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Dynamic optimal portfolio with maximum absolute deviation model

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Abstract In this paper, a new dynamic portfolio selection model is established. Different from original consideration that risk is defined as the variance of terminal wealth, the total risk is defined as the average of the sum of maximum absolute deviation of all assets in all periods. At the same time, noticing that the risk during the period is so high that the investor may go bankrupt, a maximum risk level is given to control risk in every period. By introducing an auxiliary problem, the optimal strategy is deduced via the dynamic programming method.

Keywords Portfolio optimization · Dynamic programming · Maximum absolute deviation

Mathematics Subject Classification 91B28 · 49L20 · 90C05

1 Introduction

How to extend the standard portfolio selection model to the multiperiod case is always an attractive topic in financial research. The reason for this is that contemporary financial decision making requires models which reflect an interdependent and complex reality. Since the investment period is so long that the economic situation may change drastically, the investor should rebalance their asset allocation according to the change of the economic situation during the investment period. So after Markowitz (1959) gave the well-known single period

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model via the mean-variance methodology, multiperiod decision making has become an important subject in the practical investment as well as in the academic research on finance.

One difficulty in extending Markowitz's idea to the multiperiod or continuous-time settings is that the variance of wealth involves a term which is hard to analyze due to its nonseparability in the sense of dynamic programming. Li and Ng (2000) finally solved this problem by using the idea of embedding the problem in a tractable auxiliary problem. Later, Zhou and Li (2000); Li et al. (2002); Zhu et al. (2004) further extended their method to more general case.

However there are many researchers and traders who may not be convinced that the covariance is an appropriate risk measure. They assume that the ordinary investors consider its distribution of risk may not be symmetric. In most cases, a little loss will make one very sad, while the considerably high profit can make one very happy. This implies that the classical mean variance model may serve to be some approximation to the complex portfolio problems that all investors encounter. Hence, experts in the financial area exert all possible efforts to present some new risk models and try to meet the needs of different investors.

It is worth noticing that the use of linear programming to solve the portfolio problem was first introduced by Van Moeseke (1965). The typical economic problem of allocating scarce resources can be solved by homogeneous programming. The same applies to the portfolio problem of allocating capital across risky assets. Van Moeseke developed the truncated minimax model to solve this problem. He presented the linear programming model for the allocation of scarce resources. After then, a number of journal articles followed, in particular, see Van Moeseke (1971), Van Moeseke and Hohenbalken (1973), LeBlanc and Van Moeseke (1979).

Konno and Yamazaki (1991) present another important linear risk model: Mean Absolute Deviation (MAD) model, to construct the optimal strategy in financial market. The absolute deviation model is defined as follows

$$l_1(x) = E\left|\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|.$$

The main characteristic of this model is that the risk of a portfolio is measured by the absolute deviation of the return rate of assets instead of the variance. Much attention has been focused on this risk function because the related portfolio optimization problem can be converted into a scalar parametric linear programming problem which can be easily implemented. Simplicity and computational robustness are perceived as one of the most important advantages of the MAD model. Till now, many excellent properties of this model have been found and some of them are referred to here.

It is pointed out that the MAD model takes an opportunity to make a more specific model such as a downside risk model (see Konno 1990; Feinstein and Thapa 1993).

It is known that if the return is multivariate and normally distributed, the minimization of the MAD provides similar results as the classical Markowitz formulation, and minimization of MAD is equivalent to maximization of the expected utility under risk aversion (Rudolf et al. 1999).

Markowtiz model has been criticized for not being consistent with axiomatic models of preferences for choice because it does not depend on a relation of stochastic dominance (Whitmore and Findlay 1978; Levy 1992). In contrast, the MAD model is consistent with the second degree stochastic dominance, provided that the trade-off coefficient between risk and return is bounded by a certain constant (Ogryczak and Ruszczynski 1997).

Ogryczak and Ruszczynski (1999, 2001) proved that the most optimal solutions in efficient frontier of MAD model satisfy the MEU principle no matter how the return rate of assets are distributed. At the same time, the capital asset pricing model for the l_1 risk model was derived by Konno and Yamazaki (1991). In their paper, the risk function was assumed to be differentiable at the market portfolio. Without imposing differentiability on the l_1 risk function, equilibrium relations were given by Konno and Shirakawa (1994).

Based on MAD, other new linear risk-control models are presented recently. For example, Cai et al. (2000) introduced the maximum absolute deviation risk model l_∞ as:

$$l_\infty(x) = \max_{1 \leqslant j \leqslant n} E \left| R_j x_j - E(R_j) x_j \right|.$$

In this model, the investor is assumed to minimize the maximum of individual risk. The explicit analytical solution for the model is presented and the entire efficient frontier is also plotted. The author points out that such a risk model is very conservative and it does not explicitly involve the covariance of the asset returns.

Later, the alternative H_∞^T was proposed by Teo and Yang (2001). The characteristic of these three models (l_1, l_∞, H_∞^T) is that the portfolio problems which are set up with them can be finally transformed into a linear programming problem. And this means that we can use linear programming method as a strong tool to solve finance problem (see Yu et al. 2006). Such kind of method is easier for investor to master and is quickly solved by computer.

Although the application of MAD or those linear models based on it is successful in portfolio theory, there is little literature about extending these linear models to multiperiod case (see Yu et al. 2005).The difficulty is how to use these linear models to control risk of the portfolio in mulitiperiod case.

Recently, Yu et al. (2010) constructed the dynamic portfolio model with l_∞ model. We employ a risk parameter as the maximum risk level which the investor would like to bear. By using the dynamic programming method, the closed form of solution is derived.

In this paper, we continue this topic and set up a new mulitperiod portfolio selection. Different with the classical MV model, we employ the l_∞ function to control the risk in every period. We assume that the investor wants to maximize the total wealth at the end of investment period and minimize the risk which is defined as the average of the total risk in all periods. Actually, investors are always considering the total risk during the whole investment instead of the final Tth period. So we extend the model in Yu et al. (2010) to more wider case. Moreover, considering the case of risk during the period becoming so high, the investor can not finish the whole investment period, we employ a parameter to control the risk in every period. That is, the risk in every period can not be above the given risk level. Such a consideration makes the problem more complicated, but at the same time, is more reasonable for the investor. By using dynamic programming method, we deduce the closed form solution. Our result shows that the multiperiod model with absolute deviation can be finally formulated as a linear programming problem which we can solve analytically. Our model is another example of using the linear programming method as a strong tool to solve finance problems.

The organization of this paper is as this: In Sect. 2, some important notations are introduced and the basic model is set up for the financial market. In Sect. 3, the process of solving the optimal strategy for the investment model is presented. We propose an auxiliary problem and then prove how to obtain the final optimal strategy via the solution to the auxiliary problem. In Sect. 4, the algorithm is presented. The conclusion is given in Sect. 5. The proof of main theorems can be found in Appendix.

