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A polynomial-time nearly-optimal algorithm for an edge coloring problem in outerplanar graphs

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Abstract Given a graph G, we study the problem of finding the minimum number of colors required for a proper edge coloring of G such that any pair of vertices at distance 2 have distinct sets consisting of colors of their incident edges. This minimum number is called the 2-distance vertex-distinguishing index, denoted by $\chi'_{d2}(G)$. Using the breadth first search method, this paper provides a polynomial-time algorithm producing nearly-optimal solution in outerplanar graphs. More precisely, if G is an outerplanar graph with maximum degree Δ, then the produced solution uses colors at most $\Delta + 8$. Since $\chi'_{d2}(G) \geq \Delta$ for any graph G, our solution is within eight colors from optimal.

Keywords Breadth first search · Tree · Outerplanar graph · Edge coloring · 2-Distance vertex-distinguishing index

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1 Introduction

Only simple and finite graphs are considered in this paper. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, maximum degree $\Delta(G)$ and minimum degree $\delta(G)$. For a vertex v, we use $E(v)$ to denote the set of edges incident to v. So $d_G(v) = |E(v)|$ denotes the degree of v in G. A *k-vertex* is a vertex of degree k. A *leaf* is a vertex of degree 1. The *distance* between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path connecting them if there is any. Otherwise, $d_G(u, v) = \infty$ by convention. If $d_G(u, v) = r$ for $u, v \in V(G)$, then u is called an *r-distance vertex* or an *r-neighbor* of v, and vice versa. Moreover, we use $N^r_G(v)$ to denote the set of r-neighbors of v in the graph G. In particular, we simply call a 1-neighbor of v a neighbor of v and abbreviate $N^1_G(v)$ to $N_G(v)$. If no ambiguity arises, $\Delta(G), d_G(v), d_G(u, v), N^r_G(v)$, and $N_G(v)$ are written as $\Delta, d(v), d(u, v), N^r(v)$, and $N(v)$, respectively. Let $\operatorname{diam}(G)$ denote the *diameter* of a connected graph G, i.e., the maximum of distances between any pair of different vertices in G. A graph G is called *normal* if it contains no isolated edges.

A *proper edge k-coloring* of a graph G is a mapping $\phi : E(G) \to C = \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. For a vertex $v \in V(G)$, let $C_\phi(v)$ denote the set of colors assigned to the edges in $E(v)$, i.e., $C_\phi(v) = \{\phi(uv)|uv \in E(G)\}$.

In this paper, we study the problem of finding the minimum number of colors required for a proper edge coloring of G such that any pair of vertices at distance 2 have distinct color sets. This minimum number is called the 2-distance vertex-distinguishing index, denoted by $\chi'_{d2}(G)$.

1.1 Related works

For an integer $r \geq 1$, the *r-strong edge chromatic number* $\chi'_s(G, r)$ of a graph G is the minimum number of colors required for a proper edge coloring of G such that any two vertices u and v with $d(u, v) \leq r$ have $C_\phi(u) \neq C_\phi(v)$. Note that $\chi'_s(G, r)$ is well defined if and only if G is normal. This concept was introduced by Akbari et al. [1], and independently by Zhang et al. [17]. The reader is referred to [11,12] for latest results for large r. Moreover, when $r \geq \operatorname{diam}(G)$, $\chi'_s(G, r) = \chi'_s(G)$, where $\chi'_s(G)$ is called the *strong edge chromatic number* of G and this parameter has been extensively investigated, see [3–5].

The *adjacent vertex distinguishing edge chromatic number* $\chi'_a(G)$ is precisely $\chi'_s(G, 1)$. Zhang et al. [16] first introduced this notion (*adjacent strong edge coloring* in their terminology). Among other things, they proposed the following challenging conjecture, in which C_5 denotes the cycle on five vertices.

Conjecture 1 *If G is a normal graph and $G \neq C_5$, then $\chi'_a(G) \leq \Delta + 2$.*

Conjecture 1 was confirmed for bipartite graphs and subcubic graphs [2]. Using probabilistic analysis, Hatami [9] showed that every graph G with $\Delta > 10^{20}$ has $\chi'_a(G) \leq \Delta + 300$. Wang et al. [15] showed that every graph G has $\chi'_a(G) \leq 2.5\Delta$ and every semi-regular graph G has $\chi'_a(G) \leq \frac{5}{3}\Delta + \frac{13}{3}$. A graph G is said to be *semi-regular* if each edge of G is incident to at least one Δ-vertex. If G is a planar graph, then it is shown in [10] that $\chi'_a(G) \leq \Delta + 2$ if $\Delta \geq 12$.

More recently, the first four authors considered in [13] the 2-distance vertex-distinguishing edge coloring of graphs, which can be regarded as a relaxed form of the 2-strong edge coloring. Thus, $\Delta \leq \chi'(G) \leq \chi'_{d2}(G) \leq \chi'_s(G, 2)$. In [13], the 2-distance vertex-distinguishing indices of cycles, paths, trees, complete bipartite graphs, and unicycle graphs were completely determined. Moreover, a nearly-optimal upper bound on the 2-distance vertex-distinguishing index of Halin graphs was also obtained. Especially, the following conjecture was proposed in [13]:

Conjecture 2 *For any graph G, $\chi'_{d2}(G) \leq \Delta + 2$.*

1.2 Our contribution

In this paper, we establish a nearly-optimal algorithm with running time $O(n^3)$ for the 2-distance vertex-distinguishing edge-coloring problem in outerplanar graphs. A planar graph is called *outerplanar* if it has an embedding in the Euclidean plane such that all the vertices are located on the boundary of the unbounded face. An *outerplane graph* is a particular drawing of an outerplanar graph on the Euclidean plane. A cycle C is called *separating* if both its interior and exterior contain at least one vertex of G.

Suppose that G is an outerplane graph. Then the following properties (P1)–(P3) hold. Note that (P3) follows from (P2) easily, whereas the proof of (P2) appeared in [7].

- **(P1)** $\delta(G) \leq 2$.
- **(P2)** G does not contain a subdivision of K_4 or $K_{2,3}$ as a subgraph.
- **(P3)** G does not contain a separating cycle.

Our algorithm is built on those properties. It gives an upper bound of $\Delta + 8$ for the 2-distance vertex-distinguishing index of outerplanar graphs. This means that the solution given by our algorithm is within eight colors from optimal.

2 Outerplanar graphs with $\Delta \geq 5$

In this section, we construct an algorithm of cubic time to legally color the edges of an outerplanar graph G with $\Delta \geq 5$ using at most $\Delta + 8$ colors.

2.1 Ordered breadth first search

A *rooted tree* T is a tree with a particular vertex r designated as its root. The vertices of a rooted tree can be arranged in layers, with vertices at distances i to the root r forming layer i. Hence, layer 0 consists of the root only. For a vertex v in layer $i \geq 1$, the neighbor of v in layer $i-1$ is called its *father* and all the neighbors of v in layer $i+1$ are called its *sons*. Vertices in layer i are ordered from left to right with labels $v_1^i, v_2^i, \ldots, v_{l_i}^i$ so that, for any j, either v_j^i and v_{j+1}^i have the same father, or the father of v_j^i is to the left side of the father of v_{j+1}^i.

Let G be a connected outerplane graph. Beginning with a chosen vertex r, we order all vertices clockwise. Calamoneri and Petreschi [6] constructed an algorithm OBFT for G. It is a breadth first search starting from r in such a way that vertices coming first in the cyclic ordering are visited first. Using OBFT, G can be edge-partitioned into a spanning tree T rooted at r and a subgraph H with $\Delta(H) \leq 4$, i.e., $E(G) = E(T) \cup E(H)$ and $E(T) \cap E(H) = \emptyset$. Edges in $E(T)$ and $E(H)$ are called *tree-edges* and *non-tree-edges* of G, respectively. This edge-partition is called an *OBFT partition*. Calamoneri and Petreschi [6] used OBFT partition to determined the $L(h, 1)$-labeling number of an outerplanar graph. This edge-partition technique was also successfully employed in [14] to study the surviving rate of outerplanar graphs.

The following key lemma was given in [6]:

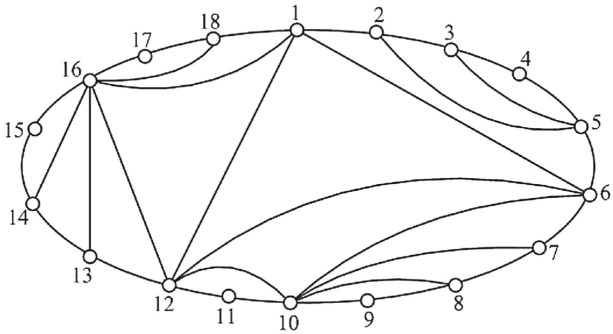
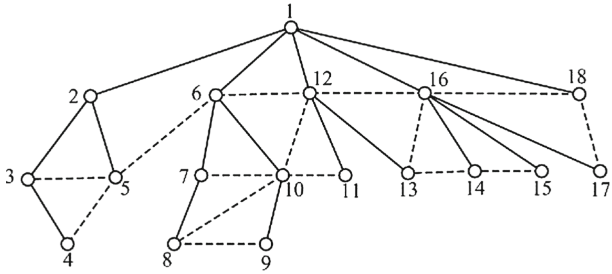


Fig. 1 An outerplane graph G^* on 18 vertices

Fig. 2 An OBFT partition of G^*



Lemma 1 *Every OBFT partition $T \cup H$ for a connected outerplane graph G has the following properties:*

(1) *If v_j^i is adjacent to v_k^i with $j < k$, then $v_j^i v_k^i$ is a non-tree-edge, and $k = j + 1$.*

(2) *If $v_j^i v_k^{i-1} \in E(H)$ and v_j^i is a son of v_r^{i-1}, then $k = r + 1$ and v_j^i is the rightmost son of v_r^{i-1}.*

Lemma 1 indicates that every vertex v_l^i has at most two neighbors in layer $i - 1$, at most two neighbors in layer i, and at most $d(v_l^i) - 1$ neighbors in layer $i + 1$.

