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Stackelberg equilibria via variational inequalities and projections

Szilárd Nagy

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Abstract Existence and location of Stackelberg equilibria is studied for two players by using appropriate variational inequalities and fixed point arguments. Both compact and non-compact strategy sets are considered in Euclidean spaces; in the non-compact case, we apply arguments from the theory of (discrete and continuous) projective dynamical systems. Some examples are also presented.

Keywords Stackelberg equilibrium · Variational inequalities · Dynamical systems

1 Introduction

The Stackelberg competition model is a game in which the leader player moves first and then the follower player moves sequentially. In order to solve such a game, the so-called *backward induction method* is applied. The first step is to find the best strategy/response for the follower player, considering the strategy action of the leader player as a parameter; then, having in our mind this parameter-depending response, the choice of the best strategy of the leader player concludes the problem. Comparing Stackelberg and Nash competition models, in the latter model the two players are competing with each other in the same level. For some comparison results, we refer the reader to the papers of Amir and Grilo [1], Novak, Feichtinger and Leitmann [9], and Stanford [11]. For instance, in [9] the authors show that the Stackelberg model (i.e., the leader-follower model) describes efficiently the combat against terror activities.

Concerning the Stackelberg competition model, formally, if $f_1, f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are the payoff/loss functions for the two players, and $K_1, K_2 \subset \mathbb{R}^N$ are their strategy sets, the first step is to determine the *Stackelberg equilibrium response set* as

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S. Nagy (✉)
Babeş-Bolyai University, 400591 Cluj-Napoca, Romania
e-mail: szilard_nagy85@yahoo.com

$$R_{SE}(x_1) = \{x_2 \in K_2 : f_2(x_1, y) - f_2(x_1, x_2) \geq 0,\ \forall y \in K_2\}$$

for every fixed $x_1 \in K_1$. Now, assume that $R_{SE}(x_1) \neq \emptyset$ for every $x_1 \in K_1$, the concluding step (for the leader player) is to minimize the map $x \mapsto f_1(x, r(x))$ on K_1 where r is a selection of the set-valued map R_{SE}; more precisely, the *Stackelberg equilibrium leader set* is

$$S_{SE} = \{x_1 \in K_1 : f_1(x, r(x)) - f_1(x_1, r(x_1)) \geq 0,\ \forall x \in K_1\}.$$

The primary aim of the present paper is to locate the elements of the Stackelberg equilibrium response set. To complete this purpose, we define a slightly larger set than the Stackelberg equilibrium response set by means of variational inequalities. More precisely, if $f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a function of class C^1, for every $x_1 \in K_1$, we introduce the so-called *Stackelberg variational response set* defined by

$$R_{SV}(x_1) = \left\{x_2 \in K_2 : \left\langle \frac{\partial f_2}{\partial x_2}(x_1, x_2), y - x_2 \right\rangle \geq 0,\ \forall y \in K_2\right\}.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$. First of all, we are able to compute the Stackelberg variational response set more easier than $R_{SE}(x_1)$, thus, we can locate the elements of the Stackelberg equilibrium response set among these points. Second, we may characterize the elements of the Stackelberg variational response set by the fixed points of a suitable function which involves the metric projection map into the set K_2. Due to the latter fact, we are able to guarantee not only existence but also location results (via projective dynamical systems) of the Stackelberg competition model whenever the strategy sets are compact or non-compact. Recently, projection-like methods for Nash equilibria have been developed by Cavazzuti, Pappalardo and Passacantando [2], Pang and Fukushima [10], Xia and Wang [13], Zhang, Qu and Xiu [14], and references therein. For generic equilibrium results via variational and non-variational methods, we refer the reader to the volumes [3,4,8].

Assume further that $f_1 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a function of class C^1. If $R_{SV}(x_1) \neq \emptyset$ for every $x_1 \in K_1$ and once we are able to choose a C^1-class selection $r : K_1 \to K_1$ of the set-valued map R_{SV}, we also introduce the *Stackelberg variational leader set*

$$S_{SV} = \left\{x_1 \in K_1 : \left\langle \frac{\partial f_1}{\partial x_1}(x_1, r(x_1)), y - x_1 \right\rangle \geq 0,\ \forall y \in K_1\right\}.$$

As expected, the set S_{SV} contains the best strategies of the first player, i.e., the minimizers for the map $x \mapsto f_1(x, r(x))$ on K_1.

