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Levitin–Polyak well-posedness for constrained quasiconvex vector optimization problems

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Abstract In this paper, a notion of Levitin–Polyak (LP in short) well-posedness is introduced for a vector optimization problem in terms of minimizing sequences and efficient solutions. Sufficient conditions for the LP well-posedness are studied under the assumptions of compactness of the feasible set, closedness of the set of minimal solutions and continuity of the objective function. The continuity assumption is then weakened to cone lower semicontinuity for vector-valued functions. A notion of LP minimizing sequence of sets is studied to establish another set of sufficient conditions for the LP well-posedness of the vector problem. For a quasiconvex vector optimization problem, sufficient conditions are obtained by weakening the compactness of the feasible set to a certain level-boundedness condition. This in turn leads to the equivalence of LP well-posedness and compactness of the set of efficient solutions. Some characterizations of LP well-posedness are given in terms of the upper Hausdorff convergence of the sequence of sets of approximate efficient solutions and the upper semicontinuity of an approximate efficient map by assuming the compactness of the set of efficient solutions, even when the objective function is not necessarily quasiconvex. Finally, a characterization of LP well-posedness in terms of the closedness of the approximate efficient map is provided by assuming the compactness of the feasible set.

Keywords Levitin–Polyak well-posedness · Quasiconvexity · Efficiency · Upper semicontinuity · Hausdorff convergence

Mathematics Subject Classification 49K40 · 90C26 · 90C29

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1 Introduction

The notion of well-posedness of an optimization problem is usually understood as the analysis of the solutions under perturbations of the data of the problem. Hence, this concept plays a key role in the convergence analysis of several iterative numerical methods and algorithms and is also used to investigate the sensitivity and the stability aspects of the problem. In 1966, Tykhonov [24] introduced the notion of well-posedness for minimization problems based on the fact that every minimizing sequence converges to a unique minimum solution. This notion of well-posedness and its generalizations have been studied extensively in literature for a scalar problem. The interested readers may refer to the books by Dontchev and Zolezzi [6] and Lucchetti [17] and the papers by Zolezzi [25,26] and the references therein.

A fundamental requirement in Tykhonov well-posedness is that every minimizing sequence is from within the feasible region. However, in several numerical methods such as exterior penalty methods and augmented Lagrangian methods, the minimizing sequence generated may not be feasible. Taking this into account, Levitin and Polyak [14] introduced another notion of well-posedness which does not necessarily require the feasibility of the minimizing sequence. However, it requires the distance of the minimizing sequence from the feasible set to approach to zero eventually. Konsulova and Revalski [12] studied Levitin–Polyak (LP in short) well-posedness for convex scalar optimization problems with explicit constraints which was further extended for nonconvex scalar problems with explicit constraints in [7].

In 1987, Bednarczuk [1] and Lucchetti [18] made the first attempt to generalize the notion of well-posedness to vector optimization problems. In vector optimization, there are several notions of well-posedness given in terms of convergence of minimizing sequences [11,13,15], upper semicontinuity of approximate solution sets [2,19], Hausdorff convergence of approximate solution sets [3] etc. Some of these notions are discussed and classified in [20]. Todorov [22] discussed well-posedness for linear vector semi-infinite optimization and investigated some stability properties of the solution sets. Further, in [23], he established generic well-posedness by considering lower semicontinuity and upper semicontinuity of mappings corresponding to the sets of efficient and weakly efficient points. Dentcheva and Helbig [5] established a certain kind of well-posedness for perturbed vector optimization using certain variational principles. Huang and Yang [8,9] extended the study further by considering LP well-posedness for constrained vector optimization problem. In [8], Huang and Yang introduced several types of LP well-posedness and generalized LP well-posedness and obtained criteria and characterizations for these types of well-posedness. In [9], the authors established characterizations for the nonemptiness and compactness of weakly efficient solutions for a convex vector optimization problem in a finite dimensional setting. They also derived sufficient conditions for establishing generalized LP well-posedness for a cone constrained convex vector optimization problem in Banach spaces.

It is well-known that a scalar optimization problem with convex lower semicontinuous objective function is Tykhonov well-posed in a finite dimensional setting (see [17]). This study has been extended further and it has been established that quasiconvex optimization problem is well-posed under certain compactness assumptions (see [10,17]). For the vector case, several authors (see [4,11,19]) have established that convex vector optimization problem is Tykhonov well-posed under various sufficiency conditions. Also, quasiconvex vector optimization problem has been shown to be well-posed in the sense of Tykhonov in [3,13,20]. While dealing with LP well-posedness for vector optimization problems, Huang and Yang [8,9] have also dealt with convex objective function.

In this paper, we introduce a notion of Levitin–Polyak well-posedness for a vector optimization problem in terms of minimizing sequences and efficient solutions. We present several sufficiency conditions for the LP well-posedness. We first establish LP well-posedness under the assumptions of compactness of the feasible set, closedness of the set of minimal solutions and continuity of the objective function. The continuity assumption is then weakened to cone lower semicontinuity to establish another set of sufficiency conditions. Further, we give a notion of LP minimizing sequence of sets and establish some sufficiency conditions for the LP well-posedness of the vector problem. For a quasiconvex vector optimization problem, we obtain sufficient conditions by weakening the compactness of the feasible set to a certain level-boundedness condition. This is extended further to establish the equivalence of LP well-posedness and the compactness of the set of efficient solutions. When the objective function is not necessarily quasiconvex, we give characterizations of LP well-posedness in terms of the upper Hausdorff convergence of the sequence of sets of approximate efficient solutions as well as the upper semicontinuity of an approximate efficient map by assuming the compactness of the set of efficient solutions. Finally, we present a characterization in terms of the closedness of the approximate efficient map assuming the compactness of the feasible set.