2 Notations and model

We consider a capital market with n risky assets $S_j,\ j = 1, \dots, n$, whose return rate is random. An investor is assumed to allocate his initial wealth denoted by W_0 among the n risky assets at the beginning of the 1st period and get the final wealth at the end of the Tth period. It is a dynamic investment selection, i.e. the wealth can be reallocated among the n risky assets at the beginning of each following $T-1$ consecutive time periods.

Denote by $R_t = (R_{t1}, \dots, R_{tn})'$ the vector of return rate of the risky securities at time period t, where R_{tj} is the random return rate of asset S_j at the tth stage. It is assumed that vectors $R_t,\ t = 1, \dots, T$, are statistically independent and has a known mean value $r_t = E(R_t) = (r_{t1}, \dots, r_{tn})'$.

Denote by $\boldsymbol{x}_t = (x_{t1}, \dots, x_{tn})'$, where x_{tj} is the amount invested in the asset S_j at the beginning of the tth stage. It is assumed that short selling is not allowed:

$$x_{tj} \geq 0, t = 1, \dots, T, j = 1, \dots, n \tag{2.1}$$

Denote by V_t the total wealth the investor obtains at the end of the tth period. Let $V_0 = W_0$. Clearly,

$$V_t = V_{t-1} + R_t' x_t,\ t = 1, \dots, T. \tag{2.2}$$

It is assumed that the whole investment is a self-financing process. The investor will not increase the money or put aside some money during the whole period, i.e., the amount of money allocated to every asset in the tth period is equal to the total wealth at the end of the $t-1$th period. That is,

$$\sum_{j=1}^{n} x_{tj} = V_{t-1},\ t = 1, \dots, T. \tag{2.3}$$

In the following discussion, we focus on how to control the risk in every period. We employ the l_∞ risk function to measure the risk in tth period which is denoted by $w_t(x_t)$, then we obtain $w_t(x_t) = \max_{1\leq j\leq n} E(|R_{tj}x_{tj} - r_{tj}x_{tj}|),\ t = 1, \dots, T$.

Denote by w_t' the total risk at the end of t periods, which is defined as follows:

$$w_0' = 0,\ w_t' = w_{t-1}' + \max_{1\leq j\leq n} E(|R_{tj}x_{tj} - r_{tj}x_{tj}|), t = 1, \dots, T. \tag{2.4}$$

We employ a risk parameter $\varepsilon_t,\ t = 1, \dots, T$, to control the risk in tth period. We assume that the risk in period t can not be above $\varepsilon_t E(V_{t-1})$, i.e.,

$$\max_{1\leq j\leq n} E(|R_{tj}x_{tj} - r_{tj}x_{tj}|) \leq \varepsilon_t E(V_{t-1}),\ t = 1, \dots, T. \tag{2.5}$$

In the following discussion, we will give some analysis of (2.5).

A bankruptcy occurs when an investor's total wealth falls bellow a predefined "disaster" level in any intermediate or the final time period. It is assumed that when an investor is in bankruptcy, he/she is not able to pursue further investment due to his/her high liability and low credit. We denote by b_t the disaster level at period t and label the event of a bankruptcy as BR_t at period t.

The probability of BR_t is

$$P(BR_t) = P(V_t \leqslant b_t,\ V_i > b_i,\ i = 1, \dots, t-1),\ t = 1, \dots, T$$

Now we deduce how to control $P(BR_t)$. It is reasonable to assume that b_t is less than $E(V_t)$.

Let us recall the definition of l_1 and l_∞,

$$w(x) = E\left(\left|\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} r_j x_j\right|\right), \quad w_\infty(x) = \max_j E(|R_j x_j - r_j xj|).$$

The following two propositions have been shown in Yu et al. (2010). Here we do not repeat it.

Proposition 1 $w(x) \leqslant n w_\infty(x)$.

Proposition 2 *If* $b_t < E(V_t)$ *and* $\dfrac{E(|V_t - E(V_t)|)}{E(V_t) - b_t} \leqslant \alpha_t$*, then* $P(BR_t) \leqslant \alpha_t$.

If V_l is already known, $l = 1, \ldots, t-1$, then

$$E(|V_t - E(V_t)|) = E\left(\left|\sum_{j=1}^{n} R_{tj} x_{tj} - \sum_{j=1}^{n} r_{tj} x_{tj}\right|\right) \leqslant n \max_j E(|R_{tj} x_{tj} - r_{tj} x_{tj}|),$$

and

$$\frac{E(|V_t - E(V_t)|)}{E(V_t) - b_t} \leqslant \frac{n \max_j E(|R_{tj} x_{tj} - r_{tj} x_{tj}|)}{E(V_t) - b_t} \equiv A_t.$$

Hence, if we give a bound α_t for A_t, then we can control the risk in period t.

In this paper, we assume that the bankruptcy will happen when $b_t = 0$. Notice that $E(V_t) \geq E(V_{t-1})$; hence,

$$P(V_t \leq 0, V_i > 0, i = 1, \ldots, t-1) \leq \frac{n \max_j E|R_{tj} x_{tj} - r_{tj} x_{tj}|}{E(V_{t-1})} \leq \alpha_t := n\varepsilon_t,$$

i.e., $\max_{1 \leq j \leq n} E|R_{tj} x_{tj} - r_{tj} x_{tj}| \leq \varepsilon_t E(V_{t-1})$. Thus, the risk in period t is finally controlled.

Based on the above discussion, we set up the dynamic portfolio model in the following way. We assume that the investor is risk averse. He/she wants to maximize the terminal wealth in the final period, i.e., $E(V_T)$. On the other hand, he/she wants to minimize the average value of total risk in T periods, i.e. $\frac{1}{T} w'_T$. Thus, our portfolio selection problem can be formulated as the following programming problem, which is denoted by P_1:

$$P_1 \begin{cases} \min \frac{\lambda}{T} w'_T - (1-\lambda) E(V_T) \\ \text{s.t. } (2.1) - (2.5) \end{cases}$$

Here $\lambda \in (0, 1)$ can be considered as the risk preference of the investor. The greater λ is, the more conservative the investor is.

3 Optimal strategy for the portfolio model

In this section, we will derive the analytical solution to P_1 via dynamic programming method.

Define $z_t \in R$, $w_t^1 \in R$, $t = 1, \ldots, T$. Let $w_0^1 = 0$, $w_T^1 = \sum_{t=1}^{T} z_t$. First, let's introduce the following problem.