The paper is structured as follows. In the next section we recall some basic notions and results which are needed for our investigations: metric projections, relation between the Stackelberg variational response set and fixed points of a suitable projection. In Sect. 3 we present the main results of the paper, by considering both the compact and non-compact cases for the strategy sets of the players.

2 Preliminaries

In this section we are going to state the basic properties of the Stackelberg variational response set. Throughout this section, we keep the notations from the previous section. The following result is based on standard arguments from variational inequalities.

Proposition 2.1 *Let $f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1 and $K_i \subset \mathbb{R}^N$ convex sets, $i = 1, 2$. Then, we have the following assertions:*

(a) *$R_{SE}(x_1) \subseteq R_{SV}(x_1)$ for every $x_1 \in K_1$;*
(b) *if $f_2(x_1, \cdot)$ is convex on K_2 for some $x_1 \in K_1$, then $R_{SE}(x_1) = R_{SV}(x_1)$.*

Proof (a) Let us fix $x_2 \in R_{SE}(x_1)$, i.e., $f_2(x_1, y) \geq f_2(x_1, x_2)$ for all $y \in K_2$. By definition, one has that

$$\left\langle \frac{\partial f_2}{\partial x_2}(x_1, x_2), h \right\rangle = \lim_{t \to 0^+} \frac{f_2(x_1, x_2 + th) - f_2(x_1, x_2)}{t}, \ \forall h \in \mathbb{R}^N.$$

Since K_2 is convex, then $x_2 + t(y - x_2) \in K_2$ for every $t \in [0, 1]$ whenever $y \in K_2$. If $h = y - x_2 \in \mathbb{R}^N$ in the above expression, we conclude that $\langle \frac{\partial f_2}{\partial x_2}(x_1, x_2), y - x_2 \rangle \geq 0$, which implies that $R_{SE}(x_1) \subseteq R_{SV}(x_1)$ for all $x_1 \in K_1$.

(b) Since the function $f_2(x_1, .)$ is convex and of class C^1, one has

$$f_2(x_1, y) - f_2(x_1, x_2) \geq \left\langle \frac{\partial f_2}{\partial x_2}(x_1, x_2), y - x_2 \right\rangle$$

for all $y \in K_2$. Since $x_2 \in R_{SV}(x_1)$, one has that $\langle \frac{\partial f_2}{\partial x_2}(x_1, x_2), y - x_2 \rangle \geq 0$ for all $y \in K_2$. Therefore, one has $f_2(x_1, y) - f_2(x_1, x_2) \geq 0$ for all $y \in K_2$, i.e., $x_2 \in R_{SE}(x_1)$, which concludes the first part. □

Remark 2.1 Note that (a) can be proved via a critical point argument, see Szulkin [12]. Indeed, if $f_2(x_1, y) \geq f_2(x_1, x_2)$ for all $y \in K_2$, then $x_2 \in K_2$ is a global minimum point for the function $f_2(x_1, \cdot) + \delta_{K_2}$, where δ_{K_2} is the indicator function of the set K_2. Now, on account of [12], we have that $0 \in \frac{\partial f_2}{\partial x_2}(x_1, x_2) + \partial_c \delta_{K_2}(x_2)$, where ∂_c denotes the subdifferential in the sense of the convex analysis. The latter fact implies that $\langle \frac{\partial f_2}{\partial x_2}(x_1, x_2), y - x_2 \rangle \geq 0$ for all $y \in K_2$, i.e., $x_2 \in R_{SV}(x_1)$.

Let $K \subset \mathbb{R}^N$ be a nonempty set, $x \in \mathbb{R}^N$. The metric projection $P_K : \mathbb{R}^N \to K$ of x to K is defined by

$$P_K(x) = \{y \in K : \|x - y\| = \inf_{z \in K} \|z - x\|\}.$$

It is well-known that when $K \subset \mathbb{R}^N$ is closed, then $P_K(x) \neq \emptyset$ for every $x \in \mathbb{R}^N$. Moreover, when K is closed and convex, the following characterization of the metric projection is know (see Moskowitz and Dines [7]):

$$x \in P_K(y) \Leftrightarrow \langle y - x, z - x \rangle \leq 0, \ \ \forall z \in K. \tag{2.1}$$

