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Existence results for vector equilibrium problems given by a sum of two functions

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Abstract We obtain existence results for the weak vector equilibrium problem where the function involved is a sum of two functions, and the assumptions are required separately on each of these functions. We show that some earlier results of this type contain too demanding assumptions. We relax several of these assumption without loosing the results. The special case of reflexive Banach spaces is also studied, where we make use of the fact that closed balls are weakly compact.

Keywords Vector equilibrium problem · K-upper semicontinuous function · Essentially quasimonotone bifunction · K-convex bifunction · Coercivity condition · KKM-mapping

1 Introduction

Let X be a real topological vector space, $A \subset X$ and $f : A \times A \to \mathbb{R}$ a given bifunction. By an *equilibrium problem* we understand the problem of finding an element $x \in A$, such that

$$\text{(EP)}\ \ f(x, y) \geq 0 \quad \text{for all } y \in A.$$

The problem itself was investigated in a paper of Ky Fan [21], under the name "minimax inequality problem", in connection with the so called "intersection theorems" (i.e., results stating the nonemptiness of a certain family of sets). Later on it turned out to be very useful as provides a unified approach for many interesting particular problems.

In order to stress the importance of (EP), let us recall just two examples related to global optimization.

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Scalar optimization problems: finding a global minimum of a function $F : \mathbb{R}^n \to \mathbb{R}$ over a feasible set $A \subset \mathbb{R}^n$ (the set of constraints - usually given by the intersection of level sets of a family of functions) amounts to solving (EP) with

$$f(x, y) = F(y) - F(x).$$

Pareto optimization problems: a weak Pareto global minimum of the vector function $F = (F_1, ..., F_m) : \mathbb{R}^n \to \mathbb{R}^m$ over a feasible set $A \subset \mathbb{R}^n$ is a vector $x \in A$ such that for any $y \in A$ there exists an index $i \in \{1, ..., m\}$ with $F_i(y) - F_i(x) \geq 0$. Finding a weak Pareto global minimum means to solving (EP) with

$$f(x, y) = \max_{1 \leq i \leq m} [F_i(y) - F_i(x)].$$

As far as we know the term "equilibrium problem" was coined by Muu and Oettli in [34], where three standard examples of (EP) were considered: the optimization problems, the variational inequalities and the fixed point problems. Further particular cases like saddlepoint (minimax) problems, Nash equilibria problems, convex differentiable optimization and complementarity problems have been considered by Blum and Oettli [13], while Antipin [2] formulated an inverse optimization problem as a noncooperative game. A survey of those equilibrium problems which are relevant in operations research and mathematical programming can be found in the recent paper of Bigi et al. [11], where among the existence results, solution methods are presented in a structured overview with different levels. For other recent results concerning existence and solution methods for (EP), see [1,5–7,9,10,12,16,18,19,26,27,31,35,36] and the references therein.

Apart from its theoretical interest, important problems arising from economics, mechanics, electricity, chemistry and other practical sciences motivate the study of (EP). In economics it often refers to production competition or dynamics of offer and demand, exploiting the mathematical model of noncooperative games and the corresponding equilibrium concept by Nash.

Despite of the growing interest on (EP), a seemingly interesting special case, where $f(x, y) = g(x, y) + h(x, y)$ with $g, h : A \times A \to \mathbb{R}$ captured less attention, although it was investigated already in [13], where the authors obtained existence results by imposing their assumptions separately on g and h. As stressed in [13], if $g = 0$, the result becomes a variant of Ky Fan's theorem [21], whereas for $h = 0$ it becomes a variant of the Browder-Minty theorem for variational inequalities (see [14,15,32]).

Consider another real topological vector space Y, partially ordered by a proper convex cone $K \subset Y$ with nonempty interior (denoted by $int K$) such that $0 \in K \cap (-K)$. If $\mathbb{R}$ is replaced by Y, i.e., f becomes a vector-valued bifunction of the form $f : A \times A \to Y$, one may consider the problem of finding an element $x \in A$, such that:

$$\text{(VEP)} \ f(x, y) \notin -int K \quad \text{for all } y \in A.$$

This problem is widely known as the *weak vector equilibrium problem* since replacing $-int K$ by $-K \setminus \{0\}$, provides the so-called *strong vector equilibrium problem*. Note that the strong case has not received much attention up to now. Results concerning this case can be found for instance in [8,23].

In this paper we only deal with the weak case; therefore, in the sequel, for brevity, we will simply call it *vector equilibrium problem*. (VEP) has also attracted much attention in the recent years especially due to its applications within the fields of vector optimization and vector variational inequalities (see, for instance, [17,22,25,33] and the references therein). Note that from the vector equilibrium problem defined above one can easily obtain the general form of a (weak) vector optimization problem rather than Pareto optimization problem. Indeed, when $f(x, y) = F(y) - F(x)$ with $F : X \to Y$, (VEP) amounts to find a weak efficient point of the problem $\min\{F(y) : y \in A\}$, i.e., an element $x \in A$ such that there is no $y \in A$ with $F(y) - F(x) \in -intK$.

Similarly to scalar equilibrium problems, (VEP) given by a sum of two bifunctions was less investigated than it deserves. We mention here the paper of Bianchi et al. [4] and Kazmi [29]. In the latter the author follows the idea and steps of the proofs given by Blum and Oettli [13]. However, his assumptions on the vector-valued bifunctions turn to be too strong in order to recover the result of Blum and Oettli when $Y := \mathbb{R}$ and $K := [0, \infty)$.

