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Generalized Farkas' lemma and gap-free duality for minimax DC optimization with polynomials and robust quadratic optimization

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Abstract Motivated by robust (non-convex) quadratic optimization over convex quadratic constraints, in this paper, we examine minimax difference of convex (dc) optimization over convex polynomial inequalities. By way of generalizing the celebrated Farkas' lemma to inequality systems involving the maximum of dc functions and convex polynomials, we show that there is no duality gap between a minimax DC polynomial program and its associated conjugate dual problem. We then obtain strong duality under a constraint qualification. Consequently, we present characterizations of robust solutions of uncertain general non-convex quadratic optimization problems with convex quadratic constraints, including uncertain trust-region problems.

Keywords Generalized Farkas's lemma · Difference of convex optimization · Minimax programs · Duality · Non-convex quadratic optimization · Robust optimization · Convex polynomials

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1 Introduction

In this paper, we consider the minimax difference of convex optimization problem,

$$(\mathrm{P}) \qquad \inf_{x\in\mathbb{R}^n}\left\{\max_{j=1,\ldots,l} f_j(x)-h(x)\ :\ g_i(x)\le 0,\ i=1,\ldots,m\right\}$$

where $h:\mathbb{R}^n\to\mathbb{R}$ is a convex function, $f_j:\mathbb{R}^n\to\mathbb{R}$, $j=1,\ldots,l$ and $g_i:\mathbb{R}^n\to\mathbb{R}$, $i=1,2,\ldots,m$ are convex polynomials. The minimax model (P) was motivated by robust optimization of uncertain (not necessarily convex) quadratic functions, $\langle x, Ax\rangle+\langle a,x\rangle+\alpha$, over a convex set K, where the data (A,a,α) is assumed to be uncertain and it belongs to an uncertainty set U. As an illustration, consider the simple case, where the uncertainty set U is given by the convex hull of the set $\{(A_j,a_j,\alpha_j)\ :\ j=1,2,\ldots,l\}$, A_j are $n\times n$ symmetric matrices, $a_j\in\mathbb{R}^n$ and $\alpha_j\in\mathbb{R}$. The worst-case solution of the uncertain quadratic minimization is obtained by solving the minimax problem

$$\min_{x\in K}\max_{j=1,\ldots,l}\left\{\langle x, A_jx\rangle+\langle a_j,x\rangle+\alpha_j\right\}.$$

It is a minimax dc-optimization problem of the form (P),

$$\min_{x\in K}\max_{j=1,\ldots,l}\left\{\langle x,(A_j+\rho I_n)x\rangle+\langle a_j,x\rangle+\alpha_j\right\}-\rho\langle x,x\rangle$$

where $\rho=\max\{0,\max_{j=1,\ldots,l}\{-\gamma_{\min}(A_j)\}\}$, $\gamma_{\min}(A_j)$ is the least eigenvalue of A_j, $A_j+\rho I_n$ is positive semidefinite, $f_j(x):=\langle x,(A_j+\rho I_n)x\rangle+\langle a_j,x\rangle+\alpha_j$, for $j=1,\ldots,l$ and $h(x)=\rho\langle x,x\rangle$ are convex polynomials.

Minimax optimization of this form arises, for instance, when solving uncertain quadratic minimization problems with convex quadratic constraints, such as uncertain trust-region problems by robust optimization (see Sect. 5). Moreover, the model (P) includes the minimax polynomial optimization problems, examined recently in [13,15], where $h=0$ and classes of DC-optimization problems which have been extensively studied in the literature (see [4–6,8,9,16] and other references therein).

We make the following key contributions to global non-convex optimization.

(i) We establish a generalization of the Farkas lemma by obtaining a complete dual characterization of the implication

$$g_i(x)\le 0\quad i=1,\ldots,m\Rightarrow \max_{j=1,\ldots,l} f_j(x)-h(x)\ge 0$$

in terms of conjugate functions *without any qualifications*. A numerical example is given to show that such a qualification-free generalization may fail when the functions g_i's are not polynomials. Consequently, we also obtain a form of S-procedure [18] for non-convex quadratic systems. For a recent survey of generalizations of the Farkas lemma, see [7] and S-procedure, see [18].

(ii) As an application of our generalized Farkas' lemma, we show that there is *no duality gap* between (P) and its associated conjugate dual problem, which, in particular, establishes that there is no duality gap between a dc polynomial program and its associated standard conjugate dual program. We also obtain strong duality between the problems under a normal cone constraint qualification, instead of the commonly used closed cone epigraph constraint qualification (see [5,6,8,11,12,16]). We then derive necessary and sufficient global optimality conditions for classes of problems including fractional programming problems.

(iii) Finally, we present complete dual characterizations of robust solutions of various classes of non-convex quadratic optimization problems in the face of data uncertainty in the objective as well as constraints. They include uncertain trust-region problems and non-convex quadratic problems with convex quadratic constraints. For related recent results, see [14,15], where conditions for robust solutions were given for minimax convex quadratic problems or classes of non-convex problems with no uncertainty in the objective function. The DC approach presented in this work allowed us to treat general non-convex quadratic objective functions.

The outline of the paper is as follows. In Sect. 2, we present basic definition and properties of convex polynomials and conjugate functions. In Sect. 3, we present a generalized Farkas's lemma and its corresponding results for quadratic systems. In Sect. 4, we present duality results for the minimax DC-optimization problems (P) and their applications to fractional programming problems involving convex polynomials. In Sect. 5, we derive robust solution characterizations of various classes of quadratic optimization problems.

2 Preliminaries

We begin this section by fixing notation and preliminaries of convex sets, functions and polynomials. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_i \geq 0\}$. We say that the set A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$. The convex hull of the set A is denoted by $\text{co}(A)$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for all $\mu \in [0, 1]$,

$$f((1-\mu)x + \mu y) \ \leq (1-\mu) f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$. The function f is said to be concave whenever $-f$ is convex. The epigraph of a function f, denoted by $\text{epi} f$, is defined by $\text{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}$.

The conjugate of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by, for any $u \in \mathbb{R}^n$, $f^*(u) = \sup_{x\in\mathbb{R}^n} \left\{u^T x - f(x)\right\}$ and, for any $x \in \mathbb{R}^n$, $f^{**}(x) = \sup_{u\in\mathbb{R}^n} \left\{u^T x - f^*(x)\right\}$, respectively. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then $f = f^{**}$. For details, see [17,19,21]. The domain of f^*, denoted by $\text{dom} f^*$, is defined by $\text{dom} f^* := \{u \in \mathbb{R}^n | f^*(u) < \infty\}$. The positive semi-definiteness of an $n \times n$ matrix B, denoted by $B \succeq 0$, is defined by $\langle x, Bx\rangle \geq 0$, for each $x \in \mathbb{R}^n$. If f_1 and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then for $u \in \mathbb{R}^n$,

$$(f_1 + f_2)^*(u) = \min_{v\in\mathbb{R}^n} \left\{f_1^*(v) + f_2^*(u-v)\right\}. \tag{1}$$

For details, see [17,19,21].

The following useful result of convex polynomial systems was given in [2] and will play an important role later in the paper.

Lemma 2.1 [2] *Let* $f_0, f_1, \ldots, f_m$ *be convex polynomials on* $\mathbb{R}^n$. *Let* $C := \{x \in \mathbb{R}^n \ : \ f_i(x) \leq 0, \ i = 1, \ldots, m\} \neq \emptyset$. *Suppose that* $\inf_{x\in C} f_0(x) > -\infty$. *Then,* $\arg\min_{x\in C} f_0(x) \neq \emptyset$.

The next lemma, given first in [13], follows easily from Lemma 2.1.

Lemma 2.2 *Let* $f_0, f_1, \ldots, f_m$ *be convex polynomials on* $\mathbb{R}^n$. *Then the set,*

$$\Omega = \left\{y \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, \ f_i(x) \leq y_i, \ i = 0, 1, 2, \ldots, m\right\}$$

is closed and convex.

Proof Clearly, as f_i's are convex functions, Ω is a convex set. Let $\{y_0^k, y_1^k, \ldots, y_m^k\} \subset \Omega$ and $(y_0^k, y_1^k, \ldots, y_m^k) \to (y_0, y_1, \ldots, y_m)$ as $k \to \infty$. By the definition of Ω, for each k, there exists $x^k \in \mathbb{R}^n$ such that $f_i(x^k) \le y_i^k$. Now, consider the optimization problem

$$(\mathrm{P}_0) \qquad \min_{x, z_0, z_1, \ldots, z_m} \left\{ \sum_{i=0}^{m} (z_i - y_i)^2 : f_i(x) \le z_i,\ i = 0, 1, \ldots, m \right\}.$$

Then, (P_0) is a convex polynomial optimization problem and $0 \le \inf(\mathrm{P}_0) \le \sum_{i=0}^{m} (y_i^k - y_i)^2 \to 0$, as $k \to \infty$. Hence, $\inf(\mathrm{P}_0) = 0$ and by Lemma 2.1, $\inf(\mathrm{P}_0)$ is attained, and so, there exists $x^* \in \mathbb{R}^n$ such that $f_i(x^*) \le y_i$. Thus, $(y_0, y_1, \ldots, y_m) \in \Omega$ and Ω is closed.

The following example shows that Lemma 2.2 may fail when all the functions involved in Ω are not convex polynomials.

Example 2.1 Let $f_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2} - x_1$ and $f_2(x_1, x_2) = 1 - x_2$. Then, f_1 is not a polynomial. Define the set C by

$$C := \left\{ y \in \mathbb{R}^2 : x \in \mathbb{R}^2,\ f_i(x) \le y_i \right\}.$$

Then, for each integer $k \ge 1$, $(y_1^k, y_2^k) = (\sqrt{k^2+1} - k, 0) \in C$ as $(f_1(k,1), f_2(k,1)) \le (\sqrt{k^2+1} - k, 0)$, and that $(y_1^k, y_2^k) = (\sqrt{k^2+1} - k, 0) = (\frac{1}{\sqrt{k^2+1}+k}, 0) \to (0,0)$ as $k \to \infty$. However, $(0,0) \notin C$ because the system

$$\begin{cases} \sqrt{x_1^2 + x_2^2} - x_1 \le 0 \\ 1 - x_2 \le 0 \end{cases}$$

has no solution. Therefore, C is not closed. □

3 Generalized Farkas' lemma

In this section, we present a generalization of Farkas' lemma and a corresponding result for quadratic inequality systems, called S-procedure [18]. The generalized Farkas' lemma holds without any regularity condition. Note that the simplex in $\mathbb{R}^l$ is given by $\Delta = \left\{ \delta \in \mathbb{R}_+^l : \sum_{i=1}^{l} \delta_i = 1 \right\}$.

