -D_\theta(\varepsilon)$. Consequently $S_\lambda(\ell, \alpha) \subset -D_\theta(\varepsilon)$.

By relation (25) and inclusion $\{y \in R^p \mid \langle \ell, y \rangle + \alpha \|y\| \leq -\lambda\} \subset -D_\theta(\varepsilon) \subset -K(\varepsilon)$, $\langle \ell, y \rangle + \alpha \|y\| > -\lambda$ for all $y \in A - \{\bar{y}\}$. □

Lemma 6 *Let C and K be two closed and convex coradiant sets where B_c and B_k are compact and convex minimal bases of C and K, respectively. If $C \cap K = \emptyset$, then there exists a closed and convex coradiant set D such that $C \subseteq int(D)$ and $D \cap K = \emptyset$.*

Proof Set $\bar{C} := cone(C)$ and $\bar{K} := cone(K)$. Since $C \cap K = \emptyset$ it can be easily shown that $(\bar{C} \cap \bar{K}) \backslash \{0\} = \emptyset$. On the other hand, since $B_c \subseteq \bar{C}$ we have $B_c \cap \bar{K} = \emptyset$. Since B_c is compact and $\bar{K}$ is closed, there exists $\varepsilon > 0$ such that

$$(B_c + \varepsilon \mathbb{B}) \cap \bar{K} = \emptyset, \tag{29}$$

where $\mathbb{B} := \{y \in R^p \mid \|y\| \leq 1\}$. Set $\bar{B} := (B_c + \varepsilon \mathbb{B})$ and $D := corad(\bar{B})$. It is obvious that $C \subseteq int(D)$. Show that $D \cap K = \emptyset$. Assume to the contrary that $y \in D \cap K$. Since $y \in D$, there are $\bar{\alpha} \geq 1$ and $\bar{y} \in \bar{B}$ such that $\bar{y} = \bar{\alpha} y$. On the other hand, by inclusion $y \in K$ we have $\bar{y} = \bar{\alpha} y \in \bar{K} = cone(K)$. This leads $\bar{y} \in \bar{B} \cap \bar{K}$ which contradicts the relation (29). □

The following theorem gives a sufficient condition for ε-properly efficient elements.

Theorem 7 *Let $\varepsilon, \lambda > 0$, $A \subset \mathbb{R}^p$, $\bar{y} \in A$ and C be a closed convex coradiant set with compact minimal base. If*

$$\langle \ell, y - \bar{y} \rangle + \alpha \|y - \bar{y}\| \geq -\lambda \qquad \textit{for all} \quad y \in A, \tag{30}$$

for some $(\ell, \alpha) \in C_\lambda^{a\#}(\varepsilon)$, then $\bar{y}$ is an ε-properly efficient element of A w.r.t C.

Proof Set $g(y) := \langle \ell, y \rangle + \alpha \|y\|$. Let $z \in corad(A + C(\varepsilon) - \bar{y})$, then there are $t \geq 1$, $y_1 \in A$ and $y_2 \in C(\varepsilon)$ such that $z = t(y_1 + y_2 - \bar{y})$ and

$$\begin{aligned} g(z) = \langle \ell, z \rangle + \alpha \|z\| &= t\left[\langle \ell, y_1 + y_2 - \bar{y} \rangle + \alpha \|y_1 + y_2 - \bar{y}\|\right] \\ &\geq t\left[\langle \ell, y_1 - \bar{y} \rangle + \alpha \|y_1 - \bar{y}\|\right] + t\left[\langle \ell, y_2 \rangle - \alpha \|y_2\|\right]. \end{aligned} \tag{31}$$

Since $y_2 \in C(\varepsilon)$ and $(\ell, \alpha) \in C_\lambda^{a\#}(\varepsilon)$, then $\langle \ell, y_2 \rangle - \alpha \|y_2\| \geq \lambda$ thus it can be concluded from (31) that

$$g(z) \geq t\left[\langle \ell, y_1 - \bar{y} \rangle + \alpha \|y_1 - \bar{y}\|\right] + t\lambda. \tag{32}$$