The aim of this paper is to weaken the assumptions of Kazmi in such a way that we are able to recover Blum-Oettli's results on one hand, and by assuming alternative conditions on the vector bifunctions, to deduce new existence theorems, on the other hand. The special case of reflexive Banach spaces endowed with the weak topology is separately treated; in that case, mild sufficient conditions for guaranteing coercivity are presented.

The paper is organized as follows. In Sect. 2 we recall or introduce the concepts we need for the sequel and state some basic properties and recall some auxiliary results related to them. Section 3 is devoted to the study of (VEP): here, our main results are established and proved together with the necessary lemmas. Finally, the last section discusses the special case of reflexive Banach spaces, where the weak compactness of closed balls are explored in order to assure the necessary coercivity conditions required in Sect. 3.

2 Preliminaries

Throughout this paper, if not otherwise stated, X and Y denote real topological vector spaces, $A, B \subset X$ nonempty convex sets (B being typically a compact subset of A, but not always), and $K \subset Y$ a proper convex cone with nonempty interior such that $0 \in K \cap (-K)$.

Let us first recall the following concept. If $B \subset A$, then $core_A B$, the core of B relative to A, is defined through

$$a \in core_A B \iff (a \in B \text{ and } B \cap (a, y] \neq \emptyset \quad \text{for all } y \in A \backslash B),$$

where $(a, y] = \{\lambda a + (1 - \lambda)y : \lambda \in [0, 1)\}$. Note that $core_A A = A$.

The following simple property will be useful in the sequel.

Lemma 1 *For all $x, y \in Y$ we have:*

$$x \in K,\ y \notin -intK \Rightarrow x + y \notin -intK.$$

The convexity concept for scalar functions has been extended in a natural way for vector-valued functions, according to the partial order introduced by the cone K.

Definition 1 A function $F : X \to Y$ is called K-convex, iff for each $x, y \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x) + (1 - \lambda)F(y) - F(\lambda x + (1 - \lambda)y) \in K.$$

F is said to be K-concave iff $-F$ is K-convex.

In vector optimization various relaxations and modifications of the classical *lower/upper semicontinuity* for scalar functions have been investigated for vector-valued functions to explore and characterize efficient solutions. The following version of upper semicontinuity will play a crucial role for our investigations; apparently it was first considered by D. T. Luc [30].

Definition 2 ([30], Definition 5.1) A function $F : X \to Y$ is said to be K-upper semicontinuous at $x \in X$ iff for any neighborhood $V \subset Y$ of $F(x)$ there exists a neighborhood $U \subset X$ of x such that $F(u) \in V - K$ for all $u \in U$.

F is said to be K-upper semicontinuous on X (K-upper semicontinuous in short) iff it is K-upper semicontinuous at each $x \in X$.
The function F is said to be K-lower semicontinuous iff $-F$ is K-upper semicontinuous.

The next characterizations of upper semicontinuity has been given by Tanaka [37].

Lemma 2 *The following three statements are equivalent:*

(i) *F is K-upper semicontinuous on X;*
(ii) *for any $x \in X$, for any $k \in intK$, there exists a neighborhood $U \subset X$ of x such that $F(u) \in F(x) + k - intK$ for all $u \in U$;*
(iii) *for any $a \in Y$, the set $\{x \in X : F(x) - a \in -intK\}$ is open.*

Next we present some monotonicity conditions for vector-valued bifunctions. In order to do this, let us first recall several specific monotonicity concepts for scalar bifunctions considered within the literature in the recent years. Most of these notions were inspired by similar (generalized) monotonicity concepts defined for operators acting from a topological vector space to its dual space (see, for instance, [28]).

Definition 3 The bifunction $f : A \times A \to \mathbb{R}$ is said to be

(i) *monotone* iff $f(x, y) + f(y, x) \leq 0$ for all $x, y \in A$;
(ii) *properly quasimonotone* iff for arbitrary integer $n \geq 1$, all $x_1, \ldots, x_n \in A$ and all $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ it holds that

$$\min_{1 \leq i \leq n} f\left(x_i, \sum_{j=1}^{n} \lambda_j x_j\right) \leq 0.$$

Proper quasimonotonicity was introduced by Zhou and Chen in [38] under the name of 0-*diagonal quasiconcavity* (see also [5]). Aiming to obtain existence results for (scalar) equilibrium problems, the authors of [26] introduced the following (slightly stronger) variant of proper quasimonotonicity (called by themselves *property P4*):

Definition 4 (cf. [26]) A bifunction $f : A \times A \to \mathbb{R}$ is said to be *essentially quasimonotone* iff for arbitrary integer $n \geq 1$, for every $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, it holds that

$$\sum_{i=1}^{n} \lambda_i f\left(x_i, \sum_{j=1}^{n} \lambda_j x_j\right) \leq 0.$$

The extension of monotonicity for vector-valued bifunctions has been considered in a natural way (see, for instance, Gong and Yao [24], and Kazmi [29]). Namely, $f : A \times A \to Y$ is said to be *K-monotone* iff

$$f(x, y) + f(y, x) \in -K, \quad \forall x, y \in A.$$

In order to obtain our results, we need to extend the concept of essential quasimonotonicity for vector-valued bifunctions.