Theorem 3.1 (Generalized Farkas' lemma) *Let $f_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \ldots, l,\ g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$ be convex polynomials and $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $K := \{x \in \mathbb{R}^n : g_i(x) \le 0,\ i = 1, \ldots, m\}$ be non-empty. Then, the following statements are equivalent:*

(i) $g_i(x) \le 0\ i = 1, \ldots, m \Rightarrow \max_{j=1,\ldots,l} f_j(x) - h(x) \ge 0$;
(ii) $(\forall u \in \mathrm{dom} h^*,\ \epsilon > 0)\ \left(\exists \lambda \in \mathbb{R}_+^m,\ \mu \in \Delta,\ v \in \mathbb{R}^n\right)$:

$$h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) + \epsilon \ge 0. \qquad (2)$$

Proof [(ii)] $\Rightarrow$ (i)] Note first, by the definition of a conjugate function, that (ii) is equivalent to the condition that, for any $u \in \text{dom}h^*$, $\epsilon > 0$, there exist $\lambda \in \mathbb{R}^m_+$, $v \in \mathbb{R}^n$ and $\mu \in \Delta$ such that

$$h^*(u) + \inf_{x \in \mathbb{R}^n} \left\{ -\langle v, x\rangle + \sum_{j=1}^{l} \mu_j f_j(x) \right\} + \inf_{x \in \mathbb{R}^n} \left\{ -\langle u - v, x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} + \epsilon \geq 0,$$

which shows that for any $x \in \mathbb{R}^n$,

$$h^*(u) - \langle u, x\rangle + \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \epsilon \geq 0.$$

Let $x \in K$. Then, $h^*(u) - \langle u, x\rangle + \left(\sum_{j=1}^{l} \mu_j f_j\right)(x) + \epsilon \geq 0$, as $\sum_{i=1}^{m} \lambda_i g_i(x) \leq 0$. So, for any $u \in \text{dom}h^*$ and $\epsilon > 0$, there exists $\mu \in \Delta$ such that for any $x \in K$:

$$\sum_{j=1}^{l} \mu_j f_j(x) + \epsilon \geq \langle u, x\rangle - h^*(u).$$

Moreover, $\max_{j=1,\ldots,l} f_j(x) = \max_{\delta \in \Delta} \delta_j f_j(x) \geq \sum_{j=1}^{l} \mu_j f_j(x)$. Hence, for any $u \in \text{dom}h^*$, $\epsilon > 0$, $\max_{j=1,\ldots,l} f_j(x) + \epsilon \geq \langle u, x\rangle - h^*(u)$ and so, $\max_{j=1,\ldots,l} f_j(x) \geq \sup_{u \in \text{dom}h^*} \{\langle u, x\rangle - h^*(u)\} = h(x)$. Thus (i) holds.

[(i)] $\Rightarrow$ (ii)] Assume that (i) holds. Let $u \in \text{dom}h^*$. Then, from (i), for each $x \in K$,

$$\max_{j=1,\ldots,l} f_j(x) \geq h(x) \geq \sup_{v \in \text{dom}h^*} \langle v, x\rangle - h^*(v) \geq \langle u, x\rangle - h^*(u).$$

So, for each $\epsilon > 0$, $\max_{j=1,\ldots,l} \left\{ f_j(x) + h^*(u) - \langle u, x\rangle \right\} + \epsilon > 0$. For $j \in \{1, \ldots, l\}$, let $\phi_j(x) = f_j(x) + h^*(u) - \langle u, x\rangle + \epsilon$, for $x \in \mathbb{R}^n$. Define a set C by

$$C := \left\{ (y, z) \in \mathbb{R}^l \times \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ s.t. } \phi_j(x) \leq y_j, \ \ j = 1, \ldots, l\,; \ \ g_i(x) \leq z_i, \ \ i = 1, \ldots, m \right\}.$$

Since ϕ_j, $j = 1, \ldots, l$ and g_i, $i = 1, \ldots, m$ are convex polynomials, C is a convex set. By Lemma 2.2, C is closed.

Since $\max_j \phi_j(\cdot) > 0$, we have $0 \notin C$. By the strong separation theorem ([21, Theorem1.1.5]), there exist $(\gamma_1, \ldots, \gamma_l, v_1, \ldots, v_m) \neq 0$, $\alpha \in \mathbb{R}$, $\delta_0 > 0$ such that for all $(y, z) \in C$,

$$\langle (0, \ldots, 0), (\gamma_1, \ldots, \gamma_l, v_1, \ldots, v_m)\rangle \leq \alpha < \alpha + \delta_0 \leq \langle (y_1, \ldots, y_l, z_1, \ldots, z_m), \\ (\gamma_1, \ldots, \gamma_l, v_1, \ldots, v_m)\rangle.$$

Since $C + \mathbb{R}^{m+l}_+ \subset C$, it follows that $\gamma_j \geq 0$, $j = 1, \ldots, l$ and $v_i \geq 0$, $i = 1, \ldots, m$ and for any $(y, z) \in C$,

$$0 \leq \alpha < \alpha + \delta_0 \leq \gamma_1 y_1 + \ldots \gamma_l y_l + \sum_{i=1}^{m} v_i z_i.$$

As $K \neq \emptyset$, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) \leq 0$, $i = 1, \ldots, m$ and so, if $(\gamma_1, \ldots, \gamma_l) = 0$, then $0 \leq \alpha < 0$. This is a contradiction. So, $0 \leq (\gamma_1, \ldots \gamma_l) \neq 0$.

Let $\lambda_i = \frac{v_i}{\gamma_1 + \ldots + \gamma_l} \geq 0$, $i = 1, \ldots, m$ and $\mu_i = \frac{\gamma_i}{\gamma_1 + \ldots + \gamma_l} \geq 0$, $i = 1, \ldots, m$. Then, $\sum_{j=1}^{l} \mu_j = 1$ and $\sum_{j=1}^{l} \mu_j \phi_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) > 0$, for any $x \in \mathbb{R}^n$, because $(\phi_1(x), \ldots, \phi_l(x), \ g_1(x), \ldots, g_m(x)) \in C$ for each $x \in \mathbb{R}^n$. It shows that

$$\sum_{j=1}^{l} \mu_j f_j(\cdot) + \sum_{i=1}^{m} \lambda_i g_i(\cdot) + h^*(u) - \langle u, \cdot \rangle + \epsilon > 0. \tag{3}$$

So, for each $u \in \mathrm{dom} h^*$ and for each $\epsilon > 0$, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$ such that, for each $x \in \mathbb{R}^n$,

$$h^*(u) + \epsilon > \langle u, x \rangle - \left[\sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right],$$

which gives us that

$$h^*(u) + \epsilon \geq \left(\sum_{j=1}^{l} \mu_j f_j + \sum_{i=1}^{m} \lambda_i g_i \right)^* (u).$$

On the other hand, as $f_j,\ j = 1, \ldots, l$ and $g_i,\ i = 1, \ldots, m$ are real-valued convex functions, by (1), for $u \in \mathrm{dom} h^*$, there exists $v \in \mathbb{R}^n$ such that

$$\left(\sum_{j=1}^{l} \mu_j f_j + \sum_{i=1}^{m} \lambda_i g_i \right)^* (u) = \sum_{j=1}^{l} \mu_j f_j^*(v) + \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v).$$

So, we obtain,

$$h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) + \epsilon \geq 0.$$

Hence (ii) holds. □

The following example shows that generalized Farkas's lemma may not hold for convex constraint systems, $g_i(x) \leq 0, i = 1, \ldots, m$, that are not polynomials.

Example 3.1 (Failure of generalized Farkas's lemma for non-polynomial convex constraint systems) Let $f_1(x_1, x_2) = 1 - x_2,\ f_2(x_1, x_2) = -x_1,\ h(x_1, x_2) = x_2^2$ and $g_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2} - x_1$. Then,

$$h^*(u_1, u_2) = \begin{cases} \frac{u_2^2}{4} & \text{if } u_1 = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $\{x \in \mathbb{R}^2 : g_1(x) \leq 0\} = \mathbb{R}_+ \times \{0\}$ and so, $\max\{f_1(x), f_2(x)\} - h(x) \geq 0$ for each $x \in \mathbb{R}_+ \times \{0\}$; thus, the implication

$$g_1(x) \leq 0 \Rightarrow \max\{f_1(x), f_2(x)\} - h(x) \geq 0$$

holds. Now, we show that the statement (ii):

$$\forall u \in \mathrm{dom} h^*, \quad \forall \epsilon > 0, \quad \exists \left(\lambda \geq 0,\ v \in \mathbb{R}^2, \mu \in \Delta \right), \quad h^*(u) - \left(\sum_{j=1}^{2} \mu_j f_j \right)^* (v)$$
$$-(\lambda g_1)^*(u - v) + \epsilon \geq 0,$$

fails. To see this, let $u = (0, 0) \in \operatorname{dom} h^*$. Then, $h^*(0, 0) = 0$, $\sum_{j=1}^{2} \mu_j f_j(x) = \mu_1(-x_2) + \mu_2(-x_1) + \mu_1$ and $(\sum_{j=1}^{2} \mu_j f_j)^*(v) = \sup_{x\in\mathbb{R}^2}\{v_1x_1 + v_2x_2 + \mu_1x_2 + \mu_2x_1 - \mu_1\}$. Moreover,

$$\left(\sum_{j=1}^{2} \mu_j f_j\right)^*(v) = \begin{cases} -\mu_1 & \text{if } \; v = (-\mu_2, -\mu_1) \\ +\infty & \text{otherwise.} \end{cases}$$

For $\lambda = 0$,

$$(\lambda g_1)^*(w) = \begin{cases} 0 & \text{if } w = (0, 0) \\ +\infty & \text{otherwise.} \end{cases}$$

For $\lambda > 0$, $(\lambda g_1)^*(\mu_2, \mu_1) = \sup_{x\in\mathbb{R}^2}\{\mu_2x_1 + \mu_1x_2 - \lambda\sqrt{x_1^2 + x_2^2} + \lambda x_1\} = (\lambda||x||)^*(\mu_2 + \lambda, \mu_1)$. Since $\mu_2 + \mu_1 = 1$, we get that

$$(\lambda||x||)^*(\mu_2 + \lambda, \mu_1) = \begin{cases} 0 & \text{if } \frac{\sqrt{(\mu_2+\lambda)^2+(\mu_1)^2}}{\lambda} \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$
$$= +\infty.$$

Hence, for $u = 0$, and $\epsilon > 0$, for any $\lambda \geq 0$, we can see that $(\lambda g_1)^*(-v) = +\infty$, whenever $v = (-\mu_2, -\mu_1)$, and $(\sum_{j=1}^{2} \mu_j f_j)^*(v) = +\infty$, whenever $v \neq (-\mu_2, -\mu_1)$, and so (ii) of Theorem 3.1 doesn't hold. □

The following example verifies our generalized Farkas' lemma, involving convex polynomial constraint systems.

Example 3.2 (Convex polynomial constraint systems and generalized Farkas' lemma) Let $f_1(x) = x_1^8 + x_1^2$, $f_2(x) = x_2^2 - 1$, $h(x) = \sqrt{x_1^2 + x_2^2}$, $g_1(x) = (x_1 - 1)^2 + x_2^2 - 1$, $g_2(x) = x_1$ and $g_3(x) = x_1 + x_2$. Then f_1, f_2, g_1, g_2, g_3 are convex polynomials, h is convex, but is not a polynomial function. Moreover, $h^*(u) = \delta_B(u)$, where δ_B is the indicator function of the unit ball B in $\mathbb{R}^2$.

Let us consider (i) and (ii) as in the previous theorem. We can see that $K := \{x \in \mathbb{R}^2 : g_i(x) \leq 0,\ i = 1, 2, 3\} = \{(0, 0)\}$. Then (i) holds.

Let $d(x, u, \lambda) := \mu_1 f_1(x) + \mu_2 f_2(x) + \sum_{i=1}^{3} \lambda_i g_i(x) + h^*(u) - \langle u, x\rangle$. Then,

$$d(x, u, \lambda) = \mu_1\left(x_1^8 + x_1^2\right) + (1 - \mu_1)\left(x_2^2 - 1\right) + \lambda_1\left((x_1 - 1)^2 + x_2^2 - 1\right) + \lambda_2 x_1 + \lambda_3(x_1 + x_2) - u_1x_1 - u_2x_2.$$

Thus, $d(x, u, \lambda) = \mu_1 x_1^8 + x_1^2(\mu_1 + \lambda_1) + x_1(\lambda_2 + \lambda_3 - 2\lambda_1 - u_1) + (\lambda_1 + 1 - \mu_1)\,x_2^2 + (\lambda_3 - u_2)x_2 + \mu_1 - 1$.

