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Functional inequalities, regularity and computation of the deficit and surplus variables in the financial equilibrium problem

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Abstract This paper is concerned with a general model of financial flows and prices related to individual entities, called sectors, which invest in financial instruments as assets and as liabilities. In particular, using delicate tools of Functional Analysis, besides existence results of financial equilibrium, in the dual formulation, the Lagrange functions $\rho_j^{*1}(t)$ and $\rho_j^{*2}(t)$, called "deficit" and "surplus" variables, appear and reveal to be very relevant in order to analyze the financial model and the possible insolvencies, which can lead to a financial contagion. In the paper the continuity of these Lagrange functions is proved. Finally, a procedure for the calculus of these variables is suggested.

Keywords Financial problem · Variational inequality formulation · Equilibrium conditions · Dual Lagrange formulation · Deficit and surplus variables · Financial contagion

Mathematics Subject Classification 90B50 · 49J40 · 91G80

1 Introduction

The first authors to develop a multi-sector, multi-instrument financial equilibrium model, using the variational inequality theory, were Nagurney et al. in [32]. These results were, subsequently, extended by Nagurney in [33,34] to include more general utility functions and

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by Nagurney and Siokos in [35,36] to the international domain (see also [27,37] for related papers). In [20], the authors apply for the first time the methodology of projected dynamical systems to develop a multi-sector, multi-instrument financial model, whose set of stationary points coincides with the set of solutions to the variational inequality model developed in [33], and then to study it qualitatively, providing stability analysis results.

Very recently, a general equilibrium model of financial flows and prices, evolving in time, is studied in [3,5].

The problem of financial equilibrium is a global optimization problem in the feasible set of a utility function, given by risk-aversion and profit. More precisely, we consider a financial economy consisting of m sectors, with a typical sector denoted by i, and of n instruments, with a typical financial instrument denoted by j, in the time interval $[0, T]$. If $s_i(t)$ denotes the total financial volume held by sector i at time t as assets, and $l_i(t)$ denotes the total financial volume held by sector i at time t as liabilities, then, the set of feasible assets and liabilities for each sector $i = 1, \ldots, m$, is

$$P_i = \Bigg\{(x_i(t), y_i(t)) \in L^2([0,T], \mathbb{R}^{2n}) : \sum_{j=1}^n x_{ij}(t) = s_i(t), \quad \sum_{j=1}^n y_{ij}(t) = l_i(t) \text{ a.e. in } [0,T],$$
$$x_i(t) \geq 0, \; y_i(t) \geq 0, \text{ a.e. in } [0,T]\Bigg\} \quad \forall i = 1, \ldots, m.$$

In such a way the set of all feasible assets and liabilities becomes

$$P = \Bigg\{(x(t), y(t)) \in L^2([0,T], \mathbb{R}^{2mn}) : \quad \sum_{j=1}^n x_{ij}(t) = s_i(t), \quad \sum_{j=1}^n y_{ij}(t) = l_i(t),$$
$$\forall\, i=1, \ldots, m, \text{ a.e. in } [0,T], \quad x_i(t) \geq 0, \; y_i(t) \geq 0, \quad \forall\, i = 1, \ldots, m, \text{ a.e. in } [0,T]\Bigg\},$$

whereas the set of feasible instrument prices is:

$$\mathcal{R} = \{r \in L^2([0,T], \mathbb{R}^n) : \underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t), \quad j = 1, \ldots, n, \text{ a.e. in } [0,T]\},$$

where $\underline{r}$ and $\overline{r}$ are assumed to belong to $L^2([0,T], \mathbb{R}^n)$, (for a detailed description of the model see Sect. 2).

In order to determine for each sector i the optimal composition of instruments held as assets and as liabilities, we consider, as usual, the influence due to risk-aversion and the process of optimization of each sector in the financial economy, namely the desire to maximize the value of the asset holdings while minimizing the value of liabilities. Moreover, the equilibrium condition for the prices, which expresses the equilibration of the total assets, the total liabilities and the portion of financial transactions per unit F_j employed to cover the expenses of the financial institutions including possible dividends and manager bonus r_j of instrument j is the following:

$$\sum_{i=1}^m (1 - \tau_{ij}(t)) \left[x_{ij}^*(t) - (1 + h_j(t)) y_{ij}^*(t)\right] + F_j(t) \begin{cases} \geq 0 & \text{if } r_j^*(t) = \underline{r}_j(t) \\ = 0 & \text{if } \underline{r}_j(t) < r_j^*(t) < \overline{r}_j(t) \\ \leq 0 & \text{if } r_j^*(t) = \overline{r}_j(t), \end{cases} \quad (1)$$

where (x^*, y^*, r^*) is the equilibrium solution for the investments as assets and as liabilities and for the prices.

Let us explicitly remark that the definition of equilibrium conditions (see Definition 2) of the financial model may be also expressed in the following form:

Definition 1 A vector of sector assets, liabilities and instrument prices $(x^*, y^*, r^*) \in P \times \mathcal{R}$ is an equilibrium of the dynamic financial model if and only if $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$, and a.e. in $[0, T]$, it is a solution to

$$\max_{P_i} \int_0^T \Big\{ u_i(t, x_i(t), y_i(t)) + (1 - \tau_i(t)) r^*(t) \times [x_i(t) - (1 + h(t)) y_i(t)] \Big\} dt,$$
$$\forall (x_i, y_i) \in P_i,$$

and verifies condition (1).

Let us note that the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent, $r_j(t)(1-\tau_{ij}(t))[x_i(t)-(1+h_j(t))y_i(t)]$ represents the value of the difference between the asset holdings and the liabilities.

In [5] these equilibrium conditions are expressed in terms of an evolutionary variational inequality and some existence theorems are provided. Moreover, it is possible to consider the so-called shadow market, namely the dual Lagrange problem, when the Lagrange variables $\rho_j^{*1}(t)$ and $\rho_j^{*2}(t)$, called "deficit" and "surplus" variables, appear. These variables play a fundamental role in order to analyze the model and to achieve suggestions for the management of the world economy (see the Deficit formula, the Balance Law and the Liabilities Formula).

On the other hand, in [10] by means of these Lagrange variables the authors study the possible insolvencies related to the financial instruments and analyze, in terms of $\sum_{j=1}^n \rho_j^{(1)*}(t) - \sum_{j=1}^n \rho_j^{(2)*}(t)$, when the insolvencies propagate to the entire system, producing a "financial contagion".

In [4] the regularity of the solutions of the evolutionary variational inequality, governing the financial model, is investigated. In particular, the authors are able to obtain a continuity result of the equilibrium solution (see [1,2] for the study of continuity of solutions to general evolutionary variational inequalities).

Because of the relevance and importance of the "deficit" and "surplus" variables $\rho_j^{*1}(t)$ and $\rho_j^{*2}(t)$ in the study of the evolutionary financial equilibrium problem, it remains to investigate regularity properties of these Lagrange variables. In this paper the authors, analyzing the model already introduced, in which the equilibrium is expressed in terms of a global maximum, are able to prove, first, the continuity of the equilibrium solution and then, under the same conditions, a continuity result of the "deficit" and "surplus" variables $\rho_j^{*1}(t)$, $\rho_j^{*2}(t)$. These regularity results allow us to apply a calculus procedure in order to compute the equilibrium solution and, then, the Lagrange variables.

The paper is organized as follows: in Sect. 2 the model is presented in detail, together with the equilibrium conditions and their evolutionary variational inequality formulation; in Sect. 3 we give the dual formulation in which the Lagrange variables $\rho_j^{*1}(t)$ and $\rho_j^{*2}(t)$ appear; in Sect. 4 we state the main result of the paper; in Sect. 5 we provide the proof of our result; in Sect. 6 we give a computational procedure and we stress the relevance of these variables in order to analyze the financial contagion; in Sect. 7 we provide a numerical example and finally Sect. 8 summarizes our results.

It is worth stressing that variational inequality theory has revealed to be a powerful instrument in order to study equilibrium problems, for example Walrasian equilibrium problem (see [17–19]), the oligopolistic market equilibrium problem (see [6,7]), the weighted traffic equilibrium problem (see [25,26]), Signorini problem (see [21] and references therein) and many others.

2 The financial model and the equilibrium conditions

Now, we describe the financial network we are dealing with (for more details we refer to [3,5,11]). In the general competitive financial equilibrium models here considered, the equilibrium yields both asset and liability volumes, as well as the instrument prices.