To give an example of OBFT partition, we consider the outerplane graph G^* depicted in Fig. 1. In Fig. 2, vertex 1 is the root of the tree T produced by OBFT and solid (broken) lines denote tree-edges (non-tree-edges).

2.2 A legal ($\Delta + 8$)-coloring algorithm

Given a proper edge coloring ϕ of a graph G, we say that two vertices u and v are *conflict* if $C_\phi(u) = C_\phi(v)$. Hence ϕ is 2-distance vertex-distinguishing if and only if there are no conflict pair of vertices at distance 2. For convenience, a 2-distance vertex-distinguishing edge k-coloring is abbreviated as a 2DVDE k-coloring in the following. We say that an edge $uv \in E(G)$ is *legally* colored if the color assigned to it is different from those of its adjacent edges and no pair of vertices of distance 2 are conflict. Let H be a subgraph of G. A proper edge coloring ϕ of H is called a 2DVDE *partial coloring* of G on H if $C_\phi(u) \neq C_\phi(v)$ for each pair of vertices $u, v \in V(H)$ with $d_H(u) = d_G(u)$, $d_H(v) = d_G(v)$, and $d_H(u, v) = 2$.

Theorem 1 *If G is an outerplane graph with $\Delta \geq 5$, then $\chi'_{d2}(G) \leq \Delta + 8$.*

Proof Let G be an outerplane graph with $\Delta \geq 5$. Without loss of generality, assume that G is connected. By carrying out OBFT, G is edge-partitioned into a rooted spanning tree T with r as its root and a subgraph H with $\Delta(H) \leq 4$. Then we are going to give an algorithm of finding a 2DVDE $(\Delta + 8)$-coloring of G using the color set $C = \{1, 2, \ldots, \Delta + 8\}$. Note that $|C| \geq 13$. Let u_j $(1 \leq j \leq |V(G)|)$ be the j-th vertex visited in this OBFT partition, where $u_1 = r$. Recall that $E(u_j)$ is the set of edges incident to the vertex u_j. Define $E_j = E(u_1) \cup E(u_2) \cup \cdots \cup E(u_j)$, and $G_j = G[E_j]$, which is the subgraph of G induced by edges in E_j.

At Step 1, we color $E(u_1)$ properly with distinct colors in C. Let $j \geq 2$. Suppose that E_{j-1} has been colored such that a 2DVDE partial coloring of G on G_{j-1} has been established. At Step j, we color $E(u_j)$ to form a 2DVDE partial coloring of G on G_j. Then we set $j := j + 1$ and continue our coloring procedure. Once all $E(u_j)$'s, for $1 \leq j \leq |V(G)|$, have been colored, a 2DVDE coloring of G using at most $\Delta + 8$ colors is constructed.

By Lemma 1 and the structural property of outerplane graphs, we obtain the following useful observation:

Observation 1 (i) *Let v_p^{i+1} be a son of v_l^i, and v_q^{i+1} be a son of v_h^i such that $l \leq h$. If $v_p^{i+1}v_q^{i+1} \in E(H)$, then $q = p + 1$, and $h = l$ or $h = l + 1$.*

(ii) *For $j = 1, 2, 3$, let $v_{l_j}^{i+1}$ be the son of $v_{k_j}^i$ with $l_1 < l_2 < l_3$. If $v_{l_1}^{i+1}v_{l_2}^{i+1}, v_{l_2}^{i+1}v_{l_3}^{i+1} \in E(H)$, then $k_1 \leq k_2 \leq k_3$, and exactly one of the following cases holds:*

(1) $k_1 = k_2 = k_3$;
(2) $k_2 = k_1$ *and* $k_3 = k_2 + 1$;
(3) $k_3 = k_2$ *and* $k_2 = k_1 + 1$.

Let $u_j = v_l^i$. Let v_k^{i-1} be the father of v_l^i. We say that a vertex $z \in V(G)$ is *full* if all edges in $E(z)$ have been colored in the foregoing steps.

We now begin with constructing a 2DVDE partial $(\Delta + 8)$-coloring of G on G_j using $C = \{1, 2, \ldots, \Delta + 8\}$. Let $d_{G_j}(v_l^i) = p$, $d_{G_{j-1}}(v_l^i) = q$, and $t = p - q$. Then $q \leq 3$. When $t = 0$, since all edges in $E(v_l^i)$ have been colored, we are done. Thus, in the following discussion, we may assume that $t \geq 1$. For a vertex $y \in V(G)$, let $\mathcal{F}(y)$ denote the set of fully conflict vertices of y when certain edges in $E(y)$ are considered to be colored. Obviously, $|\mathcal{F}(y)|$ is no more than the number of 2-neighbors of y in G_j.

Remark 1 No son z of v_{k+1}^{i-1} is in $\mathcal{F}(v_l^i)$, and no neighbor z^* of v_{l-1}^i in layer $i+1$ is in $\mathcal{F}(v_l^i)$.

Proof In fact, if z or $z^* \in \mathcal{F}(v_l^i)$, then it is easy to derive that $v_l^i v_{k+1}^{i+1} \in E(H)$ or $v_l^i v_{l-1}^i \in E(H)$ and hence $d_{G_j}(v_l^i) \geq 3$. However, z or z^* is a leaf or a 2-vertex in G_j. □

The following Claim 1 deals with the number of fully conflict vertices of v_l^i in the subgraph G_j. By Remark 1, we need to consider the number of 2-neighbors of v_l^i in layer $i-2$, $i-1$, and i, respectively.

Claim 1 (a) *If v_k^{i-1} has the unique son v_l^i, then v_l^i has at most 5 fully conflict vertices.*

(b) *If v_k^{i-1} has at least two sons, and v_l^i is the leftmost son of v_k^{i-1}, then v_l^i has at most 5 fully conflict vertices.*

(c) *If v_k^{i-1} has at least two sons, and v_l^i is the rightmost son of v_k^{i-1}, then v_l^i has at most $\Delta + 1$ fully conflict vertices.*

(d) *If v_k^{i-1} has at least three sons, and v_l^i is a middle son of v_k^{i-1}, then v_l^i has at most $\Delta - 1$ fully conflict vertices.*

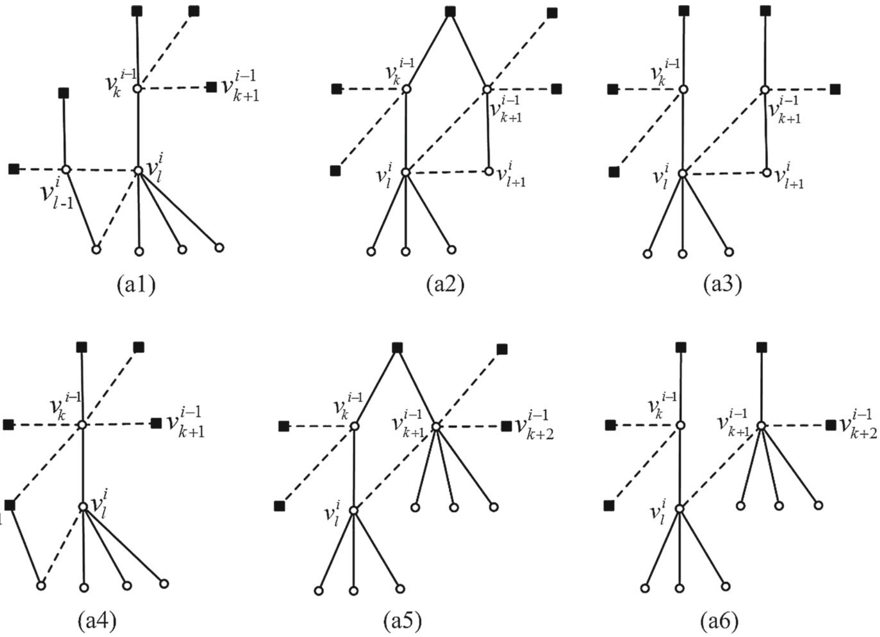


Fig. 3 Configurations **a1**–**a6** in Claim 1(a), where *black squares* denote possible conflict vertices with v_l^i

Proof We first prove (a). Note that v_l^i has at most two neighbors in layer i. If v_l^i has exactly two neighbors in layer i, then Lemma 1(1) asserts that these two neighbors are v_{l-1}^i and v_{l+1}^i, and $v_l^i v_{l-1}^i$ and $v_l^i v_{l+1}^i$ are non-tree-edges. By Observation 1(ii), v_l^i and v_{l-1}^i have a common father, or v_l^i and v_{l+1}^i have a common father, which contradicts the hypothesis that v_l^i is the unique son of v_k^{i-1}.