For $\lambda_1 > 0$, we get

$$d(x, u, \lambda) = \mu_1 x_1^8 + \left(\sqrt{\mu_1 + \lambda_1}x_1 + \frac{(\lambda_2 + \lambda_3 - 2\lambda_1 - u_1)}{2\sqrt{\lambda_1 + \mu_1}}\right)^2 + \left(\sqrt{\lambda_1 + 1 - \mu_1}x_2 + \frac{(\lambda_3 - u_2)}{2\sqrt{\lambda_1 + 1 - \mu_1}}\right)^2 - \left(\frac{(\lambda_2 + \lambda_3 - 2\lambda_1 - u_1)}{2\sqrt{\lambda_1 + \mu_1}}\right)^2 - \left(\frac{(\lambda_3 - u_2)}{2\sqrt{\lambda_1 + 1 - \mu_1}}\right)^2 + \mu_1 - 1.$$

By the above calculation, for any $u \in \mathbb{R}^2, \epsilon > 0$, taking $\mu_1 = 1$ and

$$\begin{cases} \lambda_2 = 2\lambda_1 + u_1, \lambda_3 = 0; & \text{if } (u_1, u_2) \in B, \quad u_1 \geq 0, \quad u_2 = 0; \\ \lambda_3 = 2\lambda_1 = u_2,\ \lambda_2 = u_1; & \text{if } (u_1, u_2) \in B, \quad u_1 \geq 0, \quad u_2 > 0; \\ \lambda_1 = 2(-u_1 + u_2),\ \lambda_2 = 2\lambda_1 + u_1 - u_2, \lambda_3 = u_2; & \text{if } (u_1, u_2) \in B, \quad u_1 < 0, \quad u_2 \geq 0; \\ \lambda_1 = \frac{u_2^2}{4\epsilon},\ \lambda_2 = 2\lambda_1 + u_1, \lambda_3 = 0; & \text{if } (u_1, u_2) \in B, \quad u_1 \geq 0, \quad u_2 < 0; \\ \lambda_1 = \frac{4u_1^2 + u_2^2}{4\epsilon},\ \lambda_2 = 2\lambda_1 - u_1, \lambda_3 = 0; & \text{if } (u_1, u_2) \in B, \quad u_1 < 0, \quad u_2 < 0, \end{cases}$$

we get that $-\left(\frac{\lambda_2+\lambda_3-2\lambda_1-u_1}{2\sqrt{\lambda_1+\mu_1}}\right)^2 - \left(\frac{\lambda_3-u_2}{2\sqrt{\lambda_1+1-\mu_1}}\right)^2 + \mu_1 - 1 \geq -\epsilon$ and hence

$$\inf_{x\in\mathbb{R}^2} \left\{ \sum_{j=1}^{2} \mu_j f_j(x) + \sum_{i=1}^{3} \lambda_i g_i(x) + h^*(u) - \langle u, x\rangle + \epsilon \right\} \geq 0.$$

So, we have,

$$h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j + \sum_{i=1}^{m} \lambda_i g_i\right)^*(u) + \epsilon \geq 0,$$

which is equivalent to

$$h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j\right)^*(v) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^*(u - v) + \epsilon \geq 0.$$

for some $v \in \mathbb{R}^2$ by (1). It verifies (ii). □

As a consequence of Theorem 3.1 we derive a form of the S-procedure for a quadratic system without any qualification. The S-procedure is a commonly used technique in stability analysis of nonlinear systems [18] and is a relaxation strategy for solving a quadratic inequality system or a quadratic optimization problem via a linear matrix inequality relaxation. Recall that, for a matrix C, the least eigenvalue of C is denoted by $\gamma_{\min}(C)$.

Theorem 3.2 (S-Procedure) *Let $B_1, \ldots, B_m$ be symmetric and positive semidefinite $n \times n$ matrices, $a_1, \ldots, a_l, b_1, \ldots, b_m \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \in \mathbb{R}$. Let $A_1, \ldots, A_l$ be symmetric $n \times n$ matrices. Assume that $\gamma_{\min}(A_j) < 0$ for some $j \in \{1, \ldots, l\}$. Let $\rho = \max_{j=1,\ldots,l}\{-\gamma_{\min}(A_j)\}$. Then the following statements are equivalent:*

(i) $\langle x, B_i x\rangle + \langle b_i, x\rangle + \beta_i \leq 0,\ i = 1, \ldots, m \Rightarrow \max_{j=1,\ldots,l} \langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j \geq 0$;
(ii) $(\forall u \in \mathbb{R}^n,\ \epsilon > 0)\ \left(\exists \lambda \in \mathbb{R}^m_+,\ \mu \in \Delta\right)$

$$\begin{pmatrix} 2\left(\sum_{j=1}^{l} \mu_j A_j + \rho I_n + \sum_{i=1}^{m} \lambda_i B_i\right) & \sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u \\ \left(\sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u\right)^T & 2\left(\sum_{i=1}^{m} \lambda_i \beta_i + \frac{||u||^2}{4\rho} + \sum_{j=1}^{l} \mu_j \alpha_j + \epsilon\right) \end{pmatrix} \succeq 0.$$

Proof Note that for any $j = 1, \ldots, l$ we can write [11]

$$\langle x, A_j x\rangle + \langle a_j, x\rangle = \langle x, (A_j + \rho I_n)x\rangle + \langle a_j, x\rangle - \rho||x||_2^2. \tag{4}$$

Then for any $j \in \{1, \ldots, l\}$, $A_j + \rho I_n$ is positive semidefinite. Let $f_j(x) = \langle x, (A_j + \rho I_n)x\rangle + \langle a_j, x\rangle + \alpha_j$ for $j = 1, \ldots, l$ and $h(x) = \rho||x||_2^2$. Thus, $f_j,\ j = 1, \ldots, l$ and h are convex polynomials. We have $h^*(u) = \frac{||u||^2}{4\rho}$ for any $u \in \mathbb{R}^n$. Then, the conclusion follows from Theorem 3.1 by noting that

$$
\begin{aligned}
&\sum_{j=1}^{l} \mu_j f_j(x) + \left(\sum_{i=1}^{m} \lambda_i g_i\right)(x) + h^*(u) - \langle u, x\rangle + \epsilon \\
&= \sum_{j=1}^{l} \mu_j \langle x, (A_j + \rho I_n)x\rangle + \sum_{j=1}^{l} \mu_j \langle a_j, x\rangle + \sum_{i=1}^{m} \lambda_i \left(\langle x, B_i x\rangle + \langle b_i, x\rangle + \beta_i\right) \\
&\quad + \frac{||u||^2}{4\rho} - \langle u, x\rangle + \sum_{j=1}^{l} \mu_j \alpha_j + \epsilon \\
&\geq 0,
\end{aligned}
$$

which is equivalent to for any $u \in \operatorname{dom} h^*$ and $\epsilon > 0$ there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$ such that

$$
\begin{pmatrix} 2\left(\sum_{j=1}^{l} \mu_j A_j + \rho I_n) + \sum_{i=1}^{m} \lambda_i B_i\right) & \sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u \\ \left(\sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u\right)^T & 2\left(\sum_{i=1}^{m} \lambda_i \beta_i + \frac{||u||^2}{4\rho} + \sum_{j=1}^{l} \mu_j \alpha_j + \epsilon\right) \end{pmatrix} \succeq 0.
$$

□

A special case of Theorem 3.1, where $l = 1$ is given in the following corollary which gives us a form of Farkas' lemma for different convex polynomials.

Corollary 3.1 (Qualification-free DC Farkas Lemma) *Let $h, f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ be convex polynomials. Let $K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, m\}$ be non-empty. Then the following statements are equivalent:*

(i) $g_i(x) \leq 0,\ i = 1, \ldots, m \ \Rightarrow\ f(x) \geq h(x)$;
(ii) $(\forall u \in \operatorname{dom} h^*,\ \epsilon > 0)\ \left(\exists \lambda \in \mathbb{R}^m_+,\ v \in \mathbb{R}^n\right)\ h^*(u) - f^*(v) - (\sum_{i=1}^{m} \lambda_i g_i)^*(u - v) + \epsilon \geq 0$.

Proof Applying Theorem 3.1 with $l = 1$ and $f_1 = f$, we see that $\Delta = \{1\}$ and the statement (ii) of Theorem 3.1 collapses to the statement that $(\forall u \in \operatorname{dom} h^*,\ \epsilon > 0)\ \left(\exists \lambda \in \mathbb{R}^m_+,\ \mu \in \{1\},\ v \in \mathbb{R}^n\right)$:

$$
h^*(u) - f^*(v) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^* (u - v) + \epsilon \geq 0,
$$

and so the conclusion follows. □

When $h = 0$ in Theorem 3.1, we obtain a version of Farkas's lemma for convex polynomials, given in [15].

Corollary 3.2 [15] *Let $f_j, g_i : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, l$ and $i = 1, 2, \ldots, m$ be convex polynomials. Let $K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, m\}$ be non-empty. Then the following statements are equivalent:*

(i) $g_i(x) \leq 0,\ i = 1, \ldots, m \ \Rightarrow\ \max_{j=1,\ldots,l} f_j(x) \geq 0$;
(ii) $(\forall \epsilon > 0)\ \left(\exists \lambda \in \mathbb{R}^m_+,\ \mu \in \Delta\right) : \sum_{j=1}^{l} \mu_j f_j(\cdot) + \sum_{i=1}^{m} \lambda_i g_i(\cdot) + \epsilon \geq 0$.

Proof Let $h = 0$. Then, $\mathrm{dom}h^* = \{0\}$ and Theorem 3.1 gives us that (i) is equivalent to the statement that for each $\epsilon > 0$, there exist $\lambda \in \mathbb{R}^m_+$, $\mu \in \Delta$ and $v \in \mathbb{R}^n$ such that

$$-\left(\sum_{j=1}^{l}\mu_j f_j\right)^*(v) - \left(\sum_{i=1}^{m}\lambda_i g_i\right)^*(-v) + \epsilon \geq 0.$$

Hence, by the conjugate duality Theorem (see [10]), we obtain that

$$\begin{aligned} 0 &\leq -\left(\sum_{j=1}^{l}\mu_j f_j\right)^*(v) - \left(\sum_{i=1}^{m}\lambda_i g_i\right)^*(-v) + \epsilon \\ &\leq \max_{w\in\mathbb{R}^n}\left\{-\left(\sum_{j=1}^{l}\mu_j f_j\right)^*(w) - \left(\sum_{i=1}^{m}\lambda_i g_i\right)^*(-w)\right\} + \epsilon \\ &= \inf_{x\in\mathbb{R}^n}\left\{\sum_{j=1}^{l}\mu_j f_j(x) + \sum_{i=1}^{m}\lambda_i g_i(x)\right\} + \epsilon. \end{aligned}$$

Hence, if (i) holds then (ii) also holds. The converse implication (ii) $\Rightarrow$ (i) follows by construction. □

4 Duality and minimax optimization

4.1 Duality

In this subsection, we present duality results for the minimax DC optimization problem

$$\begin{array}{lll} \text{(P)} & \inf_{x\in\mathbb{R}^n}\max_{j=1,\ldots,l} & f_j(x) - h(x) \\ & \text{s.t.} & g_i(x) \leq 0,\ i = 1,\ldots,m, \end{array}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1,\ldots,l$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,2,\ldots,m$ are convex polynomials. The conjugate dual problem associated with (P) is given by

$$\text{(D)} \quad \inf_{u\in\mathrm{dom}h^*}\ \sup_{\lambda\in\mathbb{R}^m_+,\mu\in\Delta,v\in\mathbb{R}^n}\left\{h^*(u) - \left(\sum_{j=1}^{l}\mu_j f_j\right)^*(v) - \left(\sum_{i=1}^{m}\lambda_i g_i\right)^*(u-v)\right\}.$$

We first show that there is no duality gap between problems (P) and (D).

