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Optimization reformulations of the generalized Nash equilibrium problem using regularized indicator Nikaidô–Isoda function

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Abstract In this paper, we extend the literature by adapting the Nikaidô–Isoda function as an indicator function termed as regularized indicator Nikaidô–Isoda function, and this is demonstrated to guarantee existence of a solution. Using this function, we present two constrained optimization reformulations of the generalized Nash equilibrium problem (GNEP for short). The first reformulation characterizes all the solutions of GNEP as global minima of the optimization problem. Later this approach is modified to obtain the second optimization reformulation whose global minima characterize the normalized Nash equilibria. Some numerical results are also included to illustrate the behaviour of the optimization reformulations.

Keywords Generalized Nash equilibrium problem · Regularized indicator Nikaidô–Isoda function · Optimization reformulations · Normalized Nash equilibria · Quasi-variational inequality problem

Mathematics Subject Classification (2000) 90C30 · 91A10

1 Introduction

GNEP is a generalization of the standard Nash equilibrium problem (NEP for short), in which the strategy set of each player depends on the strategies of all the other players as well as on his own strategy. GNEP has recently attracted much attention due to its applications in various fields like mathematics, computer science, economics and engineering. For more details

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we refer the reader to a recent survey paper by Facchinei and Kanzow [3] and the references therein. While the generalized Nash equilibrium is an important solution concept, solving for generalized Nash solutions is not straightforward, and efficient algorithms do not exist except under restrictive assumptions. As a consequence, practitioners have reformulated the GNEP into an optimization problem so that standard solution algorithms can potentially be applied.

The fact that NEP could be cast as a variational inequality problem (VI for short) first appeared in the work of Lions and Stampacchia [14] in infinite dimensional settings. For the finite dimensional case Gabay and Moulin [6] discussed these results in detail whereas Harker [8] considered application of these results in spatial economic theory. Bensoussan [1] was the first to recognize that GNEP can be reformulated as a quasi-variational inequality (QVI for short). Harker [9], further explored the quantitative and qualitative properties of GNEP through the use of QVI. But from a practical viewpoint this reformulation is not much of use since efficient methods for computing generalized Nash equilibrium only exist for special cases that include highly restrictive assumptions. A special class of GNEP is that of jointly convex constraints (see [20]) where the constraint functions that depend on other players variables are same for all players and are convex with respect to all variables. Certain solutions of such GNEP, namely normalized Nash equilibria can be found via a suitable VI corresponding to the GNEP instead of solving QVI, see [4] for details.

Augmented Lagrangians were introduced in order to eliminate the duality gap between a non-convex primal problem and the corresponding augmented dual problem, for instance, see [10,12,17]. In [12], zero duality gap relation is obtained by taking the indicator augmenting function which is defined in terms of indicator function of a closed ball. The benefits of indicator augmenting function are that it works as augmenting function for any problem and the existence of optimal path converging to optimal solution is guaranteed.

To study the approximation of Nash equilibria Jofré and Wets [11] employed the Nikaidô–Isoda [16] function to formulate an equivalent optimization problem. To overcome certain disadvantages while establishing the existence and uniqueness and to avoid nondifferentiability of the value function, simple regularization of Nikaidô–Isoda function has been considered in literature. The regularized Nikaidô–Isoda functions were considered by Fukushima [5] for variational inequality problems, Gürkan and Pang [7] for standard NEP and Mastronei [15] for equilibrium programming problems. Recently, Heusinger and Kanzow [20] used regularized Nikaidô–Isoda function to obtain nonsmooth constrained, smooth constrained and smooth unconstrained optimization reformulations of the GNEP in the jointly convex case.

In this paper we extend the literature by using the Nikaidô–Isoda function commonly employed in the literature, and adapting it into an indicator function based on indicator augmenting function considered in [12], which implies that the solution to the optimization problem is finite in the neighbourhood of the constraint set, so that a supremum is guaranteed to exist. We obtain two optimization reformulations of jointly convex GNEP using the regularized Nikaidô–Isoda indicator function introduced. The advantages of using this type of regularized function is that the region over which it is optimized is either an empty set or a compact set and its structure remains the same as that of Nikaidô–Isoda function in this domain. This guarantees that the value function is finite in a neighbourhood around the constraint set.

The paper comprises of six sections. In Sect. 2 we provide certain preliminaries. In Sect. 3, we introduce the notion of regularized indicator Nikaidô–Isoda function and derive a nonsmooth constrained optimization reformulation of the GNEP. In Sect. 4, we modify the regularization of the Nikaidô–Isoda function to obtain another constrained optimization reformulation of the GNEP whose solutions are the normalized Nash equilibria of the GNEP. In

Sect. 5, algorithms are presented to obtain solutions of the GNEP. We conclude with some remarks in Sect. 6.

2 Preliminaries

We first recall the definition of the GNEP. Let there be N players where each player $\nu = 1, \ldots, N$ controls the variable $x^\nu \in \mathbb{R}^{n_\nu}$. We denote by x the vector formed by all these decision variables, thus $x = (x^1, \ldots, x^N)^{\mathrm{T}} \in \mathbb{R}^n$ where $n = n_1 + \cdots + n_N$. To indicate the νth player's variables we sometimes write x as $(x^\nu, x^{-\nu})$ where $x^{-\nu}$ denotes the decision variables of all the players except the player ν.

Let $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ be the νth player's payoff function that depends on both his own variables as well as on the variables of all other players. We assume that θ_ν is continuous in x and $\theta_\nu(x^\nu, x^{-\nu})$ is convex in the variable x^ν, $\nu = 1, \ldots, N$. The strategy space of the νth player $X_\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$ depends on the rival player's strategies. We consider jointly convex GNEP where the strategy space is given by

$$X_\nu(x^{-\nu}) = \left\{ x^\nu \in \mathbb{R}^{n_\nu} \mid (x^\nu, x^{-\nu}) \in X \right\},$$

where $X \subseteq \mathbb{R}^n$ is a nonempty, closed convex set representing the joint constraints of all the players $\nu = 1, \ldots, N$. Since the set X is closed and convex so is each of the set $X_\nu(x^{-\nu})$. If the strategy space of each player is independent of the strategies of the rival players, that is, $X_\nu(x^{-\nu}) = X_\nu$ for each $\nu = 1, \ldots, N$ then GNEP reduces to the standard NEP.

Let us define the set

$$X(x) := X_1(x^{-1}) \times \cdots \times X_N(x^{-N}).$$

We say that a vector $\bar{x} \in X(\bar{x})$ is a generalized Nash equilibrium, or a solution of the GNEP, if $\bar{x}^\nu$ solves the minimization problem

$$\begin{aligned} &\min_{x^\nu} \ \theta_\nu(x^\nu, \bar{x}^{-\nu}) \\ &\text{subject to } x^\nu \in X_\nu(\bar{x}^{-\nu}), \end{aligned}$$

for each $\nu = 1, \ldots, N$. In general, for any given $x \in X$, neither $X(x)$ is a subset of X nor X is a subset of $X(x)$.