Then, we consider a financial economy consisting of m sectors, for example households, domestic business, banks and other financial institutions, as well as state and local governments, with a typical sector denoted by i, and of n instruments, for example mortgages, mutual funds, saving deposits, money market funds, with a typical financial instrument denoted by j, in the time interval $[0, T]$. Let $s_i(t)$ denote the total financial volume held by sector i at time t as assets, and let $l_i(t)$ be the total financial volume held by sector i at time t as liabilities. In general, the investments of markets in assets and liabilities, $s_i(t)$ and $l_i(t)$, may be different and depending on time, since we are in the presence of uncertainty and risk perspectives. At time t, we denote the amount of instrument j, held as an asset in sector i's portfolio, by $x_{ij}(t)$ and the amount of instrument j, held as a liability in sector i's portfolio, by $y_{ij}(t)$. The assets and liabilities in all the sectors are grouped into the matrices $x(t) = [x_1(t), \dots, x_i(t), \dots, x_n(t)]^T = \{x_{ij}(t)\}_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ and

$y(t) = [y_1(t), \dots, y_i(t), \dots, y_n(t)]^T = \{y_{ij}(t)\}_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$.

We choose as a functional setting the very general Lebesgue space $L^2([0, T], \mathbb{R}^p) = \left\{ f : [0, T] \to \mathbb{R}^p : \int_0^T \|f(t)\|_p^2 dt < +\infty \right\}$.

As it is well known, the dual space of $L^2([0, T], \mathbb{R}^p)$ is still $L^2([0, T], \mathbb{R}^p)$ and we define the canonical bilinear form in $L^2([0, T], \mathbb{R}^p) \times L^2([0, T], \mathbb{R}^p)$ as

$$\langle\langle g, x \rangle\rangle = \int_0^T \langle g(t), x(t) \rangle dt, \quad g, x \in L^2([0, T], \mathbb{R}^p).$$

Then, the set of feasible assets and liabilities, for each sector $i = 1, \dots, m$, becomes

$$P_i = \Bigg\{ (x_i(t), y_i(t)) \in L^2([0, T], \mathbb{R}^{2n}) : \sum_{j=1}^n x_{ij}(t) = s_i(t),$$

$$\sum_{j=1}^n y_{ij}(t) = l_i(t) \text{ a.e. in } [0, T],\ x_i(t) \geq 0,\ y_i(t) \geq 0, \text{ a.e. in } [0, T] \Bigg\};$$

and the set of all feasible assets and liabilities becomes

$$P = \Bigg\{ (x(t), y(t)) \in L^2([0, T], \mathbb{R}^{2mn}) : \sum_{j=1}^n x_{ij}(t) = s_i(t),$$

$$\sum_{j=1}^n y_{ij}(t) = l_i(t), \quad \forall\, i = 1, \dots, m, \text{ a.e. in } [0, T],$$

$$x_i(t) \geq 0,\ y_i(t) \geq 0, \quad \forall\, i = 1, \dots, m, \text{ a.e. in } [0, T] \Bigg\}.$$

We denote the price of instrument j, held as an asset at time t, by $r_j(t)$ and the price of instrument j, held as a liability at time t, by $(1 + h_j(t))r_j(t)$, where h_j is a nonnegative function belonging to $L^\infty([0, T])$. The term $h_j(t)$ describes the realistic behaviour of the markets, for which the liabilities are more expensive than the assets. We group the instrument prices, held as assets, into the vector $r(t) = [r_1(t), r_2(t), \ldots, r_i(t), \ldots, r_n(t)]^T$ and the instrument prices, held as liabilities, into the vector $(1 + h(t))r(t) = [(1 + h_1(t))r_1(t), (1 + h_2(t))r_2(t), \ldots, (1 + h_i(t))r_i(t), \ldots, (1 + h_n(t))r_n(t)]^T$. In our problem the prices of each instrument appear as unknown variables.

As in [5], we consider a more complete definition of equilibrium prices $r^*(t)$, based on the demand-supply law, imposing that the equilibrium prices vary between floor and ceiling prices.

To this aim, we denote the nonnegative floor price and the ceiling price, associated with instrument j, by $\underline{r}_j(t)$ and $\overline{r}_j(t)$, respectively, with $\overline{r}_j(t) > \underline{r}_j(t)$ a.e. in $[0, T]$, and we group the instrument ceiling prices $\overline{r}_j(t)$ into the column vector $\overline{r}(t) = [\overline{r}_1(t), \ldots, \overline{r}_i(t), \ldots, \overline{r}_n(t)]^T$ and the instrument floor prices $\underline{r}(t)$ into the column vector $\underline{r}_j(t) = [\underline{r}_1(t), \ldots, \underline{r}_i(t), \ldots, \underline{r}_n(t)]^T$.

The set of feasible instrument prices becomes:

$$\mathcal{R} = \{r \in L^2([0, T], \mathbb{R}^n) : \ \underline{r}_j(t) \le r_j(t) \le \overline{r}_j(t), \quad j = 1, \ldots, n, \ \text{a.e. in } [0, T]\},$$

where $\underline{r}(t)$ and $\overline{r}(t)$ are assumed to belong to $L^2([0, T], \mathbb{R}^n)$.

In this way, to each investor a minimal price $\underline{r}_j(t)$ for the assets held in the instrument j is guaranteed, whereas each investor is requested to pay for the liabilities, in any case, a minimal price $(1 + h_j)\underline{r}_j(t)$. Similarly, each investor cannot obtain, for an asset, a price greater than $\overline{r}_j(t)$ and, as a liability, the price cannot exceed the maximum price $(1 + h_j(t))\overline{r}_j(t)$.

Moreover, we incorporate policy interventions in the financial equilibrium in form of taxes and price controls. We denote the tax rate levied on sector i's net yield on financial instrument j as $\tau_{ij}(t)$ and group it into the matrix $\tau(t) = \{\tau_{ij}(t)\}_{i=1,\ldots,m, j=1,\ldots,n}$.

We assume that each sector i seeks to maximize its utility, with the utility function, $\forall i = 1, \ldots, m$, given by

$$\begin{aligned} &U_i(t, x_i(t), y_i(t), r(t)) \\ &\quad = u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} r_j(t)(1 - \tau_{ij}(t))[x_{ij}(t) - (1 + h_j(t))y_{ij}(t)], \end{aligned}$$

where the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent, $r_j(t)(1 - \tau_{ij}(t))[x_i(t) - (1 + h_j(t))y_i(t)]$ represents the value of the difference between the asset holdings and the liabilities. First, we make the following assumptions, called *Hypotheses 1*, which will be denoted by *Hp. 1*:

- *The sector's utility function* $U_i(t, x_i(t), y_i(t))$ *is defined on* $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, *is measurable in t and is continuous with respect to the other variables.*
- $\partial u_i/\partial x_{ij}$ *and* $\partial u_i/\partial y_{ij}$ *exist and they are measurable in t and continuous with respect to* x_i *and* y_i.
- $\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$, *and a.e. in* $[0, T]$, *the following growth conditions hold true:*

$$|u_i(t, x, y)| \le \alpha_i(t)\|x\|\|y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{2}$$

$$\left|\frac{\partial u_i(t, x, y)}{\partial x_{ij}}\right| \le \beta_{ij}(t)\|y\|, \quad \left|\frac{\partial u_i(t, x, y)}{\partial y_{ij}}\right| \le \gamma_{ij}(t)\|x\|, \tag{3}$$

  where $\alpha_i, \beta_{ij}, \gamma_{ij}$ *are non-negative functions of* $L^\infty([0, T])$.
- *The function* $u_i(t, x, y)$ *is concave.*

– $-\partial u_i(t, x_i(t), y_i(t))/\partial x_{ij}$, $-\partial u_i(t, x_i(t), y_i(t))/\partial y_{ij}$ *are strongly monotone functions (for the definition, see for example* [8]*).*

We remind that the time-dependent Markowitz utility function verifies conditions (2) and (3) (see [28,29]).

The equilibrium condition for the prices expresses the equilibration of the total assets, the total liabilities and the portion of financial transactions per unit F_j. Hence, the equilibrium condition for the price r_j of instrument j is the following:

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\right]+F_j(t) \begin{cases} \geq 0 & \text{if } r_j^*(t)=\underline{r}_j(t) \\ =0 & \text{if } \underline{r}_j(t)<r_j^*(t)<\overline{r}_j(t) \\ \leq 0 & \text{if } r_j^*(t)=\overline{r}_j(t) \end{cases} \tag{4}$$

where (x^*, y^*, r^*) is an equilibrium solution belonging to $P\times\mathcal{R}$ for the investments as assets and as liabilities and for the prices. $F_j(t)$ is the portion of financial transactions per unit, employed to cover the expenses of the financial institutions, including possible dividends and manager bonus.

In other words, the prices are determined taking into account the amount of the supply, the demand of an instrument and the charges F_j, namely, if there is an actual supply excess of an instrument as assets and of the charges F_j in the economy, then its price must be the floor price. If the price of an instrument is positive, but not at the ceiling, then the market of that instrument must clear. Finally, if there is an actual demand excess of an instrument as liabilities and of the charges F_j in the economy, then the price must be at the ceiling.

