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Nonemptiness and boundedness of solution sets for vector variational inequalities via topological method

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Abstract In this paper, some characterizations of nonemptiness and boundedness of solution sets for vector variational inequalities are studied in finite and infinite dimensional spaces, respectively. By using a new proof method which is different from the one used in Huang et al. (J Optim Theory Appl 162:548–558 2014), a sufficient and necessary condition for the nonemptiness and boundedness of solution sets is established. Basing on this result, some new characterizations of nonemptiness and boundedness of solution sets for vector variational inequalities are proved. Compared with the known results in Huang et al. (2014), the key assumption that $K_\infty \cap (F(K))^{w\circ}_C = \{0\}$ is not required in finite dimensional spaces. Furthermore, the corresponding result of Huang et al. (2014) is extended to the case of infinite dimensional spaces. Some examples are also given to illustrated the main results.

Keywords Vector variational inequality · Nonemptiness and boundedness · C-pseudomonotone · Connectedness · Recession cone

Mathematics Subject Classification 49J40 · 90C31

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1 Introduction

The concept of vector variational inequality (*VVI*), which was first introduced by Giannessi [2] in finite dimensional spaces, has wide applications in many problems such as finance, economics, transportation, optimization, operations research and engineering sciences. It is known that (*VVI*) is closely related to vector optimization problem (*VOP*). In recent years, various kinds of vector variational inequalities (*VVIs*) and vector optimization problems (*VOPs*) have been intensively studied in a general setting by many authors, see for example [2–8] and the references therein. An important and interesting topic for (*VVIs*) and (*VOPs*) is to study the nonemptiness and boundedness properties of solution sets, because it is an important condition to guarantee the convergence of some algorithms for solving monotone-type variational inequalities and optimization problems.

Some authors have studied the characterizations of nonemptiness and boundedness of solution sets for (*VVIs*) and (*VOPs*) by using the asymptotic analysis methods. For instance, Hu and Fang [9] investigate conditions for nonemptiness and compactness of the solution sets of pseudomonotone (*VVIs*) in finite dimensional spaces. Deng [10,11] presents various characterizations of the nonemptiness and boundedness of solution sets of convex (*VOP*) in finite dimensional spaces and reflexive Banach spaces, respectively. Huang et al. [12] characterize the nonemptiness and compactness of solution set of convex (*VOP*) with cone constraints in terms of the level-boundedness of the component functions of the objective on the perturbed sets of the original constraint set. For more related works, we refer the readers to [13–15] and the references therein. On the other hand, several characterizations of nonemptiness and compactness of the solution sets for vector equilibrium problems ($VEPs$), which include (*VVIs*) and (*VOPs*) as special cases, have been studied in [16,17].

Flores-Bazán and Vera [18] show that the characterization of the nonemptiness and compactness of solution sets for convex (*VOP*) can be expressed by that of the nonemptiness and compactness of the solution sets of a family of scalar optimization problems. This characterization is different with those obtained in the works mentioned above. Then it is natural and interesting to ask whether a parallel result of Flores-Bazán and Vera [18] still hold true for (*VVIs*).

Huang et al. [1] consider this problem in finite dimensional spaces and answer that the nonemptiness and compactness of the solution sets for (*VVI*) can also be characterized in terms of the nonemptiness and compactness of the solution sets of a family of scalar variational inequalities, but a much stringent assumption of $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$ is indispensable.

Inspired and motivated the papers [1] and [18], we further study the the nonemptiness and compactness of solution sets for *VVI* in finite and infinite dimensional spaces, respectively. First, in finite dimensional spaces, by using the connectedness property of C^{*0}, we show that the characterization of nonemptiness and compactness of solution set for *VVI* can be expressed as that of nonemptiness and boundedness of solution sets for a family of scalar variational inequalities, without the key assumption that $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$ required in [1]. Then, we extend this characterization of nonemptiness and compactness to case of infinite dimensional spaces, under some suitable assumptions. Compared with the precious results obtained in [1], our results don't require the key assumption that $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$. Furthermore, we would like to point out that the proof method here, which based on the connectedness property of C^{*0}, is topological and different with the previous methods of asymptotic analysis in [1] and [18] for the problems of (*VVIs*) and (*VOPs*).

The paper is organized as follows. In Sect. 2, we introduce some basic notations and preliminary results. In Sect. 3, by using a topological method, we establish the nonemptiness and compactness of solution set for (*VVI*) in finite and infinite dimensional spaces, respectively.