We denote the solution set of the above problem by $S_\nu(\bar{x}^{-\nu})$ and define the set $S(\bar{x}) := \prod_{\nu=1}^N S_\nu(\bar{x}^{-\nu})$. We say that $\bar{x}$ is a solution of GNEP if and if only $\bar{x} \in S(\bar{x})$, that is, if and only if $\bar{x}$ is a fixed point of the set-valued map $x \mapsto S(x)$. We denote the solution set of the GNEP by P, which is the collection of fixed points of the map S.

The *Nikaidô–Isoda function*, often referred as *Ky–Fan function*, see [4,16] for details, is the function $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$\Psi(x, y) := \sum_{\nu=1}^{N} [\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})]. \tag{1}$$

Since, for a given $x^\nu \in \mathbb{R}^{n_\nu}$ the functions $\theta_\nu(y^\nu, x^{-\nu})$ is convex in y^ν, it is clear that the Nikaidô–Isoda function $\Psi(x, y)$ is concave in $y \in \mathbb{R}^n$.

The Nikaidô–Isoda function has the following interpretation. Suppose x and y are two feasible points for the GNEP, then each summand in the definition of Nikaidô–Isoda function gives the improvement in the objective function of the νth player when he changes his strategy from x^ν to y^ν while all the other players keep their strategy unchanged. For $x \in X$, define the value function

$$V(x) := \sup_{y \in X(x)} \Psi(x, y). \tag{2}$$

Then it can be easily observed that $V(x)$ is nonnegative for every $x \in X$ and that $\overline{x}$ is a solution of GNEP if and only if $\overline{x} \in X(\overline{x})$ and $V(\overline{x}) = 0$ (see [20]). Hence $\overline{x}$ is a generalized Nash equilibrium if and only if it solves the problem

$$\begin{aligned} &\min V(x) \\ &\text{subject to} \quad x \in X(x). \end{aligned}$$

Since the set $X(x)$ is not necessarily compact the existence of the supremum in (2) is not guaranteed and even if it exists, it may not be attained at a unique point which means in general the mapping V is not differentiable.

Heusinger and Kanzow [20] used a regularization of the Nikaidô–Isoda function in order to remove some of these shortcomings. For a fixed parameter $\alpha > 0$ the following *regularized Nikaidô–Isoda function* was considered

$$\Psi_\alpha(x, y) := \sum_{\nu=1}^{N} [\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) - \frac{\alpha}{2} \left\| x^\nu - y^\nu \right\|^2], \tag{3}$$

and for $x \in X$, the value function was defined as

$$V_\alpha(x) := \sup_{y \in X(x)} \Psi_\alpha(x, y).$$

It was observed that if the supremum exists it is attained at a unique point. Further a smooth reformulation of the GNEP was derived, by taking the supremum over the convex set X instead of $X(x)$ in the definition of the value function. It was shown that the value function given by

$$\hat{V}_\alpha(x) := \sup_{y \in X} \Psi_\alpha(x, y),$$

is continuously differentiable and the supremum exists which is attained at a single point. It was established that $\hat{V}_\alpha(x)$ is nonnegative for every $x \in X$ and that $\overline{x}$ is a normalized Nash equilibrium if and only if $\overline{x} \in X$ and $\hat{V}_\alpha(\overline{x}) = 0$.

In comparison with (1) we find an additional quadratic term in (3) such that the structure of Nikaidô–Isoda function is perturbed. In the next section we introduce a regularized indicator Nikaidô–Isoda function with the same structure as the Nikaidô–Isoda function. In this direction we first recall the notion of an indicator function.

For a set A in $\mathbb{R}^n$ the indicator function $\delta_A : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is defined as

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \notin A. \end{cases}$$

Let $C = \{u \in \mathbb{R}^n \mid \|u\|_\infty \leq 1\}$ denote closed unit ball where $\|u\|_\infty$ denotes the L_∞ norm. Clearly, $C = \prod_{\nu=1}^{N} C_\nu$ where C_ν denotes the closed unit ball in $\mathbb{R}^{n_\nu}$ and $n = n_1 + \cdots + n_N$. For each $r > 0$ the set $rC = \{u \in \mathbb{R}^n \mid \|u\|_\infty \leq r\}$ is a compact convex set so that

(i) $\delta_{rC}(.)$ is a proper convex lower semicontinuous function;
(ii) $\min_{u \in \mathbb{R}^n} \delta_{rC}(u) = 0$;

(iii) $\arg\min_{u\in\mathbb{R}^n} \delta_{rC}(u) = rC$;
(iv) $\alpha\delta_{rC}(u) = \delta_{rC}(u)$, for any $\alpha > 0$.

3 Optimization reformulation for finding generalized Nash equilibria

In this section, a reformulation of the Nikaidô–Isoda function as an indicator function is defined and named as regularized indicator Nikaidô–Isoda function, so that a supremum for the resultant optimization problem is guaranteed to exist. Let $r > 0$ be fixed and define

$$\Psi_r(x, y) := \sum_{\nu=1}^{N}[\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) - \delta_{rC_\nu}(y^\nu - x^\nu)],$$

where C_ν denotes the closed unit ball in $\mathbb{R}^{n_\nu}$. Since, for a given $x^\nu \in \mathbb{R}^{n_\nu}$ the functions $\theta_\nu(y^\nu, x^{-\nu})$ and $\delta_{rC_\nu}(y^\nu - x^\nu)$ are convex in y^ν, it follows that the regularized Nikaidô–Isoda function $\Psi_r(x, y)$ is concave in $y \in \mathbb{R}^n$. Let us now define the value function

$$\begin{aligned} V_r(x) &:= \max_{y\in X(x)} \Psi_r(x, y) \\ &= \max_{y\in X(x)} \sum_{\nu=1}^{N}[\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) - \delta_{rC_\nu}(y^\nu - x^\nu)] \\ &= \sum_{\nu=1}^{N}\left[\theta_\nu(x^\nu, x^{-\nu}) - \min_{y^\nu\in X_\nu(x^{-\nu})}\{\theta_\nu(y^\nu, x^{-\nu}) + \delta_{rC_\nu}(y^\nu - x^\nu)\}\right] \\ &= \sum_{\nu=1}^{N}\left[\theta_\nu(x^\nu, x^{-\nu}) - \min_{y^\nu\in X_\nu(x^{-\nu})\cap(x^\nu+rC_\nu)} \theta_\nu(y^\nu, x^{-\nu})\right]. \end{aligned}$$

Since $\theta_\nu(., x^{-\nu})$ is now minimized over the compact set $X_\nu(x^{-\nu}) \cap (x^\nu + rC_\nu)$, the existence of the minimum is guaranteed whenever this compact set is nonempty. The following theorem gives a few properties of the function V_r.