If $K \subset \mathbb{R}^N$ is convex and closed, the set K is a Chebishev set, i.e., $P_K(x)$ is a singleton for every $x \in \mathbb{R}^N$, and the map P_K is non-expansive, i.e.,

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \ \ \forall x, y \in \mathbb{R}^N.$$

We conclude this section by an important observation, which makes a connection between the Stackelberg variational set and the fixed point of the map $A_\alpha^{x_1} : K_2 \to K_2$ defined by

$$A_\alpha^{x_1}(x) = P_{K_2}\left(x - \alpha \frac{\partial f_2}{\partial x_2}(x_1, x)\right), \tag{2.2}$$

where $x_1 \in K_1$ and $\alpha > 0$ are fixed.

Proposition 2.2 *Let $f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1 and $K_i \subset \mathbb{R}^N$ convex sets, $i = 1, 2$. Let $x_1 \in K_1$. The following statements are equivalent:*

(a) $x_2 \in R_{SV}(x_1)$;
(b) $A_\alpha^{x_1}(x_2) = x_2$ *for all* $\alpha > 0$;
(c) $A_\alpha^{x_1}(x_2) = x_2$ *for some* $\alpha > 0$.

Proof Note that $x_2 \in R_{SV}(x_1)$ is equivalent with

$$\left\langle x_2 - \alpha \frac{\partial f_2}{\partial x_2}(x_1, x_2) - x_2, y - x_2 \right\rangle \leq 0, \ \forall y \in K_2,$$

for all/some $\alpha > 0$. Therefore, on account of (2.1), one can write

$$R_{SV}(x_1) = \left\{ x_2 \in K_2 : P_{K_2}\left(x_2 - \alpha \frac{\partial f_2}{\partial x_2}(x_1, x_2) \right) = x_2 \right\}.$$

Due to (2.2), the claim is proved. □

We conclude this section by a result concerning the Stackelberg variational leader set; more precisely, the definitions imply

Proposition 2.3 *Let $f_1 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1. Assume that $x \mapsto R_{SV}(x)$ is a single-valued function of class C^1 on K_1. Then $S_{SE} \subseteq S_{SV}$.*

3 Stackelberg variational response set: existence and location

Due to Proposition 2.2, to find elements in $R_{SV}(x_1)$ is an equivalent problem to find fixed points of $A_\alpha^{x_1}$, $\alpha > 0$. To complete this aim, we distinguish two case: compact and non-compact strategy sets.

3.1 Compact case

Theorem 3.1 (Compact case) *Let $f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1 and $K_i \subset \mathbb{R}^N$ be convex and compact sets, $i = 1, 2$. Then, the following statements hold:*

(a) $R_{SV}(x_1) \neq \emptyset$ *for every* $x_1 \in K_1$;
(b) *If* $\text{card}(R_{SV}(x_1)) = 1$ *for every* $x_1 \in K_1$, *and the map* $x \mapsto R_{SV}(x)$ *is of class* C^1, *then* $S_{SV} \neq \emptyset$.

Proof (a) Fix $x_1 \in K_1$ and $\alpha > 0$. Note that the map $A_\alpha^{x_1} : K_2 \to K_2$ is continuous; therefore, due to the fixed point theorem of Brouwer, $A_\alpha^{x_1}$ has at least a fixed point $x_2 \in K_2$. According to Proposition 2.2, $x_2 \in K_2$ belongs to the Stackelberg variational response set $R_{SV}(x_1)$.

(b) Let $\beta > 0$. Since $\text{card}(R_{SV}(x)) = 1$ for every $x \in K_1$, we may introduce the function $B_\beta : K_1 \to K_1$ by

$$B_\beta(x) = P_{K_1}\left(x - \beta \frac{\partial f_1}{\partial x_1}(x, R_{SV}(x)) \right).$$

By the hypotheses, it results that the function B_β is continuous, thus it has at least a fixed point $x_1 \in K_1$. A similar argument as in Proposition 2.2 shows that $B_\beta(x_1) = x_1$ is equivalent to the fact that $x_1 \in S_{SV}$, which concludes the proof. □