Define P_2 as follows:

$$\begin{cases} \min\ \frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_T) \\ \text{s.t.}\ \ V_t = V_{t-1} + R_t' x_t \\ \qquad w_t^1 = w_{t-1}^1 + z_t,\ t = 1,\ldots,T \\ \qquad E|R_{tj}x_{tj} - r_{tj}x_{tj}| \le z_t,\ j = 1,\ldots,n, t = 1,\ldots,T \\ \qquad E|R_{tj}x_{tj} - r_{tj}x_{tj}| \le \varepsilon_t E(V_{t-1}),\ j = 1,\ldots,n, t = 1,\ldots,T \\ \qquad \sum_{j=1}^{n} x_{tj} = V_{t-1},\ t = 1,\ldots,T \\ \qquad x_{tj} \geqslant 0,\ j = 1,\ldots,n, t = 1,\ldots,T \end{cases}$$

where $V_0 = W_0$.

The following theorem sets up the relationship between P_1 and P_2. We have already proved this result (see Yu et al. 2005).

Theorem 1 (1) *If $X = (x_1,\ldots,x_T)'$ is an optimal solution to P_1, then (X, Z) is an optimal solution to P_2, where $Z = (z_1,\ldots,z_T)'$, $z_t = \max_{1\le j\le n} E|R_{tj}x_{tj} - r_{tj}x_{tj}|$.*
(2) *If (X, Z) is an optimal solution to P_2, then X is an optimal solution to P_1, where $X = (x_1,\ldots,x_T)'$.*

Now we use the dynamic programming method to solve the problem P_2. Denote by

$$f_T(V_T, w_T^1) = \frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_T),$$

and define the set

$$B_t = \{(\boldsymbol{x}, z) \in (\boldsymbol{R}^n, \boldsymbol{R}^+) \left| q_{tj}x_j \le z, q_{tj}x_j \le \varepsilon_t E(V_{t-1}), \sum_{j=1}^{n} x_j = E(V_{t-1}) \right. ,$$
$$x_j \ge 0, j = 1,\ldots,n\}$$

Define

$$f_{T-1}(V_{T-1}, w_{T-1}^1) = \min_{(x_T, z_T)\in B_T} [f_T(V_T, w_T)|(V_{T-1}, w_{T-1}^1)].$$

First we consider the $T-1$th period by assuming that (V_{T-1}, w_{T-1}^1) is known. Then

$$f_{T-1}(V_{T-1}, w_{T-1}^1) = \min_{(x_T, z_T)\in B_T} \left\{ \frac{\lambda}{T}(w_{T-1}^1 + z_T) - (1-\lambda)\left(E(V_{T-1}) + \sum_{j=1}^{n} r_{Tj}x_{Tj} \right) \right\}$$

We will solve the following problem denoted by P_T

$$P_T \begin{cases} \min \left\{ \frac{\lambda}{T}(w_{T-1}^1 + z_T) - (1-\lambda)\left(E(V_{T-1}) + \sum_{j=1}^{n} r_{Tj}x_{Tj} \right) \right\} \\ \text{s.t}\ \ q_{Tj}x_{Tj} \le z_T,\ j = 1,\ldots,n \\ \qquad q_{Tj}x_{Tj} \le \varepsilon_T E(V_{T-1}),\ j = 1,\ldots,n \\ \qquad \sum_{j=1}^{n} x_{Tj} = V_{T-1} \\ \qquad x_{Tj} \ge 0,\ j = 1,\ldots,n \end{cases}$$

where $q_{Tj} = E|R_{Tj} - r_{Tj}|$, $j = 1,\ldots,n$.

To solve this problem, first let us introduce an auxiliary problem P_T' as follows:

$$P_T' \begin{cases} \min \left\{ \frac{\lambda}{T}(w_{T-1}^1 + z_T) - (1-\lambda)\left(E(V_{T-1}) + \sum_{j=1}^n r_{Tj}x_{Tj} \right) \right\} \\ \text{s.t} \quad q_{Tj}x_{Tj} \le z_T,\ j = 1,\ldots,n \\ \qquad \sum_{j=1}^n x_{Tj} = V_{T-1} \\ \qquad x_{Tj} \ge 0,\ j = 1,\ldots,n \end{cases}$$

The difference between P_T and P_T' is that constraints $q_{Tj}x_{Tj} \le \varepsilon_T E(V_{T-1})$ is omitted in P_T', for $j = 1,\ldots,n$. The optimal solution to this problem has been obtained in Cai et al. (2000). We denote it as $(\boldsymbol{x}_T^*, z_T^*)$, where $\boldsymbol{x}_T^* = (x_{T1}^*,\ldots,x_{Tn}^*)'$. Then we will prove that the optimal solution to P_T denoted by $(\boldsymbol{x}_T^{**}, z_T^{**})$ can be obtained by $(\boldsymbol{x}_T^*, z_T^*)$ under some assumptions.

Lemma 3 *The solution to P_T' is as follows:*

$$x_{Tj}^* = \begin{cases} \frac{V_{T-1}}{q_{Tj}}\left(\sum_{i\in A_T(\lambda)} \frac{1}{q_{Ti}}\right)^{-1}, & j \in A_T(\lambda) \\ 0, & j \notin A_T(\lambda) \end{cases}, \quad z_T^* = V_{T-1}\left(\sum_{i\in A_T(\lambda)} \frac{1}{q_{Ti}}\right)^{-1} \tag{3.1}$$

where $A_T(\lambda)$ is determined by:

If there exists an integer $p_T \in [0, n-2]$ such that

$$\sum_{i=n-p_T+1}^{n} \frac{r_{Ti} - r_{Tn-p_T}}{q_{Ti}} < \frac{\lambda}{T(1-\lambda)} \text{ and } \sum_{i=n-p_T}^{n} \frac{r_{Ti} - r_{Tn-p_T-1}}{q_{Ti}} \geqslant \frac{\lambda}{T(1-\lambda)} \tag{3.2}$$

then

$$A_T(\lambda) = \{n, n-1, \ldots, n-p_T\},$$

and otherwise, if the above condition is not satisfied by any integer $p_T \in [0, n-2]$, then

$$A_T(\lambda) = \{n, n-1, \ldots, 1\}.$$

Here, in order to make the following description simplified, we denote $A_T(\lambda) = \{1,\ldots,k_T\}$, $k_T \le n$. It is necessary to notice that the rank in $A_T(\lambda)$ now is converse to the original one. That is, for $i, j \in \{n, n-1,\ldots,n-p_T\}$ or $\{n, n-1,\ldots,1\}$, with $i < j$, we have $r_i \le r_j$. For $i, j \in \{1,\ldots,k_T\}$, with $i < j$, we have $r_i \ge r_j$.

The following theorem give the optimal solution to P_T.