Since $y_1 \in A$, (30) and (32) yield

$$g(z) \geq 0 \quad \text{for all } z \in corad(A + C(\varepsilon) - \bar{y}).$$

Consequently,

$$g(z) \geq 0 \quad \text{for all } z \in cl(corad(A + C(\varepsilon) - \bar{y})).$$

Now, we show that $cl(corad(A + C(\varepsilon) - \bar{y})) \cap (-C(\varepsilon)) = \emptyset$. Assume to the contrary that $y \in cl(corad(A + C(\varepsilon) - \bar{y})) \cap (-C(\varepsilon))$. Since $y \in cl(corad(A + C(\varepsilon) - \bar{y}))$ we have $g(y) \geq 0$. On the other hand, since $y \in -C(\varepsilon)$ and $(\ell, \alpha) \in C_\lambda^{a\#}(\varepsilon)$, $\langle \ell, -y \rangle - \alpha \|-y\| \geq \lambda$ or equivalently $\langle \ell, y \rangle + \alpha \|y\| \leq -\lambda < 0$ which this means $g(y) < 0$ and hence contradicts to the relation $g(y) \geq 0$.

Since $cl(corad(A + C(\varepsilon) - \bar{y})) \cap (-C(\varepsilon)) = \emptyset$, then according to Lemma 6, there exists a closed convex coradiant set $\bar{D}$ such that

$$-C(\varepsilon) \subseteq int(\bar{D}), \tag{33}$$

$$cl(corad(A + C(\varepsilon) - \bar{y})) \cap \bar{D} = \emptyset. \tag{34}$$

From (34) it can be concluded $(A + C(\varepsilon) - \bar{y}) \cap \bar{D} = \emptyset$ or equivalently

$$(A \setminus \{\bar{y}\}) \cap (\bar{D} - C(\varepsilon)) = \emptyset, \tag{35}$$

since $-C(\varepsilon) \subseteq int(\bar{D})$,

$$(A \setminus \{\bar{y}\}) \cap \bar{D} = \emptyset. \tag{36}$$

Assuming $D := -\bar{D}$, (33) and (36) yield $C(\varepsilon) \subseteq int(D)$ and $(A \setminus \{\bar{y}\}) \cap (-D) = \emptyset$. This means $\bar{y}$ is an ε-properly efficient element in the sense of Definition 8. □

The following corollary is immediate from Theorems 6, 7 and 4.

Corollary 1 *Let $\varepsilon, \alpha, \lambda > 0$ be positive real numbers, $C(\ell, \alpha, \lambda) = \{y \in \mathbb{R}^p : \langle \ell, y \rangle - \alpha \|y\| \geq \lambda\}$ be a BP coradiant set having a compact base with $(\ell, \alpha) \in C_\lambda^{a\#}(\varepsilon)$ and $A \subset \mathbb{R}^p$. Then $\bar{y} \in A$ is an ε-properly efficient element of A w.r.t C if and only if*

$$\min_{y \in A} \{\langle \ell, y - \bar{y} \rangle + \alpha \|y - \bar{y}\|\} > -\lambda,$$

equivalently $\bar{y}$ is λ-minimizer of functional

$$\langle \lambda, \cdot - \bar{y} \rangle + \alpha \|\cdot - \bar{y}\|.$$

Proof The proof is obvious. □

5 Conclusion

Numerical algorithms often cannot obtain an exact optimal solution and obtain an approximate solution for scalar optimization problems. On the other hand, the decision maker can be satisfied by a suitable approximation solution. Hence, the study of approximate solution is of great interest.

The concept of coradiant set is a powerful tool for analyzing approximate efficiency. In this paper, we used coradiant sets to characterize $\varepsilon-$ efficient elements. We have also introduced a Bishop and Phelps coradiant set and studied some properties of these sets. The ε-dual, augmented ε-dual, quasi-interior of ε-dual coradiant set and the separation property for Bishop and Phelps coradiant sets have been derived. Using these properties, we have proposed some scalarization and characterization properties for ε-efficient and properly ε-efficient points.

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