Example 3.1 Let $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two functions defined by $f_1(x_1, x_2) = 4x_1x_2^2 - x_1^3$, $f_2(x_1, x_2) = x_2^2 + x_2(x_1 + 1)$, and the sets $K_1 = K_2 = [-1, 1]$. It is clear that Theorem 3.1(a) can be applied, and a simple computation yields that

$$R_{SV}(x_1) = -\frac{x_1 + 1}{2}, \ x_1 \in K_1.$$

Note that $f_2(x_1, \cdot)$ is convex on K_2 for every $x_1 \in K_1$; thus, on account of Proposition 2.1(a), $R_{SE}(x_1) = R_{SV}(x_1)$ for every $x_1 \in K_1$. Moreover, since $\text{card}(R_{SV}(x_1)) = 1$ for every $x_1 \in K_1$, and the map $x \mapsto R_{SV}(x)$ is of class C^1, then one has that $S_{SV} \neq \emptyset$, see Theorem 3.1(b). A simple calculation also yields that $S_{SV} = \{(-1/4, -3/8)\}$. Now, by using Proposition 2.3, we can check that the Stackelberg equilibrium leader set is $S_{SE} = \{(-1/4, -3/8)\}$.

Remark 3.1 For the same functions and sets as in Example 3.1, we state that the set of Nash equilibrium points is empty. This fact can be seen by following the arguments from the paper of Kristály [5], where a very general framework is discussed for non-smooth function on finite-dimensional Riemannian manifolds; see also the monograph of Kristaly, Radulescu and Varga [6]. More precisely, the first step is to determine the Nash critical points, i.e., the solutions for the system

$$\left\langle \frac{\partial f_i}{\partial x_i}(x_1, x_2), x - x_i \right\rangle \geq 0, \ \forall x \in K_i, \ i = 1, 2.$$

This system has the solutions $N_{SV} = \left\{(\frac{1+\sqrt{3}}{2}, -\frac{3+\sqrt{3}}{4}), (\frac{1-\sqrt{3}}{2}, -\frac{3-\sqrt{3}}{4})\right\}$. Now, the set of Nash equilibrium points is a subset of N_{SV}, due to [5]. Note however that none of the above points fulfil the system for Nash equilibria, i.e.,

$$f_1(x, x_2) \geq f_1(x_1, x_2) \text{ and } f_2(x_1, y) \geq f_2(x_1, x_2), \ \forall (x, y) \in K_1 \times K_2.$$

3.2 Non-compact case

If $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are defined by $f_1(x_1, x_2) = f_1(x_1, x_2) = e^{-x_1 - x_2}$, and $K_1 = K_2 = [0, \infty)$, then for every $x_1 \in K_1$, one has $R_{SE}(x_1) = R_{SV}(x_1) = \emptyset$. Consequently, in order to guarantee existence for the elements from the Stackelberg equilibrium/variational response set in the non-compact case, further (growth) assumptions are needed beside the regularity of the functions.

To complete the latter problem, we introduce two dynamical systems. For fixed $x_1 \in K_1$ and $\alpha > 0$, let $(DDE)_{x_1}$ be the discrete differential equation in the form

$$\begin{cases} y_{n+1} = A_\alpha^{x_1}(P_{K_2}(y_n)), \ n \geq 0, \\ y_0 \in \mathbb{R}^N. \end{cases}$$

In a similar manner, let $(CDE)_{x_1}$ be the continuous differential equation in the form

$$\begin{cases} \frac{dy}{dt} = A_\alpha^{x_1}(P_{K_2}(y(t))) - y(t), \\ y(0) = y_0 \in \mathbb{R}^N. \end{cases}$$

If X and Y are two prehilbert spaces, we recall two standard notions:

- The function $f : X \to Y$ is an L*−Lipschitz* function if

  $$\|f(x) - f(y)\| \leq L\|x - y\|, \ \forall x, y \in X.$$

  If $0 < L < 1$, then the function is an L-contraction.