Now assume that v_l^i has exactly one neighbor in layer i. We need to consider two subcases. If $v_l^i v_{l-1}^i \in E(H)$ and $v_l^i v_{l+1}^i \notin E(H)$, then the father of v_{l-1}^i is v_{k-1}^{i-1} by the outerplanarity of G, and $v_l^i v_{k+1}^{i-1} \notin E(G)$ for otherwise G will contain a separating cycle with v_k^{i-1} as an internal vertex, contradicting (P3). Hence v_l^i has at most 5 fully conflict vertices, as shown in Fig. 3a1. If $v_l^i v_{l-1}^i \notin E(H)$ and $v_l^i v_{l+1}^i \in E(H)$, then the father of v_{l+1}^i is v_{k+1}^{i-1} by Observation 1(i). Furthermore, if the father of v_k^{i-1} is v_h^{i-2}, and the father of v_{k+1}^{i-1} is v_s^{i-2}, then $s = h$, or $s = h + 1$ by the outerplanarity of G. First, we claim that v_l^i has at most two 2-neighbors in layer $i - 2$. Actually, by Lemma 1(2), v_k^{i-1} has at most two neighbors v_h^{i-2} and v_{h+1}^{i-2} in layer $i - 2$, and v_{k+1}^{i-1} has at most two neighbors v_s^{i-2} and v_{s+1}^{i-2} in layer $i - 2$. If $s = h$, then v_l^i has at most two 2-neighbors v_h^{i-2} and v_{h+1}^{i-2} in layer $i - 2$. If $s = h + 1$, then $v_{k+1}^{i-1} v_{s+1}^{i-2} \notin E(G)$ because otherwise G contains a separating cycle with v_{h+1}^{i-2} as an internal vertex, which contradicts (P3). Then v_l^i also has at most two 2-neighbors v_h^{i-2} and v_{h+1}^{i-2} in layer $i - 2$. Hence, as shown in Fig. 3a2, a3, v_l^i has at most 5 fully conflict vertices.

Finally, assume that v_l^i has no neighbor in layer i, that is, $v_l^i v_{l-1}^i, v_l^i v_{l+1}^i \notin E(H)$. It is easy to see that there do not exist a vertex x in layer $i-1$ and a vertex y in layer $i+1$ such that $x v_l^i$ and $y v_l^i$ are non-tree-edges, for otherwise G will contain a separating cycle with v_k^{i-1} as

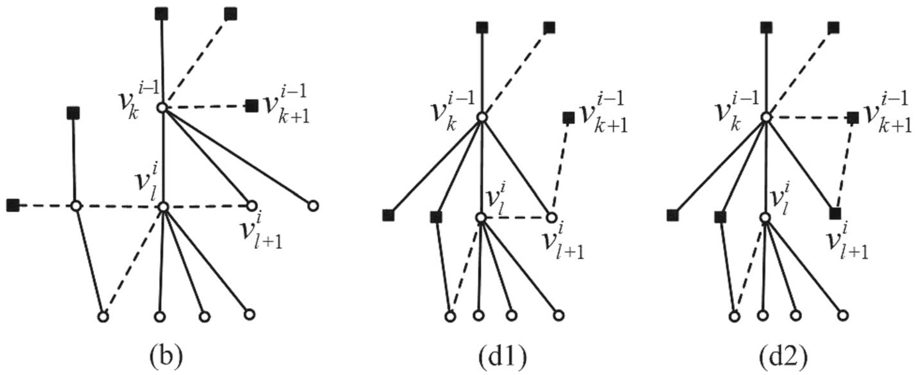


Fig. 4 Configurations **b**, **d1** and **d2** in Claim 1(b) and (d)

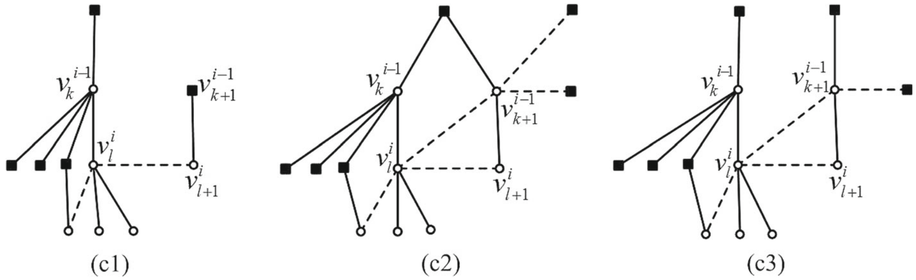


Fig. 5 Configurations **c1**–**c3** in Claim 1(c)

an internal vertex. Then v_l^i has at most 5 fully conflict vertices as shown in Fig. 3a4–a6 by Remark 1.

Next we prove (b) and (d). It is easy to inspect from Lemma 1 that v_l^i has the desired properties whatever v_l^i is adjacent to v_{l+1}^i, as shown in Fig. 4b, d1, d2. It is worth noticing that in (b), when v_{l+1}^i has two neighbors in layer $i-1$, then the second neighbor, other than v_k^{i-1}, must be v_{k+1}^{i-1} by Lemma 1(2).

Finally, we prove (c). First assume that v_l^i has exactly one neighbor in layer $i-1$, i.e., v_k^{i-1}. Thus, $v_l^i v_{k+1}^{i-1} \notin E(G)$. If $v_l^i v_{l+1}^i \notin E(G)$, then v_l^i has obviously at most $\Delta - 1$ fully conflict vertices, since each such vertex must be a neighbor of v_k^{i-1}. So assume that $v_l^i v_{l+1}^i \in E(G)$. Then v_{l+1}^i also has exactly one neighbor in layer $i-1$, i.e., v_{k+1}^{i-1}, by the outerplanarity of G. It turns out that v_l^i has at most $(\Delta - 1) + 1 = \Delta$ fully conflict vertices, as shown in Fig. 5c1.

Next assume that v_l^i has two neighbors in layer $i-1$. By Lemma 1, these two neighbors are v_k^{i-1} and v_{k+1}^{i-1}. If $v_l^i v_{l+1}^i \in E(G)$, then the father of v_{l+1}^i is v_{k+1}^{i-1} by Observation 1(i). Let v_h^{i-2} and v_s^{i-2} be the father of v_k^{i-1} and v_{k+1}^{i-1}, respectively. Then $s = h$ or $s = h + 1$ by the outerplanarity of G. If $s = h$, then v_l^i has at most $(\Delta - 1) + 2 = \Delta + 1$ fully conflict vertices, as shown in Fig. 5c2. If $s = h + 1$, then v_{k+1}^{i-1} has exactly one neighbor in layer $i-2$ by the outerplanarity of G. Thus v_l^i has at most $(\Delta - 1) + 2 = \Delta + 1$ fully conflict vertices, as shown in Fig. 5c3. If $v_l^i v_{l+1}^i \notin E(G)$, with the similar argument, we can show that v_l^i has at most $(\Delta - 1) + 2 = \Delta + 1$ fully conflict vertices. □

Since $\Delta \geq 5$, Claim 1 implies that v_l^i has at most $\Delta + 1$ fully conflict vertices in every case.

Our project is to legally color the edges in $E(u_j) \setminus \{zu_j | zu_j \in E_{j-1}\}$ in such an order if they exist: (1) $v_l^i v_{l+1}^i \in E(H)$; (2) $v_l^i v_s^{i+1} \in E(H)$; (3) other uncolored edges.

It should be pointed out that if $d_{G_j}(v_{l+1}^i) = d_G(v_{l+1}^i)$, then to construct a 2DVDE partial coloring of G on G_j, we must keep the legality of v_{l+1}^i, i.e., the color set of v_{l+1}^i is required to differ from those of its fully conflict vertices. To do this, let us estimate the number of fully conflict vertices of v_{l+1}^i or v_s^{i+1}, where $v_l^i v_s^{i+1}$ is a non-tree-edge.

Claim 2 *If $v_l^i v_{l+1}^i \in E(H)$ and $d_{G_j}(v_{l+1}^i) = d_G(v_{l+1}^i)$, then $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 3$.*

Proof If v_k^{i-1} is the root, then $v_k^{i-1} v_{l+1}^i \in E(T)$, and $\mathcal{F}(v_{l+1}^i)$ consists of at most one vertex v_{l-1}^i. Hence $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + |\mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 1$. So suppose that v_k^{i-1} is not the root. By Observation 1(i), the father of v_{l+1}^i is v_k^{i-1} or v_{k+1}^{i-1}.