Section 2 deals with the preliminaries required in the sequel. In Sect. 3, we introduce a notion of LP well-posedness and derive certain sufficiency conditions for the LP well-posedness of a vector problem. We also establish the LP well-posedness of a linear vector optimization problem under relaxed conditions. In Sect. 4, we establish the LP well-posedness of the vector optimization problem when the objective function is quasiconvex and establish the equivalence of LP well-posedness with the compactness of the set of efficient solutions. Section 5 deals with a characterization of LP well-posedness in terms of the upper Hausdorff convergence of the sequence of sets of approximate efficient solutions and in terms of the upper semicontinuity of the approximate efficient set-valued map. In Sect. 6, we establish a characterization of LP well-posedness in terms of closedness of an approximate efficient set-valued map. Finally, we provide some conclusions in Sect. 7.

2 Preliminaries

Let C be a closed, convex, pointed cone in $\mathbb{R}^p$ with nonempty interior which induces a partial ordering in $\mathbb{R}^p$ as follows: For $y_1, y_2 \in \mathbb{R}^p$, we have

$$y_1 \leq y_2 \Longleftrightarrow y_2 - y_1 \in C;$$

$$y_1 < y_2 \Longleftrightarrow y_2 - y_1 \in \operatorname{int} C.$$

For a nonempty set $A \subseteq \mathbb{R}^p$, an element $a \in A$ is said to be a *minimal element* of A if

$$(A - a) \cap (-C) = \{0\}.$$

The set of minimal elements of A is denoted by $\operatorname{Min} A$.

Consider the following vector optimization problem

$$\text{(P)} \quad \min_{x \in S} f(x)$$

where $f : \mathbb{R}^m \to \mathbb{R}^p$ is a vector-valued map and $\phi \neq S \subseteq \mathbb{R}^m$.

Based on the above notion of minimality of a set, we have the corresponding notion of minimality of problem (P) with respect to the set $f(S)$ in the space $\mathbb{R}^p$. An element $y \in f(S)$

is said to be a *minimal solution* of problem (P) if $y \in \text{Min}\, f(S)$. An element $x \in S$ is said to be an *efficient solution* of problem (P) if $f(x) \in \text{Min}\, f(S)$. The set of all efficient solutions of problem (P) is denoted by $\text{Eff}(f, S)$ and is given by

$$\text{Eff}(f, S) := \{x \in S : (f(S) - f(x)) \cap (-C) = \{0\}\}.$$

We next consider the notion of cone lower semicontinuity for a vector-valued map.

A map $f : A \subseteq \mathbb{R}^l \to \mathbb{R}^m$ is said to be *C-lower semicontinuous* [21] at x if for any $\delta > 0$, there exists an open set U in $\mathbb{R}^l$ containing x such that $f(U \cap A) \subseteq f(x) + \text{int}\, B(0, \delta) + C$, where $B(0, \delta)$ is a closed ball in $\mathbb{R}^m$ with center 0 and radius δ.

It can be easily seen that if f is continuous at x, then it is C-lower semicontinuous at x but the converse is not true in general.

We next recall the notion of upper semicontinuity of set-valued maps. A set-valued map $F : A \subseteq \mathbb{R}^l \rightrightarrows \mathbb{R}^m$ is said to be *upper semicontinuous* at $x \in \text{dom}\, F := \{x \in A : F(x) \neq \phi\}$, if for any $\delta > 0$, there exists an open set U in $\mathbb{R}^l$ containing x such that $F(U \cap A) \subseteq F(x) + \text{int}\, B(0, \delta)$.

It can be observed that the set-valued map F is upper semicontinuous at $x \in \text{dom}\, F$ if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ and any $\delta > 0$, $F(x_n) \subseteq F(x) + \text{int}\, B(0, \delta)$ for sufficiently large n. In particular, if the set-valued map is single valued, then the notion of upper semicontinuity of F coincides with the notion of $\{0\}$-lower semicontinuity.

Another notion that is closely linked with upper semicontinuity is the notion of closedness of set-valued maps. A set-valued map $F : A \subseteq \mathbb{R}^l \rightrightarrows \mathbb{R}^m$ is said to be *closed* at $x \in \text{dom}\, F$, if for any sequence $\{x_n\}$ in A such that $x_n \to x$, $y_n \in F(x_n)$ and $y_n \to y$, we have $y \in F(x)$.

It can be seen that a set-valued map F is closed at $x \in \text{dom}\, F$ if F is upper semicontinuous at x and the set $F(x)$ is closed.

We next recall the notion of upper Hausdorff convergence of a sequence of sets in $\mathbb{R}^m$ defined in terms of the distance function. For sets A, B in $\mathbb{R}^m$, we define $e(A, B) := \sup_{a \in A} d(a, B)$ where $d(a, B) := \inf_{b \in B} \|a - b\|$. Based on this notion, we have the following definition.

Definition 2.1 [19] A sequence of sets $\{S_n\}$ in $\mathbb{R}^m$ is said to converge to a set $S \subseteq \mathbb{R}^m$ in the *upper Hausdorff convergence* sense if $e(S_n, S) \to 0$ as $n \to \infty$.

We denote this convergence by $S_n \overset{H}{\rightharpoonup} S$.

We further recall the well established notions of convexity and quasiconvexity for vector-valued functions. Let $S \subseteq \mathbb{R}^m$ be a convex set and $f : \mathbb{R}^m \to \mathbb{R}^p$ be a vector-valued function.