Theorem 1 *The value function V_r has the following properties*:

(a) $V_r(x) \geq 0$ *for every* $x \in X(x)$.
(b) $\overline{x}$ *is a generalized Nash equilibrium if and only if* $\overline{x} \in X(\overline{x})$ *and* $V_r(\overline{x}) = 0$.
(c) *For every* $x \in X$ *and* $\nu = 1, \ldots, N$,

$$P_r^\nu(x) := \arg\min_{y^\nu\in X_\nu(x^{-\nu})\cap(x^\nu+rC_\nu)} \theta_\nu(y^\nu, x^{-\nu}),$$

is a nonempty convex compact subset in $\mathbb{R}^{n_\nu}$.

Proof (a) For every $x \in X(x)$, we have

$$V_r(x) = \max_{y\in X(x)} \Psi_r(x, y) \geq \Psi_r(x, x) = 0.$$

(b) Let $\overline{x}$ be a generalized Nash equilibrium, hence $\overline{x} \in X(\overline{x})$ and

$$\theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) \leq \theta_\nu(x^\nu, \overline{x}^{-\nu}), \ \forall x^\nu \in X_\nu(\overline{x}^{-\nu}),$$

for all $\nu = 1, \ldots, N$. Thus

$$\Psi_r(\overline{x}, y) = \sum_{\nu=1}^{N} [\theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) - \theta_\nu(y^\nu, \overline{x}^{-\nu}) - \delta_{rC_\nu}(y^\nu - \overline{x}^\nu)] \leq 0,$$

for every $y \in X(\overline{x})$. Hence

$$V_r(\overline{x}) = \max_{y \in X(\overline{x})} \Psi_r(\overline{x}, y) \leq 0.$$

Using part (a) of the theorem, we have $V_r(\overline{x}) = 0$.

Conversely, suppose that $\overline{x} \in X(\overline{x})$ and $V_r(\overline{x}) = 0$. Then $\Psi_r(\overline{x}, y) \leq 0$ for every $y \in X(\overline{x})$. For a fixed player $\nu \in 1, \ldots, N$, let $x^\nu \in X_\nu(\overline{x}^{-\nu})$ and $\lambda \in (0, 1)$ be arbitrary. Define $y = (y^1, \ldots, y^N) \in \mathbb{R}^n$ as

$$y^\mu := \begin{cases} \overline{x}^\mu, & \text{if } \mu \neq \nu, \\ (1-\lambda)\overline{x}^\nu + \lambda x^\nu, & \text{if } \mu = \nu. \end{cases}$$

As $X_\nu(\overline{x}^{-\nu})$ is a convex set it follows that $y^\mu \in X_\mu(\overline{x}^{-\mu})$ for every $\mu = 1, \ldots, N$, that is, $y \in X(\overline{x})$. Thus

$$\Psi_r(\overline{x}, y) = \theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) - \theta_\nu((1-\lambda)\overline{x}^\nu + \lambda x^\nu, \overline{x}^{-\nu}) - \delta_{rC_\nu}(\lambda(x^\nu - \overline{x}^\nu)).$$

Convexity of $\theta_\nu(x^\nu, \overline{x}^{-\nu})$ with respect to the variable x^ν, gives

$$\theta_\nu((1-\lambda)\overline{x}^\nu + \lambda x^\nu, \overline{x}^{-\nu}) - \theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) \leq \lambda(\theta_\nu(x^\nu, \overline{x}^{-\nu}) - \theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu})).$$

Hence, we have

$$\Psi_r(\overline{x}, y) \geq \lambda(\theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) - \theta_\nu(x^\nu, \overline{x}^{-\nu})) - \delta_{rC_\nu}(\lambda(x^\nu - \overline{x}^\nu)).$$

Since, $\lambda > 0$ and $\Psi_r(\overline{x}, y) \leq 0$ it follows that

$$\theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) - \theta_\nu(x^\nu, \overline{x}^{-\nu}) \leq \delta_{rC_\nu}(\lambda(x^\nu - \overline{x}^\nu)).$$

Since $\lambda(x^\nu - \overline{x}^\nu) \in rC_\nu$ for sufficiently small λ, we get on taking limit as $\lambda \to 0^+$

$$\theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) \leq \theta_\nu(x^\nu, \overline{x}^{-\nu}).$$

Since $x \in X_\nu(\overline{x}^{-\nu})$ is arbitrary and the above relation can be established for every player $\nu = 1, \ldots, N$ it follows that $\overline{x}$ is a generalized Nash equilibrium.

(c) Since $X_\nu(x^{-\nu}) \cap (x^\nu + rC_\nu)$ is a compact set and θ_ν is a continuous convex function, it follows that $P_r^\nu(x)$ is a nonempty convex compact set in $\mathbb{R}^{n_\nu}$. □

The above theorem implies that solving GNEP is equivalent to finding a minimum of the constrained optimization problem

$$\begin{aligned} &\min V_r(x) \\ &\text{subject to} \quad x \in X(x), \end{aligned}$$

which using Lemma 2.1 of [20] can be reformulated as

$$\begin{aligned} &\min V_r(x) \\ &\text{subject to} \quad x \in X. \end{aligned}$$

For $x \in X$, define

$$P_r(x) := \prod_{\nu=1}^{N} P_r^\nu(x).$$

Clearly, $P_r(x)$ is a compact set and for $x \in X$

$$P_r(x) = \arg\max_{y \in X(x)} \Psi_r(x, y).$$

We now give a characterization of a Nash equilibrium in terms of a solution of a quasi-variational inequality problem.

Theorem 2 *If θ_ν is continuously differentiable then x is a generalized Nash equilibrium if and only if it is a solution of $QVI(Y(x), F(x))$ where*

$$Y(x) := \prod_{\nu=1}^{N} (X_\nu(x^{-\nu}) \cap (x^\nu + rC_\nu)),$$

and $F(x) := (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N$.

Proof Let x be a generalized Nash equilibrium. Then for every $\nu = 1, \ldots, N$, x^ν solves the problem

$$\begin{aligned} &\min \theta_\nu(y^\nu, x^{-\nu}) \\ &\text{subject to} \quad y^\nu \in X_\nu(x^{-\nu}) \cap (x^\nu + rC_\nu). \end{aligned}$$

Therefore, for every $\nu = 1, \ldots, N$,

$$\left\langle \nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu}), y^\nu - x^\nu \right\rangle \geq 0, \quad \forall y^\nu \in X_\nu(x^{-\nu}) \cap (x^\nu + rC_\nu),$$

which implies

$$\sum_{\nu=1}^{N} \left\langle \nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu}), y^\nu - x^\nu \right\rangle \geq 0, \forall y = (y^1, \ldots, y^N)^{\mathrm{T}} \in \prod_{\nu=1}^{N} \left(X_\nu(x^{-\nu}) \cap (x^\nu + rC_\nu)\right), \tag{4}$$

that is,

$$\langle F(x), y - x \rangle \geq 0, \ \forall y \in Y(x).$$

Hence x is a solution of QVI($Y(x)$, $F(x)$).