- Assume that the father of v_{l+1}^i is v_k^{i-1}. If v_{l+1}^i has exactly one neighbor in layer $i-1$, i.e., v_k^{i-1}, then it is easy to see that $d_{G_j}(v_{l+1}^i) = d_G(v_{l+1}^i) = 2$. Note that every 2-neighbor of v_{l+1}^i in $N(v_k^{i-1}) \setminus \{v_{l-1}^i, v_l^i, v_{l+1}^i\}$ is also a 2-neighbor of v_l^i. This implies that $\mathcal{F}(v_{l+1}^i) \subseteq \mathcal{F}(v_l^i) \cup \{v_{l-1}^i\}$, and hence $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 1$. Now suppose that v_{l+1}^i has exactly two neighbors in layer $i-1$, i.e., v_k^{i-1} and v_{k+1}^{i-1}. It follows that $v_{l+1}^i v_{k+1}^{i-1} \in E(H)$ and $d_{G_j}(v_{l+1}^i) = d_G(v_{l+1}^i) = 3$. Note that the sons of v_{k+1}^{i-1} are of degree at most 2 in G_j. Therefore, if $x \in \mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i)$, then x is just v_{l-1}^i, or is the neighbor of v_{k+1}^{i-1} in layer $i-1$ or $i-2$. By Lemma 1, v_{k+1}^{i-1} has at most two neighbors in layer $i-2$, and at most two neighbors in layer $i-1$, whereas someone of these neighbors is v_k^{i-1}. Let v_t^{i-2} be the father of v_k^{i-1}, and v_s^{i-2} be the father of v_{k+1}^{i-1}. Then $s = t$ or $s = t+1$ by the outerplanarity of G. If $s = t$, then $v_t^{i-2} \in \mathcal{F}(v_l^i) \cap \mathcal{F}(v_{l+1}^i)$, and hence $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 3$. If $s = t+1$, then v_{k+1}^{i-1} has exactly one neighbor in layer $i-2$, for otherwise G contains a separating cycle with v_s^{i-2} as an internal vertex, contradicting (P3). We also get that $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 3$.
- Assume that the father of v_{l+1}^i is v_{k+1}^{i-1}. Then v_{l+1}^i has exactly one neighbor in layer $i-1$, for otherwise G contains a separating cycle with v_{k+1}^{i-1} as an internal vertex. Hence $v_{l+1}^i v_{k+2}^{i-1} \notin E(G)$, and $d_{G_j}(v_{l+1}^i) = d_G(v_{l+1}^i) = 2$. Let v_t^{i-2} be the father of v_k^{i-1}, and v_s^{i-2} be the father of v_{k+1}^{i-1}. Then $s = t$ or $s = t+1$ by the outerplanarity of G. We define

$$A = \left\{v_{l-1}^i, v_k^{i-1}, v_{k+2}^{i-1}, v_s^{i-2}, v_{s+1}^{i-2}\right\},$$
$$B = \left\{z | z \text{ is a son of } v_{k+1}^{i-1} \text{ other than } v_{l+1}^i \text{ with } d_{G_j}(z) = d_G(z) = 2\right\}.$$

Then it is not difficult to see that $|B| \leq 1$ and $\mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i) \subseteq A \cup B$. If $|B| = 1$, say $B = \{w\}$, then wv_{k+2}^{i-1} is a non-tree-edge by the outerplanarity of G. This implies that $v_{k+2}^{i-1} \notin \mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i)$, for otherwise $v_{k+1}^{i-1} v_{k+2}^{i-1} \in E(G)$ and $d_G(v_{k+2}^{i-1}) \geq 3$. Thus, at most one vertex in $B \cup \{v_{k+2}^{i-1}\}$ belongs to $\mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i)$. Assume that $d_{G_j}(v_k^{i-1}) \geq 3$. If at most one of v_s^{i-2} and v_{s+1}^{i-2} is in $\mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i)$, then $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 3$. Otherwise, $v_s^{i-2}, v_{s+1}^{i-2} \in \mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i)$. Then $v_{k+1}^{i-1} v_{s+1}^{i-2}$ is a non-tree-edge. By the outerplanarity of G, s must be identical to t. Hence, $v_s^{i-2} = v_t^{i-2} \in \mathcal{F}(v_l^i)$, contradicting the assumption that $v_s^{i-2} \in \mathcal{F}(v_{l+1}^i) \setminus \mathcal{F}(v_l^i)$. Now assume that $d_{G_j}(v_k^{i-1}) = 2$. Then $v_l^i v_{l-1}^i \notin E(G)$, for otherwise G would contain a separating cycle with v_k^{i-1} as an internal vertex. Thus, $v_{l-1}^i \notin \mathcal{F}(v_{l+1}^i)$. If $s = t$, then $v_t^{i-2} \in \mathcal{F}(v_l^i) \cap \mathcal{F}(v_{l+1}^i)$, and henceforth $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 3$. If $s = t+1$, then v_{k+1}^{i-1} has exactly one neighbor in layer $i-2$, for otherwise G will contain a separating cycle with v_s^{i-2} as an internal vertex. Consequently, $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \leq |\mathcal{F}(v_l^i)| + 3$. □

Claim 3 *If $v_l^i v_s^{i+1} \in E(H)$ and $d_{G_j}(v_s^{i+1}) = d_G(v_s^{i+1})$, then $|\mathcal{F}(v_s^{i+1})| \leq 2$.*

Proof By Lemma 1, v_{l-1}^i is the father of v_s^{i+1}. All the sons of v_{l-1}^i, other than v_s^{i+1}, are leaves in G_j, and $d_{G_j}(v_s^{i+1}) = d_G(v_s^{i+1}) = 2$. Every vertex in $\mathcal{F}(v_s^{i+1})$ is either a neighbor of v_l^i or v_{l-1}^i in layer $i-1$ or i, or a vertex x_0 in layer $i+1$ with $x_0 v_{l-1}^i \in E(H)$. Notice that if x_0 exists, then v_{l-2}^i is the father of x_0 by Lemma 1. Hence, either $d_{G_j}(v_{l-2}^i) \geq 3$, or $d_{G_j}(v_{l-2}^i) = 2$ and v_{l-2}^i has distance 3 to v_s^{i+1}. Therefore, at most one of x_0 and v_{l-2}^i belongs to $\mathcal{F}(v_s^{i+1})$. It is easy to see that x_0 is unique if it exists by Lemma 1. We need to consider two situations as follows.

Case 1 v_k^{i-1} is the root.

If $d_{G_j}(v_k^{i-1}) = 2$, then v_k^{i-1} has exactly two neighbors v_l^i and v_{l-1}^i in layer 1. Thus, $\mathcal{F}(v_s^{i+1})$ contains at most one vertex, i.e., v_k^{i-1}, hence $|\mathcal{F}(v_s^{i+1})| \leq 1$. If $d_{G_j}(v_k^{i-1}) \geq 3$, then since v_l^i has at most two neighbors in layer 1 (including v_{l-1}^i), we have $|\mathcal{F}(v_s^{i+1})| \leq 2$.

Case 2 v_k^{i-1} is not the root.

Case 2.1 v_k^{i-1} is not the father of v_{l-1}^i.

Then v_{k-1}^{i-1} is the father of v_{l-1}^i, and $v_l^i v_{k+1}^{i-1} \notin E(G)$, for otherwise G contains a separating cycle with v_k^{i-1} as an internal vertex.

Case 2.1.1 $v_l^i v_{l+1}^i \in E(H)$.

Then v_k^{i-1} is the father of v_{l+1}^i by the outerplanarity of G. Thus, $d_{G_j}(v_k^{i-1}) \geq 3$. If $d_{G_j}(v_{k-1}^{i-1}) \geq 3$, then $\mathcal{F}(v_s^{i+1}) \subseteq \{x_0, v_{l-2}^i, v_{l+1}^i\}$ and hence $|\mathcal{F}(v_s^{i+1})| \leq 2$ by the above discussion. So suppose that $d_{G_j}(v_{k-1}^{i-1}) = 2$. If neither x_0 nor v_{l-2}^i is in $\mathcal{F}(v_s^{i+1})$, then $\mathcal{F}(v_s^{i+1}) \subseteq \{v_{k-1}^{i-1}, v_{l+1}^i\}$ and consequently $|\mathcal{F}(v_s^{i+1})| \leq 2$. Otherwise, at least one of $x_0 v_{l-1}^i$ and $v_{l-2}^i v_{l-1}^i$ belongs to $E(H)$. Since $d_{G_j}(v_{k-1}^{i-1}) = 2$, v_{k-1}^{i-1} is not the father of v_{l-2}^i, and hence G contains a separating cycle with v_{k-1}^{i-1} as an internal vertex, contradicting (P3).

Case 2.1.2 $v_l^i v_{l+1}^i \notin E(H)$.

Then $v_{l+1}^i \notin \mathcal{F}(v_s^{i+1})$. If neither x_0 nor v_{l-2}^i is in $\mathcal{F}(v_s^{i+1})$, then $\mathcal{F}(v_s^{i+1}) \subseteq \{v_{k-1}^{i-1}, v_k^{i-1}\}$, and so $|\mathcal{F}(v_s^{i+1})| \leq 2$. Otherwise, at least one of $x_0 v_{l-1}^i$ and $v_{l-2}^i v_{l-1}^i$ belongs to $E(H)$. Hence v_{k-1}^{i-1} is the father of v_{l-2}^i, for otherwise G contains a separating cycle with v_{k-1}^{i-1} as an internal vertex, contradicting (P3). Therefore, $d_{G_j}(v_{k-1}^{i-1}) \geq 3$, and hence $\mathcal{F}(v_s^{i+1}) \subseteq \{x_0, v_{l-2}^i, v_k^{i-1}\}$. Thus $|\mathcal{F}(v_s^{i+1})| \leq 2$ by the above discussion.