Let us note that, if $\underline{r}(t)$, $\overline{r}(t)$ are continuous functions, then the equilibrium price $r^*(t)$ can be also chosen as a continuous one.

Hence, the definition of equilibrium conditions is the following one.

Definition 2 A vector of sector assets, liabilities and instrument prices $(x^*(t), y^*(t), r^*(t)) \in P\times\mathcal{R}$ is an equilibrium of the dynamic financial model if and only if, $\forall i=1,\dots,m$, $\forall j=1,\dots,n$, and a.e. in $[0,T]$, it satisfies the system of inequalities

$$-\frac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^*(t)-\mu_i^{(1)*}(t)\geq 0, \tag{5}$$

$$-\frac{\partial u_i(t,x^*,y^*)}{\partial y_{ij}}+(1-\tau_{ij}(t))(1+h_j(t))r_j^*(t)-\mu_i^{(2)*}(t)\geq 0, \tag{6}$$

and equalities

$$x_{ij}^*(t)\Big[-\frac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^*(t)-\mu_i^{(1)*}(t)\Big]=0, \tag{7}$$

$$y_{ij}^*(t)\Big[-\frac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}}+(1-\tau_{ij}(t))(1+h_j(t))r_j^*(t)-\mu_i^{(2)*}(t)\Big]=0, \tag{8}$$

where $\mu_i^{(1)*}(t), \mu_i^{(2)*}(t)\in L^2([0,T])$ are Lagrange multipliers, and it verifies conditions (4) a.e. in $[0,T]$.

It is worth remarking that the equilibrium definition is, in a sense, the same as given by Wardrop's principle, which states that, in the case of user optimization on congested transportation networks, the user rejects the less convenient (or more costly) choice.

The functions $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$ are Lagrange multipliers associated a.e. in $[0, T]$ with the constraints $\sum_{j=1}^{n} x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^{n} y_{ij}(t) - l_i(t) = 0$, respectively. They are unknown a priori, but this fact has no influence, because the following Theorem states that Definition 2 is equivalent to a variational inequality in which $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$ do not appear.

The following Theorem, which provides the variational inequality formulation of the equilibrium conditions, is proved in [5] (see Theorem 2.1).

Theorem 1 *A vector $(x^*, y^*, r^*) \in P \times \mathcal{R}$ is a dynamic financial equilibrium if and only if it satisfies the following variational inequality:*

Find $(x^*, y^*, r^*) \in P \times \mathcal{R}$:

$$\sum_{i=1}^{m} \int_0^T \left\{ \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j^*(t) \Big] \times [x_{ij}(t) - x_{ij}^*(t)] \right.$$
$$\left. + \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t)) r_j^*(t)(1+h_j(t)) \Big] \times [y_{ij}(t) - y_{ij}^*(t)] \right\} dt$$
$$+ \sum_{j=1}^{n} \int_0^T \sum_{i=1}^{m} \Big\{ (1-\tau_{ij}(t)) \Big[x_{ij}^*(t) - (1+h_j(t)) y_{ij}^*(t) \Big] + F_j(t) \Big\}$$
$$\times \big[r_j(t) - r_j^*(t) \big] dt \geq 0, \qquad \forall (x, y, r) \in P \times \mathcal{R}. \tag{9}$$

Let us observe that the first part of variational inequality (9) is equivalent to the utility maximization problem in P, whereas the second part expresses conditions (4). It is worth remarking that a solution $(x^*(t), y^*(t))$ of the first part of (9) is unique and does not depend on the equilibrium price $r^*(t)$, but only on floor and ceiling prices, $\underline{r}(t)$ and $\overline{r}(t)$, as proved in Sect. 5.

We refer to [8] for an existence theorem of the equilibrium solution for the financial model.

3 The deficit and surplus variables: the shadow market

In order to better understand the behavior of the financial equilibrium, in [5] the authors, applying to the general financial model a very recent infinite dimensional duality theory introduced and developed in [12,14–16,31], obtain an interesting and useful output as the Deficit Formula, the Balance Law and the Liability Formula, from which important suggestions and remarks follow for the management of the world economy. In this framework, the Lagrange variables surplus and deficit, $\rho_j^{(1)*}(t)$ and $\rho_j^{(2)*}(t)$, appear and play a fundamental role. For analogous results on the relevance of the Lagrange multipliers see [13,22–24].

Indeed, first the authors introduce the Lagrange functional

$$\mathcal{L}(x, y, r, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)})$$
$$= f(x, y, r) - \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \lambda_{ij}^{(1)}(t) x_{ij}(t)\, dt - \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \lambda_{ij}^{(2)} y_{ij}(t)\, dt$$

$$
-\sum_{i=1}^{m}\int_0^T \mu_i^{(1)}(t)\left(\sum_{j=1}^{n} x_{ij}(t)-s_i(t)\right)dt-\sum_{i=1}^{m}\int_0^T \mu_i^{(2)}(t)\left(\sum_{j=1}^{n} y_{ij}(t)-l_i(t)\right)dt
$$
$$
+\sum_{j=1}^{n}\int_0^T \rho_j^{(1)}(t)(\underline{r}_j(t)-r_j(t))\,dt+\sum_{j=1}^{n}\int_0^T \rho_j^{(2)}(t)(r_j(t)-\overline{r}_j(t))\,dt,
$$

where

$$
\begin{aligned}
&f(x,y,r)\\
&=\int_0^T \Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left[-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^*(t)\right]\times[x_{ij}(t)-x_{ij}^*(t)]\\
&+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial y_{ij}}+(1-\tau_{ij}(t))(1+h_j(t))r_j^*(t)\right]\times[y_{ij}(t)-y_{ij}^*(t)]\\
&+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\right]+F_j(t)\right]\times\left[r_j(t)-r_j^*(t)\right]\Bigg\}\,dt,
\end{aligned}
$$

with $(x,y,r)\in L^2([0,T],\mathbb{R}^{2mn+n})$, $\lambda^{(1)},\lambda^{(2)}\in L^2([0,T],\mathbb{R}^{mn}_+)$, $\mu^{(1)},\mu^{(2)}\in L^2([0,T],\mathbb{R}^m)$, $\rho^{(1)},\rho^{(2)}\in L^2([0,T],\mathbb{R}^n_+)$.

It is worth noting that $\lambda^{(1)},\lambda^{(2)},\rho^{(1)},\rho^{(2)}$ are the Lagrange multipliers associated, a.e. in $[0,T]$, to the sign constraints $x_i(t)\geq 0$, $y_i(t)\geq 0$, $r_j(t)-\underline{r}_j(t)\geq 0$, $\overline{r}_j(t)-r_j(t)\geq 0$, respectively. The functions $\mu^{(1)}(t)$ and $\mu^{(2)}(t)$ are the Lagrange multipliers associated, a.e. in $[0,T]$, to the equality constraints $\sum_{j=1}^{n} x_{ij}(t)-s_i(t)=0$ and $\sum_{j=1}^{n} y_{ij}(t)-l_i(t)=0$, respectively.

Then, they prove the following Theorem (Theorem 6.1 in [5]).

Theorem 2 *Let* $(x^*,y^*,r^*)\in P\times\mathcal{R}$ *be a solution to variational inequality (9) and let us consider the associated Lagrange functional (10). Then, there exist* $\lambda^{(1)*},\lambda^{(2)*}\in L^2([0,T],\mathbb{R}^{mn}_+)$, $\mu^{(1)*},\mu^{(2)*}\in L^2([0,T],\mathbb{R}^m)$, $\rho^{(1)*},\rho^{(2)*}\in L^2([0,T],\mathbb{R}^n_+)$ *such that* $(x^*,y^*,r^*,\lambda^{(1)*},\lambda^{(2)*},\mu^{(1)*},\mu^{(2)*},\rho^{(1)*},\rho^{(2)*})$ *is a saddle point of the Lagrange functional, namely*

$$
\begin{aligned}
&\mathcal{L}(x^*,y^*,r^*,\lambda^{(1)},\lambda^{(2)},\mu^{(1)},\mu^{(2)},\rho^{(1)},\rho^{(2)})\\
&\quad\leq \mathcal{L}(x^*,y^*,r^*,\lambda^{(1)*},\lambda^{(2)*},\mu^{(1)*},\mu^{(2)*},\rho^{(1)*},\rho^{(2)*})\\
&\quad\leq \mathcal{L}(x,y,r,\lambda^{(1)*},\lambda^{(2)*},\mu^{(1)*},\mu^{(2)*},\rho^{(1)*},\rho^{(2)*})
\end{aligned}\tag{10}
$$