Theorem 4.1 (Zero duality gaps & minimax DC polynomial programs) *Let $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1,\ldots,l$, let $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\ldots,m$ be convex polynomials and $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $K = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1,\ldots,m\} \neq \emptyset$. Then,*

$$\inf\text{(P)} = \inf_{u\in\mathrm{dom}h^*}\ \sup_{\lambda\in\mathbb{R}^m_+,\mu\in\Delta,v\in\mathbb{R}^n}\left\{h^*(u) - \left(\sum_{j=1}^{l}\mu_j f_j\right)^*(v) - \left(\sum_{i=1}^{m}\lambda_i g_i\right)^*(u-v)\right\}.$$

Proof Let $x \in K$. Then, by the definition of conjugate of h,

$$\max_{j=1,\ldots,l} f_j(x) - h(x) = \inf_{u\in\mathrm{dom}h^*}\max_{\mu\in\Delta}\left\{\sum_{j=1}^{l}\mu_j f_j(x) + h^*(u) - \langle u, x\rangle\right\}. \tag{5}$$

As $x \in K$, for each $\lambda \in \mathbb{R}^m_+$, $\sum_{i=1}^m \lambda_i g_i(x) \leq 0$ and so, for each $\mu \in \Delta$,

$$\sum_{j=1}^{l} \mu_j f_j(x) + h^*(u) - \langle u, x\rangle \geq \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x\rangle \right\}. \tag{6}$$

It then follows from (5) and (6) that for $x \in K$,

$$\begin{aligned} \max_{j=1,\ldots,l} f_j(x) - h(x) &= \inf_{u \in \mathrm{dom} h^*} \max_{\mu \in \Delta} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + h^*(u) - \langle u, x\rangle \right\} \\ &\geq \inf_{u \in \mathrm{dom} h^*} \max_{\mu \in \Delta} \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x\rangle \right\}. \end{aligned}$$

Now, by the definition of a conjugate function, we get that

$$\begin{aligned} &\inf_{u \in \mathrm{dom} h^*} \max_{\mu \in \Delta} \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x\rangle \right\} \\ &= \inf_{u \in \mathrm{dom} h^*} \max_{\mu \in \Delta} \sup_{\lambda \in \mathbb{R}^m_+} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j + \sum_{i=1}^{m} \lambda_i g_i \right)^* (u) \right\} \\ &= \inf_{u \in \mathrm{dom} h^*} \max_{\mu \in \Delta} \sup_{\lambda \in \mathbb{R}^m_+} \max_{v \in \mathbb{R}^n} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}. \end{aligned}$$

The last equation is obtained by the sum-conjugate formula (1). Therefore,

$$\inf (\mathrm{P}) \geq \inf_{u \in \mathrm{dom} h^*} \max_{\mu \in \Delta} \sup_{\lambda \in \mathbb{R}^m_+} \max_{v \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x\rangle \right\},$$

which shows that

$$\inf (\mathrm{P}) \geq \inf_{u \in \mathrm{dom} h^*} \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}.$$

To see the reverse inequality, we may assume, without loss of generality, that $\inf (\mathrm{P}) > -\infty$, otherwise the conclusion follows immediately. Since $K \neq \emptyset$, we have $\bar{t} := \inf (\mathrm{P}) \in \mathbb{R}$. Then, as $\max_{j=1,\ldots,l} f_j(x) - \big(h(x) + \bar{t}\big) \geq 0$ for all $x \in K$, by Theorem 3.1 we get that for any $\epsilon > 0$, $u \in \mathrm{dom} h^*$, there exist $\lambda \in \mathbb{R}^m_+$, $\mu \in \Delta$, $v \in \mathbb{R}^n$ such that

$$h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \geq \bar{t} - \epsilon.$$

Consequently

$$\inf_{u \in \mathrm{dom} h^*} \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\} \geq \bar{t} - \epsilon.$$

Since the above inequality holds for any $\epsilon > 0$, passing to the limit we obtain the desired inequality, which concludes the proof. □

The following example shows that zero duality gap between (P) and (D) may not hold for convex constraints, $g_i(x) \leq 0, i = 1, \ldots, m$, that are not polynomials.

Example 4.1 (Non-zero-duality gaps for (P) with non-polynomial convex constraints) Consider the problem

$$\begin{array}{ll} (\mathrm{EP}_1) \ \min_{x \in \mathbb{R}^2} & \max\{1 - x_2, \ -x_1\} - x_2^2 \\ \text{s.t.} & \sqrt{x_1^2 + x_2^2} - x_1 \leq 0, \end{array}$$

where $f_1(x) = 1 - x_2$, $f_2(x) = -x_1$, $h(x) = x_2^2$ and $g_1(x) = \sqrt{x_1^2 + x_2^2} - x_1$. Then, clearly, $\inf(\mathrm{EP}_1) = 1$. On the other hand,

$$\inf_{u \in \mathrm{dom} h^*} \sup_{\lambda \in \mathbb{R}_+, \mu \in \Delta, v \in \mathbb{R}^2} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - (\lambda g_1)^*(u - v) \right\}$$
$$\leq \sup_{\lambda \in \mathbb{R}_+, \mu \in \Delta, v \in \mathbb{R}^2} \left\{ h^*(0) - \left(\sum_{j=1}^{2} \mu_j f_j \right)^* (v) - (\lambda g_1)^*(-v) \right\}.$$

As we saw in Example 3.1, $h^*(0, 0) = 0$ and

$$\left(\sum_{j=1}^{2} \mu_j f_j \right)^* (v) = \begin{cases} -\mu_1 & \text{if} \quad v = (-\mu_2, -\mu_1) \\ +\infty & \text{otherwise.} \end{cases}$$

For $\lambda = 0$,

$$(\lambda g_1)^* (w) = \begin{cases} 0 & \text{if} \quad w = (0, 0) \\ +\infty & \text{otherwise.} \end{cases}$$

For $\lambda > 0$, $(\lambda g_1)^*(\mu_2, \mu_1) = +\infty$. Therefore,

$$\inf_{u \in \mathrm{dom} h^*} \sup_{\lambda \in \mathbb{R}_+, \mu \in \Delta, v \in \mathbb{R}^2} \left\{ h^*(u) - \left(\sum_{j=1}^{2} \mu_j f_j \right)^* (v) - (\lambda g_1)^*(u - v) \right\}$$
$$\leq \sup_{\lambda \in \mathbb{R}_+, \mu \in \Delta, v \in \mathbb{R}^2} \left\{ h^*(0) - \left(\sum_{j=1}^{2} \mu_j f_j \right)^* (v) - (\lambda g_1)^*(-v) \right\} = -\infty,$$

which shows that

$$\inf (\mathrm{EP}_1) = 1 \neq \inf_{u \in \mathrm{dom} h^*} \sup_{\lambda \in \mathbb{R}_+, \mu \in \Delta, v \in \mathbb{R}^2} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - (\lambda g)^*(u - v) \right\} = -\infty.$$

□

We use the same constraint system as in Example 3.2 to verify that the zero-duality gap property of a minimax DC program.

Example 4.2 (Verifying zero-duality gap for a minimax DC program). Consider the following example

$$\begin{array}{ll} (\mathrm{EP}_2) \ \min_{x \in \mathbb{R}^2} & \max\left\{x_1^8 + x_1^2, \ x_2^2 - 1\right\} - \sqrt{x_1^2 + x_2^2} \\ \text{s.t.} & (x_1 - 1)^2 + x_2^2 \leq 1 \\ & x_1 \leq 0 \\ & x_1 + x_2 \leq 0. \end{array}$$

From Example 3.2, we can see that

$$0 \geq \inf_{||u||_* \leq 1} \sup_{\lambda \in \mathbb{R}^3_+, \mu \in \Delta_2, t \in \mathbb{R}} \left\{ t \in \mathbb{R} : \sum_{j=1}^{2} \mu_j f_j(\cdot) + \sum_{i=1}^{3} \lambda_i g_i(\cdot) + h^*(u) - \langle u, \cdot \rangle - t \geq 0 \right\}$$

$$\geq \inf_{||u||_* \leq 1} \sup_{\lambda \in \mathbb{R}^3_+, \mu_1 \in [0,1]} \left\{ -\left(\frac{(\lambda_2 + \lambda_3 - 2\lambda_1 - u_1)}{2\sqrt{\lambda_1 + \mu_1}} \right)^2 - \left(\frac{(\lambda_3 - u_2)}{2\sqrt{\lambda_1 + 1 - \mu_1}} \right)^2 + \mu_1 - 1 \right\}.$$

Note that $||u||_* \leq 1$, for $u = (u_1, u_2)$. Then, $u_1, u_2 \in [-1, 1]$. Taking $\mu_1 = 1$, $\lambda_2 = 2\lambda_1$, $\lambda_3 = |u_2|$ and $\lambda_1 \to +\infty$, we get

$$\inf(\text{EP}_2) = \inf_{||u||_* \leq 1} \sup_{\lambda \in \mathbb{R}^3_+, \mu \in \Delta_2} \inf_{x \in \mathbb{R}^2} \left\{ \sum_{j=1}^{2} \mu_j f_j(x) + \sum_{i=1}^{3} \lambda_i g_i(x) + h^*(u) - \langle u, x \rangle \right\} = 0$$

$$= \inf_{||u||_* \leq 1} \sup_{\lambda \in \mathbb{R}^3_+, \mu \in \Delta_2} \left\{ h^*(u) - \left(\sum_{j=1}^{2} \mu_j f_j + \sum_{i=1}^{3} \lambda_i g_i \right)^* (u) \right\}.$$

□

A special case of Theorem 4.1 where $l = 1$ shows that there is no duality gap between a difference of convex program and its associated dual program whenever f, h and g_i's are convex polynomials.

Corollary 4.1 (Zero duality gaps for DC polynomial programs) *Let h, f, $g_i : \mathbb{R}^n \to R$, $i = 1, 2, \ldots, m$ be convex polynomials. Let $K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, m\}$ be non-empty. Then,*

$$\inf_{x \in K} \{f(x) - h(x)\} = \inf_{u \in \text{dom} h^*} \sup_{\lambda \in \mathbb{R}^m_+, v \in \mathbb{R}^n} \left\{ h^*(u) - f^*(v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}.$$

Proof Applying Theorem 4.1 with $l = 1$ and $f_1 = f$, we see that $\Delta = \{1\}$ and then the conclusion easily follows. □

Under the additional assumption that $N_K(x) = \bigcup_{\lambda_i \geq 0, \lambda_i g_i(x) = 0} \{\sum_{i=1}^m \lambda_i \nabla g_i(x)\}$, we obtain the following strong duality theorem for (P), where $N_K(x)$ is the normal cone in the sense that $N_K(x) = \{w \in \mathbb{R}^n : \langle w, y - x \rangle \leq 0,\ \forall y \in K\}$.