Theorem 2 *The optimal solution to P_T is as follows:*

(1) *If $z_T^* \le \varepsilon_T V_{T-1}$, then $(\boldsymbol{x}_T^*, z_T^*)$ is also an optimal solution to P_T;*
(2) *If $z_T^* > \varepsilon_T V_{T-1}$, then the optimal solution to P_T is as follows:*
If there exists $n \ge l_T > k_T$, such that

$$\frac{\sum_{j=1}^{k_T} \frac{1}{q_{Tj}}}{\sum_{j=1}^{l_T-1} \frac{1}{q_{Tj}}} > \frac{\varepsilon_T V_{T-1}}{z_T^*} \ge \frac{\sum_{j=1}^{k_T} \frac{1}{q_{Tj}}}{\sum_{j=1}^{l_T} \frac{1}{q_{Tj}}}, \tag{3.3}$$

then the optimal solution to P_T*, denoted by* $(\boldsymbol{x}_T^{**}, z_T^{**})$*, where* $\boldsymbol{x}_T^{**} = (x_{T1}^{**}, \ldots, x_{Tn}^{**})'$ *is as follows:*

$$z_T^{**} = \varepsilon_T V_{T-1},\ \ x_{Tj}^{**} = \begin{cases} \frac{\varepsilon_T}{q_{Tj}} V_{T-1}, & j = 1, \ldots, l_T - 1 \\ z_T^* \sum_{j=1}^{k_T} \frac{1}{q_{Tj}} - \varepsilon_T V_{T-1} \sum_{j=1}^{l_T-1} \frac{1}{q_{Tj}}, & j = l_T \\ 0 & j = l_T + 1, \ldots, n \end{cases}$$

The proof of this theorem can be found in Appendix.

Now, we substitute $(\boldsymbol{x}_T^{**}, z_T^{**})$ to f_{T-1}.

If $z_T^* \leq \varepsilon_T V_{T-1}$, then by Lemma 3, we have

$$f_{T-1}^*(V_{T-1}, w_{T-1}^1) = \frac{\lambda}{T} w_{T-1}^1 - c_T E(V_{T-1}) \tag{3.4}$$

where

$$c_T = (1-\lambda)(1 + a_T b_T) - \frac{\lambda}{T} a_T, a_T = \left(\sum_{j=1}^{k_T} \frac{1}{q_{Tj}} \right)^{-1}, b_T = \sum_{j=1}^{k_T} \frac{r_{Tj}}{q_{Tj}} \tag{3.5}$$

If $z_T^* > \varepsilon_T V_{T-1}$, then

$$f_{T-1}^*(V_{T-1}, w_{T-1}^1) = \frac{\lambda}{T} w_{T-1}^1 - \alpha_T E(V_{T-1}) \tag{3.6}$$

where

$$\alpha_T = -\frac{\lambda \varepsilon_T}{T} + (1-\lambda) \left[1 + r_{Tl_T} + \varepsilon_T \sum_{j=1}^{l_T-1} \frac{r_{Tj} - r_{Tl_T}}{q_{Tj}} \right] \tag{3.7}$$

We summarize the two cases as the following:

$$f_{T-1}^*(V_{T-1}, w_{T-1}^1) = \frac{\lambda}{T} w_{T-1}^1 - \beta_T E(V_{T-1}) \tag{3.8}$$

where

$$\beta_T = \begin{cases} c_T & \left(\sum_{l=1}^{k_T} \frac{1}{q_{Tl}} \right)^{-1} \leqslant \varepsilon_T \\ \alpha_T & \left(\sum_{l=1}^{k_T} \frac{1}{q_{Tl}} \right)^{-1} > \varepsilon_T \end{cases} \tag{3.9}$$

Suppose that at any stage t, $t \in \{1, \ldots, T-1\}$,

$$f_t^*(V_t, w_t^1) = \frac{\lambda}{T} w_t^1 - \beta_{t+1} E(V_t) \tag{3.10}$$

where

$$\beta_{t+1} = \begin{cases} c_{t+1} & \left(\sum_{l=1}^{k_{t+1}} \frac{1}{q_{t+1l}} \right)^{-1} \leqslant \varepsilon_{t+1} \\ \alpha_{t+1} & \left(\sum_{l=1}^{k_{t+1}} \frac{1}{q_{t+1l}} \right)^{-1} > \varepsilon_{t+1} \end{cases}$$

and

$$a_{t+1} = \left(\sum_{j=1}^{k_{t+1}} \frac{1}{q_{t+1j}}\right)^{-1}, \; b_{t+1} = \sum_{j=1}^{k_{t+1}} \frac{r_{t+1j}}{q_{t+1j}}, \; c_{t+1} = \beta_{t+2}(1 + a_{t+1}b_{t+1}) - \frac{\lambda}{T}a_{t+1},$$

$$\alpha_{t+1} = \begin{cases} -\frac{\lambda\varepsilon_{t+1}}{T} + \beta_{t+2}\left[1 + r_{t+1l_{t+1}} + \varepsilon_{t+1}\sum_{j=1}^{l_{t+1}-1}\frac{r_{t+1j}-r_{t+1l_{t+1}}}{q_{t+1j}}\right] & \text{if } \beta_{t+2} > 0 \\ -\frac{\lambda\varepsilon_{t+1}}{T} + \beta_{t+2}\left[1 + r_{t+1l_{t+1}} + \varepsilon_{t+1}\sum_{j=l_{t+1}+1}^{n}\frac{r_{t+1j}-r_{t+1l_{t+1}}}{q_{t+1j}}\right] & \text{if } \beta_{t+2} < 0 \\ -\frac{\lambda}{T}\sum_{j=1}^{n}\left(\frac{1}{q_{t+1j}}\right)^{-1} & \text{if } \beta_{t+2} = 0 \end{cases}$$

with boundary condition (3.9). Notice that this induction assumption holds for stage $T-1$. We will show

$$f_{t-1}^*(V_{t-1}, w_{t-1}^1) = \frac{\lambda}{T}w_{t-1}^1 - \beta_t E(V_{t-1})$$

At stage $t-1$, the optimization for a given (V_{t-1}, w_{t-1}^1) is as follows:

$$\begin{aligned} f_{t-1}(V_{t-1}, w_{t-1}^1) &= \min_{(x_t, z_t)\in B_t} E[f_t(V_t, w_t^1)|(V_{t-1}, w_{t-1}^1)] \\ &= \min_{(x_t, z_t)\in B_t}\left\{\frac{\lambda}{T}(w_{t-1}^1 + z_t) - \beta_{t+1}\left(E(V_{t-1}) + \sum_{j=1}^{n} r_{tj}x_{tj}\right)\right\} \end{aligned}$$

We need to get the optimal solution to the following problem:

$$(P_t) \quad \begin{cases} \min \; \frac{\lambda}{T}(w_{t-1}^1 + z_t) - \beta_{t+1}\left(E(V_{t-1}) + \sum_{j=1}^{n} r_{tj}x_{tj}\right) \\ \text{s.t.} \quad q_{tj}x_{tj} \le z_t, \; j = 1, \ldots, n \\ \qquad q_{tj}x_{tj} \le \varepsilon_t E(V_{t-1}), \; j = 1, \ldots, n \\ \qquad x_{tj} \ge 0, \, j = 1, \ldots, n \\ \qquad \sum_{j=1}^{n} x_{tj} = V_{t-1} \end{cases}$$

where $q_{tj} = E|R_{tj} - r_{tj}|$, $j = 1, \ldots, n$.