$\forall(x,y,r)\in L^2([0,T],\mathbb{R}^{2mn+n})$, $\forall\lambda^{(1)},\lambda^{(2)}\in L^2([0,T],\mathbb{R}^{mn}_+)$, $\forall\mu^{(1)},\mu^{(2)}\in L^2([0,T],\mathbb{R}^m)$, $\forall\rho^{(1)},\rho^{(2)}\in L^2([0,T],\mathbb{R}^n_+)$ *and, a.e. in* $[0,T]$,

$$
\begin{aligned}
&-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^*(t)-\lambda_{ij}^{(1)*}(t)-\mu_i^{(1)*}(t)=0,\\
&\quad\forall i=1,\ldots,m,\ \forall j=1\ldots,n;\\
&-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial y_{ij}}+(1-\tau_{ij}(t))(1+h_j(t))r_j^*(t)-\lambda_{ij}^{(2)*}(t)-\mu_i^{(2)*}(t)=0,\\
&\quad\forall i=1,\ldots,m,\ \forall j=1\ldots,n;
\end{aligned}
$$

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^{*}(t)-(1+h_j(t))y_{ij}^{*}(t)\right]+F_j(t)+\rho_j^{(2)*}(t)=\rho_j^{(1)*}(t),$$
$$\forall j=1,\ldots,n; \tag{11}$$
$$\lambda_{ij}^{(1)*}(t)x_{ij}^{*}(t)=0,\ \lambda_{ij}^{(2)*}(t)y_{ij}^{*}(t)=0,\quad \forall i=1,\ldots,m,\ \forall j=1,\ldots,n \tag{12}$$
$$\mu_i^{(1)*}(t)\left(\sum_{j=1}^{n}x_{ij}^{*}(t)-s_i(t)\right)=0,\quad \mu_i^{(2)*}(t)\left(\sum_{j=1}^{n}y_{ij}^{*}(t)-l_i(t)\right)=0,$$
$$\forall i=1,\ldots,m \tag{13}$$
$$\rho_j^{(1)*}(t)(\underline{r}_j(t)-r_j^{*}(t))=0,\ \rho_j^{(2)*}(t)(r_j^{*}(t)-\overline{r}_j(t))=0,\quad \forall j=1,\ldots,n. \tag{14}$$

Formula (11) represents the Deficit Formula.

$\rho_j^{(1)*}(t)$ represents the deficit per unit, whereas $\rho_j^{(2)*}(t)$ is the positive surplus per unit. From (11) it is possible to obtain the Balance Law

$$\sum_{i=1}^{m}l_i(t)=\sum_{i=1}^{m}s_i(t)-\sum_{i=1}^{m}\sum_{j=1}^{n}\tau_{ij}(t)\left[x_{ij}^{*}(t)-y_{ij}^{*}(t)\right]$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^{*}(t)+\sum_{j=1}^{n}F_j(t)-\sum_{j=1}^{n}\rho_j^{(1)*}(t)+\sum_{j=1}^{n}\rho_j^{(2)*}(t). \tag{15}$$

Finally, assuming that the taxes $\tau_{ij}(t), i=1,\ldots,m, j=1,\ldots,n$, have a common value $\theta(t)$, and the increments $h_j(t),\ j=1,\ldots,n$, have a common value $i(t)$, otherwise we can consider the average values (see Remark 7.1 in [5]), the significant Liability Formula follows

$$\sum_{i=1}^{m}l_i(t)=\frac{(1-\theta(t))\sum_{i=1}^{m}s_i(t)+\sum_{j=1}^{n}F_j(t)-\sum_{j=1}^{n}\rho_j^{(1)*}(t)+\sum_{j=1}^{n}\rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))}.$$

The financial problem can be considered from two different perspectives: one from the *Point of View of the Sectors*, which try to maximize the utility and a second point of view, that we can call *System Point of View*, which regards the whole equilibrium, namely the respect of the previous laws. For example, from the point of view of the sectors, $l_i(t)$, for $i=1,\ldots,m$, are liabilities, whereas for the economic system they are investments and, hence, the Liability Formula, from the system point of view, can be called "*Investments Formula*". The system point of view coincides with the dual Lagrange problem (the so-called "shadow market") in which $\rho_j^{(1)}(t)$ and $\rho_j^{(2)}(t)$ are the dual multipliers, representing the deficit and the surplus per unit arising from instrument j. Formally, the dual problem is given by

Find $(\rho^{(1)*},\rho^{(2)*})\in L^2([0,T],\mathbb{R}_+^{2n})$ such that

$$\sum_{j=1}^{n}\int_0^T(\rho_j^{(1)}(t)-\rho_j^{(1)*}(t))(\underline{r}_j(t)-r_j^{*}(t))dt$$
$$+\sum_{j=1}^{n}\int_0^T(\rho_j^{(2)}(t)-\rho_j^{(2)*}(t))(r_j^{*}(t)-\overline{r}_j(t))dt\le 0,$$
$$\forall(\rho^{(1)},\rho^{(2)})\in L^2([0,T],\mathbb{R}_+^{2n}). \tag{16}$$

In fact, taking into account inequality (10), we get

$$
\begin{aligned}
&-\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T (\lambda_{ij}^{(1)}(t)-\lambda_{ij}^{(1)*}(t))x_{ij}^*(t)\,dt-\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T (\lambda_{ij}^{(2)}-\lambda_{ij}^{(2)*})y_{ij}^*(t)\,dt\\
&\quad-\sum_{i=1}^{m}\int_0^T (\mu_i^{(1)}(t)-\mu_i^{(1)*}(t))\left(\sum_{j=1}^{n}x_{ij}^*(t)-s_i(t)\right)dt\\
&\quad-\sum_{i=1}^{m}\int_0^T (\mu_i^{(2)}(t)-\mu_i^{(2)*}(t))\left(\sum_{j=1}^{n}y_{ij}^*(t)-l_i(t)\right)dt\\
&\quad+\sum_{j=1}^{n}\int_0^T (\rho_j^{(1)}(t)-\rho_j^{(1)*}(t))(\underline{r}_j(t)-r_j^*(t))\,dt\\
&\quad+\sum_{j=1}^{n}\int_0^T (\rho_j^{(2)}(t)-\rho_j^{(2)*}(t))(r_j^*(t)-\overline{r}_j(t))\,dt\le 0
\end{aligned}
$$

$\forall\lambda^{(1)},\lambda^{(2)}\in L^2([0,T],\mathbb{R}_+^{mn}),\ \mu^{(1)},\mu^{(2)}\in L^2([0,T],\mathbb{R}^m),\ \rho^{(1)},\rho^{(2)}\in L^2([0,T],\mathbb{R}_+^n).$

Choosing $\lambda^{(1)}=\lambda^{(1)*}$, $\lambda^{(2)}=\lambda^{(2)*}$, $\mu^{(1)}=\mu^{(1)*}$, $\mu^{(2)}=\mu^{(2)*}$, we obtain the dual problem (16)

Note that, from the *System Point of View*, also the expenses of the institutions $F_j(t)$ are supported from the liabilities of the sectors.

Remark 1 Let us recall that, from the Liability Formula, we get the following index $E(t)$, called "Evaluation Index", that is very useful for the rating procedure:

$$
E(t)=\frac{\sum_{i=1}^{m}l_i(t)}{\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)},
$$

where we set

$$
\tilde{s}_i(t)=\frac{s_i(t)}{1+i(t)},\quad \tilde{F}_j(t)=\frac{F_j(t)}{1+i(t)-\theta(t)-\theta(t)i(t)}.
$$

From the Liability Formula we obtain

$$
\begin{aligned}
E(t)=1-&\frac{\sum_{j=1}^{n}\rho_j^{(1)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)\right)}\\
&+\frac{\sum_{j=1}^{n}\rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)\right)}
\end{aligned}
\tag{17}
$$

If $E(t)$ is greater or equal than 1, the evaluation of the financial equilibrium is positive (better if $E(t)$ is proximal to 1), whereas if $E(t)$ is less than 1, the evaluation of the financial equilibrium is negative.

Remark 2 In [10] the authors note that, if

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) > \sum_{j=1}^{n} \rho_j^{(2)*}(t), \tag{18}$$

the balance of all the financial entities is negative, the whole deficit exceeds the sum of all the surplus, a negative contagion appears and the insolvencies of individual entities propagate through the entire system. It is sufficient that only one deficit $\rho_j^{(1)*}(t)$ is large to obtain a negative balance for the entire system, even if the other ones are lightly positive.

When condition (18) is verified, we get $E(t) \leq 1$ and, hence, also $E(t)$ is a significant indicator that the financial contagion happens.

4 Main result

In the previous section the shadow market is introduced, in which the Lagrange variables $\rho_j^{(1)*}(t), \rho_j^{(2)*}(t)$ appear. These variables represent the deficit and the surplus per unit, respectively, and constitute a very important instrument in order to analyze the financial model.