Theorem 4.2 (Strong Duality) *Let $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, l$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ be convex polynomials and $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $K = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, m\} \neq \emptyset$. Assume that* inf(P) *is finite, and $N_K(x) = \bigcup_{\lambda_i \geq 0, \lambda_i g_i(x) = 0} \{\sum_{i=1}^m \lambda_i \nabla g_i(x)\}$, for each $x \in K$. Then*

$$\inf\,(\text{P}) = \inf_{u \in \text{dom} h^*} \max_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}.$$

Proof Let $f := \max_{j=1,\ldots,l} f_j$. Using the basic properties of conjugate function of h, we have

$$\inf_{x \in K} \{f(x) - h(x)\} = \inf_{x \in K} \left\{ f(x) - \sup_{u \in \text{dom} h^*} \left\{ \langle u, x \rangle - h^*(u) \right\} \right\}$$

$$= \inf_{u \in \text{dom} h^*} \inf_{x \in K} \left\{ f(x) + h^*(u) - \langle u, x \rangle \right\}$$

Let $u \in \text{dom} h^*$. As $\inf(P)$ is finite, $\inf_{x \in K} \{f(x) + h^*(u) - \langle u, x \rangle\}$ is also finite; thus,

$$\inf_{x \in K} \max_{\mu \in \Delta} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + h^*(u) - \langle u, x \rangle \right\}$$

is finite, where $\Delta = \left\{ \delta \in \mathbb{R}^l_+ : \sum_{j=1}^{l} \delta_j = 1 \right\}$. By the convex-concave minimax Theorem,

$$\max_{\mu \in \Delta} \inf_{x \in K} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + h^*(u) - \langle u, x \rangle \right\}$$

is finite, and so, there exists $\overline{\mu} \in \Delta$ such that $\inf_{x \in K} \left\{ \sum_{j=1}^{l} \overline{\mu}_j f_j(x) + h^*(u) - \langle u, x \rangle \right\}$ is finite. Now, by Lemma 2.1, $\sum_{j=1}^{l} \overline{\mu}_j f_j(x) + h^*(u) - \langle u, x \rangle$ attains its infimum. Then, there exists $\overline{x}_u \in K$ such that $\sum_{j=1}^{l} \overline{\mu}_j f_j(\overline{x}_u) + h^*(u) - \langle u, \overline{x}_u \rangle = \inf_{x \in K} \left\{ \sum_{j=1}^{l} \overline{\mu}_j f_j(x) + h^*(u) - \langle u, x \rangle \right\}$. The standard necessary optimality condition at $\overline{x}_u$ gives us that $0 \in \sum_{j=1}^{l} \overline{\mu}_j \nabla f_j(\overline{x}_u) - u + N_K(\overline{x}_u)$. Employing the assumed normal cone constraint qualification, we can find $\overline{\lambda}_i \geq 0$, $i = 1, 2, \ldots, m$ such that

$$0 = \sum_{j=1}^{l} \overline{\mu}_j \nabla f_j(\overline{x}_u) - u + \sum_{i=1}^{m} \overline{\lambda}_i \nabla g_i(\overline{x}_u) \quad \text{and} \quad \overline{\lambda}_i g_i(\overline{x}_u) = 0.$$

Since $\sum_{j=1}^{l} \overline{\mu}_j f_j(\cdot) - \langle u, \cdot \rangle + \sum_{i=1}^{m} \overline{\lambda}_i g_i(\cdot)$ is convex, for all $x \in \mathbb{R}^n$

$$\sum_{j=1}^{l} \overline{\mu}_j f_j(x) + h^*(u) - \langle u, x \rangle + \sum_{i=1}^{m} \overline{\lambda}_i g_i(x) \geq \sum_{j=1}^{l} \overline{\mu}_j f_j(\overline{x}_u) + h^*(u) - \langle u, \overline{x}_u \rangle + \sum_{i=1}^{m} \overline{\lambda}_i g_i(\overline{x}_u).$$

Then,

$$\inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \overline{\mu}_j f_j(x) + h^*(u) - \langle u, x \rangle + \sum_{i=1}^{m} \overline{\lambda}_i g_i(x) \right\} \geq \sum_{j=1}^{l} \overline{\mu}_j f_j(\overline{x}_u) + h^*(u) - \langle u, \overline{x}_u \rangle. \tag{7}$$

On the other hand, applying Theorem 4.1 by replacing $f_j(\cdot)$ by $\widetilde{f}_j(\cdot) := f_j(\cdot) + h^*(u) - \langle u, \cdot \rangle$ and $h(\cdot)$ by $\widetilde{h}(\cdot) = 0$ gives us

$$\inf_{x \in K} \max_{j=1,\ldots,l} \widetilde{f}_j(x) = \inf_{w \in \text{dom} \widetilde{h}^*} \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ \widetilde{h}^*(w) - \left(\sum_{j=1}^{l} \mu_j \widetilde{f}_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (w - v) \right\}$$

$$\inf_{x \in K} \max_{j=1,\ldots,l} \widetilde{f}_j(x) = \inf_{w \in \{0\}} \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ \widetilde{h}^*(w) - \left(\sum_{j=1}^{l} \mu_j \widetilde{f}_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (w - v) \right\}$$

$$= \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta} \left\{ \sup_{v \in \mathbb{R}^n} - \left(\sum_{j=1}^{l} \mu_j \widetilde{f}_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (-v) \right\}$$

$$= \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta} \left\{ \inf_{x \in \mathbb{R}^n} \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x \rangle \right\}. \tag{8}$$

Note that the equality in (8) follows by Fenchel's duality. As $f(\overline{x}_u) = \sum_{j=1}^{l} \overline{\mu}_j f_j(\overline{x}_u)$ and $\inf_{x \in K} \max_{j=1,\ldots,l} \widetilde{f}_j(x) = f(\overline{x}_u) + h^*(u) - \langle u, \overline{x}_u \rangle$, it follows from (7) and (8) that

$$\sum_{j=1}^{l} \overline{\mu}_j f_j(\overline{x}_u) + h^*(u) - \langle u, \overline{x}_u \rangle$$

$$= \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \overline{\mu}_j f_j(x) + h^*(u) - \langle u, x \rangle + \sum_{i=1}^{m} \overline{\lambda}_i g_i(x) \right\}$$

$$= \max_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x \rangle \right\}.$$

So, we have

$$\min_{x \in K} \{ f(x) + h^*(u) - \langle u, x \rangle \} = \max_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + h^*(u) - \langle u, x \rangle \right\}.$$

Hence, by the definition of conjugate function and (1)

$$\min_{x \in K} \{ f(x) + h^*(u) - \langle u, x \rangle \} = \max_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j + \sum_{i=1}^{m} \lambda_i g_i \right)^* (u) \right\}$$

$$= \max_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}.$$

As this is true for each $u \in \mathrm{dom} h^*$,

$$\inf (\mathrm{P}) = \inf_{u \in \mathrm{dom} h^*} \min_{x \in K} \{ f(x) + h^*(u) - \langle u, x \rangle \}$$

$$= \inf_{u \in \mathrm{dom} h^*} \max_{\lambda \in \mathbb{R}^m_+, \mu \in \Delta, v \in \mathbb{R}^n} \left\{ h^*(u) - \left(\sum_{j=1}^{l} \mu_j f_j \right)^* (v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}.$$

□

In the special case of (P) where $l = 1$, Theorem 4.2 yields that

$$\inf_{x \in K} \{ f(x) - h(x) \} = \inf_{u \in \mathrm{dom} h^*} \max_{\lambda \in \mathbb{R}^m_+, v \in \mathbb{R}^n} \left\{ h^*(u) - f^*(v) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (u - v) \right\}, \tag{9}$$

whenever the normal cone constraint qualification, $N_K(x) = \bigcup_{\lambda_i \geq 0, \lambda_i g_i(x) = 0} \{ \sum_{i=1}^{m} \lambda_i \nabla g_i(x) \}$, holds.

Note that the normal cone constraint qualification of Theorem 4.2 is guaranteed by the epigraph constraint qualification that the convex cone, $\bigcup_{\lambda\in\mathbb{R}^m_+}\mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$, is closed. The epigraph constraint qualification has extensively been used to study convex as well as difference of convex optimization problems. For details, see [5,6,8,11,12,16]. Note also that if the so-called Slater condition is satisfied in the sense that there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for all $i = 1, \ldots, m$, then both the epigraph constraint qualification and the normal cone constraint qualification hold. For details, see [12] and [10, Section 2.2].

We obtain a global optimality characterization directly from Theorem 4.2 under the normal cone constraint qualification.

Corollary 4.2 (Global optimality characterization) *Let* $f_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \ldots, l,\ g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, m$ *be convex polynomials and* $h : \mathbb{R}^n \to \mathbb{R}$ *be a convex function. Let* $K = \{x \in \mathbb{R}^n : g_i(x) \le 0,\ i = 1, \ldots, m\} \neq \emptyset$. *Assume that* $\inf(\mathrm{P})$ *is finite, and* $N_K(x) = \bigcup_{\lambda_i\ge 0, \lambda_i g_i(x)=0}\{\sum_{i=1}^m \lambda_i \nabla g_i(x)\}$ *for each* $x \in K$. *A feasible point* $\overline{x} \in K$ *is a solution of* (P) *if and only if, for any* $u \in \mathrm{dom}h^*$, *there exist* $\lambda \in \mathbb{R}^m_+$ *and* $\mu \in \Delta$ *such that*

$$\sum_{j=1}^{l}\mu_j f_j(\cdot) + \sum_{i=1}^{m}\lambda_i g_i(\cdot) + h^*(u) - \langle u, \cdot\rangle - \left(\max_{j=1,\ldots,l} f_j(\overline{x}) - h(\overline{x})\right) \ge 0. \tag{10}$$

Proof Note first from Theorem 4.2 that

$$\inf(\mathrm{P}) = \inf_{u\in\mathrm{dom}h^*}\ \max_{\lambda\in\mathbb{R}^m_+,\mu\in\Delta,v\in\mathbb{R}^n}\left\{h^*(u) - \left(\sum_{j=1}^{l}\mu_j f_j\right)^*(v) - \left(\sum_{i=1}^{m}\lambda_i g_i\right)^*(u-v)\right\}.$$

Using (1) and the definition of a conjugate function, we get that

$$\begin{aligned}\inf(\mathrm{P}) &= \inf_{u\in\mathrm{dom}h^*}\ \max_{\lambda\in\mathbb{R}^m_+,\mu\in\Delta}\left\{h^*(u) - \left(\sum_{j=1}^{l}\mu_j f_j + \sum_{i=1}^{m}\lambda_i g_i\right)^*(u)\right\}\\ &= \inf_{u\in\mathrm{dom}h^*}\ \max_{\lambda\in\mathbb{R}^m_+,\mu\in\Delta}\ \inf_{x\in\mathbb{R}^n}\left\{h^*(u) + \sum_{j=1}^{l}\mu_j f_j(x) + \sum_{i=1}^{m}\lambda_i g_i(x) - \langle u, x\rangle\right\}.\end{aligned} \tag{11}$$

[⇒] Assume that $\overline{x} \in K$ is a solution of (P). As $\max_{j=1,\ldots,l} f_j(\overline{x}) - h(\overline{x}) = \inf(\mathrm{P})$, from (11), we have for any $u \in \mathrm{dom}h^*$, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$, such that for all $x \in \mathbb{R}^n$,

$$h^*(u) + \sum_{j=1}^{l}\mu_j f_j(x) + \sum_{i=1}^{m}\lambda_i g_i(x) - \langle u, x\rangle \ge \max_{j=1,\ldots,l} f_j(\overline{x}) - h(\overline{x}),$$

and so $h^*(u) + \sum_{j=1}^{l}\mu_j f_j(\cdot) + \sum_{i=1}^{m}\lambda_i g_i(\cdot) - \langle u, \cdot\rangle - (\max_{j=1,\ldots,l} f_j(\overline{x}) - h(\overline{x})) \ge 0$. Hence, (10) holds.

[⇐] Conversely, let $\overline{x} \in K$. Assume that for any $u \in \mathrm{dom}h^*$, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$ such that (10) holds. Then, for any $u \in \mathrm{dom}h^*$,

$$\max_{\lambda\in\mathbb{R}^m_+,\mu\in\Delta}\ \inf_{x\in\mathbb{R}^n}\sum_{j=1}^{l}\mu_j f_j(x) + \sum_{i=1}^{m}\lambda_i g_i(x) + h^*(u) - \langle u, x\rangle \ge \left(\max_{j=1,\ldots,l} f_j(\overline{x}) - h(\overline{x})\right)$$

By (11),

$$\inf (\mathrm{P}) = \inf_{u\in \mathrm{dom} h^*} \max_{\lambda\in\mathbb{R}^m_+, \mu\in\Delta} \inf_{x\in\mathbb{R}^n} \left\{ h^*(u) + \sum_{j=1}^{l} \mu_j f_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) - \langle u, x\rangle \right\}$$
$$\geq \max_{j=1,\ldots,l} f_j(\overline{x}) - h(\overline{x}),$$

which shows that $\overline{x}$ is a solution of (P). □

We now give an example verifying our global optimality characterization.

