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Modified extragradient methods for variational inequality problems and fixed point problems for an infinite family of nonexpansive mappings in Banach spaces

Gang Cai · Shangquan Bu

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Abstract In this paper, we introduce a new general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of a general variational inequality for two inverse-strongly accretive mappings in Banach space. We obtain some strong convergence theorems by a modified extragradient method under suitable conditions. Our results extend the recent results announced by many others.

Keywords Strong convergence · Fixed point · Nonexpansive mapping · Variational inequality · Banach space

Mathematics Subject Classification (2000) 47H09 · 47H10

1 Introduction

Let X be a real Banach space and let J be the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad x \in X,$$

where X^* denotes the dual space of X and $\langle ., . \rangle$ denotes the generalized duality pairing. We use $F(T)$ to denote the set of fixed points of the mapping T. It is well known [1] that if X^* is strictly convex or X is a Banach space with a uniformly Gâteaux differentiable norm, then J

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G. Cai (✉) · S. Bu
Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, China
e-mail: caigang-aaaa@163.com

S. Bu
e-mail: sbu@math.tsinghua.edu.cn

is single-valued. In what follows, we denote the single-valued normalized duality mapping by j.

Let C be a nonempty convex subset of X, recall that T is a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C. \tag{1.1}$$

A mapping $f : C \to C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall\, x, y \in C. \tag{1.2}$$

A mapping $\phi : C \to C$ is a Meir-Keeler contraction if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| < \epsilon + \delta \text{ implies } \|\phi x - \phi y\| < \epsilon, \quad \forall\, x, y \in C. \tag{1.3}$$

In a Banach space X having a single-valued normalized duality mapping j, we define an operator $A : C \to C$ is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, j(x)\rangle \geq \overline{\gamma} \|x\|^2, \; \|aI - bA\| = \sup_{\|x\| \leq 1} |\langle (aI - bA)x, j(x)\rangle|, \; a \in [0, 1], \; b \in [-1, 1], \tag{1.4}$$

where I is the identity mapping.

A mapping $A : C \to X$ is said to be accretive if

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \quad \forall\, x, y \in C. \tag{1.5}$$

A mapping $A : C \to X$ is said to be α-inverse-strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall\, x, y \in C. \tag{1.6}$$

In recent years, the existence of common fixed points for a finite family of nonexpansive mappings has been considered by many authors (see [1–10]). In this direction, several iterative methods have been proposed for these problems. Recently, Marino and Xu [6] considered the following iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.7}$$

where A is a strongly positive bounded linear operator on a Hilbert space H. Under suitable conditions they proved the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution x^* in H of the variational inequality

$$\left\langle (A - \gamma f)x^*, x - x^* \right\rangle \geq 0, \quad x \in H. \tag{1.8}$$

Very recently, Wangkeeree et al. [11] extended Theorem of Marino and Xu [6] from Hilbert space to a reflexive Banach space which admits a weakly continuous duality mapping J_φ, more precisely, they introduced the following iterative algorithm:

$$\begin{cases} x_0 = x \in X, \\ y_n = \beta_n x_n + (1 - \beta_n)T_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad n \geq 0. \end{cases} \tag{1.9}$$

On the other hand, variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. This field is experiencing an explosive growth in both theory and application. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [12–20] and the references therein.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the classical variational inequality is to find x^* such that

$$\left\langle Ax^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C, \tag{1.10}$$

where $A : C \to H$ is a nonlinear mapping. The set of solutions of (1.10) is denoted by $VI(A, C)$.

Let $A, B : C \to H$ be two mappings. Ceng et al. [14] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.11}$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.11) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.12}$$

Further, if we add up the requirement that $x^* = y^*$, the problem (1.11) reduces to the classical variational inequality.

In order to find the common element of the solutions of problem (1.11) and the set of fixed points of a nonexpansive mapping T, Ceng et al. [14] studied the following algorithm: $x_1 = u \in C$ and

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n). \end{cases} \tag{1.13}$$

Under appropriate conditions they obtained a strong convergence theorem.

Very recently, in a Banach space, Yao et al. [15] considered the following variational inequality of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.14}$$

For solving the problem (1.14), Yao et al. [15] introduced the following iterative algorithm: $u, x_0 \in C$ and

$$\begin{cases} y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), \quad n \geq 0. \end{cases} \tag{1.15}$$

They proved a strong convergence theorem under suitable conditions.

Let C be a nonempty closed convex subset of a real Banach space X. For given two operators $A, B : C \to X$, we consider the following variational inequality problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.16}$$

which is called the system of general variational inequalities in a real Banach space. The set of solutions of (1.16) is denoted by Ω. If $\lambda = \mu = 1$, the problem (1.16) becomes the variational inequality problem (1.14).

In this paper, motivated by the above facts, we introduce a new general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of a general variational inequality for two inverse-strongly accretive mappings in a Banach space. Then we prove some strong convergence theorems under some suitable conditions. The results obtained in this paper improve and extend the recent ones announced by Wangkeeree et al. [11], Ceng et al. [14], Yao et al. [15] and many others.

2 Preliminaries

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{Δ}$$

exists for each $x, y \in S(X)$. In this case, X is said to be smooth. The norm of X is said to be uniformly Gâteaux differentiable, if for each $y \in S(X)$, the limit(Δ)is attained uniformly for $x \in S(X)$. The norm of the X is said to be Frêchet differentiable, if for each $x \in S(X)$, the limit(Δ)is attained uniformly for $y \in S(X)$. The norm of X is called uniformly Frêchet differentiable(or X is said to be uniformly smooth), if the limit(Δ)is attained uniformly for $x, y \in S(X)$.