Now, in order to present the main achievement of the paper, namely a regularity result of $\rho_j^{(1)*}(t), \rho_j^{(2)*}(t)$, let us set:

$$\begin{aligned}
F(t) &= [F_1(t), F_2(t), \ldots, F_n(t)]^T; \\
\nu &= (x, y, r) = \left(\left(x_{ij}\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}, \left(y_{ij}\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}, \left(r_j\right)_{j=1,\ldots,n} \right); \\
A(t, \nu) &= \Bigg(\left[-\frac{\partial u_i(t,x,y)}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j(t) \right]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}, \\
&\qquad \left[-\frac{\partial u_i(t,x,y)}{\partial y_{ij}} + (1-\tau_{ij}(t))(1+h_j(t)) r_j(t) \right]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}, \\
&\qquad \left[\sum_{i=1}^{m} (1-\tau_{ij}(t)) \left(x_{ij}(t) - (1+h_j(t)) y_{ij}(t) \right) + F_j(t) \right]_{j=1,\ldots,n} \Bigg); \\
&\qquad A : \mathcal{K} \to L^2([0,T], \mathbb{R}^{2mn+n}),
\end{aligned} \tag{19}$$

with

$$\mathcal{K} = P \times \mathcal{R}.$$

Let us note that $\mathcal{K}$ is a convex, bounded and closed subset of $L^2([0,T], \mathbb{R}^{2mn+n})$. Moreover assumption (3) implies that A is lower semicontinuous along line segments.

We are able to prove the following result:

Theorem 3 *Let $A \in C^0([0,T], \mathbb{R}^{2mn+n})$ be strongly monotone in x and y, monotone in r, namely, there exists α such that, for $t \in [0,T]$,*

$$\langle\langle A(t,\nu_1)-A(t,\nu_2),\nu_1-\nu_2\rangle\rangle \geq \alpha(\|x_1-x_2\|^2+\|y_1-y_2\|^2), \tag{20}$$

$\forall \nu_1=(x_1,y_1,r_1),\ \nu_2=(x_2,y_2,r_2)\in\mathbb{R}^{2mn+n}$.

Let $\underline{r}(t)$, $\overline{r}(t)$, $h(t)$, $F(t)\in C^0([0,T],\mathbb{R}^n_+)$, *let* $\tau(t)\in C^0([0,T],\mathbb{R}^{mn})$ *and let* $s,l\in C^0([0,T],\mathbb{R}^m)$, *satisfying the following assumption* (β)*:*

- *there exist* $\delta_1(t)\in L^2([0,T])$ *and* $c_1\in\mathbb{R}$ *such that, for a.a.* $t\in[0,T]$,

$$\|s(t)\|\leq\delta_1(t)+c_1;$$

- *there exist* $\delta_2(t)\in L^2([0,T])$ *and* $c_2\in\mathbb{R}$ *such that, for a.a.* $t\in[0,T]$,

$$\|l(t)\|\leq\delta_2(t)+c_2.$$

Then the Lagrange variables, $\rho^{(1)*}(t)$, $\rho^{(2)*}(t)$, *whose existence is ensured by Theorem 2 and which represent the deficit and the surplus per unit, respectively, are continuous too.*

5 Proof of the theorem

In order to prove the regularity of the deficit and surplus variables, we start from the following result, which ensures the continuity of the equilibrium solution only if the operator A is strongly monotone with respect to (x,y,r) (see [4]).

Theorem 4 *Let* $A\in C^0([0,T],\mathbb{R}^{2mn+n})$ *be strongly monotone. Let* $s,l\in C^0([0,T],\mathbb{R}^m)$, *let* $\underline{r}(t)$, $\overline{r}(t)\in C^0([0,T],\mathbb{R}^n_+)$. *Then variational inequality (9) admits a unique continuous solution.*

Let us note that the operator of the financial equilibrium problem (19) is strongly monotone with respect to the variables x and y, but only monotone with respect to r (see [8] Section 3). Then, Theorem 4 cannot be directly applied. However, here we are able to prove, under the assumptions of Theorem 3, the existence of a continuous equilibrium solution.

To this aim, let us consider the variational inequality (9):

$$\begin{aligned}
&\sum_{i=1}^m\int_0^T\left\{\sum_{j=1}^n\Big[-\frac{\partial u_i(t,x_i^*(t),y_i^*(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^*(t)\Big]\times[x_{ij}(t)-x_{ij}^*(t)]\right.\\
&\quad\left.+\sum_{j=1}^n\Big[-\frac{\partial u_i(t,x_i^*(t),y_i^*(t))}{\partial y_{ij}}+(1-\tau_{ij}(t))r_j^*(t)(1+h_j(t))\Big]\times[y_{ij}(t)-y_{ij}^*(t)]\right\}dt\\
&\quad+\sum_{j=1}^n\int_0^T\sum_{i=1}^m\left\{(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\right]+F_j(t)\right\}\\
&\quad\times\left[r_j(t)-r_j^*(t)\right]dt\geq 0,\qquad\forall(x,y,r)\in P\times\mathcal{R}.
\end{aligned}$$

The existence of a solution to (9) is ensured by Theorem 2.3 in [8].

Let us prove that, assuming $\underline{r}(t)$, $\overline{r}(t)$ continuous functions, we can find continuous solutions to (9).

Indeed, let $(x^*(t),y^*(t),r^*(t))$ be a solution to (9). Setting in (9) $x_{ij}=x^*_{ij}$ and $y_{ij}=y^*_{ij}$ it follows that the equilibrium price $r^*(t)$ verifies variational inequality:

Find $r^* \in R$ such that

$$\sum_{j=1}^{n} \int_0^T \sum_{i=1}^{m} \Big\{(1-\tau_{ij}(t))\Big[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\Big]+F_j(t)\Big\} \times \Big[r_j(t)-r_j^*(t)\Big]dt \geq 0, \quad \forall r \in R. \tag{21}$$

Let us note that, as in Theorem 4.4 in [3], it is possible to prove that variational inequality (21) is equivalent to conditions (4).

Now, let us define, for $j = 1, \ldots, n$,

$$\begin{aligned} \overline{E}_j &= \Big\{t \in [0,T] : r_j^*(t) = \overline{r}_j(t)\Big\}; \\ \underline{E}_j &= \Big\{t \in [0,T] : r_j^*(t) = \underline{r}_j(t)\Big\}; \\ E_j^* &= \Big\{t \in [0,T] : \underline{r}_j(t) < r_j^*(t) < \overline{r}_j(t)\Big\}. \end{aligned}$$

Since $\underline{r}(t)$, $\overline{r}(t)$ are assumed to be continuous functions, each continuous function $r^{**}(t) = (r_1^{**}(t), \ldots, r_n^{**}(t))$, that is equal to $r_j^*(t)$ in $\overline{E}_j$ and $\underline{E}_j$ and $\underline{r}_j(t) < r_j^{**}(t) < \overline{r}_j(t)$ in E_j^*, still verifies variational inequality (21), since in E_j^* the coefficient $\sum_{i=1}^{m}(1-\tau_{ij}(t))\Big[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\Big]+F_j(t)$ is equal to zero due to (4). As a consequence, $r^{**}(t)$ also verifies condition (4).

On the other hand, setting in (9) $r_j = r_j^*$, $(x^*(t), y^*(t))$ verifies variational inequality:

Find $(x^*, y^*) \in P$ such that

$$\sum_{i=1}^{m} \int_0^T \Big\{ \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t))r_j^*(t)\Big] \times [x_{ij}(t) - x_{ij}^*(t)] + \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))r_j^*(t)(1+h_j(t))\Big] \times [y_{ij}(t) - y_{ij}^*(t)] \Big\} dt \geq 0, \qquad \forall (x,y) \in P. \tag{22}$$

Arguing as in the proof of Theorem 2.1 in [5], it is possible to prove that variational inequality (22) is equivalent to conditions (5)–(8).

Moreover, following the proof of Theorem 5 in [3], we may prove the equivalence between problem (22) and the optimization problem

$$\max_P \sum_{i=1}^{m} \int_0^T (u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} r_j^*(t)\left\{(1-\tau_{ij}(t))[x_{ij}(t) - (1+h_j(t))y_{ij}(t)]\right\})dt. \tag{23}$$

If (x^*, y^*) is a solution to (23), it is still a solution to

$$\max_P \sum_{i=1}^{m} \int_0^T (u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} r_j^*(t)\left\{(1-\tau_{ij}(t))[x_{ij}(t) - (1+h_j(t))y_{ij}(t)] + F_j(t)\right\})dt. \tag{24}$$

Since r^{**} coincides with r^* in $\overline{E}_j$ and $\underline{E}_j$ and in E_j^* the coefficient

$$\sum_{i=1}^{m} r_j^{**}(t)\left\{(1-\tau_{ij}(t))[x_{ij}(t)-(1+h_j(t))y_{ij}(t)]+F_j\right\}$$

is equal to zero, it follows that (x^*, y^*) is still a solution to

$$\max_P \sum_{i=1}^{m}\int_0^T (u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} r_j^{**}(t)\left\{(1-\tau_{ij}(t))[x_{ij}(t)-(1+h_j(t))y_{ij}(t)]+F_j\right\})dt. \tag{25}$$

Then, for the already cited equivalence, $(x^*(t), y^*(t))$ verifies

$$\begin{aligned}&\sum_{i=1}^{m}\int_0^T\left\{\sum_{j=1}^{n}\Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^{**}(t)\Big]\right.\\&\quad\times[x_{ij}(t)-x_{ij}^*(t)]\\&\quad+\sum_{j=1}^{n}\Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}}+(1-\tau_{ij}(t))r_j^{**}(t)(1+h_j(t))\Big]\\&\quad\left.\times\Big[y_{ij}(t)-y_{ij}^*(t)\Big]\right\}dt\geq 0,\qquad\forall(x, y)\in P.\end{aligned}\tag{26}$$

The operator in (26) is strongly monotone with respect to (x, y) and then, by virtue of Theorem 4, the unique solution $(x^*(t), y^*(t))$ is continuous too.