2 Preliminaries

Throughout this paper, unless otherwise specified, we always assume that X is a reflexive Banach space with dual space X^*, Y is a normed space with dual space Y^* and $C \subset Y$ is a closed, convex and pointed cone with $intC \neq \emptyset$, where $intC$ denotes the interior of C in Y. Let $e \in intC$ be any given point. We define the dual cone C^* of C by

$$C^* := \left\{x^* \in Y^* : \langle x^*, x\rangle \geq 0,\ \forall x \in C\right\}$$

and the set C^{*0} with respect to C^* by

$$C^{*0} := \left\{x^* \in C^* : \langle x^*, e\rangle = 1\right\}.$$

Clearly, C^{*0} is a convex subset of C^* and so is path-connected and connected. Moreover, Lemma 3.4 of [1] showed that C^{*0} is a w^*-compact base of C^*. As defined in Definition 3.1 of [1], a subset $D_1 \subset C^*$ is said to be a base of C^* iff, $0 \notin D_1$ and $C^* \subset \cup_{t\geq 0} t D_1$.

Let $K \subset X$ be a nonempty, closed and convex subset and $F : K \to 2^{L(X,Y)}$ be a set-valued mapping, where $L(X, Y)$ denotes the family of all continuous linear mappings from X to Y. Consider the following set-valued vector variational inequality associated with (K, F):

$$VVI(K, F) \quad \text{find } x \in K \text{ and } u \in F(x) \text{ such that } \langle u, y - x\rangle \notin -intC, \quad \forall y \in K.$$

Correspondingly, consider the scalar variational inequality associated with (K, F) and $\xi \in C^{*0}$:

$$(VI)_\xi(K, F) \quad \text{find } x \in K \text{ and } u \in F(x) \text{ such that } \langle \xi(u), y - x\rangle \geq 0, \quad \forall y \in K$$

and the dual variational inequality of $(VI)_\xi(K, F)$ which is defined as

$$(DVI)_\xi(K, F) \quad \text{find } x \in K \text{ such that } \langle \xi(v), y - x\rangle \geq 0, \quad \forall y \in K, v \in F(y).$$

We denote the solution sets of $(VVI)(K, F)$, $(VI)_\xi(K, F)$ and $(DVI)_\xi(K, F)$ by $SVVI(K, F)$, $SVI_\xi(K, F)$ and $SDVI_\xi(K, F)$, respectively. In view of Theorem 2.1 of [7], it is known that

$$SVVI(K, F) = \bigcup_{\xi \in C^{*0}} SVI_\xi(K, F).$$

Now we introduce some basic notations in convex analysis. The recession cone of K is defined by

$$K_\infty := \left\{d \in X : \exists t_k \to +\infty,\ x_k \in K \text{ such that } \frac{x_k}{t_k} \rightharpoonup d\right\}.$$

From [19], we know that K_∞ can also be determined by the following formula

$$K_\infty := \{d \in X : \ x_0 + td \in K, \forall t > 0, \quad \forall x_0 \in K\}.$$

The negative polar cone K^- of K is defined by

$$K^- := \left\{x^* \in X^* : \ \langle x^*, x\rangle \ \leq 0, \quad \forall x \in K\right\} = -K^*.$$

The barrier cone of K is defined by

$$barr(K) := \left\{x^* \in X^* : \sup_{x\in K} \langle x^*, x\rangle < \infty\right\}.$$

A nonempty subset $K \subset X$ is said to be well-positioned if, there exist $x_0 \in X$ and $g \in X^*$ such that

$$\langle g, x - x_0 \rangle \geq \|x - x_0\|, \quad \forall x \in K.$$

In view of Theorem 2.1 of [20], it is showed that K is well-positioned if and only if $int(barr(K)) \neq \emptyset$. Also, some other characterizations of the class of well-positioned sets are presented in [20]. In the following, we illustrate an example in infinite dimensional spaces including a well-positioned set, which can be founded in [21].

Example 2.1 Let X be a normed space with dual space X^*. The well-known Bishop-Phelps cone K_g in functional analysis and optimization theory with respect to $g \in X^*$, is defined by

$$K_g := \{x \in X : \langle g, x \rangle \geq \|x\|\}.$$

Then, for any nonempty subset $K \subset X$ with $0 \in K$ and $K \subset K_g$, we know K is well-positioned in X.

From Example 2.1, it is easy to see that the following cone K of l^2, defined by

$$K := \left\{x = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2 : 2\eta_1 \geq \|x\|\right\},$$

is well-positioned, by taking $g = (2, 0, 0, \ldots) \in (l^2)^* = l^2$. Also, any nonempty subset $K_1 \subset K$ is well-positioned.

The following lemma which follows from Proposition 2.1 of [20] plays an important role in our proof.

Lemma 2.1 *If a nonempty, closed and convex subset $K \subset X$ is well-positioned (or $int(barr(K)) \neq \emptyset$), then there exists no sequence $\{x_n\} \subset K$ with $\|x_n\| \to \infty$ such that $\frac{x_n}{\|x_n\|} \rightharpoonup 0$.*

Using Lemma 2.1, now we illustrate an example in infinite dimensional spaces which is not a well-positioned set, which can be founded in [14].