Definition 2.2 [16] The function f is said to be

(i) *C-convex* on S if for every $x, u \in S$ and $\mu \in [0, 1]$,

$$f(\mu x + (1 - \mu)u) \leq \mu f(x) + (1 - \mu) f(u).$$

(ii) *quasi C-convex* on S if for every $x, u \in S$, $\alpha \in \mathbb{R}^p$ and $\mu \in [0, 1]$,

$$f(x) \leq \alpha,\ f(u) \leq \alpha \Rightarrow f(\mu x + (1 - \mu)u) \leq \alpha.$$

It can be easily observed that every C-convex vector-valued function is quasi C-convex but not vice-versa. For any $\alpha \in \mathbb{R}^p$, the *sublevel set* of f at height α, denoted by f^α, is defined as

$$f^\alpha := \{x \in \mathbb{R}^m : f(x) \leq \alpha\}.$$

Again, for a quasi C-convex function f defined on a convex set S, it can be seen that the set $S \cap f^{\alpha} = \{x \in S : f(x) \leq \alpha\}$ is convex for every $\alpha \in \mathbb{R}^p$. Also, if f is C-lower semicontinuous, then f^{α} is a closed set for every $\alpha \in \mathbb{R}^p$ (see [16]).

We now recall the notion of a majorized set. This notion complements the notion of a minorized set which has been defined in [16].

Definition 2.3 A set $A \subseteq \mathbb{R}^p$ is said to be *majorized* if there exists $y \in \mathbb{R}^p$ such that

$$A \subseteq y - C,$$

where C is a closed cone in $\mathbb{R}^p$.

3 LP well-posedness and sufficiency conditions

In [8], Huang and Yang discussed three types of LP well-posedness notions and their generalizations based on the notion of minimizing sequences and weak efficiency. Apart from studying the relations among the three types, they also established characterizations of these types of LP well-posednesses in terms of the Kuratowski measure of the noncompactness of a set. Several sufficiency conditions are established for the weakest form of (generalized) LP well-posedness. Further in a Banach space setting, Huang and Yang [9] established sufficiency conditions for a certain type of generalized LP well-posedness of a convex vector problem with cone constraints under the assumption of Slater's constraint qualification. In this section, we introduce a notion of LP well-posedness in terms of efficiency and obtain several sufficiency criteria for the well-posedness.

For the sake of distinction, we denote a closed ball with center 0 and radius ε in $\mathbb{R}^m$ and $\mathbb{R}^p$ by $B(0, \varepsilon)$ and $B[0, \varepsilon]$, respectively. Throughout the paper, we assume that $\text{Eff}(f, S)$ is a nonempty set.

Definition 3.1 A sequence $\{x_n\}$ in $\mathbb{R}^m$ is said to be a *LP minimizing sequence* of problem (P), if for every positive integer n

(i) there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$;
(ii) $x_n \in S + B(0, \varepsilon_n)$;
(iii) $f(x_n) \in \text{Min}\, f(S) + B[0, \varepsilon_n]$.

Definition 3.2 Problem (P) is said to be *LP well-posed* if every LP minimizing sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in \text{Eff}(f, S)$.

We now give some sufficient conditions for the LP well-posedness of problem (P).

Theorem 3.1 If the following conditions hold:

(i) S is a compact set;
(ii) $\text{Min}\, f(S)$ is a closed set;
(iii) f is a continuous function; then problem (P) is LP well-posed.

Proof Let $\{x_n\}$ be any LP minimizing sequence of problem (P). Then, there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $x_n \in S + B(0, \varepsilon_n)$ and $f(x_n) \in \text{Min}\, f(S) + B[0, \varepsilon_n]$. Since S is compact, therefore $\{x_n\}$ has a convergent subsequence. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ such that $x_{n_k} \to \widehat{x}$. Since f is a continuous function, we have $f(x_{n_k}) \to f(\widehat{x})$. Now, since $f(x_{n_k}) \in \text{Min}\, f(S) + B[0, \varepsilon_{nk}]$, $\varepsilon_{n_k} \to 0$ as $k \to \infty$ and $\text{Min}\, f(S)$ is a closed set, it follows that $f(\widehat{x}) \in \text{Min}\, f(S)$, that is, $\widehat{x} \in \text{Eff}(f, S)$. Hence, problem (P) is LP well-posed. □

We illustrate with an example that the above theorem does not hold if Min $f(S)$ is not a closed set.

Example 3.1 Let $S = [-1, 2]$ and $f : \mathbb{R} \to \mathbb{R}^2$ be defined as

$$f(x) = \begin{cases} (x, -\sqrt{1-(x-1)^2}\,), & \text{if } 0 \le x \le 2, \\ (x, 0), & \text{otherwise.} \end{cases}$$

We observe that when $C = \mathbb{R}^2_+$,

$$\text{Min } f(S) = \{(-1, 0)\} \cup \left\{(x, -\sqrt{1-(x-1)^2}\,) : 0 < x \le 1\right\},$$

and

$$\text{Eff}(f, S) = \{-1\} \cup]0, 1].$$

Clearly, $\{1/n\}$ is a LP minimizing sequence which converges to $0 \notin \text{Eff}(f, S)$. Hence, problem (P) is not LP well-posed.

The next example illustrates that the conclusion of the above theorem fails to hold if S is not compact.

Example 3.2 Let $S = [0, \infty[$ and $f : \mathbb{R} \to \mathbb{R}^2$ be defined as

$$f(x) = \begin{cases} (x, x^2), & \text{if } x \le 1, \\ (1/x^2, 1/x), & \text{if } x > 1. \end{cases}$$

We observe that f is a continuous function and when $C = \mathbb{R}^2_+$,

$$\text{Min } f(S) = \{(0, 0)\},$$

and

$$\text{Eff}(f, S) = \{0\}.$$

Here, $\{n\}$ is a LP minimizing sequence which does not have any convergent subsequence that converges to an element of $\text{Eff}(f, S)$. Hence, problem (P) is not LP well-posed.