Case 2.2 v_k^{i-1} is the father of v_{l-1}^i.

It suffices to show that at most one of v_{l+1}^i and v_{k+1}^{i-1} is in $\mathcal{F}(v_s^{i+1})$. Assume the contrary, we have $v_l^i v_{l+1}^i, v_l^i v_{k+1}^{i-1} \in E(G)$. By Observation 1(i), v_{l+1}^i is the son of v_{k+1}^{i-1}. Hence $d_{G_j}(v_{k+1}^{i-1}) \geq 3$, and so $v_{k+1}^{i-1} \notin \mathcal{F}(v_s^{i+1})$, a contradiction. Consequently, $|\mathcal{F}(v_s^{i+1})| \leq 2$. □

Now we continue to construct a 2DVDE partial $(\Delta + 8)$-coloring of G on G_j when $t \geq 1$. It suffices to discuss the following cases, depending on the size of t.

Case 1 $t = 1$.

Let x be the neighbor of v_l^i not in G_{j-1}. There are two subcases to be disposed as follows:
Case 1.1 x is in layer i.

By Lemma 1, $x = v_{l+1}^i$. Note that xv_l^i has at most $q+2$ adjacent edges in G_{j-1}, which are contained in $\{v_l^i v_{l-1}^i, v_l^i v_k^{i-1}, v_l^i v_{k+1}^{i-1}, xv_h^{i-1}, xv_{h+1}^{i-1}\}$, where v_k^{i-1} is the father of v_i^l, v_h^{i-1} is the father of x, and $k \le h$. By Observation 1(i), $h = k$ or $h = k+1$. Thus, xv_l^i has at most $q+2 \le 5$ forbidden colors when it is considered for legal coloring at Step j.

If $d_G(x) > d_{G_j}(x)$, then we need only to legally color xv_l^i with some color in C to form a 2DVDE partial coloring of G on G_j. This can be done since $|C| - (q+2+|\mathcal{F}(v_l^i)|) \ge |C| - (q+2+(\Delta+1)) = \Delta+8-(\Delta+6) = 2$ by Claim 1. It is worth mentioning that, in this case, we do not need to consider the legality of the vertex x. Now suppose that $d_G(x) = d_{G_j}(x)$. Let us further handle two possibilities:

- Assume that $h = k$. Then $q \le 2$. If v_l^i is the leftmost son of v_k^{i-1}, then $|\mathcal{F}(v_l^i)| \le 5$ by Claim 1(b). Since $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \le |\mathcal{F}(v_l^i)|+3 \le 8$ by Claim 2, and $|C|-(q+2+8) \ge 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j. Otherwise, v_l^i is a middle son of v_k^{i-1}, we have $|\mathcal{F}(v_l^i)| \le \Delta - 1$ by Claim 1(d), and $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)| \le (\Delta-1)+3 = \Delta+2$ by Claim 2. Since $|C| - (q+2+(\Delta+2)) \ge 2$, we can legally color xv_l^i with some color in C to form a 2DVDE partial coloring of G on G_j.
- Assume that $h = k+1$. Then v_l^i is the rightmost son of v_k^{i-1}, and x is the leftmost son of v_{k+1}^{i-1}. Then $v_{l+1}^i v_{k+2}^{i-1} \notin E(G)$, for otherwise G has a separating cycle with v_{k+1}^{i-1} as an internal vertex. Hence, xv_l^i has at most $q+1$ adjacent edges in G_{j-1} and $d_G(x) = d_{G_j}(x) = 2$. If v_l^i is the unique son of v_k^{i-1}, then $|\mathcal{F}(v_l^i)| \le 5$ by Claim 1(a), $|\mathcal{F}(v_l^i) \cup \mathcal{F}(v_{l+1}^i)| \le |\mathcal{F}(v_l^i)|+3 \le 8$ by Claim 2, and $|C|-(q+1+8) \ge 1$. We can legally color xv_l^i with some color in C. Otherwise, v_k^{i-1} has at least two sons, and v_l^i is the rightmost son of v_k^{i-1}. By Claim 1(c), $|\mathcal{F}(v_l^i)| \le \Delta+1$. We note that if $q = 3$, then $v_l^i v_{l-1}^i$ is a non-tree-edge, which implies that $|\mathcal{F}(v_l^i)| \le \Delta$ by the proof of Claim 1(c). Thus, we always have that $|\mathcal{F}(v_l^i)|+q \le \Delta+3$. Since $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|+q \le |\mathcal{F}(v_l^i)|+3+q \le \Delta+3+3 = \Delta+6$ by Claim 2, and $|C| - ((\Delta+6)+1) \ge 1$, we can legally color xv_l^i with some color in C to set up a 2DVDE partial coloring of G on G_j.

Case 1.2 x is in layer $i+1$.

Then xv_l^i has at most $q+1 \le 4$ forbidden colors. If $d_G(x) > d_{G_j}(x)$, then we can legally color xv_l^i with some color in C, because $|C|-(q+1+|\mathcal{F}(v_l^i)|) \ge |C|-(q+1+(\Delta+1)) \ge 3$ by Claim 1. Now assume that $d_G(x) = d_{G_j}(x)$. If $d_G(x) = d_{G_j}(x) = 1$, then we can give xv_l^i a legal coloring as in the previous case. Otherwise, $d_G(x) = d_{G_j}(x) = 2$. By Claim 1, $|\mathcal{F}(v_l^i)| \le \Delta+1$. By Claim 3, $|\mathcal{F}(x)| \le 2$. Thus, $|C| - (|\mathcal{F}(v_l^i)| + |\mathcal{F}(x)| + q + 1) \ge |C| - ((\Delta+1)+2+4) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.
Case 2 $t = 2$.

Let x and y be the neighbors of v_l^i not in G_{j-1}. Note that at least one of x and y differs from v_{l+1}^i, say $y \ne v_{l+1}^i$. Our proof splits into the following two subcases.
Case 2.1 $x = v_{l+1}^i$.

Since xv_l^i has at most $q+2 \le 5$ forbidden colors, and $|\mathcal{F}(x)| \le \Delta+1$ by Claim 1, we can legally color xv_l^i with a color a in C (at the moment, we do not consider whether or not v_l^i has fully conflict vertices because yv_l^i has not been colored). Now yv_l^i still has at most

$q+1+1 \leq 5$ forbidden colors. We assert that $|C|-(|\mathcal{F}(v_l^i)|+|\mathcal{F}(y)|+q+2) > 0$, so that yv_l^i can be legally colored with some color in C. Otherwise, by Claims 2-3, we derive that $|\mathcal{F}(v_l^i)| = \Delta+1$, $|\mathcal{F}(y)| = 2$, and $q = 3$. Since $q = 3$, we see that $v_l^i v_{l-1}^i \in E(H)$, hence $v_{l-1}^i \notin \mathcal{F}(v_l^i)$. Recalling the proof of Claim 1(c), we have $|\mathcal{F}(v_l^i)| \leq \Delta$, a contradiction.
Case 2.2 $x \neq v_{l+1}^i$.

Both x and y are in layer $i+1$, say $x = v_h^{i+1}$, $y = v_s^{i+1}$ with $h > s$. Indeed, $h = s+1$ by Lemma 1. We first legally color yv_l^i with some color $a \in C$, since yv_l^i has at most $q+1 \leq 4$ forbidden colors, and $|\mathcal{F}(y)| \leq 2$ by Claim 3. Then we color legally xv_l^i with some color in $C \setminus \{a\}$, since xv_l^i has at most $q+1 \leq 4$ forbidden colors, and $|\mathcal{F}(v_l^i)| \leq \Delta+1$ by Claim 1.
Case 3 $t \geq 3$.

Let $x_1, x_2, \ldots, x_t$ be the neighbors of v_l^i not in G_{j-1}. Without loss of generality, we may assume that $x_1 = v_{l+1}^i$, $x_2 = v_s^{i+1}$, …, $x_t = v_{s+t-2}^{i+1}$ by Lemma 1. (If $x_1 = v_{s-1}^{i+1}$, we have a similar discussion). First, since $x_1 v_l^i$ has at most $q+2 \leq 5$ forbidden colors, and $|\mathcal{F}(x_1)| \leq \Delta+1$ by Claim 1, we may legally color $x_1 v_l^i$. Second, since $x_2 v_l^i$ has at most $q+2 \leq 5$ forbidden colors and $|\mathcal{F}(x_2)| \leq 2$ by Claim 3, we may legally color $x_2 v_l^i$. Finally, since $\binom{\Delta+8-q-2}{t-2} \geq \binom{\Delta+3}{t-2} \geq \Delta+3$, there exists a subset C' of available colors in C with $|C'| = t-2$, such that $v_l^i v_{s+1}^{i+1}, \ldots, v_l^i v_{s+t-2}^{i+1}$ can be legally colored with C'. □

Using the result of Theorem 1, we can describe the following algorithm of finding a 2DVDE $(\Delta+8)$-coloring of an outerplanar graph G with $\Delta \geq 5$:

Algorithm: 2DVDE-Color Outerplanar Graphs (I).
Input: A connected outerplanar graph G with maximum degree $\Delta \geq 5$;
Output: A 2DVDE $(\Delta+8)$-coloring of G;

1. Choose a Δ-vertex u_1 and run an OBFT partition starting from u_1.
2. Color properly $E(u_1)$ with colors $1, 2, \ldots, \Delta$.
3. Repeat for each vertex u_j, from left to right, from top to down: Coloring the edges in $E(u_j) \setminus (E(u_1) \cup E(u_2) \cup \cdots \cup E(u_{j-1}))$ according to Theorem 1.