Definition 5 A bifunction $f : A \times A \to Y$ is said to be $K-$*essentially quasimonotone* iff for arbitrary integer $n \geq 1$, for all $x_1, ...x_n \in A$ and all $\lambda_1, ...\lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ it holds that

$$\sum_{i=1}^n \lambda_i f\left(x_i, \sum_{j=1}^n \lambda_j x_j\right) \notin int K.$$

Let us conclude this section with the following simple property.

Proposition 1 *Suppose that $f : A \times A \to Y$ is K-monotone and K-convex in the second argument. Then f is K-essentially quasimonotone.*

Proof Take $x_1, ...x_n \in A$ and $\lambda_1, ...\lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and set $z := \sum_{j=1}^n \lambda_j x_j$. Then

$$\sum_{i=1}^n \lambda_i f(x_i, z) \leq_K \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j f(x_i, x_j) =$$
$$\frac{1}{2} \sum_{i,j=1}^n \lambda_i \lambda_j (f(x_i, x_j) + f(x_j, x_i)) \leq_K 0.$$

The next example shows that a K-essentially quasimonotone bifunction is not necessarily K-monotone, even if it is K-convex in the second argument.

Example 1 Let $f : [0, 1] \times [0, 1] \to \mathbb{R}^2$ given by $f = (f_1, f_2)$, where $f_1(x, y) = |x - y|$, $f_2(x, y) = 0$ for every $x, y \in [0, 1]$. It is easy to see that f is $\mathbb{R}^2_+$-essentially quasimonotone, $\mathbb{R}^2_+$-convex in the second argument, but not $\mathbb{R}^2_+$-monotone, since $f(1, 0)+f(0, 1) = (2, 0) \notin -\mathbb{R}^2_+$.

3 Main results

In what follows we are interested to obtain existence results for (VEP) when the bifunction $f : A \times A \to Y$ is given by $f(x, y) = g(x, y) + h(x, y)$ where $g, h : A \times A \to Y$. Such a situation was already explored by Blum and Oetlli [13] in the particular case when $Y := \mathbb{R}$ and $K := [0, \infty)$, i.e., the scalar equilibrium problem (EP). Their result was obtained with g monotone satisfying a mild upper semicontinuity in the first argument, whereas h is not necessarily monotone, but has to satisfy a much stronger upper semicontinuity condition in the first argument. Later, Kazmi [29], made an attempt to extend these results to the general case of (VEP) by following the ideas of [13], but many of his assumptions are too strong, therefore he couldn't recover the results of Blum and Oettli. For example, while the latter assumes semicontinuity for scalar functions, Kazmi requires continuity of the corresponding vector functions. The aim of this section is to improve Kazmi's results by weakening their conditions (both topological and algebraical) in such a way to recover Blum and Oettli's results (the three Lemmas and Theorem 1 below), and to obtain a variant without monotonicity requirement upon g; this is substituted by assuming K-concavity of g in its first variable (Corollary 3 below). We emphasize that $f(x, x) = 0$ for every $x \in A$ is generally an important assumption on equilibrium problems and holds in almost all particular cases, although there are some exceptions. It is our aim to keep the conditions upon $h(x, x)$ and $g(x, x)$ as general as possible, and this, in some situations allows us to assume $g(x, x) \in K$ (or $g(x, x) \in K \cap (-K)$), respectively $h(x, x) \in K$ rather than $g(x, x) = h(x, x) = 0$. All these assumptions are merely technical details.

To start, let us first prove the following three lemmas. All of them serve as tools for the proofs of the main results. Moreover, the first one can also be seen as an existence result for a special vector equilibrium problem, therefore it seems interesting on its own.

Lemma 3 *Suppose that B is a compact subset of X, let $g : B \times B \to Y$ and $h : B \times B \to Y$ be given bifunctions satisfying:*

(i) g is K-essentially quasimonotone and K-lower semicontinuous in the second argument;
(ii) h is K-upper semicontinuous in the first argument and K-convex in the second argument; $h(x, x) \in K$ for all $x \in B$.

Then there exists $\overline{x} \in B$ such that

$$h(\overline{x}, y) - g(y, \overline{x}) \notin -intK,$$

for all $y \in B$.

Proof Let, for each fixed $y \in B$,

$$S(y) := \{x \in B : h(x, y) - g(y, x) \notin -intK\}. \tag{1}$$

Let us show that $\cap_{y \in B} S(y) \neq \emptyset$. Indeed, let $\{y_1, y_2, ..., y_n\} \subset B$ and set $I := \{1, 2, ..., n\}$. Take arbitrary $z \in co\{y_i : i \in I\}$. Then $z = \sum_{i \in I} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i \in I} \lambda_i = 1$. Assume, by contradiction, that

$$h(z, y_i) - g(y_i, z) \in -intK,$$

for all $i \in I$. From this we have

$$\sum_{i \in I} \lambda_i h(z, y_i) - \sum_{i \in I} \lambda_i g(y_i, z) \in -intK. \tag{2}$$

Since g is K-essentially quasimonotone it follows that

$$\sum_{i \in I} \lambda_i g\left(y_i, \sum_{i \in I} \lambda_i y_i\right) \notin intK. \tag{3}$$