Example 2.2 Let $X = l^2$ and $K = \{x = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2 : |\eta_n| \leq n, \forall n \in N\}$. We show that K is not a well-positioned set in l^2. Suppose to the contrary that K is a well-positioned set. Taking $x_n = ne_n \in K$, where e_n has 1 on the n-th coordinate and zeros elsewhere. Clearly, we have $\|x_n\| = n$ and $\frac{x_n}{\|x_n\|} = e_n$. Since $e_n \rightharpoonup 0$, it follows that $\frac{x_n}{\|x_n\|} \rightharpoonup 0$, which is a contradiction with Lemma 2.1. Therefore, K is not a well-positioned set in l^2.

In [17], the authors propose the concepts of weak and strong C-polar cones associated with a set $\mathcal{L} \subset L(X, Y)$, which are defined by

$$L_C^{w\circ} := \{x \in X : \langle l, x \rangle \notin intC, \ \forall l \in \mathcal{L}\}$$

and

$$L_C^{s\circ} := \{x \in X : \langle l, x \rangle \in -C, \ \forall l \in \mathcal{L}\},$$

respectively. The weak and strong C-polar cones are useful to discuss the solvability of vector variational inequalities [9] and vector equilibrium problems [17].

Definition 2.1 Let $F : K \to 2^{L(X,Y)}$ be a set-valued mapping with nonempty values and $C \subset Y$ is a closed, convex and pointed cone with $intC \neq \emptyset$. F is said to be

(i) C-monotone on K iff, for any $(x, u), (y, v) \in graph(F)$, one has

$$\langle v - u, y - x \rangle \in C;$$

(ii) C-pseudomonotone on K iff, for any $(x, u), (y, v) \in graph(F)$, one has

$$\langle u, y - x \rangle \notin -intC \Rightarrow \langle v, y - x \rangle \in C;$$

(iii) scalar C-pseudomonotone on K iff, for any $\xi \in C^* \setminus \{0\}$ and for any $(x, u), (y, v) \in graph(F)$, one has

$$\langle \xi(u), y - x \rangle \geq 0 \Longrightarrow \langle \xi(v), y - x \rangle \geq 0.$$

Remark 2.1 (i) Obviously, a C-monotone mapping is scalar C-pseudomonotone.

(ii) A scalar C-pseudomonotone mapping is weaker than C-pseudomonotone mapping. Indeed, for any $\xi \in C^*\setminus\{0\}$ and for any $(x, u), (y, v) \in graph(F)$ satisfying $\langle \xi(u), y - x \rangle \geq 0$, we have $\langle u, y - x \rangle \notin -intC$ and so $\langle v, y - x \rangle \in C$, which yields that $\langle \xi(v), y - x \rangle \geq 0$.

Definition 2.2 A topological space E is said to be connected iff, it is not the union of two disjoint nonempty open sets. Moreover, E is said to be path-connected iff, any two points of E can be joined by a path.

The following lemma, which gives an equivalent characterization of connected spaces, plays an important role in our proof.

Lemma 2.2 *A topological space E is connected if and only if the only subsets of E which are both open and closed are E and $\emptyset$ (empty set).*

Definition 2.3 Let $F : K \to 2^{L(X,Y)}$ be a set-valued mapping with nonempty values. F is said to be

(i1) upper semicontinuous on K iff, for every $x \in K$ and every neighborhood $\mathcal{N}(F(x))$ of $F(x)$, there exists a neighborhood $\mathcal{N}(x)$ of x such that $F(\mathcal{N}(x)) \subset \mathcal{N}(F(x))$;

(ii) lower semicontinuous on K iff, for every $x \in K$, $u \in F(x)$ and every neighborhood $\mathcal{N}(u)$ of u, there exists a neighborhood $\mathcal{N}(x)$ of x such that $F(x') \cap \mathcal{N}(u) \neq \emptyset$ for every $x' \in \mathcal{N}(x)$.

The following lemma, which establishes the nonemptiness and boundedness property of solution set for scalar variational inequality $VI_\xi(K, F)$, is due to Theorem 3.2 of [14].

Lemma 2.3 *Let X be a reflexive Banach space, Y be a norm space and $K \subset X$ be a nonempty, closed and convex subset with $int(barr(K)) \neq \emptyset$. Let $\xi \in C^{*0}$ be any given point. Let $F : K \to 2^{L(X,Y)}$ be a set-valued mapping with nonempty, compact and convex values. Suppose that F is scalar pseudomonotone and upper semicontinuous on K. Then the following two conclusions are equivalent:*

(i) $SVI_\xi(K, F)$ is nonempty and bounded;

(ii) $K_\infty \cap [\xi(F(K))]^- = \{0\}$.