Then, we may conclude that the solution $(x^*(t), y^*(t), r^{**}(t))$ is continuous, generalizing Theorem 4.1 to operators which are strongly monotone with respect to (x, y) and only monotone with respect to r.

Let us stress that $(x^*(t), y^*(t))$ is unique and does not depend on the equilibrium price $r^{**}(t)$, but only on floor and ceiling prices, $\underline{r}(t)$ and $\overline{r}(t)$.

Now, we are able to prove our main result, Theorem 3.

Let us set, for a fixed $j = 1, \ldots, n$,

$$I_{1,j}^{+} = \{t\in[0, T]:\ \rho_j^{(1)*}(t)>0\}$$
$$I_{1,j}^{0} = \{t\in[0, T]:\ \rho_j^{(1)*}(t)=0\}.$$

From the Deficit Formula (11) we have, $\forall j = 1, \ldots, n$,

$$B_j(t)=\sum_{i=1}^{m}(1-\tau_{ij}(t))\Big[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\Big]+F_j(t)=\rho_j^{(1)*}(t)-\rho_j^{(2)*}(t).$$

$\rho_j^{(1)*}(t)$ is a continuous function in $I_{1,j}^{+}$. In fact, in $I_{1,j}^{+}$, from (14) it follows $\rho_j^{(2)*}(t)=0$ and then, from Deficit Formula (11), in $I_{1,j}^{+}$

$$\rho_j^{(1)*}(t)=B_j(t)=\sum_{i=1}^{m}(1-\tau_{ij}(t))\Big[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\Big]+F_j(t), \tag{27}$$

namely, $\rho_j^{(1)*}(t)$ is the sum of continuous functions.

Let us remark that the continuity of $\rho_j^{(1)*}(t)$ in $I_{1,j}^+$ implies that $I_{1,j}^+$ is an open set.

In the interior of the closed set $I_{1,j}^0$ $\rho_j^{(1)*}(t)$ is a continuous function, since in this set $\rho_j^{(1)*} \equiv 0$.

Let us examine the continuity of $\rho_j^{(1)*}(t)$ in $\partial I_{1,j}^+ = \partial I_{1,j}^0$. Let $t_0 \in \partial I_{1,j}^0$ and let us prove the continuity of $\rho_j^{(1)*}(t)$ in t_0. Since $\partial I_{1,j}^0 \subseteq I_{1,j}^0$, it results $\rho_j^{(1)*}(t_0) = 0$. Because of the continuity of $B_j(t)$, it results to be

$$\lim_{\substack{t \to t_0 \\ t \in I_{1,j}^0}} B_j(t) = \lim_{\substack{t \to t_0 \\ t \in I_{1,j}^0}} (-\rho_j^{(2)*}(t)) = B_j(t_0)$$

and

$$\lim_{\substack{t \to t_0 \\ t \in I_{1,j}^+}} B_j(t) = \lim_{\substack{t \to t_0 \\ t \in I_{1,j}^+}} \rho_j^{(1)*}(t) = B_j(t_0).$$

Then, we may conclude

$$B_j(t_0) = \lim_{\substack{t \to t_0 \\ t \in I_{1,j}^+}} \rho_j^{(1)*}(t) = \lim_{\substack{t \to t_0 \\ t \in I_{1,j}^0}} (-\rho_j^{(2)*}(t)),$$

and the only possibility, taking into account the nonnegativity of $\rho_j^{(1)*}(t)$ and $\rho_j^{(2)*}(t)$, is that $B_j(t_0) = 0$. This means that

$$\lim_{\substack{t \to t_0 \\ t \in I_{1,j}^+}} \rho_j^{(1)*}(t) = \lim_{\substack{t \to t_0 \\ t \in I_{1,j}^0}} \rho_j^{(1)*}(t) = 0 = \rho_j^{(1)*}(t_0).$$

In a similar way, we may prove the continuity of $\rho_j^{(2)*}(t)$ in $[0, T]$.

In the following Section we also provide a procedure in order to compute numerically the deficit and surplus variables (see [9]).

6 Computational procedure

First, it is worth noting that variational inequality (9) is equivalent to the problem: Find $\nu^* \in \mathcal{K} = P \times \mathcal{R}$ such that

$$\langle A(t, \nu^*(t)), \nu(t) - \nu^*(t) \rangle \geq 0 \quad \forall \nu \in \mathcal{K}, \quad a.e.\ in\ [0, T], \tag{28}$$

(see [9]).

As we already proved in Sect. 5, we may construct a continuous equilibrium solution to (28). For example, we may find a continuous solution $r^{**}(t)$ to (4) in the following form

$$r_j^{**}(t) = \begin{cases} \underline{r}_j(t) & t \in \overline{E}_j \\ \underline{r}_j(t_1) + \frac{\overline{r}_j(t_2) - \underline{r}_j(t_1)}{t_2 - t_1}(t - t_1) & t \in E_j^* \\ \overline{r}_j(t) & t \in \underline{E}_j, \end{cases}$$

($\overline{E}_j$ or $\underline{E}_j$ may eventually be empty).

Then, if $\nu^* = (x^*, y^*, r^{**})$ is the continuous equilibrium solution constructed, it satisfies:

$$\langle A(t, \nu^*(t)), \nu(t) - \nu^*(t) \rangle \geq 0 \qquad \forall t \in [0, T]. \tag{29}$$

Consider, now, a sequence of partitions π_n of $[0, T]$, such that:

$$\pi_n = (t_n^0, t_n^1, \ldots, t_n^{N_n}), \quad 0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$$

and

$$\delta_n = \max\left\{t_n^k - t_n^{k-1} : k = 1, \ldots, N_n\right\}$$

with $\lim_{n\to\infty} \delta_n = 0$. Then, for each value t_n^{k-1}, we consider the variational inequality

$$\langle A(t_n^{k-1}, \nu^*(t_n^{k-1})), \nu - \nu^*(t_n^{k-1})\rangle \geq 0 \quad \forall \nu \in \mathcal{K}(t_n^{k-1}) \tag{30}$$

where

$$\mathcal{K}(t_n^{k-1}) = \Bigg\{\nu = (x, y, r) \in \mathbb{R}^{2mn+n} : \sum_{j=1}^{n} x_{ij} = s_i(t_n^{k-1}),$$

$$\sum_{j=1}^{n} y_{ij} = l_i(t_n^{k-1}),\ x_i \geq 0,\ y_i \geq 0,\ \forall\, i = 1, \ldots, m,$$

$$\underline{r}_j(t_n^{k-1}) \leq r_j \leq \overline{r}_j(t_n^{k-1}),\ j = 1, \ldots, n\Bigg\}.$$

Using the well-known methods for the finite dimensional variational inequality, we can construct an interpolation function $\nu_n(t)$ such that

$$\lim_n \|\nu_n(t) - \nu^*(t)\|_{L^\infty([0,T],\mathbb{R}^{2mn+n})} = 0.$$

Once achieved the equilibrium solution, we may obtain the surplus and deficit variables from (11) and (14). In fact, if $r_j^{**}(t) = \underline{r}_j(t)$, from (14) it follows $\rho_j^{(2)*}(t) = 0$ and we may calculate $\rho_j^{(1)*}(t)$ from (11). Analogously, if $r_j^{**}(t) = \overline{r}_j(t)$, from (14) it follows $\rho_j^{(1)*}(t) = 0$ and we may calculate $\rho_j^{(2)*}(t)$ from (11). Finally, if $\underline{r}_j(t) < r_j^{**}(t) < \overline{r}_j(t)$ from (11) it follows $\rho_j^{(1)*}(t) = \rho_j^{(2)*}(t) = 0$.