The method of solving this problem is similar to that of P_T. First, we employ the following auxiliary problem:

$$(P_t') \quad \begin{cases} \min \; \frac{\lambda}{T}(w_{t-1}^1 + z_t) - \beta_{t+1}\left(E(V_{t-1}) + \sum_{j=1}^{n} r_{tj}x_{tj}\right) \\ \text{s.t.} \quad q_{tj}x_{tj} \le z_t, \; j = 1, \ldots, n \\ \qquad x_{tj} \ge 0, \, j = 1, \ldots, n \\ \qquad \sum_{j=1}^{n} x_{tj} = V_{t-1} \end{cases}$$

The following lemma has already been proved in Yu et al. (2005).

Lemma 4 *The optimal solution to P_t' is as follows:*

$$x_{tj}^* = \begin{cases} V_{t-1}\left(\sum_{l\in A_t(\lambda)} \frac{1}{q_{tl}}\right)^{-1} \cdot \frac{1}{q_{tj}}, & j \in A_t(\lambda) \\ 0, & j \notin A_t(\lambda) \end{cases}$$

$$z_t^* = V_{t-1}\left(\sum_{l\in A_t(\lambda)} \frac{1}{q_{tl}}\right)^{-1}$$

where $A_t(\lambda)$ is determined by:

(1) *If $\beta_{t+1} = 0$, then $A_t(\lambda) = \{1, \ldots, n\}$.*
(2) *If $\beta_{t+1} > 0$, if there exists an integer $k \in [0, n-2]$ such that*

$$\sum_{i=n-k+1}^{n} \frac{r_{ti} - r_{tn-k}}{q_{ti}} < \frac{\lambda}{T\beta_{t+1}} \text{ and } \sum_{i=n-k}^{n} \frac{r_{ti} - r_{tn-k-1}}{q_{ti}} \geqslant \frac{\lambda}{T\beta_{t+1}}$$

then

$$A_t(\lambda) = \{n, n-1, \ldots, n-k\},$$

and otherwise, if the above condition is not satisfied by any integer $k \in [0, n-2]$, then

$$A_t(\lambda) = \{n, n-1, \ldots, 1\}.$$

(3) *If $\beta_{t+1} < 0$. If there exists an integer $k \in [0, n-2]$ such that*

$$\sum_{i=1}^{n-k-2} \frac{r_{ti} - r_{tn-k-1}}{q_{ti}} > \frac{\lambda}{T\beta_{t+1}} \text{ and } \sum_{i=1}^{n-k-1} \frac{r_{ti} - r_{tn-k-1}}{q_{ti}} \leqslant \frac{\lambda}{T\beta_{t+1}}$$

then

$$A_t(\lambda) = \{1, 2, \ldots, n-k-1\},$$

and otherwise, if the above condition is not satisfied by any integer $k \in [0, n-2]$, then

$$A_t(\lambda) = \{1, 2, \ldots, n\}.$$

Here, in order to make the following description simplified, we denote $A_t(\lambda) = \{1, \ldots, k_t\}$, $k_t \le n$. For $i, j \in \{1, \ldots, k_t\}$, if $i \le j$, $r_i \ge r_j$.

Theorem 3 *The optimal solution to P_t denoted by $(\boldsymbol{x}_t^{**}, z_t^{**})$ is as follows:*

(1) *If $z_t^* \le \varepsilon_t V_{t-1}$, then $(\boldsymbol{x}_t^*, z_t^*)$ is also an optimal solution to P_t;*
(2) *If $z_t^* > \varepsilon_t V_{t-1}$, then the optimal solution to P_t is as follows:*

Case 1, if $\beta_{t+1} > 0$, then if there exists $n \ge l_t > k_t$, such that

$$\frac{\sum_{j=1}^{k_t} \frac{1}{q_j}}{\sum_{j=1}^{l_t-1} \frac{1}{q_{tj}}} > \frac{\varepsilon_t}{z_t^*} V_{t-1} \ge \frac{\sum_{j=1}^{k_t} \frac{1}{q_{tj}}}{\sum_{j=1}^{l_t} \frac{1}{q_{tj}}}$$

then the optimal solution to P_t is as follows:

$$z_t^{**} = \varepsilon_t V_{t-1},$$
$$x_{tj}^{**} = \begin{cases} \frac{\varepsilon_t}{q_{tj}} V_{t-1}, & j = 1, \ldots, l_t - 1 \\ z_t^* \sum_{j=1}^{k_t} \frac{1}{q_{tj}} - \varepsilon_t V_{t-1} \sum_{j=1}^{l_t - 1} \frac{1}{q_{tj}}, & j = l_t \\ 0 & j = l_t + 1, \ldots, n \end{cases}$$

Case 2, if $\beta_{t+1} < 0$, then if there exists $n \geq l_t > k_t$, such that

$$\frac{\sum_{j=1}^{k_t} \frac{1}{q_{tj}}}{\sum_{j=l_t+1}^{n} \frac{1}{q_{tj}}} > \frac{\varepsilon_t}{z_t^*} V_{t-1} \geq \frac{\sum_{j=1}^{k_t} \frac{1}{q_{tj}}}{\sum_{j=l_t}^{n} \frac{1}{q_{tj}}}$$

then the optimal solution to P_t is as follows:

$$z_t^{**} = \varepsilon_t V_{t-1},$$
$$x_{tj}^{**} = \begin{cases} \frac{\varepsilon_t}{q_{tj}} V_{t-1}, & j = l_t + 1, \ldots, n \\ z_t^* \sum_{j=1}^{k_t} \frac{1}{q_{tj}} - \varepsilon_t V_{t-1} \sum_{j=l_t+1}^{n} \frac{1}{q_{tj}}, & j = l_t \\ 0 & j = 1, \ldots, l_t - 1 \end{cases}$$

Case 3 If $\beta_{t+1} = 0$, then if P_t has the optimal solution, then $x_{tj}^{**} = \frac{1}{q_{tj}} (\sum_{j=1}^{n} \frac{1}{q_{tj}})^{-1}$, $j = 1, \ldots, n$.