Particularly, if X is finite dimensional, then the condition that $int(barr(K)) \neq \emptyset$ can be omitted.

In finite dimensional spaces case, Huang et al. [1] show that the nonemptiness and boundedness property of solution set for $VVI(K, F)$ can be characterized by that of a family of scalar variational inequalities. We list some main results as follows (see Theorems 3.1 and 3.3 of [1]).

Lemma 2.4 *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex subset and Y be a norm space. Let $F : K \to 2^{L(\mathbb{R}^n, Y)}$ be a set-valued mapping with nonempty, compact and convex values. Suppose that F is scalar pseudomonotone and upper semicontinuous on K. Consider the following two statements:*

(i) for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and compact;
(ii) $SVVI(K, F)$ is nonempty and compact.

Then (i)⇒(ii). If in addition that $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$, then (i)⇔(ii).

Remark 2.2 In the proof of [1], the assumption that $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$ is indispensable to ensure the equivalence between (i) and (ii).

The following lemma shows that the condition of $\bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\}$ is weaker than that of $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$.

Lemma 2.5 The following conclusions hold:

(i) $\bigcup_{\xi \in C^{*0}} [\xi(F(K))]^- \subset (F(K))_C^{w\circ}$;
(ii) $K_\infty \cap (F(K))_C^{w\circ} = \{0\} \Rightarrow \bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\}$.

Proof (i) For any $d \notin (F(K))_C^{w\circ}$, there exists some $x_0 \in K$ and $y_0 \in F(K)$ such that

$$\langle y_0, d \rangle \in int\, C.$$

Then, for any $\xi \in C^{*0}$, we have $\langle \xi(y_0), d \rangle > 0$ and so

$$d \notin \bigcup_{\xi \in C^{*0}} [\xi(F(K))]^-,$$

which implies that $\bigcup_{\xi \in C^{*0}} [\xi(F(K))]^- \subset (F(K))_C^{w\circ}$.
(ii) The conclusion (ii) follows directly from (i). This complete the proof. □

The following example shows that the inclusion $\bigcup_{\xi \in C^{*0}} [\xi(F(K))]^- \subset (F(K))_C^{w\circ}$ in Lemma 2.5 may be proper and the inverse of implication (ii) may not be true.

Example 2.3 Let

$$X = \mathbb{R}, \quad K = \mathbb{R}^+, \quad C = \mathbb{R}^2_+, \quad e = (1, 1) \in int\, C, \quad F(x) = \begin{cases} (0, 1 - x), & 0 \le x \le 1, \\ (x - 1, 0), & x > 1. \end{cases}$$

Then $C^{*0} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \ge 0\}$. By a simple computation, we have

$$\bigcup_{\xi \in C^{*0}} [\xi(F(K))]^- = (-\infty, 0]$$

and $(F(K))_C^{w\circ} = \mathbb{R}$, which shows the inclusion in (i) may be strict.
Moreover, since $K_\infty = \mathbb{R}^+$, we obtain that

$$\bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\}$$

and

$$K_\infty \cap (F(K))_C^{w\circ} = \mathbb{R}^+,$$

which means the reverse implication in (ii) is not true.

3 Nonemptiness and boundedness of solution sets for (VVI)

In this section, we study the characterizations of nonemptiness and boundedness of solution sets for $VVI(K, F)$ in finite and infinite dimensional spaces, respectively.

First, when the space X is finite dimensional, we obtain the following Theorem 3.1, which shows that the characterization of nonemptiness and boundedness of solution set for $VVI(K, F)$ can be expressed as that of nonemptiness and boundedness of solution sets for a family of scalar variational inequality, without the assumption that $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$.

Theorem 3.1 *Let X be a finite dimensional space and K be a closed convex subset of X. Let Y be a normed space. Let $F : K \to 2^{L(X,Y)}$ be scalar C-pseudomonotone and upper semicontinuous with nonempty, compact and convex values. Then SVVI(K, F) is nonempty and bounded if and only if for any $\xi \in C^{*0}$, SVI$_\xi(K, F)$ is nonempty and bounded.*