Conversely, suppose x is a solution of QVI($Y(x)$, $F(x)$) then $x \in Y(x) \subseteq X(x)$ and (4) holds. Since each θ_ν is convex in x^ν, we have

$$\theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu}) \geq \left\langle \nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu}), y^\nu - x^\nu \right\rangle, \ \forall y^\nu \in X_\nu(x^{-\nu}).$$

Using (4) we have

$$\sum_{\nu=1}^{N} \left[\theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu})\right] \geq 0, \ \forall y \in Y(x).$$

Also, since $\delta_{rC_\nu}(y^\nu - x^\nu) = -\infty$ for any $y^\nu \notin (x^\nu + rC_\nu)$ it follows that $\Psi_r(x, y) \leq 0$ which in turn implies $V_r(x) \leq 0$. By Theorem 1(a) we have $V_r(x) = 0$ and hence by Theorem 1(b) it follows that x is a generalized Nash equilibrium. □

The next result provides a characterization for the solution of the GNEP as a fixed point of the set-valued map P_r.

Theorem 3 *$\overline{x}$ is a generalized Nash equilibrium if and only if $\overline{x}$ is a fixed point of the map $x \mapsto P_r(x)$.*

Proof Suppose $\overline{x}$ is a generalized Nash equilibrium, then by Theorem 1(b), we have $\overline{x} \in X(\overline{x})$ and $V_r(\overline{x}) = 0$. Therefore, for every $p_r(\overline{x}) \in P_r(\overline{x})$

$$0 = V_r(\overline{x}) = \max_{y \in X(\overline{x})} \Psi_r(\overline{x}, y) = \Psi_r(\overline{x}, p_r(\overline{x})).$$

Also, as $\Psi_r(\overline{x}, \overline{x}) = 0$ it follows that $\overline{x} \in P_r(\overline{x})$.

Conversely, let $\overline{x}$ be a fixed point of the set-valued map $x \mapsto P_r(x)$. Then $\overline{x} \in P_r(\overline{x})$, that is,

$$\Psi_r(\overline{x}, \overline{x}) = \max_{y \in X(\overline{x})} \Psi_r(\overline{x}, y) = V_r(\overline{x}).$$

Since $\Psi_r(\overline{x}, \overline{x}) = 0$ we have $\overline{x} \in X(\overline{x})$ and $V_r(\overline{x}) = 0$. Hence by Theorem 1(b), $\overline{x}$ is a generalized Nash equilibrium. □

We now give an example which illustrates that $V_r(x)$ is finite for $x \in X$ (in fact it is finite in a neighbourhood of X) and $P_r^\nu(x)$ is not necessarily a singleton.

Example 1 Consider the GNEP with two players where problem for the first player is

$$\begin{aligned} &\min_{x_1} \theta_1(x_1, x_2) := 0 \\ &\text{subject to } x_1 + x_2 \le 1, \end{aligned}$$

and that of the second player is

$$\begin{aligned} &\min_{x_2} \theta_2(x_1, x_2) := x_2^2 \\ &\text{subject to } x_1 + x_2 \le 1. \end{aligned}$$

Here, the set $X = \{(x_1, x_2)^{\mathrm{T}} \mid x_1 + x_2 \le 1\}$ and the strategy sets for the two players are $X_1(x_2) = \{x_1 \mid x_1 \le 1 - x_2\}$ and $X_2(x_1) = \{x_2 \mid x_2 \le 1 - x_1\}$, respectively. The solution sets of the two players are

$$S_1(x_2) = \{x_1 \mid x_1 \le 1 - x_2\},$$

and

$$S_2(x_1) = \begin{cases} 0, & \text{if } x_1 \le 1, \\ 1 - x_1, & \text{if } x_1 > 1, \end{cases}$$

respectively. Hence the solution set of the GNEP is given by

$$P = \{(\alpha, 0) \mid \alpha \le 1\} \cup \{(\alpha, 1 - \alpha) \mid \alpha \ge 1\}.$$

The value function

$$V_r(x) = \theta_1(x_1, x_2) - \min_{y_1 \in X_1(x_2) \cap (x_1 + rC_1)} \theta_1(y_1, x_2) + \theta_2(x_1, x_2) - \min_{y_2 \in X_2(x_1) \cap (x_2 + rC_2)} \theta_2(x_1, y_2)$$

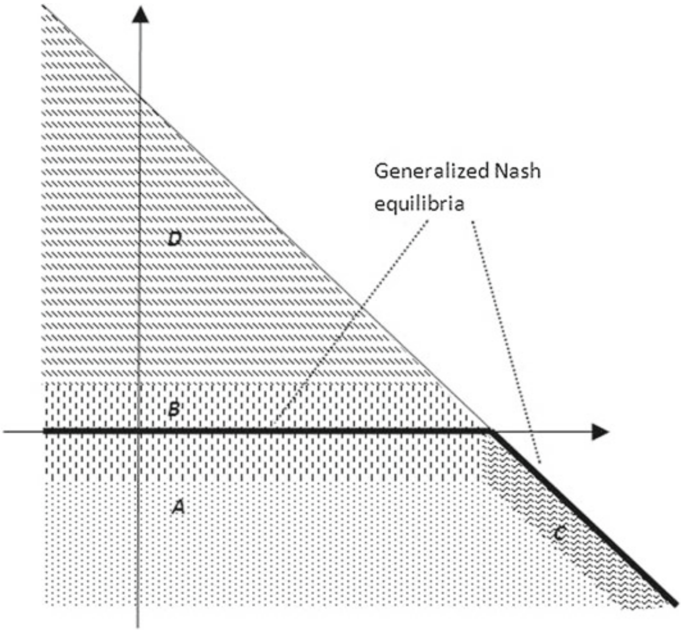

Fig. 1 Generalized Nash equilibria for GNEP in Example 1

where $C_1 = C_2 = [-1, 1]$. It can be easily seen that for sufficiently small r and for $x \in X$,

$$V_r(x) = \begin{cases} {x_2}^2 - (x_2 + r)^2, & \text{if } x_1 + x_2 \leq 1 - r, x_2 < -r, \\ {x_2}^2, & \text{if } x_1 + x_2 \leq 1, -r \leq x_2 \leq r, x_1 \leq 1, \\ {x_2}^2 - (1 - x_1)^2, & \text{if } 1 - r < x_1 + x_2 \leq 1, x_1 > 1, \\ {x_2}^2 - (x_2 - r)^2, & \text{if } x_1 + x_2 \leq 1, x_2 > r. \end{cases}$$

The regions in the feasible set X represented by $x_1 + x_2 \leq 1 - r, x_2 < -r$; $x_1 + x_2 \leq 1, -r \leq x_2 \leq r, x_1 \leq 1$; $1 - r < x_1 + x_2 \leq 1, x_1 > 1$; $x_1 + x_2 \leq 1, x_2 > r$ are represented by the shaded regions A, B, C and D, respectively in the Fig. 1. It may be noted here that $V_r(x)$ has finite value for any $x \in \{(x_1, x_2 \mid x_1 + x_2 \leq 1 + r\}$ and is continuous on X.