We would like to remind the importance of the calculus of the variables deficit and surplus. In fact, from the difference

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) - \sum_{j=1}^{n} \rho_j^{(2)*}(t)$$

it depends if a negative contagion appears on the financial system (see [10]). In fact, if

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) > \sum_{j=1}^{n} \rho_j^{(2)*}(t), \tag{31}$$

the balance of all the financial entities is negative, the whole deficit exceeds the sum of all the surplus, a negative contagion appears and the insolvencies of individual entities propagate through the entire system. As we can see, it is sufficient that only one deficit $\rho_j^{(1)*}(t)$ is large to obtain a negative balance for the entire system, even if the other ones are ligthly positive. Moreover we would like to stress that, even if for only a sector there is a big insolvency related to the instrument j, we can have $\rho_j^{(1)*}(t) > 0$.

7 Numerical example

Let us analyze a numerical financial example, in which we consider a utility function of Markowitz type with memory, namely in which the risk aversion function u_i is given by the sum of the one suggested by H.M. Markowitz in [28] and [29], which expresses at each instant $t \in [0, T]$ the risk aversion by means of variance-covariance matrices, denoting the sector's assessment of the standard deviation of prices for each instrument, and of the memory term. As it happens in real life, we even assume in the example that the amount of the investments as assets and liabilities depends on the expected solution (see [8]).
In detail, we consider an economy with two agents and two financial instruments. The variance-covariance matrices of the two agents are:

$$Q^1 = \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the time interval [0, 1], the risk aversion function $u_i(t, x_i(t), y_i(t))$ is given by:

$$u_i(t, x_i(t), y_i(t)) = -\begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + \int_0^t \begin{bmatrix} x_i(t-z) \\ y_i(t-z) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t-z) \\ y_i(t-z) \end{bmatrix} dz.$$

We choose as the feasible set for assets, liabilities and prices:

$$\begin{aligned} \mathbb{K}(w^*) = \Big\{ & w = (x(t), y(t), r(t)) \in L^2([0, 1], \mathbb{R}^{10}_+) : \\ & x_{11}(t) + x_{12}(t) = 6\int_0^1 r_1^*(s)\, ds + 3,\ \ x_{21}(t) + x_{22}(t) = 6\int_0^1 r_2^*(s)\, ds + 3, \\ & y_{11}(t) + y_{12}(t) = 11,\ \ y_{21}(t) + y_{22}(t) = 1 \text{ a.e. in } [0, 1] \\ & \text{and } 2t \le r_1(t) \le 3t,\ \ t \le r_2(t) \le \frac{3}{2}t \text{ a.e. in } [0, 1] \Big\} \end{aligned}$$

Moreover, we choose $\tau_{ij} = \frac{1}{4}, \forall i, j = 1, 2$ and $h_j = 1\ \forall j = 1, 2$.
It follows that quasi-variational inequality (9) becomes: Find $w^* \in \mathbb{K}(w^*)$:

$$\begin{aligned} \int_0^1 \Bigg\{ & \left[2(x_{11}^*(t) - 0.5y_{11}^*(t)) + \int_0^t 2(x_{11}^*(\tau) - 0.5y_{11}^*(\tau))d\tau - \frac{3}{4} r_1^*(t) \right] \left(x_{11}(t) - x_{11}^*(t) \right) \\ & + \left[2x_{12}^*(t) + \int_0^t 2x_{12}^*(\tau)d\tau - \frac{3}{4} r_2^*(t) \right] \left(x_{12}(t) - x_{12}^*(t) \right) \\ & + \left[2x_{21}^*(t) + \int_0^t 2x_{21}^*(\tau)d\tau - \frac{3}{4} r_1^*(t) \right] \left(x_{21}(t) - x_{21}^*(t) \right) \\ & + \left[2(x_{22}^*(t) - 0.5y_{21}^*(t)) + \int_0^t 2(x_{22}^*(\tau) - 0.5y_{21}^*(\tau))d\tau - \frac{3}{4} r_2^*(t) \right] \left(x_{22}(t) - x_{22}^*(t) \right) \\ & + \left[2(y_{11}^*(t) - 0.5x_{11}^*(t)) + \int_0^t 2(y_{11}^*(\tau) - 0.5x_{11}^*(\tau))d\tau + \frac{3}{2} r_1^*(t) \right] \left(y_{11}(t) - y_{11}^*(t) \right) \\ & + \left[2y_{12}^*(t) + \int_0^t 2y_{12}^*(\tau)d\tau + \frac{3}{2} r_2^*(t) \right] \left(y_{12}(t) - y_{12}^*(t) \right) \\ & + \left[2(y_{21}^*(t) - 0.5x_{22}^*(t)) + \int_0^t 2(y_{21}^*(\tau) - 0.5x_{22}^*(\tau))d\tau + \frac{3}{2} r_1^*(t) \right] \left(y_{21}(t) - y_{21}^*(t) \right) \end{aligned}$$

$$
\begin{aligned}
&+\left[2y_{22}^*(t)+\int_0^t 2y_{22}^*(\tau)d\tau+\frac{3}{2}\,r_2^*(t)\right]\,\big(y_{22}(t)-y_{22}^*(t)\big)\\
&+\left\{\frac{3}{4}\left[x_{11}^*(t)-2\,y_{11}^*(t)\right]+\frac{3}{4}\,\left[x_{21}^*(t)-2\,y_{21}^*(t)\right]+F_1(t)\right\}\,\big(r_1(t)-r_1^*(t)\big)\\
&+\left\{\frac{3}{4}\left[x_{12}^*(t)-2\,y_{12}^*(t)\right]+\frac{3}{4}\,\left[x_{22}^*(t)-2\,y_{22}^*(t)\right]+F_2(t)\right\}\,\big(r_2(t)-r_2^*(t)\big)\,\Bigg\}\mathrm{d}t\ge 0\,,\\
&\forall w\in\mathbb{K}(w^*).
\end{aligned}
\tag{32}
$$

We can apply to problem (32) the method described in Sect. 6, but, in this case, it is more convenient to apply the direct method (see [9,30]). So, we can first derive from the constraints of the convex set $\mathbb{K}(w^*)$ the values of some variables in terms of the others, namely, a.e. in [0, 1],

$$
\begin{aligned}
&x_{12}(t)=6\int_0^1 r_1^*(s)\,\mathrm{d}s-x_{11}(t)+3, && x_{21}(t)=6\int_0^1 r_2^*(s)\,\mathrm{d}s-x_{22}(t)+3,\\
&y_{12}(t)=-\,y_{11}(t)+11, && y_{21}(t)=-\,y_{22}(t)+1.
\end{aligned}
$$

Substituting in the quasi variational inequality (32), we have:

$$
\begin{aligned}
&\int_0^1\Bigg(\Bigg[4\,x_{11}^*(t)\,+\,4\int_0^t x_{11}^*(\tau)\,\mathrm{d}\tau\,-\,y_{11}^*(t)\,-\,\int_0^t y_{11}^*(\tau)\,\mathrm{d}\tau\,-\,12\int_0^1 r_1^*(s)\,\mathrm{d}s\\
&-12\int_0^t\int_0^1 r_1^*(s)\,\mathrm{d}\tau\mathrm{d}s\,-6-6t-\frac{3}{4}r_1^*(t)+\frac{3}{4}r_2^*(t)\Bigg](x_{11}(t)-x_{11}^*(t))\\
&+\Bigg[4\,x_{22}^*(t)\,+\,4\int_0^t x_{22}^*(\tau)\,\mathrm{d}\tau\,+\,y_{22}^*(t)\,+\,\int_0^t y_{22}^*(\tau)\,\mathrm{d}\tau\,-\,12\int_0^1 r_2^*(s)\,\mathrm{d}s\\
&-12\int_0^t\int_0^1 r_2^*(s)\,\mathrm{d}\tau\mathrm{d}s\,-7-7t+\frac{3}{4}r_1^*(t)-\frac{3}{4}r_2^*(t)\Bigg](x_{22}(t)-x_{22}^*(t))\\
&+\Bigg[4\,y_{11}^*(t)\,+\,4\int_0^t y_{11}^*(\tau)\,\mathrm{d}\tau\,-\,x_{11}^*(t)-\int_0^t x_{11}^*(\tau)\,\mathrm{d}\tau\,-\,22\,-\,22t\\
&+\frac{3}{2}\,r_1^*(t)-\frac{3}{2}\,r_2^*(t)\Bigg](y_{11}(t)-y_{11}^*(t))+\Bigg[4\,y_{22}^*(t)\,+\,4\int_0^t y_{22}^*(\tau)\,\mathrm{d}\tau\,+\,x_{22}^*(t)\\
&+\int_0^t x_{22}^*(\tau)\,\mathrm{d}\tau\,-\,2\,-\,2t-\frac{3}{2}\,r_1^*(t)+\frac{3}{2}\,r_2^*(t)\Bigg](y_{22}(t)-y_{22}^*(t))\\
&+\Bigg\{\frac{3}{4}\Bigg[6\int_0^1 r_2^*(s)\,\mathrm{d}s-x_{22}^*(t)+3-2\Big(-y_{22}^*(t)+1\Big)\Bigg]\\
&+\frac{3}{4}[x_{11}^*(t)-2y_{11}^*(t)]+F_1(t)\Bigg\}(r_1(t)-r_1^*(t))\\
&+\Bigg\{\frac{3}{4}\Bigg[6\int_0^1 r_1^*(s)\,\mathrm{d}s-x_{11}^*(t)+3-2\Big(-y_{11}^*(t)+11\Big)\Bigg]\\
&+\frac{3}{4}[x_{22}^*(t)-2y_{22}^*(t)]+F_2(t)\Bigg\}(r_2(t)-r_2^*(t))\,\Bigg)\mathrm{d}t\ge 0\,,
\end{aligned}
$$