Proof Suppose that for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded. Then for any $\xi \in C^{*0}$, $K_\infty \cap [\xi(F(K))]^- = \{0\}$. We claim that $SVVI(K, F)$ is nonempty and bounded. The nonemptiness of $SVVI(K, F)$ is obvious. We only need to claim that $SVVI(K, F)$ is bounded. Otherwise, there exists a sequence $x^n \in SVVI(K, F)$ such that $\|x^n\| \to +\infty$. Since $x^n \in SVVI(K, F)$, there exists $u^n \in F(x^n)$ such that

$$\langle u^n, x - x^n\rangle \notin -intC, \quad \forall x \in K$$

and so there exists $\xi^n \in C^{*0}$ such that

$$\langle \xi^n(u^n), x - x^n\rangle \geq 0, \quad \forall x \in K.$$

By the pseudomonotonicity of F, it follows that

$$\langle \xi^n(v), x - x^n\rangle \geq 0, \quad \forall x \in K, v \in F(x)$$

and so

$$\left\langle \xi^n(v), \frac{x}{\|x^n\|} - \frac{x^n}{\|x^n\|} \right\rangle \geq 0, \quad \forall x \in K, v \in F(x). \tag{3.1}$$

Since in finite dimensional spaces C^{*0} is a w^*-compact base of C and $\|\frac{x^n}{\|x^n\|}\| = 1$, without loss of generality, we can assume that $w^*\text{-}\lim_{n\to\infty} \xi^n = \xi^0 \in C^{*0}$ and $\lim_{n\to\infty} \frac{x^n}{\|x^n\|} = d \in K_\infty$. Clearly, $d \neq 0$. Letting $n \to \infty$ in (3.1), we obtain that

$$\langle \xi^0(v), d\rangle \leq 0, \quad \forall x \in K, v \in F(x)$$

and so

$$d \in K_\infty \cap (\xi^0(F(K)))^-.$$

Note that $d \neq 0$, this is a contradiction with $K_\infty \cap [\xi^0(F(K))]^- = \{0\}$.

Conversely. Suppose that $SVVI(K, F)$ is nonempty and bounded, we claim that $SVI_\xi(K, F)$ is nonempty and bounded for any $\xi \in C^{*0}$. Define a set A as follows

$$A := \left\{\xi \in C^{*0} : SVI_\xi(K, F) \text{ is nonempty and bounded}\right\}.$$

Clearly, A is nonempty and $A \subset C^{*0}$. We claim that A is both open and closed in C^{*0}. First, we claim that A is open in C^{*0}. Otherwise, there exists $\xi^0 \in A$ and a sequence $\xi^n \in C^{*0}$ with $\xi^n \to \xi^0$ such that $\xi^n \notin A$. This implies that

$$K_\infty \cap [\xi^n(F(K))]^- \neq \{0\}$$

and so there exists $d_n \in K_\infty \cap [\xi^n(F(K))]^-$ with $\|d_n\| = 1$. Since $\|d_n\| = 1$, without loss of generality, we may assume that $d_n \to d_0 \in K_\infty \backslash \{0\}$. Since $d_n \in K_\infty \cap [\xi^n(F(K))]^-$, we have

$$\langle d_n, \xi^n(v) \rangle \leq 0, \quad \forall x \in K, v \in F(x).$$

Letting $n \to \infty$, we have

$$\langle d_0, \xi^0(v) \rangle \leq 0, \quad \forall x \in K, v \in F(x)$$

and so

$$d_0 \in K_\infty \cap (\xi^0(F(K)))^-.$$

Note that $d_0 \neq 0$, we have $SVI_{\xi^0}(K, F)$ is not nonempty and bounded, which contradicts with the fact that $\xi^0 \in A$. Thus, the set A is open in C^{*0}.

Next, we claim that A is closed in C^{*0}. Let $\xi^n \in A$ with $\xi^n \to \xi^0$. We claim that $\xi^0 \in A$. Since $\xi^n \in A$, we have $SVI_{\xi^n}(K, F)$ is nonempty and bounded. Let $x_n \in SVI_{\xi^n}(K, F)$. Since $SVI_{\xi^n}(K, F) \subset SVVI(K, F)$ and $SVVI(K, F)$ is bounded, clearly $\{x_n\}$ is bounded. Without loss of generality, we may assume that $x_n \to x^0 \in K$. Since $x_n \in SVI_{\xi^n}(K, F)$, there exists $x^n \in K, u^n \in F(x^n)$ and $\xi^n \in C^{*0}$ such that

$$\langle \xi^n(u^n), x - x^n \rangle \geq 0, \quad \forall x \in K.$$

By the pseudomonotonicity of F, it follows that

$$\langle \xi^n(v), x - x^n \rangle \geq 0, \quad \forall x \in K, v \in F(x)$$

Letting $n \to \infty$, we yields that

$$\langle \xi^0(v), x - x^0 \rangle \geq 0, \quad \forall x \in K, v \in F(x).$$

This implies that x_0 is a solution of dual variational inequality $(DVI)_{\xi^0}(K, F)$, i.e., $x_0 \in SDVI_{\xi^0}(K, F)$ and so $x_0 \in SVI_{\xi^0}(K, F)$ by Proposition 1 in [13]. Thus, $SVI_{\xi^0}(K, F)$ is nonempty. Moreover, the boundedness of $SVVI(K, F)$ implies that $SVI_{\xi^0}(K, F)$ is bounded. This yields that $\xi^0 \in A$ and so A is closed in C^{*0}.