By Theorem 1(b), a point $\overline{x}$ in the set X is a generalized Nash equilibrium if and only if $V_r(\overline{x}) = 0$. The darkened line in the Fig. 1 represents the solutions of the GNEP. It can be easily seen that the collection of such points is given by the solution set P. Also,

$$\begin{aligned} P_r^1(x) &= \operatorname*{arg\,min}_{y_1 \in X_1(x_2) \cap (x_1 + rC_1)} \theta_1(y_1, x_2) \\ &= \begin{cases} [x_1 - r, x_1 + r], & \text{if } x_1 + x_2 \leq 1 - r, \\ [x_1 - r, 1 - x_2], & \text{if } 1 - r < x_1 + x_2 \leq 1, \end{cases} \end{aligned}$$

and

$$P_r^2(x) = \underset{y_2 \in X_2(x_1) \cap (x_2 + rC_2)}{\arg\min} \theta_2(x_1, y_2)$$
$$= \begin{cases} \{x_2 + r\}, & \text{if } x_1 + x_2 \leq 1 - r, x_2 < -r, \\ \{0\}, & \text{if } x_1 + x_2 \leq 1, -r \leq x_2 \leq r, x_1 \leq 1, \\ \{1 - x_1\}, & \text{if } 1 - r < x_1 + x_2 \leq 1, x_1 > 1, \\ \{x_2 - r\}, & \text{if } x_1 + x_2 \leq 1, x_2 > r. \end{cases}$$

Also, it can be easily seen the set of fixed points of the map $x \mapsto P_r(x) = P_r^1(x) \times P_r^2(x)$ coincides with the solution set P.

4 Optimization reformulation for finding normalized Nash equilibria

In this section we present another constrained optimization reformulation of the GNEP which gives a characterization of the normalized Nash equilibria. For details one may refer to [4,19].

A vector $\overline{x} \in X$ is called a *normalized Nash equilibrium* of the GNEP, if

$$\sup_{y \in X} \Psi(\overline{x}, y) \leq 0,$$

where Ψ is the Nikaidô–Isoda function.

It can be easily seen that, for jointly convex GNEP, a normalized Nash equilibrium is a solution of the GNEP but the converse is not always true. This fact is illustrated later in this section for the GNEP considered in Example 1.

In order to obtain the second optimization reformulation we maximize the regularized indicator Nikaidô–Isoda function $\Psi_r(x, y)$ in the second variable over the feasible set X instead of the set $X(x)$. We define the value function as

$$\hat{V}_r(x) := \max_{y \in X} \Psi_r(x, y).$$

Since,

$$\Psi_r(x, y) = \sum_{\nu=1}^{N} [\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) - \delta_{rC_\nu}(y^\nu - x^\nu)]$$
$$= \Psi(x, y) - \sum_{\nu=1}^{N} \delta_{rC_\nu}(y^\nu - x^\nu),$$

it follows that

$$\hat{V}_r(x) = \max_{y \in X \cap (x + rC)} \Psi(x, y),$$

where $C = \prod_{\nu=1}^{N} C_\nu$.

We next present a few properties of the value function $\hat{V}_r$.

Theorem 4 *The function $\hat{V}_r$ satisfies the following properties*:

(a) $\hat{V}_r(x) \geq 0$ *for all* $x \in X$.
(b) $\overline{x}$ *is a normalized Nash equilibrium if and only if* $\overline{x} \in X$ *and* $\hat{V}_r(\overline{x}) = 0$.

(c) *For every* $x \in X$,

$$\hat{P}_r(x) := \arg\max_{y \in X} \Psi_r(x, y),$$

is a nonempty convex compact set in $\mathbb{R}^n$.

Proof (a) Let $x \in X$, then we have

$$\hat{V}_r(x) = \max_{y \in X} \Psi_r(x, y) \geq \Psi_r(x, x) = 0.$$

(b) Suppose, $\overline{x}$ is a normalized Nash equilibrium. Then by definition, we have $\overline{x} \in X$ and $\sup_{y \in X} \Psi(\overline{x}, y) \leq 0$ which implies that $\Psi(\overline{x}, y) \leq 0$ for every $y \in X$. Therefore,

$$\Psi_r(\overline{x}, y) = \Psi(\overline{x}, y) - \sum_{\nu=1}^{N} \delta_{rC_\nu}(y^\nu - \overline{x}^\nu) \leq \Psi(\overline{x}, y) \leq 0,$$

for every $y \in X$. Hence

$$\hat{V}_r(\overline{x}) = \max_{y \in X} \Psi_r(\overline{x}, y) \leq 0.$$

Combining this with part (a) of the theorem we get that $\hat{V}_r(\overline{x}) = 0$.

Conversely, suppose $\overline{x} \in X$ and $\hat{V}_r(\overline{x}) = 0$. This implies

$$\Psi_r(\overline{x}, y) \leq 0, \ \forall y \in X. \tag{5}$$

To establish that $\overline{x}$ is a normalized Nash equilibrium, it is enough to show $\Psi(\overline{x}, y) \leq 0$, for every $y \in X$. Suppose on the contrary, there exists a vector $\hat{y} \in X$ such that $\Psi(\overline{x}, \hat{y}) > 0$. Then, $(1-\lambda)\overline{x} + \lambda\hat{y} \in X$ for every $\lambda \in (0, 1)$, as X is a convex set. As,

$$\Psi_r(\overline{x}, (1-\lambda)\overline{x} + \lambda\hat{y}) = \Psi(\overline{x}, (1-\lambda)\overline{x} + \lambda\hat{y}) - \sum_{\nu=1}^{N} \delta_{rC_\nu}((1-\lambda)\overline{x}^\nu + \lambda\hat{y}^\nu - \overline{x}^\nu),$$

$\Psi(\overline{x}, y)$ is concave in y and $\Psi(\overline{x}, \overline{x}) = 0$, we have

$$\Psi_r(\overline{x}, (1-\lambda)\overline{x} + \lambda\hat{y}) \geq \lambda\Psi(\overline{x}, \hat{y}) - \sum_{\nu=1}^{N} \delta_{rC_\nu}(\lambda(\hat{y}^\nu - \overline{x}^\nu)).$$

Hence, for sufficiently small λ we have $\lambda(\hat{y}^\nu - \overline{x}^\nu) \in rC_\nu$ and hence

$$\Psi_r(\overline{x}, (1-\lambda)\overline{x} + \lambda\hat{y}) > 0,$$

which contradicts (5).

(c) Since, $\hat{P}_r(x)$ can be rewritten as

$$\hat{P}_r(x) = \arg\max_{y \in X \cap (x + rC)} \Psi(x, y),$$

where $X \cap (x + rC)$ is a compact set and $\Psi(x, y)$ is a continuous concave function of y it follows that $\hat{P}_r(x)$ is a nonempty convex compact set in $\mathbb{R}^n$. □

The above theorem shows that the normalized Nash equilibria of the GNEP can be characterized as the global minima of the constrained optimization problem

$$\begin{aligned} &\min \ \hat{V}_r(x) \\ &\text{subject to} \quad x \in X. \end{aligned} \tag{6}$$

We now give a characterization for normalized equilibrium as a solution of a related quasi-variational inequality problem.