On the other hand, since $h(x, x) \in K$ for all $x \in B$, and h is K-convex in the second argument,

$$\sum_{i \in I} \lambda_i h(z, y_i) \in K. \tag{4}$$

Therefore, by Lemma 1, (3) and (4) it follows that

$$\sum_{i \in I} \lambda_i h(z, y_i) - \sum_{i \in I} \lambda_i g(y_i, z) \notin -intK, \tag{5}$$

which contradicts (2). Hence we obtain

$$co\{y_i : i \in I\} \subset \bigcup_{i \in I} S(y_i).$$

Now, since h is K-upper semicontinuous in the first argument, and g is K-lower semicontinuous in the second argument, it follows that the function F defined by $F(x) = h(x, y) - g(y, x)$ is K-upper semicontinuous. By Lemma 2 (iii), the sets $S(y)$ are closed for every $y \in B$, and since B is compact, they are compact too. Hence, by the KKM theorem [20], $\cap_{y \in B} S(y) \neq \emptyset$. Therefore, there exists at least one $\overline{x} \in B$ such that

$$h(\overline{x}, y) - g(y, \overline{x}) \notin -intK,$$

for all $y \in B$.

The next lemma makes the connection between the special equilibrium problem considered in Lemma 3 and the equilibrium problem we are interested in.

Lemma 4 *Let $g : B \times B \to Y$ and $h : B \times B \to Y$ be given bifunctions satisfying:*

(i) g is K-convex in the second argument, $g(x, x) \in K$ for all $x \in B$, and for all $x, y \in B$ the function $t \in [0, 1] \to g(ty + (1 - t)x, y)$ is K-upper semicontinuous at 0;
(ii) h is K-convex in the second argument; $h(x, x) = 0$ for all $x \in B$.

If there exists $\overline{x} \in B$ such that $h(\overline{x}, y) - g(y, \overline{x}) \notin -intK$ for all $y \in B$, then $h(\overline{x}, y) + g(\overline{x}, y) \notin -intK$ for all $y \in B$.

Proof Let $y \in B$ be arbitrary and let $x_\lambda := \lambda y + (1 - \lambda)\overline{x}$, $0 < \lambda \leq 1$. Then $x_\lambda \in B$. By the hypothesis we obtain

$$(1 - \lambda)h(\overline{x}, x_\lambda) - (1 - \lambda)g(x_\lambda, \overline{x}) \notin -intK. \tag{6}$$

Since g is K-convex in the second argument and $g(x, x) \in K$ for all $x \in B$, then for all $0 < \lambda \leq 1$,

$$\lambda g(x_\lambda, y) + (1 - \lambda)g(x_\lambda, \overline{x}) \in K. \tag{7}$$

By Lemma 1, (6) and (7) we have

$$(1 - \lambda)h(\overline{x}, x_\lambda) + \lambda g(x_\lambda, y) \notin -intK. \tag{8}$$

Since h is K-convex in the second argument and $h(x, x) = 0$ for all $x \in B$, then

$$\lambda h(\overline{x}, y) + (1 - \lambda)h(\overline{x}, \overline{x}) - h(\overline{x}, x_\lambda) = \lambda h(\overline{x}, y) - h(\overline{x}, x_\lambda) \in K.$$

Thus

$$(1 - \lambda)\lambda h(\overline{x}, y) - (1 - \lambda)h(\overline{x}, x_\lambda) \in K. \tag{9}$$

From (8) and (9) and using Lemma 1, we have

$$(1 - \lambda)\lambda h(\overline{x}, y) + \lambda g(x_\lambda, y) \notin -intK.$$

Dividing the last relation by $\lambda > 0$ we obtain

$$(1 - \lambda)h(\overline{x}, y) + g(x_\lambda, y) \notin -intK, \tag{10}$$

for all $0 < \lambda \leq 1$. We will prove that (10) implies

$$h(\overline{x}, y) + g(\overline{x}, y) \notin -intK.$$

Suppose by contradiction that

$$h(\overline{x}, y) + g(\overline{x}, y) \in -intK.$$

Hence there exists $k \in intK$ such that

$$h(\overline{x}, y) + g(\overline{x}, y) + k \in -intK. \tag{11}$$

Since the set $\frac{k}{2} - intK$ is open and $0 \in \frac{k}{2} - intK$, there exists $\mu > 0$ such that

$$-\lambda h(\overline{x}, y) \in \frac{k}{2} - intK, \tag{12}$$

for all $0 \leq \lambda \leq \mu$. Since for all $x, y \in B$ the function $t \in [0, 1] \to g(ty + (1 - t)x, y)$ is K-upper semicontinuous at 0, it follows that there exists $\delta > 0$ such that

$$g(x_\lambda, y) \in g(\overline{x}, y) + \frac{k}{2} - intK, \tag{13}$$

for all $0 \le \lambda \le \delta$. Let us take $\eta = \min\{\mu, \delta\}$. By (11), (12), and (13) we obtain that

$$g(x_\lambda, y) + (1-\lambda)h(\overline{x}, y) \in -intK,$$

for all $0 \le \lambda \le \eta$, which contradicts (10). Therefore,

$$h(\overline{x}, y) + g(\overline{x}, y) \notin -intK.$$

Finally, the technical result below serves for exploiting the coercivity condition we are going to assume in Theorem 1 below (assumption (iii)).