for all $x_{11}(t),\ x_{22}(t),\ y_{11}(t),\ y_{22}(t),\ r_1(t),\ r_2(t)$ such that

$$
\begin{aligned}
&0\le x_{11}(t)\le 6\int_0^1 r_1^*(s)\,\mathrm{d}s+3,\quad 0\le x_{22}(t)\le 6\int_0^1 r_2^*(s)\,\mathrm{d}s+3,\quad 0\le y_{11}(t)\le 11,\\
&0\le y_{22}(t)\le 1,\ \ 2t\le r_1(t)\le 3t,\ \ t\le r_2(t)\le\frac{3}{2}t,\ \text{a.e.in } [0,1].
\end{aligned}
\tag{33}
$$

Then, the solution to the quasivariational inequality (33) is obtained by solving the system:

$$\begin{cases} \Gamma_1 = 4\,x^*_{11}(t) \,+\, 4\displaystyle\int_0^t x^*_{11}(t)\,\mathrm{d}\tau \,-\, y^*_{11}(t) \,-\, \int_0^t y^*_{11}(t)\,\mathrm{d}\tau \,-\, 12\int_0^1 r^*_1(s)\,\mathrm{d}s + \\ \quad -\, 12\displaystyle\int_0^t\int_0^1 r^*_1(s)\,\mathrm{d}\tau\mathrm{d}s \,-6-6t-\frac{3}{4}r^*_1+\frac{3}{4}r^*_2(t) \,=\, 0 \\ \Gamma_2 = 4\,x^*_{22}(t) \,+\, 4\displaystyle\int_0^t x^*_{22}(t)\,\mathrm{d}\tau \,+\, y^*_{22}(t) \,+\, \int_0^t y^*_{22}(t)\,\mathrm{d}\tau \,-\, 12\int_0^1 r^*_2(s)\,\mathrm{d}s + \\ \quad -\, 12\displaystyle\int_0^t\int_0^1 r^*_2(s)\,\mathrm{d}\tau\mathrm{d}s \,-7-7t+\frac{3}{4}r^*_1-\frac{3}{4}r^*_2(t) \,=\, 0 \\ \Gamma_3 = 4\,y^*_{11}(t)+4\displaystyle\int_0^t y^*_{11}(t)\,\mathrm{d}\tau - x^*_{11}(t) - \int_0^t x^*_{11}(t)\,\mathrm{d}\tau - 22 - 22t + \frac{3}{2}r^*_1 - \frac{3}{2}r^*_2 = 0 \\ \Gamma_4 = 4\,y^*_{22}(t)+4\displaystyle\int_0^t y^*_{22}(t)\,\mathrm{d}\tau + x^*_{22}(t) + \int_0^t x^*_{22}(t)\,\mathrm{d}\tau - 2 - 2t - \frac{3}{2}r^*_1 + \frac{3}{2}r^*_2 > 0 \\ \Gamma_5 = \dfrac{3}{4}\left[x^*_{11}(t)-2y^*_{11}(t)\right] + \dfrac{3}{4}\left[6\displaystyle\int_0^1 r^*_2(s)\,\mathrm{d}s - x^*_{22} + 3 - 2\Big(-y^*_{22}+1\Big)\right] + F_1(t) > 0 \\ \Gamma_6 = \dfrac{3}{4}\left[6\displaystyle\int_0^1 r^*_1(s)\,\mathrm{d}s - x^*_{11} + 3 - 2\Big(-y^*_{11}+11\Big)\right] + \dfrac{3}{4}\left[x^*_{22}(t)-2y^*_{22}(t)\right] + F_2(t) > 0. \end{cases}$$

Since $\Gamma_4 > 0$, $\Gamma_5 > 0$ and $\Gamma_6 > 0$, the direct method allows to find the following solution, a.e. in [0, 1],

$$\begin{cases} x^*_{11}(t) = -\dfrac{1}{10}\,e^{-t} \,+\, \dfrac{191}{30} \\ x^*_{22}(t) = \dfrac{3}{16}\,e^{-t} \,+\, \dfrac{49}{16} \\ y^*_{11}(t) = \dfrac{7}{20}\,e^{-t} \,+\, \dfrac{403}{60} \\ y^*_{22}(t) = 0 \\ r^*_1(t) \;= 2t \\ r^*_2(t) \;= t \end{cases} \quad \text{and} \quad \begin{cases} x^*_{12}(t) = \dfrac{1}{10}\,e^{-t} \,+\, \dfrac{79}{30} \\ x^*_{21}(t) = -\dfrac{3}{16}\,e^{-t} \,+\, \dfrac{47}{16} \\ y^*_{12}(t) = -\dfrac{7}{20}\,e^{-t} \,+\, \dfrac{257}{60} \\ y^*_{21}(t) = 1. \end{cases}$$

From (11) and (14), taking into account that $r^*_1(t) = \underline{r}_1(t)$, and $r^*_2(t) = \underline{r}_2(t)$, we get:

$$\begin{aligned} \Gamma_5 &= \frac{3}{4}\left[x^*_{11}(t) - x^*_{22}(t) - 2y^*_{11}(t) + 2y^*_{22}(t) + 4\right] + F_1(t) \\ &= -\frac{3}{4}\left[\frac{79}{80}\,e^{-t} + \frac{1471}{240}\right] + F_1(t) = \rho_1^{(1)*}(t) > 0, \\ \Gamma_6 &= \frac{3}{4}\left[-x^*_{11}(t) + x^*_{22}(t) + 2y^*_{11}(t) - 2y^*_{22}(t) - 13\right] + F_2(t) \\ &= \frac{3}{4}\left[\frac{79}{80}\,e^{-t} - \frac{689}{240}\right] + F_2(t) = \rho_2^{(1)*}(t) > 0, \end{aligned}$$

provided that, a.e. in [0, 1],

$$F_1(t) > \frac{3}{4}\left[\frac{79}{80}e^{-t} + \frac{1471}{240}\right],$$
$$F_2(t) > \frac{3}{4}\left[-\frac{79}{80}e^{-t} + \frac{689}{240}\right].$$

Since the deficit variables $\rho_1^{(1)*}(t)$, $\rho_2^{(1)*}(t)$ are positive, then the economy is in a phase of regression. The same conclusion is confirmed by the evaluation index. In fact, setting, a.e. in [0, 1],

$$F_1 = \frac{3}{4}\left[\frac{79}{80}e^{-t} + \frac{1471}{240}\right] + \phi_1(t),$$
$$F_2 = \frac{3}{4}\left[-\frac{79}{80}e^{-t} + \frac{689}{240}\right] + \phi_2(t),$$
$$\phi_1(t), \phi_2(t) > 0 \quad \text{in} \quad [0, 1],$$

and, taking into account that,

$$E(t) = \frac{\sum_{i=1}^{2} l_i(t)}{\sum_{i=1}^{2} \tilde{s}_i(t) + \sum_{j=1}^{2} \tilde{F}_j(t)}$$

with $\tilde{F}_j(t) = \dfrac{F_j(t)}{1 + i(t) - \theta(t) - \theta(t)i(t)}$; $\tilde{s}_i(t) = \dfrac{s_i(t)}{1 + i(t)}$, $i(t) = 1, \theta(t) = \frac{1}{4}$, a simple calculation gives

$$E(t) = \frac{12}{12 + \frac{2}{3}(\phi_1(t) + \phi_2(t))} < 1.$$

Finally, since it results

$$\sum_{j=1}^{2} \rho_j^{(1)*}(t) > \sum_{j=1}^{2} \rho_j^{(2)*}(t),$$

as noted in Remark 2, a negative contagion can appear and the insolvencies of individual entities can mutually propagate through the entire system.

8 Conclusions

In the paper a general equilibrium model of financial flows and prices is studied. In particular, considering the Lagrange dual formulation of the financial model, the Lagrange variables, which represent the deficit and the surplus per unit, are considered and the continuity of these variables is proved. Finally, a procedure in order to compute numerically these variables is provided.

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