Substitute the result in Theorem 3 back to f_{t-1}, we have:
If $z_t^* \leq \varepsilon_t V_{t-1}$, then

$$f_{t-1}^*(V_{t-1}, w_{t-1}^1) = \frac{\lambda}{T} w_{t-1}^1 - c_t E(V_{t-1})$$

where

$$c_t = \beta_{t+1}(1 + a_t b_t) - \frac{\lambda}{T} a_t, a_t = \left(\sum_{j=1}^{k_t} \frac{1}{q_{tj}} \right)^{-1} . b_t = \sum_{j=1}^{k_t} \frac{r_{tj}}{q_{tj}}$$

If $z_t^* > \varepsilon_t V_{t-1}$, then

$$f_{t-1}^*(V_{t-1}, w_{t-1}^1) = \frac{\lambda}{T} w_{t-1}^1 - \alpha_t E(V_{t-1})$$

where

$$\alpha_t = \begin{cases} -\frac{\lambda \varepsilon_t}{T} + \beta_{t+1} \left[1 + r_{tl_t} + \varepsilon_t \sum_{j=1}^{l_t - 1} \frac{r_{tj} - r_{tl_t}}{q_{tj}} \right] & \text{if } \beta_{t+1} > 0 \\ -\frac{\lambda \varepsilon_t}{T} + \beta_{t+1} \left[1 + r_{tl_t} + \varepsilon_t \sum_{j=l_t+1}^{n} \frac{r_{tj} - r_{tl_t}}{q_{tj}} \right] & \text{if } \beta_{t+1} < 0 \\ -\frac{\lambda}{T} \left(\sum_{j=1}^{n} \frac{1}{q_{tj}} \right)^{-1} & \text{if } \beta_{t+1} = 0 \end{cases}$$

We can summarize the above two cases into the following formula:

$$f_{t-1}(V_{t-1}, w_{t-1}^1) = \frac{\lambda}{T} w_{t-1}^1 - \beta_t E(V_{t-1})$$

where

$$\beta_t = \begin{cases} c_t \left(\sum_{l=1}^{k_t} \frac{1}{q_{tl}} \right)^{-1} & \leqslant \varepsilon_t \\ \alpha_t \left(\sum_{l=1}^{k_t} \frac{1}{q_{tl}} \right)^{-1} & > \varepsilon_t \end{cases}$$

Based on the above result, we can get the expected total wealth at the end of the T periods:

$$E(V_T) = (1 + \xi_T)(1 + \xi_{T-1}) \ldots (1 + \xi_1) V_0$$

where for $t = 1, \ldots, T$,

$$\xi_t = \begin{cases} a_t b_t \left(\sum_{l=1}^{k_t} \frac{1}{q_{tl}} \right)^{-1} & \leqslant \varepsilon_t \\ \zeta_t \quad \left(\sum_{l=1}^{k_t} \frac{1}{q_{tl}} \right)^{-1} & > \varepsilon_t \end{cases}$$

where

$$\zeta_t = \begin{cases} r_{tl_t} + \varepsilon_t \sum_{j=1}^{l_t-1} \frac{r_{tj} - r_{tl_t}}{q_{tj}} & \beta_t > 0 \\ r_{tl_t} + \varepsilon_t \sum_{j=l_t+1}^{n} \frac{r_{tj} - r_{tl_t}}{q_{tj}} & \beta_t < 0 \\ \sum_{j=1}^{n} \frac{r_{tj}}{q_{tj}} \left(\sum_{j=1}^{n} \frac{1}{q_{tj}} \right)^{-1} & \beta_t = 0 \end{cases}$$

4 Algorithm

In this section, we give the algorithm of solving the multi-period model and one example is presented at the end of this section.

Define

$$B_t^1 = \left\{ (\boldsymbol{x}, z) \in (\boldsymbol{R}^n, \boldsymbol{R}^+) \middle| q_{tj} x_j \le z, \sum_{j=1}^{n} x_j = E(V_{t-1}), x_j \ge 0, j = 1, \ldots, n \right\}$$

$$B_t^2 = \left\{ \boldsymbol{x} \in \boldsymbol{R}^n \middle| q_{tj} x_j \le \varepsilon_t E(V_{t-1}), \sum_{j=1}^{n} x_j = E(V_{t-1}), x_j \ge 0, j = 1, \ldots, n \right\}$$

The algorithm is presented as follows. Many types of software can be used to solve this problem. In our paper, we use Matlab 7.0.

Algorithm
Step 0: Given V_0, r_{ti}, q_{ti}. $i = 1, \ldots, n, t = 1, \ldots, T$. Let $E(V_0) = V_0$, $w_0 = 0$.

Step 1: $t = 1$.

Step 2: Given $\varepsilon_t > 0$.

Step 3: solve the following problem $P(t)$,

$$P(t) \qquad \min_{(x_t, z_t) \in B_t^1} \left\{ \frac{\lambda}{T}(w_{t-1} + z_t) - (1 - \lambda) \left(E(V_{t-1}) + \sum_{j=1}^{n} r_{tj} x_{tj} \right) \right\}$$

The optimal solution is denoted by $(\boldsymbol{x}_t^*, z_t^*)$.

Step 4: Let $E(V_t) = E(V_{t-1}) + \sum_{j=1}^{n} r_{tj} x_{tj}^*$, $w_t = w_{t-1} + z_t^*$.

Step 5: If $z_t^* < \varepsilon_t E(V_t)$, then $\boldsymbol{x}_t^{**} = \boldsymbol{x}_t^*$, $z_t^{**} = z_t^*$. If $z_t^* \geq \varepsilon_t E(V_t)$, Goto Step 7.

Step 6: let $t = t + 1$. If $t \leq T$, then goto Step 2. If $t = T + 1$, goto Step 10.

Step 7: Solve the following problem:

$$P'(t) \qquad \min_{x_t \in B_t^2} \left\{ \frac{\lambda}{T} w_{t-1} - (1 - \lambda) \left(E(V_{t-1}) + \sum_{j=1}^{n} r_{tj} x_{tj} \right) \right\}$$

Denote the solution to $P'(t)$ is $(\widehat{\boldsymbol{x}}_t^*, \widehat{z}_t^*)$. Then $\boldsymbol{x}_t^{**} = \widehat{\boldsymbol{x}}_t^*$, $z_t^{**} = \widehat{z}_t^*$.

Step 8: Let $E(V_t) = E(V_{t-1}) + \sum_{j=1}^{n} r_{tj} \widehat{\boldsymbol{x}}_{tj}^*$, $w_t = w_{t-1} + \widehat{z}_{tj}^*$. And $t = t + 1$.
Step 9: If $t \leq T$, then goto Step 2.
Step 10: stop: the optimal solution is $(\boldsymbol{x}_t^{**}, z_t^{**})$.