Theorem 5 *If θ_ν is continuously differentiable then x is a normalized Nash equilibrium of the GNEP if and only if it is a solution of QVI($\hat{Y}(x), F(x)$) where $\hat{Y}(x) := X \cap (x + rC)$.*

Proof Suppose x is a normalized Nash equilibrium of the GNEP. Let $y \in \hat{Y}(x)$ and $\lambda \in [0, 1]$, then convexity of $\hat{Y}(x)$ implies that $(1-\lambda)x + \lambda y \in \hat{Y}(x)$. Since x is a normalized Nash equilibrium it follows that $\sup_{y \in \hat{Y}(x)} \Psi(x, y) \leq 0$ and hence

$$\sum_{\nu=1}^{N} \left[\theta_\nu(x^\nu,\ x^{-\nu}) - \theta_\nu((1-\lambda)x^\nu + \lambda y^\nu,\ x^{-\nu})\right] \leq 0.$$

Since θ_ν is differentiable we have

$$\sum_{\nu=1}^{N} \left\langle \nabla_{x^\nu}\theta_\nu(x^\nu,\ x^{-\nu}), y^\nu - x^\nu \right\rangle \geq 0, \quad \forall y \in \hat{Y}(x), \tag{7}$$

which implies that x is solution of QVI($\hat{Y}(x), F(x)$).

Conversely, suppose x is a solution of QVI($\hat{Y}(x), F(x)$) then $x \in \hat{Y}(x) \subseteq X$ and (7) holds. By convexity of θ_ν is convex in x^ν, we have

$$\sum_{\nu=1}^{N} \left[\theta_\nu(y^\nu,\ x^{-\nu}) - \theta_\nu(x^\nu,\ x^{-\nu})\right] \geq 0, \quad \forall y = (y^1, \ldots, y^N)^{\mathrm{T}} \in \hat{Y}(x).$$

Also, since $\delta_{rC_\nu}(y^\nu - x^\nu) = -\infty$ for any $y^\nu \notin (x^\nu + rC_\nu)$ it follows that $\Psi_r(x, y) \leq 0$ for every $y \in X$ which in turn implies that $\hat{V}_r(x) \leq 0$. By Theorem 4(a) we have $\hat{V}_r(x) = 0$ and hence by Theorem 4(b) it follows that x is a normalized Nash equilibrium of the GNEP. □

We next present a result similar to Theorem 3.

Theorem 6 *$\overline{x}$ is a normalized Nash equilibrium of the GNEP if and only if $\overline{x}$ is a fixed point of the map $x \mapsto \hat{P}_r(x)$.*

Proof Suppose that $\overline{x}$ is a normalized Nash equilibrium. Then by definition, we have

$$\Psi(\overline{x}, y) \leq 0,\ \forall y \in X. \tag{8}$$

Since $\Psi(\overline{x}, \overline{x}) = 0$ and $\overline{x} \in X \cap (\overline{x} + rC)$, it follows that for any $y \in X \cap (\overline{x} + rC)$

$$\Psi(\overline{x}, y) \leq \Psi(\overline{x}, \overline{x}),$$

that is, $\overline{x} \in \hat{P}_r(\overline{x})$.

Conversely, suppose $\overline{x}$ is a fixed point of the map $x \mapsto \hat{P}_r(x)$. Then, for every $y \in X$

$$\Psi(\overline{x}, y) \leq \Psi(\overline{x}, \overline{x}) = 0.$$

Hence, $\hat{V}_r(\overline{x}) = 0$ and therefore by Theorem 4(b), $\overline{x}$ is a normalized Nash equilibrium of the GNEP. □

We again consider Example 1 to find the normalized Nash equilibria of the GNEP. For $x \in X$

$$\hat{V}_r(x) = \max_{y \in X \cap (x + rC)} \left[\theta_1(x_1, x_2) - \theta_1(y_1, x_2) + \theta_2(x_1, x_2) - \theta_1(x_1, y_2)\right],$$

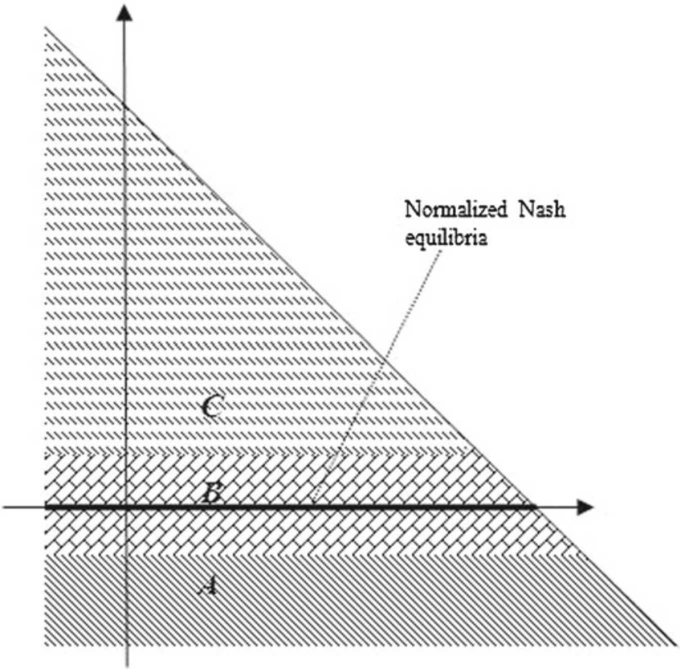

Fig. 2 Normalized Nash equilibria for GNEP in Example 1

where $C = [-1, 1] \times [-1, 1]$. Here, it can be easily shown that for $x \in X$

$$\hat{V}_r(x) = \begin{cases} x_2^2 - (x_2 + r)^2, & \text{if } x_2 < -r, \\ x_2^2, & \text{if } -r \le x_2 \le r, \\ x_2^2 - (x_2 - r)^2, & \text{if } x_2 > r. \end{cases}$$

In the above expression every x satisfies the condition $x_1 + x_2 \le 1$. The regions in the feasible set X represented by $x_2 < -r$; $-r \le x_2 \le r$; $x_2 > r$ are represented by the shaded regions A, B and C in the Fig. 2.

By Theorem 4(b), a feasible point $\overline{x}$ is a normalized Nash equilibrium of the GNEP if and only if $\hat{V}_r(\overline{x}) = 0$. So the set of the normalized Nash equilibrium points is given by $\{(\alpha, 0) \mid \alpha \le 1\}$ and the set $\hat{P}_r(x)$ for $x \in X$ is given by

$$\hat{P}_r(x) = \begin{cases} I \times \{x_2 + r\}, & \text{if } x_2 < -r, \\ I \times \{0\}, & \text{if } -r \le x_2 \le r, \\ I \times \{x_2 - r\}, & \text{if } x_2 > r. \end{cases}$$

where $I = [x_1 - r, \min\{x_1 + r, 1 - x_2\}]$. As observed earlier every x satisfies the condition $x_1 + x_2 \le 1$. The darkened line in the Fig. 2 represents the normalized Nash equilibrium points.