Theorem 2 *Let G be a connected outerplanar graph with $n \geq 2$ vertices and $\Delta \geq 5$. The algorithm* **2DVDE-Color Outerplanar Graphs (I)** *runs in $O(n^3)$ time.*

Proof First, according to a result of [8], the algorithm of searching a spanning tree T in a graph G by using an OBFT partiton needs $O(n)$ time. Next, the algorithm performed in Theorem 1 is iterated n times. At each iteration, we legally color the edge subset $E_j^* := E(u_j) \setminus E_{j-1}$. The time complexity to consider in this part includes performing the following four tasks.

(a) Computing the total number τ_1 of forbidden colors for the edges in E_j^*;
(b) Computing the total number τ_2 of fully conflict vertices for at most three fixed vertices;
(c) Selecting a subset C' of available colors from $C = \{1, 2, \ldots, \Delta+8\}$;
(d) Coloring properly E_j^* with C'.

For (a), it follows from the proof of Theorem 1 that $\tau_1 \leq 5|E_j^*| \leq 5d_G(u_j) \leq 5\Delta$. For (b), Claims 1–3 showed that each vertex x has at most $\Delta+1$ fully conflict vertices in G_{j-1}. Thus, $\tau_2 \leq 3(\Delta+1)$. For (c), we need to eliminate at most $\Delta+7$ color subsets from C, whereas every color subset has at most Δ elements. For (d), at most Δ edges are required for a legal coloring. The above analysis shows that the total running time of our algorithm is at most $n(5\Delta+3(\Delta+1)+\Delta(\Delta+7)+\Delta) = n(\Delta^2+16\Delta+3) = O(n\Delta^2)$. Since $\Delta \leq n-1$, our algorithm runs in $O(n^3)$ time. □

3 Outerplanar graphs with $\Delta \le 4$

In this section, we will present a 2DVDE 10-coloring for an outerplanar graph with maximum degree at most 4 using the color set $C = \{1, 2, \ldots, 10\}$. The discussion is similar to the proof of Theorem 1.

Theorem 3 *If G is an outerplane graph with $\Delta \le 4$, then $\chi'_{d2}(G) \le 10$.*

Proof Let G be a connected outerplane graph with $\Delta \le 4$. Carrying out an OBFT partiton, we edge-partition G into a rooted spanning tree T with r as its root and a subgraph H with $\Delta(H) \le 4$, where $d_G(r) = \Delta$. Let u_j $(1 \le j \le |V(G)|)$ be the j-th vertex visited in OBFT, where $u_1 = r$. Define $E_j = E(u_1) \cup E(u_2) \cup \cdots \cup E(u_j)$, and $G_j = G[E_j]$, which is the subgraph of G induced by the edges in E_j.

At Step 1, we color $u_1u_2, u_1u_3, \ldots, u_1u_{\Delta+1}$ with colors $1, 2, \ldots, \Delta$, respectively. Then we color successively $u_2u_3, u_3u_4, \ldots, u_{\Delta}u_{\Delta+1}$ with 5, 6, 7 if existed. This is available since $G[\{u_2, u_3, \ldots, u_{\Delta+1}\}]$ contains at most three edges by Lemma 1.

At Step j, for $j = 2, 3, \ldots, \Delta+1$, we properly color the uncolored edges in $E(u_j)$ from left to right such that the used colors are selected as follows:

- If j is even, the foregoing colors in the set $\{8, j, j+1, \ldots, \Delta, 1, 2, \ldots, j-2\}$ are used;
- If j is odd, the foregoing colors in the set $\{9, j, j+1, \ldots, \Delta, 1, 2, \ldots, j-2\}$ are used.

Let $j \ge \Delta + 2$. Assume that E_{j-1} has been colored such that a 2DVDE partial coloring ϕ_{j-1} of G on G_{j-1} has been established. At Step j, we will color $E(u_j)$ to find a 2DVDE partial coloring of G on G_j. As soon as all $E(u_j)$'s, for $1 \le j \le |V(G)|$, have been colored, a 2DVDE coloring of G using at most 10 colors is constructed.

Let $u_j = v_l^i$. Then $i \ge 2$. Let v_k^{i-1} be the father of v_l^i, and v_h^{i-2} be the father of v_k^{i-1}. Set $d_{G_j}(v_l^i) = d_G(v_l^i) = p$, $d_{G_{j-1}}(v_l^i) = q$, and $t = p - q$. Then $q \le 3$ and $p = t + q \le \Delta \le 4$. Let us consider the following cases, depending on the size of t.

Case 1 $t = 0$.

Since all edges in $E(v_l^i)$ have been colored, we are done.

Case 2 $t = 1$.

Let x be the neighbor of v_l^i not in G_{j-1}. Without loss of generality, assume that $d_{G_j}(x) = d_G(x)$. In fact, if $d_{G_j}(x) < d_G(x)$, the proof is easier since we do not need to consider whether or not x has fully conflict vertices. There are two subcases as follows:

Case 2.1 x is in layer i.

By Lemma 1, $x = v_{l+1}^i$. Note that xv_l^i has at most $q + 2$ adjacent edges in G_{j-1}.

Case 2.1.1 $q = 3$, i.e., $v_l^i v_{l-1}^i, v_l^i v_{k+1}^{i-1} \in E(H)$.

Then v_{k+1}^{i-1} is the father of x, and $v_{l-1}^i v_k^{i-1} \in E(T)$ by Observation 1. So $d_{G_j}(v_k^{i-1}) \ge 3$. Note that $xv_{k+2}^{i-1} \notin E(G)$ for else G contains a separating cycle with v_{k+1}^{i-1} as internal vertex, contradicting (P3). Let z^* be the forth neighbor of v_k^{i-1} if $d_{G_j}(v_k^{i-1}) = 4$. If $v_{l-1}^i v_{l-2}^i \in E(G)$, then $z^* = v_{l-2}^i$ for otherwise G will contain a separating cycle with v_k^{i-1} as internal vertex. Since v_{k+1}^{i-1} has at most two other neighbors except v_l^i and x, say y_1 and y_2, by the fact that $\Delta \le 4$, we know that $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, y_1, y_2\}$ and $\mathcal{F}(x) \subseteq \{v_{l-1}^i, y_1, y_2\}$ by Remark 1. Since $|C| - (q + 1 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \ge 10 - (3 + 1 + 5) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Case 2.1.2 $q = 2$ with $v_l^i v_{k+1}^{i-1} \in E(H)$.

Then $v_l^i v_{l-1}^i \notin E(G)$, and v_{k+1}^{i-1} is the father of x by Observation 1. Again, $xv_{k+2}^{i-1} \notin E(G)$ by (P3). Similarly to the previous discussion, v_{k+1}^{i-1} has at most two other neighbors y_1

and y_2, and v_k^{i-1} has at most two other neighbors z_1 and z_2. This implies that $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, y_1, y_2, z_1, z_2\}$ and $\mathcal{F}(x) \subseteq \{v_k^{i-1}, y_1, y_2\}$. Since $|C| - (q + 1 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \geq 10 - (2 + 1 + 6) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Case 2.1.3 $q = 2$ with $v_l^i v_{l-1}^i \in E(H)$.

Then $v_l^i v_{k+1}^{i-1} \notin E(G)$. Now the proof splits into the following two cases.

Case 2.1.3.1 v_k^{i-1} is the father of x.

Suppose that $xv_{k+1}^{i-1} \notin E(G)$. Then every 2-neighbor of v_{l+1}^i in $N(v_k^{i-1}) \setminus \{v_{l-1}^i, v_l^i, v_{l+1}^i\}$ is also a 2-neighbor of v_l^i. This implies that $\mathcal{F}(x) \subseteq \mathcal{F}(v_l^i) \cup \{v_{l-1}^i\}$, and hence $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)| \leq |\mathcal{F}(v_l^i)| + 1$. Since $|\mathcal{F}(v_l^i)| \leq 5$ by Claim 1, and $|C| - (q + 1 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \geq 10 - (2 + 1 + 5 + 1) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Suppose that $xv_{k+1}^{i-1} \in E(G)$. Then $d_{G_j}(x) = 3$, and v_k^{i-1} is the father of v_{l-1}^i by (P3). Moreover, $v_{l-1}^i v_{l-2}^i \notin E(G)$ by (P3) and the assumption that $\Delta \leq 4$, which implies that $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, v_{k+1}^{i-1}\}$ by Remark 1. By Claim 2, $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)| \leq |\mathcal{F}(v_l^i)| + 3 \leq 5$. Since $|C| - (q + 2 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \geq 10 - (2 + 2 + 5) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Case 2.1.3.2 v_{k+1}^{i-1} is the father of x.