Finally, we give an example to demonstrate the adoption of the above model.

Example 1 We choose 7 assets in the Tokyo Stock market. These are T4523(eisai), T4063(Shin-Etsu Chemical),T4503(astellas), T4901(Fuji film),T4452(Kao), T4543(terumo), T4502(takeda). We consider the historical data of these seven companies' stock from 1995.1 to 1999.7. We use the data from 1995.1 to 1998.12 to calculate the expected return rate for these 7 stocks. The expected return rate for these seven assets are:

$$r_t = (0.0093, 0.0141, 0.0155, 0.0190, 0.0213, 0.0308, 0.0337).$$

We assume that there are two investors who start their investment from the beginning of 1999.1 and end in 1999.7. We assume that these two investors employ the model mentioned above to find their optimal strategies. For simplicity, we assume that $\lambda = 1/2$ for both two investors. It is assumed that every month is one period. We assume that these two investors have different risk preference in each period: for $t = 1, \ldots, 7$,

$$\varepsilon_A = (0.0045, 0.005, 0.0015, 0.003, 0.005, 0.01, 0.003).$$
$$\varepsilon_B = (0.005, 0.0055, 0.0017, 0.0032, 0.01, 0.1, 0.006).$$

Table 1 Optimal strategy for investor A

Month	S_1	S_2	S_3	S_4	S_5	S_6	S_7
1999.1	0	0.0205	0.2278	0.5562	0.0584	0.0835	0.0536
1999.2	0.0457	0.0273	0.6749	0.0943	0.0404	0.0798	0.0572
1999.3	0.1883	0.0412	0.0138	0.0310	0.0245	0.0257	0.7224
1999.4	0.0660	0.0439	0.0552	0.0950	0.3792	0.0371	0.3994
1999.5	0.1995	0.4396	0.0457	0.0977	0.0653	0.1835	0.0800
1999.6	0.3364	0.2040	0.1132	0.0970	0.2421	0.0668	0.0721
1999.7	0	0.1847	0.5105	0.0312	0.0639	0.0372	0.3236

Table 2 Optimal strategy for investor B

Month	S_1	S_2	S_3	S_4	S_5	S_6	S_7
1999.1	0	0	0.1648	0.6180	0.0649	0.0928	0.0595
1999.2	0.0504	0.0301	0.7427	0.0017	0.0445	0.0879	0.0630
1999.3	0.0740	0.0468	0.0157	0.0353	0.0278	0.0293	0.8214
1999.4	0.0708	0.0470	0.0592	0.0298	0.4067	0.0398	0.4284
1999.5	0	0.1681	0.0922	0.1969	0.1316	0.3701	0.1613
1999.6	0	0	0	0.2328	0.5812	0.1604	0.1732
1999.7	0	0	0.2442	0.0637	0.1305	0.0760	0.6609

Then by using the model presented above, we find the expected wealth is $E(V_A) = 1.1758$ and $E(V_B) = 1.2080$. It is easy to see that the investor A got less final expected wealth than investor B because he/she requires lower risk level in each period. The optimal strategies are given in Tables 1 and 2, respectively.

5 Conclusion

In this paper, we present a new multiperiod portfolio model with a mean absolute deviation risk model. The investor is assumed to seek an investment strategy to maximize his/her final wealth and minimize risk. The optimal portfolio strategy is obtained via the dynamic programming method.

In the future, we will study the risk free assets involved in the portfolio selection. Moreover, how to set up the dynamic portfolio model when $b_t > 0$ is another meaningful and challengeable work which will be further studied.

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Appendix

We give the proof of Theorem 2. Theorem 3 can be proved by a similar method and we omit it in this appendix.

First we simplify the problem P_T as the following problem:

$$P_I \begin{cases} \min \frac{\lambda}{T} y - (1-\lambda) \sum_{j=1}^{n} r_j x_j \\ \text{s.t.} \quad \sum_{j=1}^{n} x_j = U \\ \qquad a_j x_j \leq y, \ j = 1, \ldots, n \\ \qquad a_j x_j \leq \varepsilon U, \ j = 1, \ldots, n \\ \qquad x_j \geq 0, \ j = 1, \ldots, n \end{cases}$$

where $r_1 \leq r_2 \leq \ldots \leq r_n \leq 0$, $a_j > 0$, $j = 1, \ldots, n$ and $U > 0$.

To solve this problem, we can introduce the following auxiliary problem:

$$P_{II} \begin{cases} \min \frac{\lambda}{T} y - (1-\lambda) \sum_{j=1}^{n} r_j x_j \\ \text{s.t.} \quad \sum_{j=1}^{n} x_j = U \\ \qquad a_j x_j \leq y, \ j = 1, \ldots, n; \\ \qquad x_j \geq 0, \ j = 1, \ldots, n; \end{cases}$$

We can get an optimal solution to P_I by solving P_{II}. Noticing that an optimal solution to P_{II} is known (see Cai et al. 2000), we denote the solution as $(\boldsymbol{x}^*, y^*)$.

Lemma A.1 *The solution to P_I denoted by $(\boldsymbol{x}^*, z^*)$, where $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)'$, is as follows:*

$$x_j^* = \begin{cases} \frac{U}{a_j} \left(\sum_{i \in A} \frac{1}{a_i} \right)^{-1}, & j \in A, \\ 0, & j \notin A, \end{cases} \qquad y^* = U \left(\sum_{j \in A} \frac{1}{a_j} \right)^{-1}$$

where A is determined by:

(1) *If there exists an integer $p \in [0, n-2]$ such that*

$$\sum_{i=n-p+1}^{n} \frac{r_i - r_{n-p}}{a_i} < \frac{\lambda}{T(1-\lambda)} \quad \text{and} \quad \sum_{i=n-p}^{n} \frac{r_i - r_{n-p-1}}{a_i} \geq \frac{\lambda}{T(1-\lambda)}$$

then

$$A = \{n, n-1, \ldots, n-p\}.$$

(2) *Otherwise, if the above condition is not satisfied by any integer $p \in [0, n-2]$, then*

$$A = \{n, n-1, \ldots, 1\}.$$

Obviously, x_j^* satisfies that $a_j x_j^* = y^*$, $j \in A$, $a_j x_j^* = 0$ for $j \notin A$. Here, $A = \{n, n-1, \ldots, n-p\}$ or $\{n, n-1, \ldots, 1\}$. In the following discussion, for convenience of notation, we denote $A = \{1, \ldots, k\}$, $k \leq n$. It is worth noticing that the rank of assets in A is converse to the original one. That is for $i, j \in \{1, \ldots, k\}$, if $i < j$, then $r_i \geq r_j$.