The next example illustrates that continuity of f cannot be relaxed in the above theorem.

Example 3.3 Let $S = [0, 2]$ and $f : \mathbb{R} \to \mathbb{R}^2$ be defined as

$$f(x) = \begin{cases} (x, x), & \text{if } x \le 1, \\ (x-1, x-1), & \text{if } x > 1. \end{cases}$$

We observe that f is not a continuous function and when $C = \mathbb{R}^2_+$,

$$\text{Min } f(S) = \{(0, 0)\},$$

and

$$\text{Eff}(f, S) = \{0\}.$$

It can be seen that the sequence $\{x_n\}$ where $x_n = 1 + 1/n$ is a LP minimizing sequence. Since every convergent subsequence of $\{x_n\}$ converges to $1 \notin \text{Eff}(f, S)$, we conclude that problem (P) is not LP well-posed.

In the next theorem, we relax the continuity assumption on f by C-lower semicontinuity and establish another set of sufficient conditions for the LP well-posedness of (P).

Theorem 3.2 If the following conditions hold:

(i) S is a compact set;
(ii) Min $f(S)$ is a closed, majorized set;
(iii) f is a C-lower semicontinuous function; then problem (P) is LP well-posed.

Proof Let $\{x_n\}$ be any LP minimizing sequence. Then, there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $x_n \in S + B(0, \varepsilon_n)$ and $f(x_n) \in \text{Min } f(S) + B[0, \varepsilon_n]$. Since S is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \widehat{x} \in S$.

Let δ be a positive number, then there exists $m \in \mathbb{N}$ such that

$$f(x_{n_k}) \in \text{Min } f(S) + B[0, \varepsilon_{n_k}] \subseteq \text{Min } f(S) + B[0, \delta] \tag{1}$$

for every $k \geq m$. Now, since f is C-lower semicontinuous at $\widehat{x}$, therefore for any neighbourhood V of $f(\widehat{x})$, $f(x_{n_k}) \in V + C$ for sufficiently large k. We can choose a neighbourhood $V_1 \subseteq V$ and $m_1 \in \mathbb{N}$ such that

$$f(x_{n_k}) \in \overline{V_1} + C \tag{2}$$

for $k \geq m_1$ where $\overline{V_1}$ denotes the closure of V_1. Using (1) and (2), we get

$$f(x_{n_k}) \in (\text{Min } f(S) + B[0, \delta]) \cap (\overline{V_1} + C)$$

for $k \geq \max\{m, m_1\}$. Since Min $f(S) + B[0, \delta]$ is a majorized set, therefore we observe that $\{f(x_{n_k})\}$ is a bounded sequence. Without loss of generality, assume that $f(x_{n_k}) \to \widehat{y} \in$ Min $f(S)$. We claim that $f(\widehat{x}) = \widehat{y}$. Suppose to the contrary, $f(\widehat{x}) \neq \widehat{y}$. Since $\widehat{y} \in$ Min $f(S)$, it follows that $f(\widehat{x}) \notin \widehat{y} - C$. Hence, there exists some $\eta > 0$ such that

$$B[f(\widehat{x}), \eta] \cap (\widehat{y} - C) = \phi$$

which further leads to $(B[f(\widehat{x}), \eta] + C) \cap (\widehat{y} - C) = \phi$. Now $\widehat{y} \notin B[f(\widehat{x}), \eta] + C$, therefore $f(x_{n_k}) \notin B[f(\widehat{x}), \eta] + C$ for sufficiently large k, which is a contradiction. Hence, $f(\widehat{x}) = \widehat{y}$, that is, $\widehat{x} \in \text{Eff}(f, S)$. □

We next define a notion of LP minimizing sequence of sets.

Definition 3.3 A sequence of sets $\{S_n\}$ in $\mathbb{R}^m$ is said to be a *LP minimizing sequence of sets* of problem (P) if there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $S_n \subseteq S + B(0, \varepsilon_n)$ and $f(S_n) \xrightarrow{H}$ Min $f(S)$.

We next give another set of sufficient conditions for the LP well-posedness of problem (P). The conditions are obtained by replacing the continuity of f in Theorem 3.1 by the upper Hausdorff convergence of any LP minimizing sequence of sets to Eff(f, S), where Eff(f, S) is a closed set.

Theorem 3.3 If the following conditions hold:

(i) S is a compact set;
(ii) Eff(f, S) is a closed set;
(iii) $S_n \xrightarrow{H}$ Eff(f, S) for any LP minimizing sequence of sets $\{S_n\}$ of problem (P); then problem (P) is LP well-posed.

Proof Let $\{x_n\}$ be any LP minimizing sequence. Then, there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $x_n \in S + B(0, \varepsilon_n)$ and $f(x_n) \in \text{Min } f(S) + B[0, \varepsilon_n]$. By choosing $S_n = \{x_n\}$, we find that $\{S_n\}$ is a LP minimizing sequence of sets. Hence, from (iii) we get

$$d(x_n, \text{Eff}(f, S)) \to 0 \tag{3}$$

as $n \to \infty$. Now since S is compact, therefore $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \widehat{x} \in S$. Using the fact that $\mathrm{Eff}(f, S)$ is closed and hence compact, we conclude from (3) that $\widehat{x} \in \mathrm{Eff}(f, S)$. □

We now consider a linear vector optimization problem

$$(\mathrm{P}_0)\ \min_{x\in S_0} f_0(x) = (c_1x,\ c_2x, ...c_px)$$

where $S_0 = \{x \in \mathbb{R}^m : Bx \leq b\}$, $c_i \in \mathbb{R}^m$, $i = 1, 2, ...p$, B is a $l \times m$ matrix and $b \in \mathbb{R}^l$.