- The function f is $\kappa-$*strictly monotone* if

$$\langle f(x) - f(y), x - y\rangle \geq \kappa\|x - y\|^2.$$

The main theorem of the present section reads as follows:

Theorem 3.2 (Non-compact case) *Let $f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function of class C^1 and $K_i \subset X$ convex (not necessarily compact) sets, $i = 1, 2$. Let $x_1 \in K_1$ be fixed and assume that $\frac{\partial f_2}{\partial x_2}(x_1, \cdot)$ is an $L-$Lipschitz and $\kappa-$strictly monotone function. Then* $\text{card}(R_{SV}(x_1)) = 1$*; moreover, both dynamical systems, $(DDE)_{x_1}$ and $(CDE)_{x_1}$, exponentially converge to the unique element of $R_{SV}(x_1)$.*

Proof Let us choose

$$0 < \alpha < \frac{\kappa - \sqrt{(\kappa^2 - L^2)_+}}{L^2}, \tag{3.1}$$

where $s_+ = \max(0, s)$. First, some estimates for the map $A_\alpha^{x_1}$ are in order. More precisely, exploiting the non-expansiveness of the projection P_{K_2} and basic estimates, one has

$$\begin{aligned}
\|A_\alpha^{x_1}(x) - A_\alpha^{x_1}(y)\|^2 &= \left\| P_{K_2}\left(x - \alpha\frac{\partial f_2}{\partial x_2}(x_1, x)\right) - P_{K_2}\left(y - \alpha\frac{\partial f_2}{\partial x_2}(x_1, y)\right)\right\|^2 \\
&\leq \left\| x - y - \alpha\left(\frac{\partial f_2}{\partial x_2}(x_1, x) - \frac{\partial f_2}{\partial x_2}(x_1, y)\right)\right\|^2 \\
&= \|x - y\|^2 - 2\alpha\left\langle x - y, \frac{\partial f_2}{\partial x_2}(x_1, x) - \frac{\partial f_2}{\partial x_2}(x_1, y)\right\rangle \\
&\quad + \alpha^2\left\|\frac{\partial f_2}{\partial x_2}(x_1, x) - \frac{\partial f_2}{\partial x_2}(x_1, y)\right\|^2 \\
&\leq \|x - y\|^2 - 2\alpha\kappa\|x - y\|^2 + \alpha^2 L^2\|x - y\|^2 \\
&= (1 - 2\alpha\kappa + \alpha^2 L^2)\|x - y\|^2.
\end{aligned}$$

Due to the choice of α, see (3.1), we have that $\rho = \sqrt{1 - 2\alpha\kappa + \alpha^2 L^2} \in (0, 1)$, and $A_\alpha^{x_1}$ is a ρ-contraction.

(I) *Discrete case* In the case of the discrete dynamical system $(DDE)_{x_1}$, the Banach fixed point theorem provides us the required result, i.e., the existence of the unique fixed point of $A_\alpha^{x_1}$ for every $x_1 \in K_1$. Moreover, every orbit (with an arbitrarily fixed initial data) in the dynamical system $(DDE)_{x_1}$ converges exponentially to the unique fixed point of $A_\alpha^{x_1}$. On account of Proposition 2.2, $R_{SV}(x_1)$ is a singleton.

(II) *Continuous case* In the case of the continuous dynamical system $(CDE)_{x_1}$, standard ODE shows that the system has a (local) solution in $[0, T)$. Assume that $T < \infty$. Let us introduce the Lyapunov function which has the form

$$h_{x_1}(t) = \frac{1}{2}\|y(t) - x_2\|^2,$$

where x_2 is the unique element of $R_{SV}(x_1)$. Consequently, for a.e. $t \in [0, T)$, we have

$$\begin{aligned}\frac{d}{dt}h_{x_1}(t) &= \left\langle y(t) - x_2, \frac{dy}{dt}\right\rangle \\ &= \langle y(t) - x_2, A_\alpha^{x_1}(P_{K_2}(y(t))) - y(t)\rangle \\ &= \langle y(t) - x_2, -y(t) + x_2 - x_2 + A_\alpha^{x_1}(P_{K_2}(y(t)))\rangle \\ &= -\|y(t) - x_2\|^2 + \langle y(t) - x_2, A_\alpha^{x_1}(P_{K_2}(y(t))) - x_2\rangle \\ &\le -\|y(t) - x_2\|^2 + \|y(t) - x_2\|\|A_\alpha^{x_1}(P_{K_2}(y(t))) - x_2\| \\ &= -\|y(t) - x_2\|^2 + \|y(t) - x_2\|\|A_\alpha^{x_1}(P_{K_2}(y(t))) - A_\alpha^{x_1}(x_2)\| \\ &\le \|y(t) - x_2\|^2[-1 + \sqrt{1 - 2\alpha\kappa + \alpha^2 L^2}].\end{aligned}$$