Lemma 5 *Let $B \subset A$. Assume that $F : A \to Y$ is K-convex, $x_0 \in core_A B$, $F(x_0) \in -K$, and $F(y) \notin -intK$ for all $y \in B$. Then $F(y) \notin -intK$ for all $y \in A$.*

Proof Assume that there exists a $y \in A$ such that $f(y) \in -intK$ and set for each $\lambda \in [0, 1]$ $x_\lambda := \lambda y + (1-\lambda)x_0$. By K-convexity

$$F(x_\lambda) \le_K \lambda F(y) + (1-\lambda)F(x_0) \in -intK - K \subset -intK, \quad \forall \lambda \in (0, 1]. \tag{14}$$

On the other hand, since $x_0 \in core_A B$, there exists a sufficiently small $\lambda > 0$ for which $x_\lambda \in B$, and this contradicts (14).

Remark 1 As the next example shows, the assumption $F(x_0) \in -K$ within Lemma 5 cannot be weakened to $F(x_0) \notin intK$, and in this way, Lemma 10 in Kazmi [29] is false.

Example 2 Let $X = A := \mathbb{R}$, $B := [-1, 1]$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $F(x) = (x+1, x-1)$, and $x_0 = 0$. Then F is obviously $\mathbb{R}^2_+$-convex (furthermore, both components are affine functions), $x_0 \in core_A B$, $F(x_0) = (1, -1) \notin intK$ and $F(y) \notin -intK$ for each $y \in B$. However, for instance, for $y = -2$ we get $F(-2) = (-1, -3) \in -intK$.

Now we are in a position to state our main result. Note that the assumption of compactness upon the domain in Lemma 3 was essential for its proof. However, this condition appears rather demanding when dealing with equilibrium problems since it cannot be guaranteed in many applications. To overcome this difficulty, it is common to assume different kinds of *coercivity conditions*. We take over the coercivity condition used in [29] which is originated in the scalar case from [13].

Theorem 1 *Suppose that $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfy:*

(i) g is K-essentially quasimonotone, K-convex and K-lower semicontinuous in the second argument; $g(x, x) \in K \cap (-K)$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0, 1] \to g(ty + (1-t)x, y)$ is K-upper semicontinuous at 0;
(ii) h is K-upper semicontinuous in the first argument and K-convex in the second argument; $h(x, x) = 0$ for all $x \in A$;
(iii) There exists a nonempty compact convex subset C of A such that for every $x \in C \backslash core_A C$ there exists an $a \in core_A C$ such that

$$g(x, a) + h(x, a) \in -K.$$

Then there exists $\overline{x} \in C$ such that

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -intK,$$

for all $y \in A$.

Proof By Lemma 3, it follows that there exists at least one $\overline{x} \in C$ such that

$$h(\overline{x}, y) - g(y, \overline{x}) \notin -intK,$$

for all $y \in C$. By Lemma 4, we obtain that

$$h(\overline{x}, y) + g(\overline{x}, y) \notin -intK,$$

for all $y \in C$. Set $F(\cdot) = h(\overline{x}, \cdot) + g(\overline{x}, \cdot)$. It is obvious that F is K-convex and

$$F(y) \notin -intK,$$

for all $y \in C$. If $\overline{x} \in core_A C$, then set $x_0 := \overline{x}$. If $\overline{x} \in C \backslash core_A C$, then set $x_0 := a$, where a is as in assumption (iii). In both cases $x_0 \in core_A C$, and $F(x_0) \in -K$. Hence, by Lemma 5, it follows that

$$F(y) \notin -intK,$$

for all $y \in A$. Thus, there exists $\overline{x} \in C$ such that

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -intK,$$

for all $y \in A$.

The next result due to Kazmi [29] is a particular case of Theorem 1. X and Y are the same as before, $A \subset X$ is a nonempty closed convex set and $K \subset Y$ is a proper pointed closed convex cone with nonempty interior.

Corollary 1 *([29], Theorem 7) Suppose that $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfy:*

(i) g is K-monotone, K-convex and continuous in the second argument; $g(x, x) = 0$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0, 1] \to g(ty + (1-t)x, y)$ is continuous at 0;
(ii) h is continuous in the first argument and K-convex in the second argument; $h(x, x) = 0$ for all $x \in A$;
(iii) There exists a nonempty compact convex subset C of A such that for every $x \in C \backslash core_A C$ there exists an $a \in core_A C$ such that

$$g(x, a) + h(x, a) \in -K.$$

Then there exists $\overline{x} \in C$ such that

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -intK,$$

for all $y \in A$.

Proof The continuity assumptions obviously imply K-lower (upper) semicontinuity, while the essential quasimonotonicity of g follows by Proposition 1.

Remark 2 Theorem 1 improves from several points of view the above result of Kazmi. With respect to the monotonicity, notice that the bifunction f defined in Example 1 satisfies all requirements demanded upon g in item (i) of Theorem 1, but since it is not $\mathbb{R}^2_+$-monotone, it doesn't satisfy item (i) of Corollary 1. Another improvement, related to the continuity assumptions, can be identified within the next example, obtained by a slight modification of Example 9.27 in [3].