Example 4.3 Consider the problem

$$(\mathrm{EP}_3) \quad \begin{array}{ll} \min_{x\in\mathbb{R}^2} \max & \left\{x_1^8 + x_1x_2 + x_1^2 + x_2^2,\ x_1^2 - 3\right\} - \left(x_1^2 + x_2^2\right) \\ \text{s.t.} & x_1^2 + x_2^2 + 4x_1 + 3 \leq 0 \\ & x_2 \leq 0. \end{array}$$

Let $f_1(x_1, x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2$, $f_2(x_1, x_2) = x_1^2 - 3$, $f(x_1, x_2) = \max_{j=1,2} f_j(x_1, x_2)$, $h(x_1, x_2) = x_1^2+x_2^2$, $g_1(x_1, x_2) = x_1^2+x_2^2+4x_1+3$ and $g_2(x_1, x_2) = x_2$. Then f_1, f_2, h, g_1, g_2 are convex polynomials. Moreover, $h^*(u_1, u_2) = \frac{1}{4}(u_1^2 + u_2^2)$. We can see that $\overline{x} = (-1, 0)$ is a solution of (EP_3) and $\inf\,(\mathrm{EP}_3) = 1$.

Let $\overline{\lambda} = (4, 1)$, $\overline{\mu}_1 = 1$. Then, for any $u \in \mathbb{R}^2$ and $x \in \mathbb{R}^2$,

$$\overline{\mu}_1 f_1(x) + (1 - \overline{\mu}_1) f_2(x) + \sum_{i=1}^{2} \overline{\lambda}_i g_i(x) + h^*(u) - \langle u, x\rangle$$
$$= \left(\tfrac{1}{2}u_1 - x_1\right)^2 + \left(\tfrac{1}{2}u_2 - x_2\right)^2 + \left(x_1^4 - 1\right)^2 + 2\left(x_1^2 - 1\right)^2 + 5(x_1 + 1)^2 + (x_1 + x_2 + 1)^2$$
$$+ (x_1 - x_2 + 1)^2 + \left(x_1 + \tfrac{1}{2}x_2 + 1\right)^2 + \tfrac{7}{4}x_2^2 + 1.$$

So, for each $u \in \mathrm{dom} h^*$

$$\overline{\mu}_1 f_1(x) + (1 - \overline{\mu}_1) f_2(x) + \sum_{i=1}^{2} \overline{\lambda}_i g_i(x) + h^*(u) - \langle u, x\rangle - f(\overline{x}) + h(\overline{x}) \geq 0.$$

It verifies the optimality condition of Corollary 4.2. □

4.2 Minimax fractional programs

In this subsection, we derive global optimality characterizations for fractional programs, involving convex polynomials.

Corollary 4.3 *Let f_j, $j = 1, \ldots, l$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ be convex polynomials. Assume that $f_j(x) > 0$ over the feasible set $K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ \ i = 1, \ldots, m\}$, $K \neq \emptyset$ and the Slater condition holds. Let $\overline{x} \in K$ and $\overline{t} = \max_{j=1,\ldots,l} \frac{f_j(\overline{x})}{||\overline{x}||+1}$. Then, $\overline{x}$ is a solution of*

$$(\mathrm{FP}_1)\ \inf_{x\in\mathbb{R}^n} \max_{j=1,\ldots,l} \left\{ \frac{f_j(x)}{||x|| + 1} \ :\ g_i(x) \leq 0,\ \ i = 1, \ldots, m \right\},$$

if and only if, for any $u \in \mathbb{R}^n$, $||\frac{u}{\overline{t}}||_* \leq 1$, there exist $\lambda \in \mathbb{R}^m_+$, and $\mu \in \Delta$ such that

$$\sum_{j=1}^{l} \mu_j f_j(\cdot) + \sum_{i=1}^{m} \lambda_i g_i(\cdot) - \overline{t} - \langle u, \cdot\rangle \geq 0.$$

Proof As $\bar{t} = \max_{j=1,\dots,l} \frac{f_j(\bar{x})}{||\bar{x}||+1}$, $\bar{t} = \inf(\text{FP}_1) \in \mathbb{R}_+$, if and only if,

$$\inf_{x\in K} \left\{ \max_{j=1,\dots,l} f_j(x) - \bar{t}h(x) \right\} = 0. \tag{12}$$

Let $h(x) = ||x|| + 1$. Then,

$$(\bar{t}h)^*(u) = \begin{cases} -\bar{t} & \text{if } ||\frac{u}{\bar{t}}||_* \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

By Corollary 4.2, $\bar{x} \in K$ is a solution of (12), if and only if, for any $u \in \text{dom}(\bar{t}h)^*$, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$ such that

$$\sum_{j=1}^{l} \mu_j f_j(\cdot) + \sum_{i=1}^{m} \lambda_i g_i(\cdot) + \left(\bar{t}h\right)^* (u) - \langle u, \cdot\rangle \ge 0.$$

Thus, $\bar{x} \in K$ is a solution of (FP_1) if and only if, for any $u \in \text{dom}(\bar{t}h)^*$, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$ such that $\sum_{j=1}^{l} \mu_j f_j(\cdot) + \sum_{i=1}^{m} \lambda_i g_i(\cdot) - \bar{t} - \langle u, \cdot\rangle \ge 0$. □

Now, consider the minimax fractional quadratic programs of the form

$$(\text{FP}_2) \quad \inf_{x\in\mathbb{R}^n} \max_{j=1,\dots,l} \left\{ \frac{\langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j}{\langle x, Cx\rangle + \langle c, x\rangle + \delta} \ :\ \langle x, B_i x\rangle + \langle b_i, x\rangle + \beta_i \le 0,\ \ i = 1, \dots, m \right\},$$

where $A_1, \dots, A_l, B_1, \dots, B_m$ are symmetric and positive semidefinite $n \times n$ matrices, $a_1, \dots, a_l,\ b_1, \dots, b_m \in \mathbb{R}^n$ and $\alpha_1, \dots, \alpha_l,\ \beta_1, \dots, \beta_m \in \mathbb{R}$. The matrix C is $n \times n$ symmetric and positive definite, $c \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$. As before, we assume that $K := \{x \in \mathbb{R}^n : \langle x, B_i x\rangle + \langle b_i, x\rangle + \beta_i \le 0,\ i = 1, \dots, m\} \ne \emptyset$.

Corollary 4.4 *For problem* (FP_2)*, assume that* $f_j(x) = \langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j > 0$, $h(x) = \langle x, Cx\rangle + \langle c, x\rangle + \delta > 0$ *over the feasible set* K *and the Slater condition holds. Assume that C is positive definite. Let* $\bar{x} \in K$ *and* $\bar{t} = \max_{j=1,\dots,l} \left\{ \frac{\langle \bar{x}, A_j\bar{x}\rangle + \langle a_j, \bar{x}\rangle + \alpha_j}{\langle \bar{x}, C\bar{x}\rangle + \langle c, \bar{x}\rangle + \delta} \right\}$*. Then,* $\bar{x}$ *is a solution of* (FP_2)*, if and only if,* $\forall u \in \mathbb{R}^n$, $\exists \lambda \in \mathbb{R}^m_+$, $\mu \in \Delta$ *such that*

$$\begin{pmatrix} 2\left[\sum_{j=1}^{l} \mu_j A_j + \sum_{i=1}^{m} \lambda_i B_i\right] & \left[\sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u - \bar{t}c\right] \\ \left[(\sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u - \bar{t}c)^T\right] & 2\left[\sum_{j=1}^{l} \mu_j \alpha_j + \sum_{i=1}^{m} \lambda_i \beta_i + \frac{1}{4\bar{t}}\langle u, C^{-1}u\rangle - \bar{t}\delta\right] \end{pmatrix} \succeq 0.$$

Proof Note that, $\inf(\text{FP}_2) = \bar{t} \in \mathbb{R}$, if and only if, $\inf(\mathcal{P}_{\bar{t}}) = 0$, where

$$\begin{aligned} \left(\mathcal{P}_{\bar{t}}\right) \qquad & \inf_{x\in K} \left\{ \max_{j=1,\dots,l} f_j(x) - \bar{t}h(x) \right\} \\ &= \inf_{x\in K} \left\{ \max_{j=1,\dots,l} \langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j - \bar{t}\left(\langle x, Cx\rangle + \langle c, x\rangle + \delta\right) \right\} \\ &= \inf_{x\in K} \left\{ \max_{j=1,\dots,l} \left[\langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j - \bar{t}\left(\langle c, x\rangle + \delta\right)\right] - \bar{t}\langle x, Cx\rangle \right\}. \end{aligned}$$

Let $\widehat{f_j}(x) = \left[\langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j - \bar{t}\left(\langle c, x\rangle + \delta\right)\right]$, $j = 1, \dots, l$ and $\widehat{h}(x) = \bar{t}\langle x, Cx\rangle$. We have $\widehat{h}^*(u) = \frac{1}{4\bar{t}}\langle u, C^{-1}u\rangle$.

Then, $\overline{x}$ is a solution of $(\mathcal{P}_{\overline{t}})$, if and only if, by Corollary 4.2, for any $u \in \mathbb{R}^n$, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \Delta$ such that

$$\sum_{j=1}^{l} \mu_j \left[\langle x, A_j x\rangle + \langle a_j, x\rangle + \alpha_j - \overline{t}\,(\langle c, x\rangle + \delta)\right] + \sum_{i=1}^{m} \lambda_i (\langle x, B_i x\rangle + \langle b_i, x\rangle + \beta_i)$$
$$+ \frac{1}{4\overline{t}}\langle u, C^{-1}u\rangle - \langle u, x\rangle \geq 0,$$

which is equivalent to

$$\begin{pmatrix} 2\left[\sum_{j=1}^{l} \mu_j A_j + \sum_{i=1}^{m} \lambda_i B_i\right] & \left[\sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u - \overline{t}c\right] \\ \left[(\sum_{j=1}^{l} \mu_j a_j + \sum_{i=1}^{m} \lambda_i b_i - u - \overline{t}c)^T\right] & 2\left(\sum_{j=1}^{l} \mu_j \alpha_j + \sum_{i=1}^{m} \lambda_i \beta_i + \frac{1}{4\overline{t}}\langle u, C^{-1}u\rangle - \overline{t}\delta\right) \end{pmatrix} \succeq 0.$$

□

5 Robust solutions of uncertain quadratic problems

In this section, we present characterizations of robust solutions of classes of quadratic optimization problems under data uncertainty.