For establishing LP well-posedness of problem (P_0), we first state a sufficient condition for the closedness of the efficient set in terms of property (P^*) for the feasible set S_0. For more details, refer Todorov [22,23].

Definition 3.4 The feasible set S_0 has property (P^*) if for each $x, u \in S_0$, there exists $\varepsilon > 0$ such that for each $v \in B(0, \varepsilon) \cap S_0$, there exists $\alpha > 0$ such that $v + \alpha(u - x) \in S_0$.

Lemma 3.1 [23] If the feasible set S_0 of problem (P_0) has property (P^*), then the set $\mathrm{Eff}(f_0, S_0)$ is a closed set.

Remark 3.1 It can be seen that polyhedrons in $\mathbb{R}^n$ satisfy property (P^*)(see [22]). Hence, $\mathrm{Eff}(f_0, S_0)$ is a closed set.

Using this fact, we have the following result.

Theorem 3.4 If the following conditions hold:

(i) S_0 is a bounded set;
(ii) $S_n \overset{H}{\rightharpoonup} \mathrm{Eff}(f_0, S_0)$ for any LP minimizing sequence of sets $\{S_n\}$ of problem (P_0); then problem (P_0) is LP well-posed.

Proof The proof follows from Theorem 3.3 and Remark 3.1. □

4 LP well-posedness for quasiconvex vector problem and the compactness of the set of efficient solutions

Huang and Yang [8] established the equivalence of a certain type of LP well-posedness with the nonemptiness and compactness of the weakly efficient solution set. Later in [9], they established a similar characterization for a convex vector problem with cone constraints in a finite dimensional setting. In this paper, we establish some equivalence criteria in terms of compactness of the set of efficient solutions for a vector problem with quasiconvex objective function.

The quasiconvexity assumption allows us to weaken the compactness of S by a certain level-boundedness condition in the sufficiency conditions for establishing LP well-posedness which further leads to a characterization in terms of compactness of the set of efficient solutions.

We require the following lemma to establish the sufficiency of LP well-posedness for quasiconvex vector optimization problem.

Lemma 4.1 If the following conditions hold:

(i) S is a closed, convex set;
(ii) $S \cap f^\alpha$ is bounded for every $\alpha \in \mathbb{R}^p$;

(iii) f is a C-lower semicontinuous quasi C-convex function; then $(S + B(0, \varepsilon_n)) \cap f^\alpha$ is bounded for sufficiently large n where $\varepsilon_n \to 0$ as $n \to \infty$.

Proof Suppose to the contrary, there exist $x_{n_k} \in (S + B(0, \varepsilon_{n_k})) \cap f^\alpha$ such that $\|x_{n_k}\| \to \infty$ as $\varepsilon_{n_k} \to 0$. For a fixed $x \in S$, let $f(x) = \alpha$. Hence, $S \cap f^\alpha \neq \phi$. Now, $x \in S \cap f^\alpha \subseteq (S + B(0, \varepsilon_{n_k})) \cap f^\alpha$. For any $\mu \geq 0$ and sufficiently large k, consider

$$\left(1 - \frac{\mu}{\|x_{n_k}\|}\right) x + \frac{\mu}{\|x_{n_k}\|} x_{n_k} \to x + \mu d \in S$$

where d is a unit vector in $\mathbb{R}^m$. Since f is a quasi C-convex function, therefore

$$f\left(\left(1 - \frac{\mu}{\|x_{n_k}\|}\right) x + \frac{\mu}{\|x_{n_k}\|} x_{n_k}\right) \leq \alpha$$

that is,

$$\left(1 - \frac{\mu}{\|x_{n_k}\|}\right) x + \frac{\mu}{\|x_{n_k}\|} x_{n_k} \in (S + B(0, \varepsilon_{n_k})) \cap f^\alpha$$

for sufficiently large k. Again since f is C-lower semicontinuous, therefore by using Corollary 5.10 in [16], we get $x + \mu d \in S \cap f^\alpha$. Since $\mu \geq 0$ is arbitrary, it follows that $S \cap f^\alpha$ is unbounded which contradicts (ii). □

The next theorem justifies the fact that a quasiconvex vector problem is LP well-posed, which in turn implies that a convex problem is LP well-posed.

Theorem 4.1 If the following conditions hold:

(i) S is a closed, convex set;
(ii) Min $f(S)$ is a closed, majorized set;
(iii) $S \cap f^\alpha$ is bounded for every $\alpha \in \mathbb{R}^p$;
(iv) f is a C-lower semicontinuous quasi C-convex function; then problem (P) is LP well-posed.

Proof Let $\{x_n\}$ be any LP minimizing sequence. Then, there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $x_n \in S + B(0, \varepsilon_n)$ and $f(x_n) \in \text{Min } f(S) + B[0, \varepsilon_n]$. Let δ be a positive number, then there exists $m \in \mathbb{N}$ such that

$$f(x_n) \in \text{Min } f(S) + B[0, \varepsilon_n] \subseteq \text{Min } f(S) + B[0, \delta],$$

for every $n \geq m$. Since Min $f(S)$ is a majorized set, there exists $\alpha \in \mathbb{R}^p$ such that

$$f(x_n) \leq \alpha,$$

for every $n \geq m$. Hence, $x_n \in (S + B(0, \varepsilon_n)) \cap f^\alpha$ for every $n \geq m$. By Lemma 4.1, there exists $k \in \mathbb{N}$, $k \geq m$ such that $(S + B(0, \varepsilon_n)) \cap f^\alpha$ is bounded and hence compact for every $n \geq k$. As

$$x_n \in (S + B(0, \varepsilon_n)) \cap f^\alpha \subseteq (S + B(0, \varepsilon_k)) \cap f^\alpha,$$

for every $n \geq k$, it follows that $\{x_n\}$ has a convergent subsequence. Let $\{x_{n_k}\}$ be a convergent subsequence such that $x_{n_k} \to \widehat{x}$. Using similar arguments as given in the proof of Theorem 3.2, we conclude that (P) is LP well-posed. □

Corollary 4.1 If Eff(f, S) is a compact set and the following conditions hold:

(i) S is a closed, convex set;
(ii) $S \cap f^{\alpha}$ is bounded for every $\alpha \in \mathbb{R}^p$;
(iii) f is a continuous quasi C-convex function; then problem (P) is LP well-posed.