Example 3 Let $A := \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(0, 0)\}$ and consider the function $F : A \to \mathbb{R}$ given by

$$F(x, y) = \begin{cases} \frac{y^2}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

It is obvious that F is convex and lower semicontinuous, but not continuous at $(0, 0)$. Indeed, take any sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n > 0$ and $x_n \to 0$ whenever $n \to \infty$. Then $F(x_n^2, x_n) = 1$ for each n, but $F(0, 0) = 0$, i.e., F is not continuous at $(0, 0)$. Now let $Y := \mathbb{R}$ and $K := [0, \infty)$, and consider the bifunction $g : A \times A \to \mathbb{R}$ given by $g(a, b) := F(b) - F(a)$, where $a = (x, y)$, $b = (u, v) \in A$. Then g satisfies all assumptions of Theorem 1 (i), but not item (i) of Corollary 1, due to the lack of continuity.

Thanks to the improvements made for Theorem 7 of Kazmi [29], the next result of Blum and Oettli [13] becomes a particular case of Theorem 1.

Corollary 2 *([13], Theorem 1) Let X be a real topological vector space, $A \subset X$ a nonempty closed convex set, $g : A \times A \to \mathbb{R}$ and $h : A \times A \to \mathbb{R}$ satisfying:*

(i) g is monotone (Definition 3 (i)), convex and lower semicontinuous in the second argument; $g(x, x) = 0$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0, 1] \to g(ty + (1 - t)x, y)$ is upper semicontinuous at 0;
(ii) h is upper semicontinuous in the first argument and convex in the second argument; $h(x, x) = 0$ for all $x \in A$;
(iii) There exists a nonempty compact convex subset C of A such that for every $x \in C \backslash core_A C$ there exists an $a \in core_A C$ such that

$$g(x, a) + h(x, a) \leq 0.$$

Then there exists $\overline{x} \in C$ such that

$$g(\overline{x}, y) + h(\overline{x}, y) \geq 0,$$

for all $y \in A$.

Proof Set $Y := \mathbb{R}$ and $K := [0, \infty)$ and apply Theorem 1.

We conclude this section with a corollary of Theorem 1 in which no monotonicity assumptions are made on the bifunction g; the lack of this requirement is substituted by assuming K-concavity of g in its first argument. In this way, the algebraic conditions upon g become symmetric. Apparently, this provides us a new result even in the particular case of scalar functions (i.e., where $Y := \mathbb{R}$ and $K := [0, \infty)$).

Corollary 3 *Suppose that $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfy:*

(i) g is K-concave in the first argument, K-convex and K-lower semicontinuous in the second argument; $g(x, x) \in K \cap (-K)$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0, 1] \to g(ty + (1 - t)x, y)$ is K-upper semicontinuous at 0;
(ii) h is K-upper semicontinuous in the first argument and K-convex in the second argument; $h(x, x) = 0$ for all $x \in A$;
(iii) There exists a nonempty compact convex subset C of A such that for every $x \in C \backslash core_A C$ there exists an $a \in core_A C$ such that

$$g(x, a) + h(x, a) \in -K.$$

Then there exists $\overline{x} \in C$ *such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -int K,$$

for all $y \in A$.

Proof Let $n \geq 1$, $x_1, ..., x_n \in A$, $\lambda_1, ..., \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ be arbitrary and set $z := \sum_{i=1}^{n} \lambda_i x_i$. Then by concavity of g with respect to its first variable, we have

$$\sum_{i=1}^{n} \lambda_i g(x_i, z) - g\left(\sum_{i=1}^{n} \lambda_i x_i, z\right) \in -K.$$

Since

$$g\left(\sum_{i=1}^{n} \lambda_i x_i, z\right) = g(z, z) \in -K,$$

it follows by summing up these relations that

$$\sum_{i=1}^{n} \lambda_i g(x_i, z) \in -K,$$

implying that

$$\sum_{i=1}^{n} \lambda_i g(x_i, z) \notin int K,$$

which means that g is K-essentially quasimonotonotone, and such the statement follows by Theorem 1.

Let us remark that the assumptions of Corollary 3 do not imply the monotonicity of g even in the simplest case when $X = Y = \mathbb{R}$ and $K = \mathbb{R}_+$. The next example shows an instance when all assumptions of Corollary 3 hold, but g is not monotone.

Example 4 Let $F : \mathbb{R} \to \mathbb{R}$ be a convex and lower bounded function and consider the numbers a and b such that $0 < b < a < 2b$. Set

$$g(x, y) = ax(y-x) + b(y-x)^2, \quad h(x, y) = F(y) - F(x) \text{ and } f(x, y) = g(x, y) + h(x, y).$$

It is obvious that $g(x, x) = 0$ and the function g is concave in the first, convex in the second argument; moreover, it is (globally) continuous. The concavity of g follows by $b < a$. The same properties are valid for h. Moreover, the lower boundedness of F assures that $\lim_{|x| \to +\infty} [F(x) + (a-b)x^2] = +\infty$, and from this we conclude that $\lim_{|x| \to +\infty} f(x, 0) = -\infty$, showing that the coercivity assumption (iii) of Corollary 3 holds with $a = 0$. Thus, all hypotheses of this corollary hold. On the other hand, g is not monotone, since

$$g(x, y) + g(y, x) = (2b - a)(y - x)^2 > 0, \quad \forall x \neq y.$$

We conclude that Corollary 3 can be applied for the bifunctions given in Example 4, but it is not the case either for Corollary 1, or Corollary 2.