Then v_k^{i-1} is the father of v_{l-1}^i by Observation 1(ii), so $d_{G_j}(v_k^{i-1}) \geq 3$. Note that $xv_{k+2}^{i-1} \notin E(G)$ by (P3). Let z^* be the forth neighbor of v_k^{i-1} if $d_{G_j}(v_k^{i-1}) = 4$. If $v_{l-1}^i v_{l-2}^i \in E(G)$, then $z^* = v_{l-2}^i$ by (P3). By Claim 2, $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)| \leq |\mathcal{F}(v_l^i)| + 3 \leq 6$. Since $|C| - (q + 1 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \geq 10 - (2 + 1 + 6) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Case 2.1.4 $q = 1$, i.e., $v_l^i v_{k+1}^{i-1}, v_l^i v_{l-1}^i \notin E(G)$.

Suppose that v_k^{i-1} is the father of x. Let z^* be the forth neighbor of v_k^{i-1} if exists. Then $|\mathcal{F}(v_l^i)| \leq |\{z^*, v_h^{i-2}, v_{k+1}^{i-1}\}| = 3$. By Claim 2, $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)| \leq |\mathcal{F}(v_l^i)| + 3 \leq 6$. Since $|C| - (q + 2 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \geq 10 - (1 + 2 + 6) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Suppose that v_{k+1}^{i-1} is the father of x. Then $v_{l+1}^i v_{k+2}^{i-1} \notin E(G)$, and $|\mathcal{F}(v_l^i)| \leq |(N(v_k^{i-1}) \setminus \{v_l^i\}) \cup \{v_{k+1}^{i-1}\}| \leq 4$. By Claim 2, $|\mathcal{F}(v_l^i) \cup \mathcal{F}(x)| \leq |\mathcal{F}(v_l^i)| + 3 \leq 7$. Since $|C| - (q + 1 + |\mathcal{F}(v_l^i) \cup \mathcal{F}(x)|) \geq 10 - (1 + 1 + 7) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Case 2.2 x is in layer $i + 1$.

Then xv_l^i has at most $q + 1 \leq 4$ forbidden colors. If $d_G(x) = d_{G_j}(x) = 1$, then we can legally color xv_l^i with some color in C, because $|C| - (q + 1 + |\mathcal{F}(v_l^i)|) \geq 10 - (q + 1 + 5) \geq 1$ by Claim 1. Otherwise, $d_G(x) = d_{G_j}(x) = 2$, i.e., $xv_l^i \in E(H)$, and v_{l-1}^i is the father of x.

Case 2.2.1 $v_l^i v_{k+1}^{i-1} \in E(G)$.

Then v_k^{i-1} is the father of v_{l-1}^i by (P3), which implies that $d_{G_j}(v_k^{i-1}) \geq 3$. Let z^* be the forth neighbor of v_k^{i-1} if $d_{G_j}(v_k^{i-1}) = 4$.

Suppose that $d_{G_j}(v_{k+1}^{i-1}) = 2$. Let y^* be the neighbor of v_{k+1}^{i-1} other than v_l^i. Then $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, y^*, v_{l-1}^i\}$. Moreover, when $q = 3$, we have $v_l^i v_{l-1}^i \in E(G)$ and so $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, y^*\}$. Consequently, $q + |\mathcal{F}(v_l^i)| \leq 6$. By Claim 3, $|\mathcal{F}(x)| \leq 2$. Since $|C| - (q + 1 + |\mathcal{F}(v_l^i)| + |\mathcal{F}(x)|) \geq 10 - (6 + 1 + 2) = 1$, we can legally color xv_l^i with some color in C.

Suppose that $d_{G_j}(v_{k+1}^{i-1}) \geq 3$. By the proof of Claim 3, $|\mathcal{F}(x)| \leq 1$. By Claim 1, $|\mathcal{F}(v_l^i)| \leq 5$. Moreover, if $q = 3$, then $v_l^i v_{l-1}^i \in E(G)$ and $v_{l-1}^i \notin \mathcal{F}(v_l^i)$, implying $|\mathcal{F}(v_l^i)| \leq 4$. Consequently, $q+|\mathcal{F}(v_l^i)| \leq 7$. Since $|C|-(q+1+|\mathcal{F}(v_l^i)|+|\mathcal{F}(x)|) \geq 10-(7+1+1) = 1$, we can legally color xv_l^i with some color in C.

Case 2.2.2 $v_l^i v_{k+1}^{i-1} \notin E(G)$.

Then $1 \leq q \leq 2$. Assume that $q = 1$. By Claim 1, $|\mathcal{F}(v_l^i)| \leq 5$. By Claim 3, $|\mathcal{F}(x)| \leq 2$. Since $|C| - (q + 1 + |\mathcal{F}(v_l^i)| + |\mathcal{F}(x)|) \geq 10 - (1 + 1 + 5 + 2) = 1$, we can legally color xv_l^i with some color in C to get a 2DVDE partial coloring of G on G_j.

Assume that $q = 2$, i.e., $v_l^i v_{l-1}^i \in E(G)$. We assert that $|C| - (q + 1 + |\mathcal{F}(v_l^i)| + |\mathcal{F}(x)|) \geq 1$, so that xv_l^i can be legally colored with some color in C to get a 2DVDE partial coloring of G on G_j. Otherwise, we derive that $|\mathcal{F}(v_l^i)| = 5$, $|\mathcal{F}(x)| = 2$, and $\phi_{j-1}(xv_{l-1}^i) \neq \phi_{j-1}(v_l^i v_k^{i-1})$. Hence $\phi_{j-1}(xv_{l-1}^i) \notin \{\phi_{j-1}(v_l^i v_{l-1}^i), \phi_{j-1}(v_l^i v_k^{i-1})\}$, which implies that $v_k^{i-1} \notin \mathcal{F}(x)$. In view of the proof of Claim 3, we have $|\mathcal{F}(x)| \leq 1$, a contradiction.

Case 3 $t = 2$.

Then $q \leq 2$. Let x and y be the neighbors of v_l^i not in G_{j-1}. Note that at least one of x and y differs from v_{l+1}^i, say $y \neq v_{l+1}^i$. Our proof splits into the following two subcases.

Case 3.1 $x = v_{l+1}^i$.

Since xv_l^i has at most $q + 2 \leq 4$ forbidden colors, and $|\mathcal{F}(x)| \leq 5$ by Claim 1, we can legally color xv_l^i with a color a in C (at the moment, we do not consider whether or not v_l^i has fully conflict vertices because yv_l^i has not been colored). Now yv_l^i still has at most $q + 1 + 1 \leq 4$ forbidden colors.

If $d_G(y) > d_{G_j}(y)$, then we can legally color yv_l^i with some color in C, because $|C|-(q+1+1+|\mathcal{F}(v_l^i)|) \geq 10-(q+2+5) \geq 1$ by Claim 1. Now assume that $d_G(y) = d_{G_j}(y)$. If $d_G(y) = d_{G_j}(y) = 1$, then we can give yv_l^i a legal coloring as in the previous case. Otherwise, $d_G(y) = d_{G_j}(y) = 2$, i.e., $yv_l^i \in E(H)$. Then v_{l-1}^i is the father of y.

Case 3.1.1 v_{k+1}^{i-1} is the father of x.

Then v_k^{i-1} is the father of v_{l-1}^i, and hence $d_{G_j}(v_k^{i-1}) \geq 3$. Let z^* be the forth neighbor of v_k^{i-1} if $d_{G_j}(v_k^{i-1}) = 4$.

Suppose that $q = 1$. Then $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, v_{l-1}^i, v_{k+1}^{i-1}\}$ by Remark 1. By Claim 3, $|\mathcal{F}(y)| \leq 2$. Since $|C| - (q + 1 + 1 + |\mathcal{F}(v_l^i)| + |\mathcal{F}(y)|) \geq 10 - (1 + 1 + 1 + 4 + 2) = 1$, we can legally color yv_l^i with some color in C.

Suppose that $q = 2$ and $v_l^i v_{l-1}^i \in E(G)$. If $v_{l-1}^i v_{l-2}^i \in E(G)$, then $z^* = v_{l-2}^i$ by (P3). So $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, v_{k+1}^{i-1}\}$ by Remark 1. By Claim 3, $|\mathcal{F}(y)| \leq 2$. Since $|C| - (q + 1 + 1 + |\mathcal{F}(v_l^i)| + |\mathcal{F}(y)|) \geq 10 - (2 + 1 + 1 + 3 + 2) = 1$, we can legally color yv_l^i with some color in C. If $v_{l-1}^i v_{l-2}^i \notin E(G)$, then we have a similar discussion.