Proof Since Eff(f, S) is a compact set and f is a continuous function, it follows that Min $f(S)$ is a compact set and hence closed and majorized. As every continuous vector-valued function is C-lower semicontinuous, therefore the proof follows from Theorem 4.1.□

Remark 4.1 The assumption of Min $f(S)$ being a majorized set cannot be relaxed in the above theorem. This is illustrated in the following example.

Example 4.1 Let $S =]-\infty, 1]$ and $f : \mathbb{R} \to \mathbb{R}^2$ be defined as

$$f(x) = (x, |x|),$$

Here, f is a C-lower semicontinuous quasi C-convex function where $C = \mathbb{R}^2_+$. We observe that

$$\text{Min } f(S) = \{(x, -x) : x \leq 0\},$$

which is not a majorized set and Eff$(f, S) =]-\infty, 0]$. It can be seen that $\{-n\}$ is a LP minimizing sequence which does not have any convergent subsequence that converges to an element of Eff(f, S). Hence, problem (P) is not LP well-posed.

Corollary 4.2 If Eff(f_0, S_0) is a bounded set and $S_0 \cap f_0^{\alpha}$ is bounded for every $\alpha \in \mathbb{R}^p$, then problem (P$_0$) is LP well-posed.

Proof The proof follows from Remark 3.1 and Corollary 4.1.

The next lemma establishes that the set of efficient solutions is a closed set if the problem (P) is LP well-posed.

Lemma 4.2 If problem (P) is LP well-posed, then Eff(f, S) is a closed set.

Proof Let $\{x_n\}$ be any sequence in Eff(f, S) such that $x_n \to \widehat{x}$. Then, $\{x_n\}$ is a LP minimizing sequence of problem (P) and hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in$ Eff(f, S). Since $x = \widehat{x}$, it follows that $\widehat{x} \in$ Eff(f, S), that is, Eff(f, S) is a closed set.□

The next lemma presents sufficient conditions for the boundedness of the set of efficient solutions.

Lemma 4.3 If the following conditions hold:

(i) S is a closed, convex set;
(ii) Min $f(S)$ is a majorized set;
(iii) $S \cap f^{\alpha}$ is bounded for every $\alpha \in \mathbb{R}^p$;
(iv) f is a quasi C-convex function; then Eff(f, S) is a bounded set.

Proof Suppose to the contrary, there exists a sequence $\{x_n\}$ in Eff(f, S) such that $\|x_n\| \to \infty$ as $n \to \infty$. Since $f(x_n) \in$ Min $f(S)$ for every n and Min $f(S)$ is a majorized set, therefore there exists $\alpha \in \mathbb{R}^p$ such that $f(x_n) \leq \alpha$. Thus, $x_n \in S \cap f^{\alpha}$ for every n, that is the unbounded sequence $\{x_n\}$ is contained in $S \cap f^{\alpha}$. This contradicts (iii), hence Eff(f, S) is a bounded set.

Combining the above two lemmas, we have the converse implication of Corollary 4.1.

Theorem 4.2 If problem (P) is LP well-posed and the following conditions hold:

(i) S is a closed, convex set;
(ii) Min $f(S)$ is a majorized set;
(iii) $S \cap f^{\alpha}$ is bounded for every $\alpha \in \mathbb{R}^p$;
(iv) f is a quasi C-convex function; then Eff(f, S) is a compact set.

Proof The proof follows from Lemmas 4.2 and 4.3. □

Combining Corollary 4.1 and Theorem 4.2, we obtain a characterization of LP well-posedness of problem (P) in terms of compactness of the efficient solution set.

Theorem 4.3 If the following conditions hold:

(i) S is a closed, convex set;
(ii) Min $f(S)$ is a majorized set;
(iii) $S \cap f^{\alpha}$ is bounded for every $\alpha \in \mathbb{R}^p$;
(iv) f is a continuous quasi C-convex function; then, Eff(f, S) is a compact set if and only if problem (P) is LP well-posed.

5 Characterizations of LP well-posedness in terms of upper Hausdorff convergence and upper semicontinuity of approximate solutions

In this section, we establish characterizations of LP well-posedness of problem (P) in terms of upper Hausdorff convergence of a sequence of sets of approximate efficient solutions and in terms of the upper semicontinuity of the approximate efficient map.

Define a set-valued map $D : \mathbb{R}_+ \to \mathbb{R}^m$ as

$$D(\varepsilon) := \bigcup_{y \in \text{Min}\, f(S)} \{x \in S + B(0, \varepsilon) : f(x) \in y + B[0, \varepsilon]\},$$

where $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. We refer to $D(\varepsilon)$ as the set of approximate efficient solutions and the map D as the *approximate efficient map.*

It can be easily observed that $D(0) = \text{Eff}(f, S)$. Since we have already assumed that the set of efficient solutions is a nonempty set, it is clear that $0 \in \text{dom}\, D$. We first establish a lemma.

Lemma 5.1 If $D(\varepsilon_n) \overset{H}{\rightharpoonup} D(0)$ as $\varepsilon_n \to 0$, then D is upper semicontinuous at $\varepsilon = 0$.