4 The case of reflexive Banach spaces

The aim of this section is to provide several sufficient conditions for the coercivity required in assumption (iii) of Theorem 1 (or Corollary 3) when X is a reflexive Banach space endowed with the weak topology. It is well-known that in this setting every closed, convex and bounded set (in particular closed balls) are (weakly) compact. Hence, all conditions which we formulate below are vacuously satisfied when the (closed and convex) set $A \subset X$ is bounded. Thus, in the sequel we shall suppose that A is closed, convex and unbounded. Let $a \in A$ be a fixed element, $g, h : A \times A \to Y$ be given bifunctions. Let us start with the following:

(C) there exists $\rho > 0$ such that for all $x \in A : \ \|x - a\| = \rho$ one has $g(x, a) + h(x, a) \in -K$.

Proposition 2 *Under condition (C), assumption (iii) in Theorem 1 is satisfied.*

Proof Let $B := \{x \in A : \|x - a\| \leq \rho\}$. Then B is weakly compact and $a \in core_A B$. Moreover, $x \in B \backslash core_A B$ iff $\|x - a\| = \rho$. Thus $g(x, a) + h(x, a) \in -K$, hence (iii) is fulfilled.

Next we give mild sufficient conditions separately on g and h for (C). Consider the following assumptions.

(G) (upper boundedness of $g(\cdot, a)$ on a closed ball): there exist $M \in Y$ and $r > 0$ such that

$$M - g(x, a) \in K, \quad \text{whenever} \quad x \in A, \ \|x - a\| \leq r,$$

and

(H) there exists an element $u \in -int K$ such that for all $t > 0$ there is an $R > 0$ satisfying

$$\forall x \in A : \ \|x - a\| \geq R : \ tu\|x - a\| - h(x, a) \in K.$$

Remark 3 (H) is obviously fulfilled when $Y := \mathbb{R}$, $K := [0, \infty)$ and

$$\frac{h(x, a)}{\|x - a\|} \to -\infty \quad \text{whenever} \quad \|x - a\| \to \infty, \quad x \in A.$$

(Condition (c) in [13].) Indeed, take $u = -1$ and arbitrary $t > 0$. Since

$$\frac{h(x, a)}{\|x - a\|} \to -\infty \quad \text{whenever} \quad \|x - a\| \to \infty,$$

we can find $R > 0$ such that for all $x \in A$:

$$\|x - a\| \geq R, \quad \frac{h(x, a)}{\|x - a\|} \leq -t,$$

proving the assertion.

Next we need the following technical lemma concerning the function g.

Lemma 6 *Suppose that g satisfies (G) and $g(\cdot, a)$ is K-concave with $g(a, a) \in K$. Then*

$$\frac{M}{r} - \frac{g(x, a)}{\|x - a\|} \in K, \quad \forall x \in A, \quad \|x - a\| \geq r,$$

where the vector M and the number r are defined in (G).

Proof Let $x \in A$ such that $\|x-a\| \geq r$. Set

$$y := \frac{r}{\|x-a\|}x + \left(1 - \frac{r}{\|x-a\|}\right)a.$$

Since A is convex, it follows that $y \in A$ and $\|y-a\| = r$. Therefore,

$$M - g(y, a) \in K. \tag{15}$$

Since $g(\cdot, a)$ is K-concave,

$$g(y, a) - \frac{r}{\|x-a\|}g(x, a) - \left(1 - \frac{r}{\|x-a\|}\right)g(a, a) \in K. \tag{16}$$

Summing up (15) and (16) we obtain that

$$M - \frac{r}{\|x-a\|}g(x, a) - \left(1 - \frac{r}{\|x-a\|}\right)g(a, a) \in K.$$

This implies by

$$\left(1 - \frac{r}{\|x-a\|}\right)g(a, a) \in K$$

that

$$M - \frac{r}{\|x-a\|}g(x, a) \in K,$$

which proves the assertion.

Now we are able to provide sufficient conditions, separately on g and on h, for the coercivity assumption (iii) in Theorem 1.

Proposition 3 *Assume that*

(i) g satisfies (G) and $g(\cdot, a)$ is K-concave with $g(a, a) \in K$;
(ii) h satisfies (H).

Then condition (iii) of Theorem 1 is fulfilled.

Proof Let $v := M/r$, with M and r provided by (G) and take $u \in -intK$ given by (H). It is easy to see that there exists a sufficiently large $t > 0$ such that

$$v + tu \in -intK. \tag{17}$$

Take $R > 0$ according to (H) and let $x \in A$ such that $\|x-a\| \geq \max\{R, r\}$. By Lemma 6

$$v - \frac{g(x, a)}{\|x-a\|} \in K,$$

which, together with (17) implies that

$$\frac{g(x, a)}{\|x-a\|} + tu \in -intK. \tag{18}$$

On the other hand, by (H),

$$tu - \frac{h(x, a)}{\|x-a\|} \in K,$$

which, together with (18) implies that

$$\frac{g(x,a)}{\|x-a\|} + \frac{h(x,a)}{\|x-a\|} \in -int K.$$

This implies that $g(x,a) + h(x,a) \in -int K$, i.e. condition (C) is satisfied. Hence, the statement follows by Proposition 2.

The next result, which might have some interest on its own, gives sufficient conditions for lower (upper) boundedness of a K-lower (K-upper) semicontinuous function in the sense of Definition 2.