Suppose that $q = 2$ and $v_l^i v_{k+1}^{i-1} \in E(G)$. Then $d_{G_j}(v_{k+1}^{i-1}) \geq 3$. Note that v_{k+1}^{i-1} has at most two other neighbors, say y_1, y_2. Therefore, $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, v_{l-1}^i, z^*, y_1, y_2\}$. By Claim 3, $|\mathcal{F}(y)| \leq 2$. Without loss of generality, assume that $\phi_{j-1}(v_l^i v_k^{i-1}) = 1$, $\phi_{j-1}(v_l^i v_{k+1}^{i-1}) = 2$, $a = 3$, and $\phi_{j-1}(v_{l-1}^i y) = \alpha$.

- $\alpha \in \{1, 2, 3\}$. Suppose that yv_l^i cannot be legally colored. It follows that $|\mathcal{F}(y)| = 2$ and $|\mathcal{F}(v_l^i)| = 5$. In this case, $x \in \mathcal{F}(y)$, and the second vertex in $\mathcal{F}(y)$ is v_{l-2}^i or a neighbor of v_{l-1}^i in layer $i + 1$, say u^*, with $d_{G_j}(v_{l-2}^i) = 2$ or $d_{G_j}(u^*) = 2$. If $v_{l-2}^i \in \mathcal{F}(y)$, then

$v_{l-2}^i v_{l-1}^i \in E(H)$, and the father of v_{l-2}^i is v_k^{i-1} by (P3). That is $v_{l-2}^i \in \mathcal{F}(v_l^i)$. Hence, $|\mathcal{F}(y) \cup \mathcal{F}(v_l^i)| \le |\mathcal{F}(y)| + |\mathcal{F}(v_l^i)| - 1 = 6$. Since $|C| - (q + 1 + |\mathcal{F}(y) \cup \mathcal{F}(v_l^i)|) \ge 10 - (2 + 1 + 6) = 1$, yv_l^i can be legally colored, a contradiction. Next suppose that $u^* \in \mathcal{F}(y)$. Then the father of u^* is v_{l-2}^i by Lemma 1, and v_k^{i-1} is the father of v_{l-2}^i by (P3). That is $z^* = v_{l-2}^i$, which implies that $v_{l-2}^i = z^* \in \mathcal{F}(v_l^i)$. Without loss of generality, assume that $C_{\phi_{j-1}}(x) = \{3, 4\}$, $C_{\phi_{j-1}}(y_i) = \{1, 2, 3, i + 4\}$ for $i = 1, 2$, $C_{\phi_{j-1}}(v_h^{i-2}) = \{1, 2, 3, 7\}$, $C_{\phi_{j-1}}(z^*) = \{1, 2, 3, 8\}$, $C_{\phi_{j-1}}(v_{l-1}^i) = \{1, 2, 3, 9\}$, and $C_{\phi_{j-1}}(u^*) = \{\alpha, 10\}$. Then $10 \in C_{\phi_{j-1}}(v_{l-1}^i)$, which is impossible.

- $\alpha \notin \{1, 2, 3\}$, say $\alpha = 4$. Then $x \notin \mathcal{F}(y)$, since when yv_l^i is properly colored, x and y have different color sets. Similarly, we can show that $v_{l-1}^i \notin \mathcal{F}(v_l^i)$. Therefore $|\mathcal{F}(y)| \le 1$ and $|\mathcal{F}(v_l^i)| \le 4$ by the proof of Claims 1 and 3. Since $|C| - (4 + |\mathcal{F}(y)| + |\mathcal{F}(v_l^i)|) \ge 10 - (4 + 1 + 4) = 1$, we can get a 2DVDE partial coloring of G on G_j.

Case 3.1.2 v_k^{i-1} is the father of x.

Then $d_{G_j}(v_k^{i-1}) \ge 3$. Let z^* be the forth neighbor of v_k^{i-1} when $d_{G_j}(v_k^{i-1}) = 4$.

If $xv_{k+1}^{i-1} \in E(G)$, then $z^* = v_{l-1}^i$, for else G contains a separating cycle with v_k^{i-1} as an internal vertex, contradicting (P3). Hence, $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, v_{k+1}^{i-1}\}$. Since $|C| - (q + 1 + 1 + |\mathcal{F}(y)| + |\mathcal{F}(v_l^i)|) \ge 10 - (2 + 1 + 1 + 2 + 3) = 1$ by Claim 3, we can get a 2DVDE partial coloring of G on G_j. If $xv_{k+1}^{i-1} \notin E(G)$, then $\mathcal{F}(v_l^i) \subseteq \{v_h^{i-2}, z^*, v_{l-1}^i\}$ when $v_l^i v_{l-1}^i \notin E(G)$, or $\mathcal{F}(v_l^i) \cup \mathcal{F}(y) \subseteq \{v_h^{i-2}, z^*, z_1, z_2, x\}$ when $v_l^i v_{l-1}^i \in E(G)$, where z_1, z_2 denote the other two neighbors of v_{l-1}^i. Since $|C| - (q + 1 + 1 + |\mathcal{F}(y)| + |\mathcal{F}(v_l^i)|) \ge 10 - (2 + 1 + 1 + 2 + 3) = 1$ by Claim 3, we can get a 2DVDE partial coloring of G on G_j.

Case 3.2 $x \ne v_{l+1}^i$.

Both x and y are in layer $i + 1$, say $x = v_h^{i+1}$, $y = v_s^{i+1}$ with $h > s$. Indeed, $h = s + 1$ by Lemma 1. We first legally color yv_l^i with some color $a \in C$, since yv_l^i has at most $q + 1 \le 3$ forbidden colors, and $|\mathcal{F}(y)| \le 2$ by Claim 3. Then we color legally xv_l^i with some color in $C \setminus \{a\}$, since xv_l^i has at most $q + 1 \le 3$ forbidden colors, and $|\mathcal{F}(v_l^i)| \le 5$ by Claim 1.

Case 4 $t = 3$.

Then $q = 1$. Let x_1, x_2, x_3 be the neighbors of v_l^i not in G_{j-1}. Without loss of generality, we may assume that $x_1 = v_{l+1}^i$, $x_2 = v_s^{i+1}$, $x_3 = v_{s+1}^{i+1}$ by Lemma 1. (If $x_1 = v_{s-1}^{i+1}$, we have a similar discussion). First, since $x_1 v_l^i$ has at most $q + 2 \le 3$ forbidden colors, and $|\mathcal{F}(x_1)| \le 5$ by Claim 1, we may legally color $x_1 v_l^i$. Second, since $x_2 v_l^i$ has at most $q + 1 \le 2$ forbidden colors and $|\mathcal{F}(x_2)| \le 2$ by Claim 3, we may legally color $x_2 v_l^i$. Finally, since $|C| - (q + 2) = 7 > |\mathcal{F}(v_l^i)|$, we color legally $x_3 v_l^i$ with some color in C. □

Using the result of Theorem 3, we now describe an algorithm of finding a 2DVDE 10-coloring in an outerplanar graph G with $\Delta \le 4$:

Algorithm: 2DVDE-Color Outerplanar Graphs (II).
Input: A connected outerplanar graph G with maximum degree $\Delta \le 4$;
Output: A 2DVDE 10-coloring of G;

1. Choose a Δ-vertex u_1 and run an OBFT partition starting from u_1.
2. Coloring $E(u_1)$ with colors $1, 2, \ldots, \Delta$.
3. Coloring the edges in $G[\{u_2, u_3, \ldots, u_{\Delta+1}\}]$ with colors $5, 6, 7$.
4. Coloring the uncolored edges in $E(u_j)$ for $j = 2, 3, \ldots, \Delta + 1$ from left to right, where the used colors are selected as follows:

- If j is even, the foregoing colors in the set $\{8, j, j+1, \ldots, \Delta, 1, 2, \ldots, j-2\}$ are used;
- If j is odd, the foregoing colors in the set $\{9, j, j+1, \ldots, \Delta, 1, 2, \ldots, j-2\}$ are used.

5. Repeat for each vertex u_j with $j \geq \Delta+2$, from left to right, from top to down: Coloring the edges in $E(u_j) \setminus (E(u_1) \cup E(u_2) \cup \cdots \cup E(u_{j-1}))$ according to Theorem 3.

Similarly to the proof of Theorem 2, we obtain the following algorithmic complexity:

Theorem 4 *Let G be a connected outerplanar graph with $n \geq 2$ vertices and $\Delta \leq 4$. The algorithm* **2DVDE-Color Outerplanar Graphs (II)** *runs in $O(n^3)$ time.*

4 Conclusion

In this paper, we focus on the study of 2-distance vertex-distinguishing index of the class of outerplanar graphs. Combining Theorems 1 and 3, we have the following consequence:

Theorem 5 *If G is an outerplane graph, then $\chi'_{d2}(G) \leq \Delta + 8$.*

The upper bound $\Delta + 8$ in Theorem 5 seems not to be best possible. We like to conclude this paper by raising the following conjecture:

Conjecture 3 *If G is an outerplane graph, then $\chi'_{d2}(G) \leq \Delta + 2$.*

Note that if a graph G contains two Δ-vertices at distance 2, then $\chi'_{d2}(G) \geq \Delta+1$. On the other hand, it is easy to construct infinitely many outerplanar graphs G with $\chi'_{d2}(G) = \Delta+1$. These two facts imply that the upper bound $\Delta + 1$ is tight, if Conjecture 3 is true.

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