We begin with a quadratic problem where the data of both the constraints and the objective functions are uncertain. Note that for a $(n \times n)$ matrix Ω, $||\Omega||_{\text{spec}}$ is the spectral norm of Ω and is defined by $||\Omega||_{\text{spec}} = \sqrt{\lambda_{\max}(\Omega^T \Omega)}$. Consider the quadratic problem

$$\begin{array}{lll} \text{(QP)} & \inf_{x\in\mathbb{R}^n} & \langle x, Ax\rangle + \langle a, x\rangle + \alpha \\ & \text{s.t.} & \langle x, B_i x\rangle + 2\langle b_i, x\rangle + \beta_i \leq 0, \quad i = 1, \ldots, m. \end{array}$$

where the data $(B_i, b_i, \beta_i) \in S^n \times \mathbb{R}^n \times \mathbb{R}$, $i = 1, \ldots, m$ are uncertain and belong to the spectral norm uncertainty set

$$\mathcal{W}_i = \left\{ (B_i, b_i, \beta_i) \in S^n \times \mathbb{R}^n \times \mathbb{R} : \left\| \begin{pmatrix} B_i & b_i \\ b_i^T & \beta_i \end{pmatrix} - \begin{pmatrix} \overline{B}_i & \overline{b}_i \\ \overline{b}_i^T & \overline{\beta}_i \end{pmatrix} \right\|_{\text{spec}} \leq \delta_i \right\},$$

$\delta_i > 0$, and $(A, a, \alpha) \in S^n \times \mathbb{R}^n \times \mathbb{R}$ is uncertain and it belongs to the uncertainty set

$$\mathcal{V} = \left\{ (A_0, a_0, \alpha_0) + \sum_{k=1}^{l} \omega_k (A_k, a_k, \alpha_k) : \ \omega = (\omega_1, \ldots, \omega_l) \in \mathbb{R}^l \right\},$$

$\omega = \sum_{j=1}^{p} \tau_j v^j$, $\tau \in \Delta_p$ for some l, $p \in \mathbb{N}$, $v^1, \ldots, v^p \in \mathbb{R}^l_+$ and $(A_k, a_k, \alpha_k) \in S^n \times \mathbb{R}^n \times \mathbb{R}$, $k = 1, \ldots, l$.

The robust counterpart of (QP), that finds a worst-case solution of (QP), is

$$\begin{array}{lll} \text{(RQP)} & \inf_{x\in\mathbb{R}^n} \max_{(A,a,\alpha)\in\mathcal{V}} & \langle x, Ax\rangle + \langle a, x\rangle + \alpha \\ & \text{s.t.} & \langle x, B_i x\rangle + 2\langle b_i, x\rangle + \beta_i \leq 0, \quad \forall (B_i, b_i, \beta_i) \in \mathcal{W}_i, \ i = 1, \ldots, m. \end{array}$$

Note that the constraints are enforced for all (B_i, b_i, β_i) in the uncertainty set $\mathcal{W}_i$, $i = 1, \ldots, m$. Following robust optimization [3,14], an optimal solution of the robust counterpart of an uncertain problem is a **robust solution** of the uncertain problem.

Let $K := \{x \in \mathbb{R}^n : \langle x, B_i x\rangle + 2\langle b_i, x\rangle + \beta_i \leq 0, \ \forall (B_i, b_i, \beta_i) \in \mathcal{W}_i, \ i = 1, \ldots, m\}$.

Theorem 5.1 (Characterization of Robust Solutions) *For problem* (QP)*, assume that for* $i = 1, \dots, m$*,* $\overline{B}_i$ *is positive semidefinite and the Slater condition holds. Let* $\widehat{A}_j = A_0 + \sum_{k=1}^{l} v_k^j A_k$*,* $\widehat{a}_j = a_0 + \sum_{k=1}^{l} v_k^j a_k$ *and* $\widehat{\alpha}_j = \alpha_0 + \sum_{k=1}^{l} v_k^j \alpha_k$ *for* $j = 1, \dots, p$*.*

Let $\rho = \max_{j=1,\dots,p}\{-\gamma_{\min}(\widehat{A}_j)\}$*. Assume that* $\gamma_{\min}(\widehat{A}_j) < 0$ *for some* $j \in \{1, \dots, p\}$*. Then* $x^* \in K$ *is a robust global minimizer of* (QP)*, if and only if, for any* $u \in \mathbb{R}^n$*, there exist* $\lambda \in \mathbb{R}_+^m$*, and* $\mu \in \Delta$ *such that*

$$\begin{pmatrix} 2\left[\sum_{j=1}^{p}\mu_j\widehat{A}_j+\rho I_n+\sum_{i=1}^{m}\lambda_i(\overline{B}_i+\delta_i I_n)\right] & \left[\sum_{j=1}^{p}\mu_j\widehat{a}_j+2\sum_{i=1}^{m}\lambda_i\overline{b}_i-u\right] \\ \left[\sum_{j=1}^{p}\mu_j\widehat{a}_j+2\sum_{i=1}^{m}\lambda_i\overline{b}_i-u\right]^T & 2\left[\sum_{i=1}^{m}\lambda_i\left(\overline{\beta}_i+\delta_i\right)+\frac{\|u\|^2}{4\rho}+\sum_{j=1}^{p}\mu_j\widehat{\alpha}_j-\overline{t}\right] \end{pmatrix} \succeq 0, \tag{13}$$

where $\overline{t} = \max_{(A,a,\alpha)\in\mathcal{V}} \langle x^*, Ax^*\rangle + \langle a, x^*\rangle + \alpha$*.*

Proof We first note that for each $x \in \mathbb{R}^n$

$$\max_{(A,a,\alpha)\in\mathcal{V}} \langle x, Ax\rangle + \langle a, x\rangle + \alpha$$
$$= \max_{j=1,\dots,p}\left\{\left\langle x, \left(A_0 + \sum_{k=1}^{l} v_k^j A_k\right)x\right\rangle + \left\langle a_0 + \sum_{k=1}^{l} v_k^j a_k, x\right\rangle + \alpha_0 + \sum_{k=1}^{l} v_k^j\alpha_k\right\}.$$

Since $\widehat{A}_j = A_0 + \sum_{k=1}^{l} v_k^j A_k$, $\widehat{a}_j = a_0 + \sum_{k=1}^{l} v_k^j a_k$ and $\widehat{\alpha}_j = \alpha_0 + \sum_{k=1}^{l} v_k^j \alpha_k$, we have

$$\max_{(A,a,\alpha)\in\mathcal{V}} \langle x, Ax\rangle + \langle a, x\rangle + \alpha = \max_{j=1,\dots,p}\left\{\langle x, \widehat{A}_j x\rangle + \langle \widehat{a}_j, x\rangle + \widehat{\alpha}_j\right\}. \tag{14}$$

Note also that $K \subseteq \{x \in \mathbb{R}^n : \langle x, \left(\overline{B}_i + \delta_i I_n\right)x\rangle + 2\langle \overline{b}_i, x\rangle + \overline{\beta}_i + \delta_i \le 0,\ i = 1, \dots, m\}$ because $\left(\overline{B}_i + \delta_i I_n, \overline{b}_i, \overline{\beta}_i + \delta_i\right) \in \mathcal{W}_i,\ i = 1, \dots, m$.

On the other hand, it is easy to check that, for each $(B_i, b_i, \beta_i) \in \mathcal{W}_i$, all the eigenvalues of the matrix $\begin{pmatrix} \overline{B}_i + \delta_i I_n & \overline{b}_i \\ \overline{b}_i^T & \overline{\beta}_i + \delta_i \end{pmatrix} - \begin{pmatrix} B_i & b_i \\ b_i^T & \beta_i \end{pmatrix}$ are non-negative, and so, $\begin{pmatrix} \overline{B}_i + \delta_i I_n & \overline{b}_i \\ \overline{b}_i^T & \overline{\beta}_i + \delta_i \end{pmatrix} - \begin{pmatrix} B_i & b_i \\ b_i^T & \beta_i \end{pmatrix}$ is a positive semidefinite matrix. Hence,

$$\begin{pmatrix} x \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} \overline{B}_i + \delta_i I_n & \overline{b}_i \\ \overline{b}_i^T & \overline{\beta}_i + \delta_i \end{pmatrix} - \begin{pmatrix} B_i & b_i \\ b_i^T & \beta_i \end{pmatrix}\right)\begin{pmatrix} x \\ 1 \end{pmatrix} \ge 0.$$

This shows that $\{x \in \mathbb{R}^n : \langle x, \left(\overline{B}_i + \delta_i I_n\right)x\rangle + 2\langle \overline{b}_i, x\rangle + \overline{\beta}_i + \delta_i \le 0,\ i = 1, \dots, m\} \subseteq K$. Thus, $K = \{x \in \mathbb{R}^n : \langle x, \left(\overline{B}_i + \delta_i I_n\right)x\rangle + 2\langle \overline{b}_i, x\rangle + \overline{\beta}_i + \delta_i \le 0,\ i = 1, \dots, m\}$.

Using (14) and the fact that $K = \{x \in \mathbb{R}^n : \langle x, \left(\overline{B}_i + \delta_i I_n\right)x\rangle + 2\langle \overline{b}_i, x\rangle + \overline{\beta}_i + \delta_i \le 0,\ i = 1, \dots, m\}$, the problem (RQP) can be written as

$$\begin{array}{ll} \inf_{x\in\mathbb{R}^n} \max_{j=1,\dots,p} & \langle x, \widehat{A}_j x\rangle + \langle \widehat{a}_j, x\rangle + \widehat{\alpha}_j \\ \text{s.t.} & \langle x, \left(\overline{B}_i + \delta_i I_n\right)x\rangle + 2\langle \overline{b}_i, x\rangle + \overline{\beta}_i + \delta_i \le 0,\ i = 1, \dots, m. \end{array}$$

The quadratic function $\langle x, \widehat{A}_j x\rangle + \langle \widehat{a}_j, x\rangle + \widehat{\alpha}_j$ can be written as $\langle x, \left(\widehat{A}_j + \rho I_n\right)x\rangle + \langle \widehat{a}_j, x\rangle + \widehat{\alpha}_j - \rho\|x\|^2$ where $\langle x, \left(\widehat{A}_j + \rho I_n\right)x\rangle + \langle \widehat{a}_j, x\rangle + \widehat{\alpha}_j$ is convex for any $j = 1, \dots, p$ and $\rho = \max_{j=1,\dots,p}\{-\gamma_{\min}(\widehat{A}_j)\}$. By our assumption, $\rho > 0$. Now, the problem (RQP) becomes

$$\begin{array}{ll}\inf_{x\in\mathbb{R}^n}\max_{j=1,\ldots,p} & \left\{\langle x,\left(\widehat{A}_j+\rho I_n\right)x\rangle+\left\langle\widehat{a}_j,x\right\rangle+\widehat{\alpha}_j-\rho\|x\|^2\right\}\\ \text{s.t.} & \langle x,\left(\overline{B}_i+\delta_i I_n\right)x\rangle+2\langle\overline{b}_i,x\rangle+\overline{\beta}_i+\delta_i\le 0,\ i=1,\ldots,m.\end{array}\tag{15}$$

By Corollary 4.2, $x^*\in K$ is a solution of (15) if and only if for any $u\in\mathbb{R}^n$, there exist $\lambda\in\mathbb{R}_+$, $\mu\in\Delta$ such that

$$\begin{aligned}&\sum_{j=1}^{p}\mu_j\left(\langle x,\left(\widehat{A}_j+\rho I_n\right)x\rangle+\langle\widehat{a}_j,x\rangle+\widehat{\alpha}_j\right)\\&\quad+\sum_{i=1}^{m}\lambda_i\left[\langle x,\left(\overline{B}_i+\delta_i I_n\right)x\rangle+2\langle\overline{b}_i,x\rangle+\overline{\beta}_i+\delta_i\right]+\frac{\|u\|^2}{4\rho}-\langle u,x\rangle-\overline{t}\ge 0,\end{aligned}\tag{16}$$

as required.

Conversely, assume for any $u\in\mathbb{R}^n$, there exist $\lambda\in\mathbb{R}_+$, $\mu\in\Delta$ relation (13) holds. The linear matrix inequality (13) can be equivalent to relation (16). Hence, x^* is a solution of problem in (15) and then, x^* is a robust global minimizer of (QP). □

Let us consider the uncertain trust region problem as a special case of problem (QP)

$$\begin{array}{lll}\text{(TP)} & \inf_{x\in\mathbb{R}^n} & \langle x,Ax\rangle+\langle a,x\rangle+\alpha\\ & \text{s.t.} & \|x-x_0\|\le s,\end{array}\tag{17}$$

where the data (A,a,α) is uncertain and it belongs to the uncertainty set $\mathcal{V}$, which is given by

$$\mathcal{V}=\left\{(A_0,a_0,\alpha_0)+\sum_{k=1}^{l}\omega_k(A_k,a_k,\alpha_k):\ \omega=(\omega_1,\ldots,\omega_l)\in\mathbb{R}^l\text{ and }\omega=\sum_{j=1}^{p}\tau_j v^j,\ \tau\in\Delta_p\right\},$$

$l,\ p\in\mathbb{N}$ and $v^1,\ldots,v^p\in\mathbb{R}^l_+$. We assume that $s>0$ and $x_0\in\mathbb{R}^n$.