In the above example every normalized Nash equilibrium is a solution of the GNEP but there are many solutions of the GNEP which are not normalized Nash equilibria. Also it may be noted that the mapping $\hat{V}_r(x)$ is continuous on X and has finite values for all those points x that satisfy $x_1 + x_2 \le 1 + 2r$, for instance if in the above example we consider point $x = (1 + 2r, 0)$ then $\hat{V}_r(x) = -r^2$.

5 Algorithms to compute generalized and normalized Nash equilibria

In this section we give algorithms to solve the GNEP using the equivalent optimization reformulations defined in Sects. 3 and 4.

In view of Theorem 1 we first illustrate the performance of the reformulation function V_r using the nonlinear Jacobi-type method. We recall the notion of projection of a point $x \in \mathbb{R}^n$ onto a closed set $X \subseteq \mathbb{R}^n$, denoted by $\mathrm{Proj}(x \mid X)$, is defined as

$$\mathrm{Proj}(x \mid X) := \left\{ y \in X \mid \|y - x\| = \inf_{u \in X} \|u - x\| \right\}.$$

Algorithm 1

Step 1 Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, set $x_0 = ({x_0}^1, \ldots, {x_0}^N) = \mathrm{Proj}(x \mid X)$ and $k = 0$.
Step 2 For $\nu = 1, \ldots, N$, compute a solution x_{k+1}^{ν} of the problem

$$\min_{y^\nu} \theta_\nu(y^\nu, {x_k}^{-\nu})$$

$$\text{subject to } y^\nu \in X_\nu({x_k}^{-\nu}) \cap ({x_k}^{\nu} + rC_\nu).$$

Step 3 Set $x_{k+1} = (x_{k+1}^1, \ldots, x_{k+1}^N)$.
Step 4 Compute $V_r(x_k)$ using

$$V_r(x_k) = \sum_{\nu=1}^{N} [\theta_\nu({x_k}^{\nu}, {x_k}^{-\nu}) - \theta_\nu(x_{k+1}^{\nu}, {x_k}^{-\nu})].$$

If $V_r(x_k) < \varepsilon$, STOP and return x_k is a solution of the GNEP, else set $k = k + 1$ and go to Step 2.

In the above algorithm the iteration is terminated if the value of $V_r(x_k)$ becomes less than ε, where ε is a sufficiently small positive real number.

Consider again the problem described in Example 1 with $r = 0.5$. For instance, if we take $x = (1, 1)$ then its projection on the feasible set is $x_0 = (0.5, 0.5)$ and from that point we reach a solution point in one iteration with one of the sequence being $(0.5, 0.5) \to (0.5, 0)$. If we take $x_0 = (0, 0.8)$ it takes two iterations to obtain a solution of the GNEP and a sequence obtained is $(0, 0.8) \to (0, 0.3) \to (0, 0)$. Let us now take $x_0 = (1.4, -0.9)$ then one iteration is performed and the sequence generated is $(1.4, -0.9) \to (1.4, -0.4)$. If we start with $x_0 = (0, -1.3)$ we observe that one of the sequences generated is $(0, -1.3) \to (0.5, -0.8) \to (0.7, -0.3) \to (0.8, 0)$. We observe that the solution can be obtained in one iteration if we start with a feasible point in the shaded region B or C but more iterations are required for points lying in the regions A and D in Fig. 1.

At each iteration k, the above algorithm solves N optimization problems in Step 2 and for each $\nu \in 1, \ldots, N$ the objective function

$$\theta_\nu({x_k}^1, \ldots, {x_k}^{\nu-1}, y^\nu, {x_k}^{\nu+1}, \ldots, {x_k}^N),$$

is minimized subject to $y^\nu \in X_\nu({x_k}^{-\nu}) \cap ({x_k}^{\nu} + rC_\nu)$. But in this version we did not use the latest information, that is before solving νth player's problem we have already solved $\nu - 1$ problems of the players $1, 2, \ldots, \nu - 1$ and we have the new variables $x_{k+1}^1, \ldots, x_{k+1}^{\nu-1}$ and use them instead of ${x_k}^1, \ldots, {x_k}^{\nu-1}$ both in θ_ν and in the feasible sets. This modification is employed in the following Gauss–Siedal type method.

Algorithm 2

Step 1 Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, set $x_0 = ({x_0}^1, \ldots, {x_0}^N) = \text{Proj}(x \mid X)$ and $k = 0$.
Step 2 For $\nu = 1, \ldots, N$, compute a solution x_{k+1}^{ν} of the problem

$$\min_{y^\nu} \theta_\nu(x_{k+1}^1, \ldots, x_{k+1}^{\nu-1}, y^\nu, {x_k}^{\nu+1}, \ldots, {x_k}^N)$$

$$\text{subject to } y^\nu \in X_\nu(x_{k+1}^1, \ldots, x_{k+1}^{\nu-1}, {x_k}^{\nu+1}, \ldots, {x_k}^N) \cap ({x_k}^\nu + rC_\nu).$$

Step 3 Set $x_{k+1} = (x_{k+1}^1, \ldots, x_{k+1}^N)$.
Step 4 Compute $V_r(x_k)$ using

$$V_r(x_k) = \sum_{\nu=1}^{N} [\theta_\nu({x_k}^\nu, {x_k}^{-\nu}) - \theta_\nu(x_{k+1}^\nu, {x_k}^{-\nu})].$$

If $V_r(x_k) < \varepsilon$, STOP and return x_k is a solution of the GNEP, else set $k = k + 1$ and go to Step 2.

See [18] for instance, for the well-known counterparts of these methods in the case of system of linear equations. The two methods described above are straightforward and easy to implement. Similarly we can check the performance of the second regularized function $\hat{V}_r$ also. The next algorithm is the counterpart of the Algorithm 1 for the optimization reformulation (6).

Algorithm 3

Step 1 Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, set $x_0 = ({x_0}^1, \ldots, {x_0}^N) = \text{Proj}(x \mid X)$ and $k = 0$.
Step 2 Compute a solution x_{k+1}^{ν} of the problem

$$\min_{y} \sum_{\nu=1}^{N} \theta_\nu(y^\nu, {x_k}^{-\nu})$$

$$\text{subject to } y \in X \cap (x_k + rC).$$

Step 3 Set $x_{k+1} = (x_{k+1}^1, \ldots, x_{k+1}^N)$.
Step 4 Compute $\hat{V}_r(x_k)$ using

$$\hat{V}_r(x_k) = \sum_{\nu=1}^{N} [\theta_\nu({x_k}^\nu, {x_k}^{-\nu}) - \theta_\nu(x_{k+1}^\nu, {x_k}^{-\nu})].$$

If $\hat{V}_r(x_k) < \varepsilon$, STOP and return x_k is a solution of the GNEP, else set $k = k + 1$ and go to Step 2.