From the above discussion, we know that $A \subset C^{*0}$ and A is both open and closed in C^{*0}. Since the base C^{*0} of C is connected, from Lemma 2.2 we further obtain that $A = C^{*0}$, which means that for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded. This completes the proof. □

Remark 3.1 Theorem 3.1 presents a new proof method to characterizes the nonemptiness and boundedness of solution set for $VVI(F, K)$ in finite dimensional spaces. The proof method employed here is basing on the connectedness of of C^{*0} and the openness and closeness for the subset $A \subset C^{*0}$. Lemma 2.2 is critical, which allows us to show that $A = C^{*0}$ and so for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded.

Remark 3.2 (i) In Theorem 3.3 of [1], Huang, Fang and Yang obtain a corresponding result of Theorem 3.1 in finite dimensional spaces with the assumption that $K_\infty \cap (F(K))_C^{w\circ} = \{0\}$. We do not require this condition here.

(ii) The proof method is topological and it is different with the previous methods in [1,18] where some asymptotic analysis techniques are used.

As a consequence of Theorem 3.1, we have the following result which establishes some characterizations for $SVVI(K, F)$ being nonempty and bounded in finite dimensional spaces.

Theorem 3.2 Let X be a finite dimensional space and K be a closed convex subset of X. Let Y be a normed space. Let $F : K \to 2^{L(X,Y)}$ be scalar C-pseudomonotone and upper semicontinuous with nonempty, compact and convex values. Then the following statements are equivalent.

(i) $SVVI(K, F)$ is nonempty and bounded;
(ii) For any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded;
(iii) $\bigcup_{\xi \in C^{*0}}(K_\infty \cap [\xi(F(K))]^-) = \{0\}$.

Proof The conclusion follows directly from Theorem 3.1 and Lemma 2.3. This completes the proof. □

The following example is used to illustrate Theorems 3.1 and 3.2.

Example 3.1 Let

$$X = \mathbb{R}, \quad K = \mathbb{R}^+, \quad C = \mathbb{R}^2_+, \quad e = (1, 1) \in int\, C, \quad F(x) = (F_1(x), F_2(x)), \ \forall x \in K,$$

where $F_1(x) = x^2$ for any $x \in K$ and $F_2(x) = \begin{cases} 0, & 0 \le x \le 1, \\ x^2 - 1, & x > 1. \end{cases}$

Then $K_\infty = \mathbb{R}^+$ and $C^{*0} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \ge 0\}$. It is not hard to verify that F is scalar $\mathbb{R}^2_+$-pseudomonotone and upper semicontinuous with nonempty, compact and convex values. Thus, all the conditions of Theorem 3.1 are satisfied. By a simple computation, we have

$$SVVI(K, F) = [0, 1]$$

and

$$SVI_\xi(K, F) = \begin{cases} [0, 1], & \xi = (0, 1), \\ \{0\}, & \xi = (1, 0), \\ \{0\}, & \text{otherwise}. \end{cases}$$

Furthermore, we obtain that

$$[\xi(F(K))]^- = -R^+, \quad \forall \xi \in C^{*0}$$

and so

$$\bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\}.$$

From the above discussion, we know that the conclusions of Theorems 3.1 and 3.2 hold.

When the space X is infinite dimensional, a similar result of Theorem 3.1 can also be obtained under some additional assumptions that C^{*0} is a compact base of C^* and $int(barr K) \neq \emptyset$.

Theorem 3.3 *Let X be a reflexive Banach space and K be a closed convex subset of X with $int(barr K) \neq \emptyset$. Let Y be a normed space. Let $F : K \to 2^{L(X,Y)}$ be scalar C-pseudomonotone and upper semicontinuous with nonempty, compact and convex values. Suppose that C^{*0} is a compact base of C^*. Then SVVI(K, F) is nonempty and bounded if and only if for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded.*