Therefore,

$$\frac{d}{dt}h_{x_1}(t) \le 2h_{x_1}(t)[-1 + \sqrt{1 - 2\alpha\kappa + \alpha^2 L^2}], \forall t \in [0, T).$$

Let $\rho^* = -\sqrt{1 - 2\alpha\kappa + \alpha^2 L^2} + 1$. Then, $\rho^* > 0$, and one has

$$\frac{d}{dt}[h_{x_1}(t)e^{2\rho^* t}] = \left(\frac{d}{dt}h_{x_1}(t) + 2\rho^* h_{x_1}\right)e^{2\rho^* t} \le 0.$$

This inequality implies that the function $t \mapsto h_{x_1}(t)e^{2\rho^* t}$ is non-increasing, thus $h_{x_1}(t)e^{2\rho^* t} \le h_{x_1}(0)$ for all $t \in [0, T)$. In particular, the orbit $t \mapsto y(t)$ can be extended beyond T, which contradicts the initial assumption. Therefore, $T = \infty$.

From this estimate one has for every $t \ge 0$ that

$$h_{x_1}(t) \le h_{x_1}(0)e^{-2\rho^* t}.$$

In particular, it yields that

$$\|y(t) - x_2\| \le \|y_0 - x_2\|e^{-\rho^* t},$$

which concludes the proof. □

Remark 3.2 The argument based on projective dynamical systems has been exploited in the papers of Cavazzuti, Pappalardo and Passacantando [2], Xia and Wang [13], Zhang, Qu and Xiu [14] for Nash-type equilibria. Note that the present result for Stackelberg equilibria is slightly general than those in the above works. A systematic approach to this topic can be found also in Kristály, Rădulescu and Varga [6, Chapter III] in the context of Nash equilibria on curved spaces.

Example 3.2 Fix $n \ge 2$. Let $M_n(\mathbb{R})$ be the set of symmetric $n \times n$ matrices. The standard inner product on $M_n(\mathbb{R})$ is defined as $\langle U, V\rangle = \text{tr}(UV)$. Here, $\text{tr}(Y)$ denotes the trace of $Y \in M_n(\mathbb{R})$. It is well-known that $(M_n(\mathbb{R}), \langle\cdot,\cdot\rangle)$ is an Euclidean space, the unique segment between $X, Y \in M_n(\mathbb{R})$ is $\gamma_{X,Y}(s) = (1-s)X + sY,\ s \in [0, 1]$. Let us consider the functions $f_1, f_2 : \mathbb{R} \times M_n(\mathbb{R}) \to \mathbb{R}$ defined by

$$f_1(t, X) = t^3 - t\det(X),\ \ f_2(t, X) = \text{tr}((X - tA)^2),$$

where $A \in M_n(\mathbb{R})$ is fixed, and the sets $K_1 = [0, \infty),\ K_2 = \{X \in M_n(\mathbb{R}) : \text{tr}(X) \ge 1\}$. It is clear that both sets are non-compact, and convex. Moreover, one has that for every $t \in K_1$, the function $X \mapsto \frac{\partial f_2}{\partial x_2}(t, X) = 2(X - tA)$ is 2−Lipschitz and 2−strictly monotone.

Then, on account of Theorem 3.2, $\text{card}(R_{SV}(t)) = 1$ for every $t \in K_1$, and both dynamical systems, $(DDE)_t$ and $(CDE)_t$, exponentially converge to the unique element of $R_{SV}(t)$. In this particular example, on account of (2.1), one can see that $R_{SV}(t) = P_{K_2}(tA)$ for every $t \in K_1$. In order to obtain the Stackelberg equilibrium leader set S_{SE}, it remains to minimize the function $t \mapsto f_1(t, P_{K_2}(tA)) = t^3 - t\det(P_{K_2}(tA))$ on $K_1 = [0, \infty)$.

Remark 3.3 The Stackelberg competition model described above via variational arguments can be studied also for n players ($n > 2$), they being situated in different levels of the game; for instance, n_1 player(s) in a leading position, n_2 player(s) at the second level, etc, n_k player(s) at the kth level, where $n_1 + \cdots + n_k = n$.

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