Lemma 7 *Let C be a compact subset of X and $F : C \to Y$.*

(i) if F is K-lower semicontinuous on C, then it is lower bounded, i.e. there exists a vector $m \in Y$ such that $F(x) - m \in int K$ for all $x \in C$;
(ii) if F is K-upper semicontinuous on C, then it is upper bounded, i.e. there exists a vector $M \in Y$ such that $M - F(x) \in int K$ for all $x \in C$.

Proof It is enough to prove item (i), (ii) follows by a similar argument. Supposing the contrary, for each $v \in Y$ there exists $x_v \in C$ such that $F(x_v) - v \notin int K$. Consider for each $v \in Y$ the level sets

$$L_v := \{x \in C : F(x) - v \notin int K\}.$$

By Lemma 2 item (iii), it follows that the nonempty set L_v is closed for every $v \in Y$. Since C is compact, L_v is compact too. Let $v \in -int K$ be arbitrary. Then

$$u \in Y \text{ with } u - v \in K \Rightarrow L_v \subset L_u. \tag{19}$$

Indeed, let $x \in L_v$, i.e., $F(x) - v \notin int K$. By Lemma 1, it follows that $u - F(x) \notin -int K$, i.e., $x \in L_u$.

Now take for every $n \in \mathbb{N}$ a vector $x_n \in C$ such that $x_n \in L_{nv}$. By compactness, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to the element $x \in C$. Let us show that $x \in L_{tv}$ for every $t > 0$. To do this, take any $k \in \mathbb{N}$ such that $n_k > t$. Then by (19),

$$x_{n_k} \in L_{n_k v} \subset L_{tv}.$$

Since L_{tv} is a compact set, it follows that $x \in L_{tv}$, i.e.,

$$F(x) - tv \notin int K. \tag{20}$$

On the other hand, since $v \in -int K$, there exists a sufficiently large $\tau > 0$ such that

$$F(x) - \tau v \in int K,$$

contradicting (20).

Summarizing, we have the following existence result in reflexive Banach spaces. The basic assumptions upon the set A, the space Y, and the cone K remain the same.

Theorem 2 *Suppose that X is a reflexive Banach space. Let $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfying the following properties:*

(i) g is K-concave and weakly K-upper semicontinuous in the first argument; K-convex and weakly K-lower semicontinuous in the second argument; $g(x,x) \in K \cap (-K)$ for all $x \in A$;

(ii) h is weakly K-upper semicontinuous in the first argument; K-convex in the second argument; $h(x, x) = 0$*, for all* $x \in A$ *and (H) holds.*

Then there exists $\overline{x} \in A$ *such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -intK, \quad \forall y \in A.$$

Proof We shall verify the assumptions of Corollary 3 in the reflexive Banach space setting, endowed with the weak topology. Conditions (i) and (ii) are trivially satisfied, thus we only have to verify (iii). To do this, we shall make use of Proposition 3. Let us prove that g satisfies condition (G). Take any $a \in A$ and a sufficiently large $r > 0$ such that the set $C := \{x \in A : \|x - a\| \le r\}$ is nonempty. This set is also weakly compact and $g(\cdot, a)$ being weakly K-upper semicontinuous, by Lemma 7 (ii) there exists a vector $M \in Y$ such that $M - g(x, a) \in intK$ for all $x \in C$, i.e., (G) holds. This completes the proof.

In the scalar case, i.e., when $Y := \mathbb{R}$ and $K := [0, \infty]$, we obtain a simplified form of Theorem 2.

Corollary 4 *Let* $g : A \times A \to \mathbb{R}$ *and* $h : A \times A \to \mathbb{R}$ *satisfying the following properties:*

(i) g is concave and upper semicontinuous in the first argument, convex and lower semicontinuous in the second argument, and $g(x, x) = 0$ *for all* $x \in A$;
(ii) h is weakly upper semicontinuous in the first argument, convex in the second argument, $h(x, x) = 0$ *for all* $x \in A$*, and*

$$\frac{h(x, a)}{\|x - a\|} \to -\infty \text{ whenever } \|x - a\| \to \infty, \quad x \in A.$$

Then there exists $\overline{x} \in A$ *such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \ge 0, \quad \forall y \in A. \tag{21}$$

Proof Since X is reflexive, each concave and upper semicontinuos function is weakly upper semicontinuous, and similarly, each convex and lower semicontinuos function is weakly lower semicontinuous. By Remark 3, condition (H) holds. Thus, the assertion follows by Theorem 2.

Example 5 Let X be a reflexive Banach space and suppose that $F : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function with dom$F$ unbounded. It is well-known from convex analysis that F has a global minimum point whenever it satisfies the following coercivity condition

$$\lim_{\|x\| \to +\infty} \frac{F(x)}{\|x\|} \to +\infty.$$

This fact can be easily re-obtained by Corollary 4. Indeed, take $g \equiv 0$, $A :=$ dom$F$ and $h(x, y) := F(y) - F(x)$. Since F is convex and lower semicontinuous, it is weakly lower semicontinuous as well, thus h is weakly upper semicontinuous in the first argument. Therefore, assumption (ii) of Corollary 4 is satisfied. It is obvious that the solution $\overline{x} \in A$ of (21) is a global minimum point of F.

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