Proof Suppose that for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded. Then for any $\xi \in C^{*0}$, $K_\infty \cap [\xi(F(K))]^- = \{0\}$. We claim that $SVVI(K, F)$ is nonempty and bounded. The nonemptiness of $SVVI(K, F)$ is obvious. We only need to claim that $SVVI(K, F)$ is bounded. Otherwise, there exists a sequence $x^n \in SVVI(K, F)$ such that $\|x^n\| \to +\infty$. Since $x^n \in SVVI(K, F)$, there exists $u^n \in F(x^n)$ such that

$$\langle u^n, x - x^n\rangle \notin -intC, \quad \forall x \in K.$$

Then there exists $\xi^n \in C^{*0}$ such that

$$\langle \xi^n(u^n), x - x^n\rangle \geq 0, \quad \forall x \in K.$$

By the pseudomonotonicity of F, we have

$$\langle \xi^n(v), x - x^n\rangle \geq 0, \quad \forall x \in K, v \in F(x)$$

and so

$$\left\langle \xi^n(v), \frac{x}{\|x^n\|} - \frac{x^n}{\|x^n\|}\right\rangle \geq 0, \quad \forall x \in K, v \in F(x). \tag{3.2}$$

Since C^{*0} is compact and $\|\frac{x^n}{\|x^n\|}\| = 1$, without loss of generality, we can assume that $\xi^n \to \xi^0 \in C^{*0}$ and $\frac{x^n}{\|x^n\|} \rightharpoonup d \in K_\infty$. By Lemma 2.1, $d \neq 0$. Letting $n \to \infty$ in (3.2), we obtain that

$$\langle \xi^0(v), d\rangle \leq 0, \quad \forall x \in K, v \in F(x)$$

and so

$$d \in K_\infty \cap (\xi^0(F(K)))^-$$

with $d \neq 0$, which is a contradiction.

Conversely. Suppose that $SVVI(K, F)$ is nonempty and bounded, we claim that $SVI_\xi(K, F)$ is nonempty and bounded for any $\xi \in C^{*0}$. Define a set $A \subset C^{*0}$ as follows

$$A := \{\xi \in C^{*0} : SVI_\xi(K, F) \text{ is nonempty and bounded}\}.$$

Clearly, A is nonempty and $A \subset C^{*0}$. We claim that A is both open and closed in C^{*0}. First, we claim that A is open in C^{*0}. Otherwise, there exists $\xi^0 \in A$ and a sequence $\xi^n \in C^{*0}$ with $\xi^n \to \xi^0$ such that $\xi^n \notin A$. This means that

$$K_\infty \cap [\xi^n(F(K))]^- \neq \{0\}$$

and so there exists $d_n \in K_\infty \cap [\xi^n(F(K))]^-$ such that $\|d_n\| = 1$. Since C^{*0} is compact and $\|d_n\| = 1$, without loss of generality, we may assume that $d_n \rightharpoonup d_0 \in K_\infty \backslash \{0\}$. Since $d_n \in K_\infty \cap [\xi^n(F(K))]^-$, we have

$$\langle d_n, \xi^n(v)\rangle \leq 0, \quad \forall x \in K, v \in F(x).$$

Letting $n \to \infty$, we have

$$\langle d_0, \xi^0(v)\rangle \leq 0, \quad \forall x \in K, v \in F(x)$$

and so

$$d_0 \in K_\infty \cap (\xi^0(F(K)))^-$$

with $d_0 \neq 0$. This implies that $SVI_{\xi^0}(K, F)$ is not nonempty and bounded, which contradicts with the fact that $\xi^0 \in A$. Thus, the set A is open in C^{*0}.

Next, we claim that A is closed in C^{*0}. Let $\xi^n \in A$ with $\xi^n \to \xi^0$. We claim that $\xi^0 \in A$. Since $\xi^n \in A$, we have $SVI_{\xi^n}(K, F)$ is nonempty and bounded. Let $x_n \in SVI_{\xi^n}(K, F)$. Since

$SVI_{\xi^n}(K, F) \subset SVVI(K, F)$ and $SVVI(K, F)$ is bounded, clearly $\{x_n\}$ is bounded. Without loss of generality, we may assume that $x_n \rightharpoonup x^0 \in K$. Since $x_n \in SVI_{\xi^n}(K, F)$, there exists $x^n \in K$, $u^n \in F(x^n)$ and $\xi^n \in C^{*0}$ such that

$$\langle \xi^n(u^n), x - x^n \rangle \geq 0, \quad \forall x \in K.$$

By the pseudomonotonicity of F, it follows that

$$\langle \xi^n(v), x - x^n \rangle \geq 0, \quad \forall x \in K, v \in F(x).$$

Letting $n \to \infty$, we obtain that

$$\langle \xi^0(v), x - x^0 \rangle \geq 0, \quad \forall x \in K, v \in F(x).$$

This implies that $x_0 \in SDVI_{\xi^0}(K, F)$ and so $x_0 \in SVI_{\xi^0}(K, F)$ by Proposition 1 in [13]. Thus, $SVI_{\xi^0}(K, F)$ is nonempty. Then, the boundedness of $SVVI(K, F)$ implies that $SVI_{\xi^0}(K, F)$ is bounded. This yields that $\xi^0 \in A$ and so A is closed.