The robust counterpart, that finds a worst-case solution of (TP), is

$$\begin{array}{lll}\text{(RP)} & \inf_{x\in\mathbb{R}^n}\max_{(A,a,\alpha)\in\mathcal{V}} & \langle x,Ax\rangle+\langle a,x\rangle+\alpha\\ & \text{s.t.} & \|x-x_0\|\le s.\end{array}$$

Corollary 5.1 (Uncertain Trust-Region Problem) *For problem* (TP) *with data uncertainty* $\mathcal{V}$ *as above, let* $\widehat{A}_j=A_0+\sum_{k=1}^{l}v_k^jA_k$, $\widehat{a}_j=a_0+\sum_{k=1}^{l}v_k^ja_k$ *and let* $\widehat{\alpha}_j=\alpha_0+\sum_{k=1}^{l}v_k^j\alpha_k$ *for* $j=1,\ldots,p$. *Let* $\rho=\max_{j=1,\ldots,p}\{-\gamma_{\min}(\widehat{A}_j)\}$.

(i) *If* $\rho>0$, *then a feasible point* x^* *of* (TP) *is a robust global minimizer of* (TP) *if and only if, for any* $u\in\mathbb{R}^n$, *there exist* $\lambda\in\mathbb{R}_+$ *and* $\mu\in\Delta_p$ *such that*

$$\begin{pmatrix}2\left[\sum_{j=1}^{p}\mu_j\widehat{A}_j+(\rho+\lambda)I_n\right] & \left[\sum_{j=1}^{p}\mu_j\widehat{a}_j-u-2\lambda x_0\right]\\ \left[\sum_{j=1}^{p}\mu_j\widehat{a}_j-u-2\lambda x_0\right]^T & 2\left[\lambda\left(\|x_0\|^2-s\right)+\frac{\|u\|^2}{4\rho}-\overline{t}+\sum_{j=1}^{p}\mu_j\widehat{\alpha}_j\right]\end{pmatrix}\succeq 0,$$

where $\overline{t}=\max_{(A,a,\alpha)\in\mathcal{V}}\langle x^*,Ax^*\rangle+\langle a,x^*\rangle+\alpha$.

(ii) *If* $\rho\le 0$, *then a feasible point* x^* *of* (TP) *is a robust global minimizer of* (TP) *if and only if, there exist* $\lambda\in\mathbb{R}_+$ *and* $\mu\in\Delta_p$ *such that*

$$\begin{pmatrix} 2\left[\sum_{j=1}^{p}\mu_j\widehat{A}_j+\lambda I_n\right] & \left[\sum_{j=1}^{p}\mu_j\widehat{a}_j-2\lambda x_0\right] \\ \left[\sum_{j=1}^{p}\mu_j\widehat{a}_j-2\lambda x_0\right]^T & 2\left[\lambda\left(\|x_0\|^2-s\right)+\sum_{j=1}^{p}\mu_j\widehat{\alpha}_j-\bar{t}\right]\end{pmatrix}\succeq 0,$$

where $\bar{t}=\max_{(A,a,\alpha)\in\mathcal{V}}\langle x^*,Ax^*\rangle+\langle a,x^*\rangle+\alpha$.

Proof Note as before that for each $x\in\mathbb{R}^n$

$$\begin{aligned}&\max_{(A,a,\alpha)\in\mathcal{V}}\langle x,Ax\rangle+\langle a,x\rangle+\alpha\\ &=\max_{j=1,\ldots,p}\left\{\left\langle x,(A_0+\sum_{k=1}^{l}v_k^jA_k)x\right\rangle+\left\langle a_0+\sum_{k=1}^{l}v_k^ja_k,x\right\rangle+\alpha_0+\sum_{k=1}^{l}v_k^j\alpha_k\right\}.\end{aligned}$$

Then, the problem (RP) becomes

$$\begin{array}{ll}\inf_{x\in\mathbb{R}^n}\max_{j=1,\ldots,p} & \langle x,\widehat{A}_jx\rangle+\langle\widehat{a}_j,x\rangle+\widehat{\alpha}_j\\ \text{s.t.} & \|x-x_0\|\le s.\end{array}$$

and the Slater condition holds.

(i) Assume that $\rho>0$. Let

$$\mathcal{W}=\left\{(B,b,\beta)\in S^n\times\mathbb{R}^n\times\mathbb{R}:\left\|\begin{pmatrix}B & b\\ b^T & \beta\end{pmatrix}-\begin{pmatrix}I & -x_0\\ -x_0^T & \|x_0\|^2-s^2\end{pmatrix}\right\|_{\text{spec}}\le 0\right\}$$

The constraint $\|x-x_0\|\le s$ is equivalent to the quadratic constraint $\langle x,B_1x\rangle+2\langle b_1,x\rangle+\beta_1\le 0$, where $B_1=I$, $b_1=-x_0$ and $\beta_1=\|x_0\|^2-s^2$. Applying the previous theorem with $m=1$, we obtain the required condition characterizing the robust global minimizer.

(ii) If $\rho\le 0$, then $\gamma_{\min}(A_j)\ge 0$ for all $j=1,\ldots,p$. Let $f_j(x)=\langle x,\widehat{A}_jx\rangle+\langle\widehat{a}_j,x\rangle+\widehat{\alpha}_j$, and $h(x)=0$. Then f_j is convex. By Corollary 4.2, the conclusion follows. □

A special case of (TP) occurs when $(A,a,\alpha)\in\mathcal{V}_1:=\text{co}\{(A_j,a_j,\alpha_j)\in S^n\times\mathbb{R}^n\times\mathbb{R},\ j=1,\ldots,p\}$, where co denotes the convex hull. The robust counterpart of (TP) in this case is

$$\begin{array}{lll}(\text{RP}_1) & \inf_{x\in\mathbb{R}^n}\max_{(A,a,\alpha)\in\mathcal{V}_1} & \langle x,Ax\rangle+\langle a,x\rangle+\alpha\\ & \text{s.t.} & \|x-x_0\|\le s.\end{array}$$

Corollary 5.2 (Polytope uncertainty) *For the problem* (TP) *with the data uncertainty* $\mathcal{V}_1$, *let* $\bar{t}=\inf$ (TP) *and let* $\rho=\max\{-\gamma_{\min}(A_j):j=1,\ldots,p\}$.

(i) *If* $\rho>0$, *then a feasible point* x^* *of* (TP) *is a robust global minimizer of* (TP), *if and only if, for any* $u\in\mathbb{R}^n$, *there exist* $\lambda\in\mathbb{R}_+$ *and* $\mu\in\Delta_p$ *such that*

$$\begin{pmatrix} 2\left[\sum_{j=1}^{p}\mu_jA_j+(\rho+\lambda)I_n\right] & \left[\sum_{j=1}^{p}\mu_ja_j-u-2\lambda x_0\right] \\ \left[\sum_{j=1}^{p}\mu_ja_j-u-2\lambda x_0\right]^T & 2\left[\lambda\left(\|x_0\|^2-s\right)+\frac{\|u\|^2}{4\rho}+\sum_{j=1}^{p}\mu_j\alpha_j-\bar{t}\right]\end{pmatrix}\succeq 0,$$

where $\bar{t}=\max_{(A,a,\alpha)\in\mathcal{V}_1}\langle x^*,Ax^*\rangle+\langle a,x^*\rangle+\alpha$.

(ii) *If* $\rho\le 0$, *then a feasible point* x^* *of* (TP) *is a robust global minimizer of* (TP), *if and only if, there exist* $\lambda\in\mathbb{R}_+$ *and* $\mu\in\Delta_p$ *such that*

$$\begin{pmatrix} 2\left[\sum_{j=1}^{p}\mu_j A_j+\lambda I_n\right] & \left[\sum_{j=1}^{p}\mu_j a_j-2\lambda x_0\right] \\ \left[\sum_{j=1}^{p}\mu_j a_j-2\lambda x_0\right]^T & 2\left[\lambda\left(||x_0||^2-s\right)+\sum_{j=1}^{p}\mu_j\alpha_j-\bar{t}\right] \end{pmatrix}$$

where $\bar{t}=\max_{(A,a,\alpha)\in\mathcal{V}_1}\langle x^*, Ax^*\rangle+\langle a, x^*\rangle+\alpha.$

Proof Note that for each $x\in\mathbb{R}^n$, we have

$$\max_{(A,a,\alpha)\in\mathcal{V}_1}\langle x, Ax\rangle+\langle a, x\rangle+\alpha=\max_{j=1,\ldots,p}\langle x, A_j x\rangle+\langle a_j, x\rangle+\alpha_j.$$

The conclusion follows from the same line of arguments as in the proof of the previous Corollary. □

We give an example illustrating Corollary 5.2(i).

Example 5.1 Consider the uncertain problem

$$\begin{array}{ll} \min_{x\in\mathbb{R}} & -x^2+ax \\ \text{s.t.} & x^2-\frac{1}{4}\leq 0, \end{array}$$

where a is uncertain and it belongs to the uncertainty set $\mathcal{V}=[-1, 1]$. Its robust counterpart is given by

$$(\text{EP}_4)\quad \begin{array}{ll} \min_{x\in\mathbb{R}}\max_{a\in[-1,1]} & \{-x^2+ax\} \\ \text{s.t.} & x^2-\frac{1}{4}\leq 0. \end{array}$$

Then the problem (EP_4) can be written as $\inf_{x\in K}\max\{-x^2+x,\ -x^2-x\}=\inf_{x\in K}\max\{x,\ -x\}-x^2$. We can check that $\inf\,(\text{EP}_4)=0$.

Let $h(x)=x^2$, we have $h^*(u)=\frac{u^2}{4}$ and let

$$\begin{aligned} d(x,u,\lambda,\mu)&=\mu_1(x)+\mu_2(-x)+\lambda\left(x^2-\frac{1}{4}\right)+\frac{u^2}{4}-\langle u, x\rangle \\ &=x(2\mu_1-1-u)+\lambda\left(x^2-\frac{1}{4}\right)+\frac{u^2}{4}. \end{aligned}$$

Then d can be written as

$$d=\begin{cases} \frac{u^2}{4}, & \text{if } u\in[-1,1], \quad \text{with } \lambda=0 \quad \text{and} \quad \mu_1=\frac{1+u}{2} \\ \left(\sqrt{\lambda}x+\frac{2\mu_1-1-u}{2\sqrt{\lambda}}\right)^2, & \text{if } u>1, \quad \text{with } \lambda=\frac{u^2+\sqrt{u^4-4(2\mu_1-1-u)^2}}{2}\geq 0 \quad \text{and} \quad \mu_1=1 \\ \left(\sqrt{\lambda}x+\frac{2\mu_1-1-u}{2\sqrt{\lambda}}\right)^2, & \text{if } u<-1, \quad \text{with } \lambda=\frac{u^2+\sqrt{u^4-4(2\mu_1-1-u)^2}}{2}\geq 0 \quad \text{and} \quad \mu_1=0. \end{cases}$$

Hence, for any $u\in\mathbb{R}^2$, there exist $\lambda\in\mathbb{R}_+$ and $\mu\in\Delta$ such that $d(x,u,\lambda,\mu)\geq 0$, which is equivalent to the linear matrix inequality of Corollary 5.2 (i).

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