Proof Suppose D is not upper semicontinuous at $\varepsilon = 0$. Then, there exist $\delta > 0$ and a sequence $\varepsilon_k \to 0$ such that

$$D(\varepsilon_k) \not\subseteq D(0) + \text{int}\, B(0, \delta).$$

Hence, there exist $x_k \in D(\varepsilon_k)$ such that

$$x_k \notin D(0) + \text{int}\, B(0, \delta)$$

which contradicts $D(\varepsilon_k) \overset{H}{\rightharpoonup} D(0)$ as $\varepsilon_k \to 0$. □

We first establish the upper Hausdorff convergence of a sequence of approximate solutions as a necessary condition for the LP well-posedness of (P).

Theorem 5.1 If problem (P) is LP well-posed, then $D(\varepsilon_n) \overset{H}{\rightharpoonup} D(0)$ where $\varepsilon_n \to 0$ as $n \to \infty$.

Proof Let problem (P) be LP well-posed. Suppose to the contrary, $D(\varepsilon_n)$ does not tend to $D(0)$ in the upper Hausdorff sense for some sequence $\varepsilon_n \to 0$. Then, there exists an open set W containing 0, such that for any $n \in \mathbb{N}$, there exists $n_k \geq n$ with

$$D(\varepsilon_{n_k}) \cap (D(0) + W)^c \neq \phi. \tag{4}$$

Let $x_{n_k} \in D(\varepsilon_{n_k}) \cap (D(0) + W)^c$, then $\{x_{n_k}\}$ is a LP minimizing sequence. Since (P) is LP well-posed, therefore $\{x_{n_k}\}$ has a convergent subsequence converging to an element of $\mathrm{Eff}(f, S) = D(0)$. Without loss of generality, assume $x_{n_k} \to \widehat{x} \in D(0)$. This clearly contradicts (4). □

The following corollary establishes that the upper semicontinuity of the approximate efficient map D at $\varepsilon = 0$ is also a necessary condition for problem (P) to be LP well-posed.

Corollary 5.1 If problem (P) is LP well-posed then D is upper semicontinuous at $\varepsilon = 0$.

Proof The result follows trivially using Lemma 5.1 and Theorem 5.1. □

We next give sufficient conditions for problem (P) to be LP well-posed in terms of the upper semicontinuity of the approximate efficient map D at $\varepsilon = 0$.

Theorem 5.2 If $\mathrm{Eff}(f, S)$ is a compact set and D is upper semicontinuous at $\varepsilon = 0$, then problem (P) is LP well-posed.

Proof Suppose to the contrary, problem (P) is not LP well-posed. Then, by definition there exists a LP minimizing sequence $\{x_n\}$ such that no subsequence of $\{x_n\}$ converges to an element of $\mathrm{Eff}(f, S)$. Since $\{x_n\}$ is a LP minimizing sequence, therefore there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $x_n \in S + B(0, \varepsilon_n)$ and $f(x_n) \in \mathrm{Min}\, f(S) + B[0, \varepsilon_n]$, that is $x_n \in D(\varepsilon_n)$. We have two possibilities here.

Case 1: $\{x_n\}$ has atleast one convergent subsequence whose limit does not belong to $\mathrm{Eff}(f, S)$. Let $\{x_{n_k}\}$ be a convergent subsequence whose limit lies outside $\mathrm{Eff}(f, S)$. Since $\mathrm{Eff}(f, S)$ is a closed set, there exists $\delta > 0$ and $m \in \mathbb{N}$, such that $x_{n_k} \notin \mathrm{Eff}(f, S) + \mathrm{int}\, B(0, \delta)$ for every $k \geq m$.

Case 2: $\{x_n\}$ has no convergent subsequence. Since $\mathrm{Eff}(f, S)$ is compact, it follows that for every $n \in \mathbb{N}$, there exists $n_k \geq n$ such that

$$x_{n_k} \notin \mathrm{Eff}(f, S) + \mathrm{int}\, B(0, \delta).$$

Hence, in either of the two situations, there exists $\delta > 0$ such that for every $n \in \mathbb{N}$, there exists $n_k \geq n$ with

$$x_{n_k} \notin \mathrm{Eff}(f, S) + \mathrm{int}\, B(0, \delta) = D(0) + \mathrm{int}\, B(0, \delta).$$

Thus, $D(\varepsilon_{n_k}) \nsubseteq D(0) + \mathrm{int}\, B(0, \delta)$. This contradicts the fact that D is upper semicontinuous at $\varepsilon = 0$. □

Remark 5.1 It can be observed that the boundedness assumption on $\mathrm{Eff}(f, S)$ cannot be dropped in the above theorem. For the problem considered in Example 4.1, we observe that $\mathrm{Eff}(f, S) =]-\infty, 0] = D(0)$ is not a bounded set and for any $\delta > 0, V =]-\infty, \delta[$ is an open set containing $D(0)$. Let $\{\varepsilon_n\}$ be any sequence such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Clearly, $D(\varepsilon_n) \subseteq V$ for sufficiently large n which implies that D is upper semicontinuous at $\varepsilon = 0$. It has been observed that the problem in this example is not LP well-posed.

Remark 5.2 We also observe that the closedness of Eff(f, S) cannot be relaxed in the above theorem. In the problem considered in Example 3.1,

$$D(\varepsilon) =]-1-\varepsilon, -1+\varepsilon[\cup]-\varepsilon, 1+\varepsilon[,$$

for $0 < \varepsilon < 1/2$. Clearly, D is upper semicontinuous at $\varepsilon = 0$, but Eff(f, S) is not a closed set. It was observed that the problem is not LP well-posed.

Combining Corollary 5.1 and Theorem 5.2, we obtain a characterization of LP well-posedness in terms of upper semicontinuity of D at $\varepsilon = 0$.