We again consider Example 1 with $r = 0.5$ to illustrate the performance of the above algorithm. For instance, if we take $x = (1, 1)$, then $x_0 = (0.5, 0.5)$ and the convergence to a normalized Nash equilibrium occurs in one iteration and a sequence generated is $(0.5, 0.5) \to (0.5, 0)$ and if we take $x_0 = (0, 0.8)$ the convergence is attained in two iterations and one of the sequences is $(0, 0.8) \to (0, 0.3) \to (0, 0)$. For the point $x_0 = (1.4, -0.9)$ the convergence to a normalized Nash equilibrium point is attained in two iterations and a sequence

Table 1 Iterative details for Algorithm 1

k	$x_{1,k}$	$x_{2,k}$	$x_{3,k}$	$V_r(x_k)$
0	19.9965833550	0	0	8,686.7746982
1	29.9965833550	10	10	5,866.7746982
2	39.9965833550	20	20	3,046.7746982
3	49.9965833550	30	30	501.3334276
4	59.9965833550	40	28.4621646297	51.58862589
5	59.2761555215	45.1322940852	18.6370058204	11.16021827
6	61.5993563161	45.4104236504	18.7166652405	2.74377269
7	61.4222323587	44.5104442762	18.4629004916	0.67455874
8	61.9933937632	44.5785239178	18.4827928909	0.16583715
9	61.9498431665	44.3575851722	18.4200913482	0.04076686
10	62.0902587103	44.3688486671	18.4303907359	0.01000639
11	62.0795842103	44.3196871984	18.4098851053	0.00245910
12	62.1140731979	44.3224692767	18.4123843792	0.00060279
13	62.1114581809	44.3136761988	18.4041009509	0.00014783
14	62.1199125873	44.3141214452	18.4049467655	0.00003634
15	62.1192726227	44.3117455761	18.4031354399	0.00000900
16	62.1213463714	44.3116505215	18.4035435841	0.00000221
17	62.1213963714	44.3108806900	18.4032840456	0.00000015

generated is $(1.4, -0.9) \rightarrow (1.4, -0.4) \rightarrow (1, 0)$. If we start with $x_0 = (0, -1.3)$ we observe that one of the sequences generated is $(0, -1.3) \rightarrow (0.5, -0.8) \rightarrow (0.7, -0.3) \rightarrow (0.8, 0)$. We observe that the solution can be obtained in one iteration if we start with a feasible point in the shaded region B but more iterations are required for points lying in the shaded regions A and C in Fig. 2.

Example 2 We illustrate the algorithms for the electricity market model considered in [2] which was also studied in [13] for the case of multi-leader-common-follower games. There are two players (generating companies) and three variables (generators). The first player controls one variable (owns one generator), namely $x^1 = (x_1)$ and the second player controls two variables (owns two generators), namely $x^2 := (x_2, x_3)$. The objective function of the first player is

$$\min_{x_1} \theta_1(x_1, x_2, x_3) := 0.02{x_1}^2 + 2x_1 - [378.4 - 2(x_1 + x_2 + x_3)]\, x_1$$

and that of the second player is

$$\min_{x_2,\, x_3} \theta_2(x_1, x_2, x_3) := 0.0175{x_2}^2 + 1.75x_2 + 0.0625{x_3}^2 + x_3 \\ - [378.4 - 2(x_1 + x_2 + x_3)]\, (x_2 + x_3)$$

and the set X is given by

$$X := \{(x_1, x_2, x_3) \mid 0 \le x_1 \le 80;\ 0 \le x_2 \le 80;\ 0 \le x_3 \le 50\}.$$

We apply all the three algorithms to this problem with $r = 10$ and $\varepsilon = 10^{-6}$. The implementation is done in MATLAB using the in built function *fmincon* from the optimization

Table 2 Iterative details for Algorithm 2

k	$x_{1,k}$	$x_{2,k}$	$x_{3,k}$	$V_r(x_k)$
0	80	80	50	7,208.00000000
1	70	70	40	4,388.00000000
2	60	60	30	1,568.00000000
3	50	50	20	113.67870687
4	58.5148517242	45.7077471318	18.7973488145	11.27265238
5	61.2351015445	44.6518906582	18.5023380532	0.68132095
6	61.9038471897	44.3922667314	18.4298968548	0.04116776
7	62.0682355741	44.3285277854	18.4119806334	0.00249177
8	62.1086592232	44.3180705478	18.4024194170	0.00015017
9	62.1185692547	44.3150886889	18.4004902276	0.00000948
10	62.1210069524	44.3139733872	18.4003959669	0.00000097

Table 3 Iterative details for Algorithm 3

k	$x_{1,k}$	$x_{2,k}$	$x_{3,k}$	$\hat{V}_r(x_k)$
0	49.9999416090	49.9999416090	0	1,260.69068539
1	59.9999416090	58.6493473245	10	55.90197241
2	59.1834675901	48.6493473245	15.2249238720	12.41182013
3	61.5473993613	45.4471604018	18.7259784109	2.81953061
4	61.3995307034	44.5306110933	18.4684275993	0.69310258
5	61.9805270909	44.5869765334	18.4852284683	0.16981374
6	61.9447269440	44.3629133013	18.4208044347	0.04170308
7	62.0876906992	44.3729040935	18.4288987071	0.01024203
8	62.0783936292	44.3211684534	18.4097967685	0.00251374
9	62.1133834796	44.3236591710	18.4117933957	0.00061640
10	62.1111499582	44.3143489927	18.4037917031	0.00015139
11	62.1197316623	44.3144036186	18.4048207202	0.00003737
12	62.1191869470	44.3116499511	18.4033144472	0.00000943
13	62.1213037079	44.3112838825	18.4039500284	0.00000236
14	62.1211429373	44.3104256124	18.4036690196	0.00000057

toolbox in order to find $\mathrm{Proj}(x \mid X)$ and the solution x_{k+1}^{ν} for each player ν at each iteration k.

We first start with the point $x = (20, -10, -10)$, its projection on X is

$$x_0 = (19.9965833550, 0, 0).$$

The iterative details of Algorithm 1 are given in Table 1.

We now find a solution using Algorithm 2. We take point $x = (100, 100, 100)$, then the projection point $x_0 = (80, 80, 50)$. The iterative details of Algorithm 2 are given in Table 2.

We then take point $x = (50, 50, -10)$, and apply Algorithm 3. The projection point $x_0 = (49.9999416090, 49.9999416090, 0)$. The iterative details of Algorithm 3 are given in Table 3.

6 Conclusions

We presented two constrained optimization reformulations of a jointly convex GNEP using regularized indicator Nikaidô–Isoda function. The global minima of these two reformulations correspond to generalized Nash equilibria and normalized Nash equilibria, respectively of the given GNEP. The regularization using indicator function not only retains the same structure as that of Nikaidô–Isoda function, it also ensures that the region on which the function is to be optimized, in the definition of value function, is a compact subset of the constraint set. Thus the computation of value function is easier and remains finite in a neighbourhood around the constraint set. We also provide numerical illustations to find solutions of GNEP by using nonlinear Jacobi and Gauss–Seidel type algorithms for equivalent optimization reformulations. It would be worthwhile to investigate investigate the continuity and differentiability properties of the value functions V_r and $\hat{V}_r$.

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