An optimal solution to P_I can be found by the following steps:

Step 1: If there exists l with $n \geq l > k$ such that

$$\frac{\sum_{j=1}^{k} \frac{1}{a_j}}{\sum_{j=1}^{l-1} \frac{1}{a_j}} > \frac{\varepsilon U}{y^*} \geq \frac{\sum_{j=1}^{k} \frac{1}{a_j}}{\sum_{j=1}^{l} \frac{1}{a_j}}$$

Step 2: An optimal solution to P_I is:

$$x_j^{**} = \frac{\varepsilon U}{a_j},\ j = 1, \ldots, l-1$$
$$x_l^{**} = y^* \sum_{j=1}^{k} \frac{1}{a_j} - \varepsilon U \sum_{j=1}^{l-1} \frac{1}{a_j}$$
$$x_j^{**} = 0,\ j = l+1, \ldots, n$$
$$y^{**} = \varepsilon U$$

The following discussion will show that the solution in Steps 1 and 2 is an optimal solution to P_I. We separate the proof into three theorems. First, we will show that a solution which satisfy Steps 1 and 2 is a feasible solution to the problem P_I. And then we will prove that if we can not find l which satisfy Steps 1 and 2, then the problem P_I has no feasible solution. Third, we will show that the solution in Steps 1 and 2 is the optimal solution to P_I.

Theorem A.1 *The solution in Step 2 is a feasible solution to* P_I.

Proof Since $j = 1, \ldots, k$,

$$x_j^{**} = \frac{\varepsilon U}{a_j} = x_j^* - \frac{y^* - \varepsilon U}{a_j}$$

We have

$$\begin{aligned}\sum_{j=1}^{n} x_j^{**} &= U - \sum_{j=1}^{k} \frac{y^* - \varepsilon U}{a_j} + \sum_{j=k+1}^{l-1} \frac{\varepsilon U}{a_j} + x_l^{**} \\ &= U - y^* \sum_{j=1}^{k} \frac{1}{a_j} + \sum_{j=1}^{l-1} \frac{\varepsilon U}{a_j} + x_l^{**} \\ &= U\end{aligned}$$

Obviously, $x_l^{**} \geq 0$, and $a_l x_l^{**} \leq \varepsilon U$. Hence, the solution in Step 2 is a feasible solution to P_I. □

Theorem A.2 *If we can not find l satisfy Step 1, then the feasible solution to* P_I *is empty.*

Proof If there is no l satisfy Step 1, then

$$y^* \sum_{j=1}^{k} \frac{1}{a_j} > \varepsilon U \sum_{j=1}^{n} \frac{1}{a_j}$$

Noticing that $a_j x_j^* = y^*$, $j = 1, \ldots, k$ and $\sum_{j=1}^{k} x_j^* = y^* \sum_{j=1}^{k} \frac{1}{a_j} = U$.

If x_j is a feasible solution to P_I, then $a_j x_j \leq \varepsilon U$, $j = 1, \ldots, n$.

Noticing that $U = \sum_{j=1}^{n} x_j \leq \varepsilon U \sum_{j=1}^{n} \frac{1}{a_j}$, we have

$$y^* \sum_{j=1}^{k} \frac{1}{a_j} \leq \varepsilon U \sum_{j=1}^{n} \frac{1}{a_j}$$

which is a contradiction. Hence, P_I has no feasible solution. □

Theorem A.3 *Feasible solution in Steps 1 and 2 is an optimal solution to P_I.*

Proof For problem P_{II}, we have

$$\sum_{j=1}^{k} \frac{r_j - r_{k+1}}{a_j} \geq \frac{\lambda}{T(1-\lambda)}$$

Considering that $l > k$, $r_l \leq r_{k+1}$, we have

$$\sum_{j=1}^{l} \frac{r_j - r_l}{a_j} > \sum_{j=1}^{k} \frac{r_j - r_l}{a_j} \geq \sum_{j=1}^{k} \frac{r_j - r_{k+1}}{a_j} \geq \frac{\lambda}{T(1-\lambda)}$$

The Lagrangian function of problem P_{II} is

$$L_f(x_j, y, \mu_j, p_j, s) = \frac{\lambda}{T} y - (1-\lambda)\sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n} \mu_j (a_j x_j - y) + \sum_{j=1}^{n} l_j (a_j x_j - \varepsilon U) - \sum_{j=1}^{n} p_j x_j + s\left(\sum_{j=1}^{n} x_j - U\right)$$

The Kuhn-Tucker condition is as follows:

$$\frac{\partial L}{\partial y} = \frac{\lambda}{T} - \sum_{j=1}^{n} \mu_j = 0,\ \frac{\partial L}{\partial x_j} = -(1-\lambda) r_j + a_j \mu_j + l_j a_j + s - p_j = 0$$

$$\sum_{j=1}^{n} x_j = U,\ \mu_j (a_j x_j - y) = 0,\ l_j (a_j x_j - \varepsilon U) = 0,$$

$$p_j x_j = 0,\ \mu_j \geq 0,\ l_j \geq 0,\ p_j \geq 0,\ x_j \geq 0\ j = 1, \ldots, n$$

Since P_{II} is a linear program, the Kuhn-Tucker point is an optimal solution.

According to Steps 1 and 2, we have $y^{**} = \varepsilon U$, $a_j x_j^{**} = \varepsilon U$, for $j = 1, \ldots, l-1$ and $a_l x_l^{**} \leq \varepsilon U$, $x_j^{**} = 0$ for $j > l$ and $\sum_{j=1}^{n} x_j^{**} = U$.

It is easy to verify that the following Lagrange multipliers and KT point satisfy the Kuhn-Tucker condition as follows:

$$p_j = 0,\ j = 1, \ldots, l$$

$$p_j = (1-\lambda)(r_l - r_j) \geq 0,\ j = l+1, \ldots, n$$

$$s = (1-\lambda) r_l$$

$$\mu_j = \frac{\lambda}{T} \frac{\frac{r_j - r_{l-1}}{a_j}}{\sum_{j=1}^{l-1} \frac{r_j - r_{l-1}}{a_j}},\ j = 1, \ldots, l-1$$

$$\mu_j = 0,\ j = l, \ldots, n$$

$$l_j = (1-\lambda)\frac{r_j - r_l}{a_j} - \frac{\lambda}{T} \frac{\frac{r_j - r_{l-1}}{a_j}}{\sum_{j=1}^{l-1} \frac{r_j - r_{l-1}}{a_j}},\ j = 1, \ldots, l-1$$

$$l_j = 0,\ j = l, \ldots, n$$

Hence the proof is completed. □

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