Corollary 5.2 If Eff(f, S) is a compact set, then problem (P) is LP well-posed if and only if D is upper semicontinuous at $\varepsilon = 0$.

In the next corollary, we give sufficient conditions for the LP well-posedness of (P) in terms of the upper Hausdorff convergence of the sequence of sets $D(\varepsilon_n)$ to $D(0)$ where $\varepsilon_n \to 0$.

Corollary 5.3 If Eff(f, S) is a compact set and $D(\varepsilon_n) \overset{H}{\rightharpoonup} D(0)$ where $\varepsilon_n \to 0$ as $n \to \infty$, then problem (P) is LP well-posed.

Proof The proof follow trivially from Lemma 5.1 and Theorem 5.2. □

Remark 5.3 The compactness assumption cannot be dropped in Corollary 5.3. In the problem considered in Example 4.1, it can be seen that

$$D(\varepsilon) = \begin{cases}]-\infty, \varepsilon], & \text{if } 0 \le \varepsilon \le 1, \\ ]-\infty, 1], & \text{if } \varepsilon > 1. \end{cases}$$

Here, $D(\varepsilon_n)$ tends $D(0) = \text{Eff}(f, S)$ in the upper Hausdorff sense for any $\varepsilon_n \to 0$. As observed earlier, the efficient set is not compact and the problem is not LP well-posed.

Remark 5.4 It can be seen that Corollary 5.3 also holds if we replace the compactness of Eff(f, S) by the closedness of Eff(f, S) and conditions (i)–(iv) of Lemma 4.3.

Combining Theorem 5.1 and Corollary 5.3, we obtain a characterization of LP well-posedness in terms of upper Hausdorff convergence of a sequence of approximate efficient sets to the set $D(0)$.

Corollary 5.4 If Eff(f, S) is a compact set, then problem (P) is LP well-posed if and only if $D(\varepsilon_n) \overset{H}{\rightharpoonup} D(0)$ where $\varepsilon_n \to 0$ as $n \to \infty$.

6 LP well-posedness and closedness of approximate efficient map

In this section, we establish a characterization of LP well-posedness in terms of the closedness of the approximate efficient map at the origin. We first discuss sufficient conditions for LP well-posedness of problem (P) in terms of closedness of the map D at $\varepsilon = 0$.

Theorem 6.1 If S is a compact set and D is closed at $\varepsilon = 0$, then problem (P) is LP well-posed.

Proof Let $\{x_n\}$ be any LP minimizing sequence. Then, there exist $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $x_n \in S + B(0, \varepsilon_n)$ and $f(x_n) \in \text{Min } f(S) + B[0, \varepsilon_n]$. Since S is compact, therefore $\{x_n\}$ has a convergent subsequence. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ such that $x_{n_k} \to \widehat{x}$. Now, $x_{n_k} \in D(\varepsilon_{n_k})$ and $\varepsilon_{n_k} \to 0$, which by the closedness of D at $\varepsilon = 0$ implies that $\widehat{x} \in D(0) = \text{Eff}(f, S)$. Hence, problem (P) is LP well-posed. □

Remark 6.1 It can be observed that while establishing the LP well-posedness of (P) under the upper semicontinuity assumption of D in Theorem 5.2, it was enough to assume the compactness of Eff(f, S). However, the above theorem fails to hold if the compactness assumption on S is replaced by the compactness of Eff(f, S). This is clear from the problem considered in Example 3.2 where Eff(f, S) is compact and D is closed at $\varepsilon = 0$ but the feasible S is not compact.

The following theorem gives converse implication of the above theorem.

Theorem 6.2 If problem (P) is LP well-posed, then D is closed at $\varepsilon = 0$.

Proof Let $\{\varepsilon_n\}$ and $\{x_n\}$ be sequences such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $x_n \in D(\varepsilon_n)$ and $x_n \to x$. Clearly, $\{x_n\}$ is a LP minimizing sequence and hence it has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \widehat{x} \in \text{Eff}\,(f, S) = D(0)$. As $x_n \to x$, it follows that $x = \widehat{x} \in D(0)$, that is D is closed at $\varepsilon = 0$. □

Combining Theorems 6.1 and 6.2, we obtain a characterization of LP well-posedness in terms of closedness of the approximate efficient map D at $\varepsilon = 0$.

Corollary 6.1 If S is a compact set, then problem (P) is LP well-posed if and only if D is closed at $\varepsilon = 0$.

7 Conclusions

In this paper, we have established several sufficiency conditions for the LP well-posedness of the vector optimization problem (P). By establishing the converse implications, we are able to provide complete characterizations for LP well-posedness. For a quasiconvex vector problem, we have established a characterization in terms of the compactness of the set of efficient solutions, Eff(f, S). This characterization justifies that the class of quasiconvex (and in particular convex) vector optimization problem is LP well-posed. For a vector optimization problem where the objective function is not necessarily quasiconvex, we are able to provide characterizations of LP well-posedness in terms of the upper Hausdorff convergence of a sequence of sets of approximate efficient solutions and in terms of the upper semicontinuity of an approximate efficient map by assuming the compactness of Eff(f, S). Characterization of well-posedness in terms of closedness of approximate solution map is not usually found in literature. Assuming the compactness of the feasible set, we are able to establish a characterization in terms of the closedness of the approximate efficient map. There are many aspects of LP well-posednes which need to be investigated. One of the possibilities is to investigate a suitable scalarization of the vector optimization problem and study the LP well-posedness through the scalarized problem. Another possibility is to study the LP well-posedness for a vector problem with explicit constraints. It would also be interesting to establish characterizations of LP well-posedness in terms of different types of continuity like Hausdorff semicontinuity, Lipschitz continuity and Aubin continuity.

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