Then using a similar discussion as in Theorem 3.1, we obtain that for any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded. This completes the proof. □

From Theorem 3.3, we have the following conclusion.

Theorem 3.4 Let X be a reflexive Banach space and K be a closed convex subset of X with $int(barr K) \neq \emptyset$. Let Y be a normed space. Let $F : K \to 2^{L(X,Y)}$ be scalar C-pseudomonotone and upper semicontinuous with nonempty, compact and convex values. Suppose that C^{*0} is a compact base of C^*. Then the following statements are equivalent.

(i) $SVVI(K, F)$ is nonempty and bounded;
(ii) For any $\xi \in C^{*0}$, $SVI_\xi(K, F)$ is nonempty and bounded;
(iii) $\bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\}$.

Proof The conclusion follows directly from Theorem 3.3 and Lemma 2.3. This completes the proof. □

The following example is used to illustrate Theorems 3.3 and 3.4.

Example 3.2 Let

$$X = l^2, \quad K := \{x = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2 : 2\eta_1 \geq \|x\|, \eta_n \geq 0, \forall n \in N\}$$

and

$$C = \mathbb{R}^2_+, \quad e = (1, 1) \in int\, C, \quad F(x) = (F_1(x), F_2(x)), \quad \forall x \in K,$$

where $F_1(x) := x$ for each $x \in K$ and $F_2(x) := (\eta_1^2, \eta_2^2, \ldots, \eta_n^2, \ldots)$ for each $x = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in K$. Then for any $\xi = (\xi_1, \xi_2) \in C^* \backslash \{0\}$, the mapping $\xi_1 F_1 + \xi_2 F_2$ is monotone and so $F = (F_1, F_2)$ is C-pseudomonotone and upper semicontinuous on K with weakly compact convex values. Moreover, from Example 2.1, we know that K is a well-positioned set in l^2 and so $int(barr K) \neq \emptyset$. Thus, all the conditions of Theorem 3.4 are satisfied. By a simple computation, we have

$$SVVI(K, F) = \{0\}$$

and

$$SVI_\xi(K, F) = 0, \quad \forall \xi \in C^{*0}.$$

Since K is a cone, then $K_\infty = K$. Furthermore, we obtain that

$$K_\infty \cap [\xi(F(K))]^- = K \cap [\xi(F(K))]^- = \{0\}, \quad \forall \xi \in C^{*0}$$

and so

$$\bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\}.$$

From the above discussion, we know that all the conclusions of Theorems 3.3 and 3.4 hold.

The following example shows that the assumption of $int(barr K) \neq \emptyset$ (i.e., the well-positionedness of K) in Theorem 3.4 can not be dropped (in the setting of infinite dimensional spaces).

Example 3.3 Let

$$X = l^2, \quad K = \{x = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2 : |\eta_n| \leq n, \quad \forall n \in N\}$$

and

$$C = \mathbb{R}^2_+, \quad e = (1, 1) \in int\, C, \quad F(x) = (F_1(x), F_2(x)), \quad \forall x \in K,$$

where $F_1(x) := N_K(x) \cap B = \partial d_K(x)$ for each $x \in K$ (here, B denotes the closed unit ball in X), and $F_2(x) := x$ for each $x \in K$. It is not hard to see that K is an unbounded closed convex set. Moreover, for any $\xi = (\xi_1, \xi_2) \in C^* \backslash \{0\}$, the mapping $\xi_1 F_1 + \xi_2 F_2$ is monotone and so $F = (F_1, F_2)$ is C-pseudomonotone and upper semicontinuous on K with weakly compact convex values. As pointed out in Example 2.2, K is not a well-positioned set in l^2 and so $int(barr K) = \emptyset$. Now we show that $K_\infty = \{0\}$. Otherwise, there exists some $d \in l^2$ with $d \neq 0$ such that

$$x_0 + td \in K, \quad \forall t > 0, \quad \forall x_0 \in l^2.$$

If $t > 0$ large enough, it is a contradiction with $|\eta_n| \leq n$ and so $K_\infty = \{0\}$.

Since $K_\infty = \{0\}$ (here 0 denotes the zero point in l^2), we have

$$\bigcup_{\xi \in C^{*0}} (K_\infty \cap [\xi(F(K))]^-) = \{0\},$$

which means that conclusion (iii) of Theorem 3.4 holds.

However, by a simple computation, we obtain that

$$SVVI(K, F) = K$$

and

$$SVI_\xi(K, F) = \begin{cases} \{0\}, & \xi = (0, 1), \\ K, & \xi = (1, 0), \\ \{0\}, & \text{otherwise}, \end{cases}$$

which implies that conclusions (i) and (ii) of Theorem 3.4 doesn't hold.

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