A Banach space X is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $\|x\| = \|y\| = 1, x \neq y$; uniformly convex if for all $\epsilon \in [0, 2]$, $\exists \delta_\epsilon > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta_\epsilon$ for $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$.

Let $\rho_X : [0, \infty) \longrightarrow [0, \infty)$ be the modulus of smoothness of X defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(X),\ \|y\| \leq t\right\}.$$

A Banach space X is said to be uniformly smooth if $\dfrac{\rho_X(t)}{t} \to 0$ as $t \to 0$. A Banach space X is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$. It is well known that each uniformly convex Banach space X is reflexive and strictly convex and every uniformly smooth Banach space X is a reflexive Banach space with uniformly Gâteaux differentiable norm.

Recall that, if C and D are nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \longrightarrow D$ is sunny [21] provided

$$P(x + t(x - P(x))) = P(x) \quad \text{for all } x \in C \text{ and } t \geq 0,$$

whenever $x + t(x - P(x)) \in C$. A mapping $P : C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following propositions concern the sunny nonexpansive retraction.

Proposition 2.1 [21] *Let C be a closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C. Let $P : C \to D$ be a retraction and let J be the normalized duality mapping on X. Then the following are equivalent:*

(a) *P is sunny and nonexpansive.*
(b) $\|Px - Py\|^2 \le \langle x - y, J(Px - Py)\rangle, \quad \forall x, y \in C.$
(c) $\langle x - Px, J(y - Px)\rangle \le 0, \quad \forall x \in C, y \in D.$

Proposition 2.2 [22] *If X is strictly convex and uniformly smooth and $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then the set $F(T)$ is a sunny nonexpansive retraction of C.*

We need the following lemmas for the proof of our main results.

Lemma 2.3 [7] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [23] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.5 [24] *Let X be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x + y\|^q \le \|x\|^q + q\left\langle y, j_q x\right\rangle + C_q\|y\|^q,$$

for all $x, y \in X$. In particular, if X is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2,$$

for all $x, y \in X$.

Lemma 2.6 [25] *Let X be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \le \|x\|^2 - 2\langle x, jy\rangle + \|y\|^2$, for all $x, y \in B_r$, where $B_r = \{z \in X : \|z\| \le r\}$.*

Lemma 2.7 ([26], Lemma 2.1) *In a Banach space X, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad x, y \in X,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.8 [27] *Let C be a closed convex subset of a strictly convex Banach space X. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \ne \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1 - \lambda)T_2 x, \quad \forall x \in C,$$

where λ is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.9 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $A : C \to X$ be a α-inverse-strongly accretive. Then the following inequality holds:*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \le \|x-y\|^2 - 2\lambda(\alpha - K^2\lambda)\|Ax-Ay\|^2. \tag{2.1}$$

In particular, if $0<\lambda\le\frac{\alpha}{K^2}$, then $I-\lambda A$ is nonexpansive.

Proof Indeed, for all $x, y \in C$, it follows from Lemma 2.5 that

$$\begin{aligned}
&\|(I-\lambda A)x-(I-\lambda A)y\|^2\\
&\quad = \|(x-y)-\lambda(Ax-Ay)\|^2\\
&\quad \le \|x-y\|^2 - 2\lambda\langle Ax-Ay, j(x-y)\rangle + 2K^2\lambda^2\|Ax-Ay\|^2\\
&\quad \le \|x-y\|^2 - 2\alpha\lambda\|Ax-Ay\|^2 + 2K^2\lambda^2\|Ax-Ay\|^2\\
&\quad = \|x-y\|^2 - 2\lambda(\alpha-K^2\lambda)\|Ax-Ay\|^2.
\end{aligned}$$

It is clear that if $0<\lambda\le\frac{\alpha}{K^2}$, then $I-\lambda A$ is nonexpansive. This completes the proof. □

Lemma 2.10 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let P_C be the sunny nonexpansive retraction from X onto C. Let the mapping $A : C \to X$ be α-inverse-strongly accretive and let $B : C \to X$ be β-inverse-strongly accretive. Let $G : C \to C$ be a mapping defined by*

$$G(x) = P_C\left[P_C(x-\mu Bx) - \lambda A P_C(x-\mu Bx)\right], \quad \forall x \in C.$$

If $0<\lambda\le\frac{\alpha}{K^2}$ and $0<\mu\le\frac{\beta}{K^2}$, then $G : C\to C$ is nonexpansive.

Proof For all $x, y \in C$, by Lemma 2.9, we have

$$\begin{aligned}
&\|G(x)-G(y)\|\\
&\quad = \|P_C[P_C(x-\mu Bx)-\lambda AP_C(y-\mu By)] - P_C[P_C(y-\mu By)-\lambda AP_C(y-\mu By)]\|\\
&\quad \le \|(I-\lambda A)P_C(I-\mu B)x-(I-\lambda A)P_C(I-\mu B)y\|\\
&\quad \le \|P_C(I-\mu B)x - P_C(I-\mu B)y\|\\
&\quad \le \|(I-\mu B)x-(I-\mu B)y\|\\
&\quad \le \|x-y\|,
\end{aligned}$$

which implies that G is nonexpansive. This completes the proof. □

Lemma 2.11 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let P_C be the sunny nonexpansive retraction from X onto C. Let $A, B : C \to X$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.16) if and only if $x^* = P_C(y^*-\lambda Ay^*)$ where $y^* = P_C(x^*-\mu Bx^*)$, that is $x^* = Gx^*$, where G is defined by Lemma 2.10.*

Proof We rewrite (1.16) as

$$\begin{cases}\langle (y^*-\lambda Ay^*)-x^*, j(x-x^*)\rangle \le 0, & \forall x\in C,\\ \langle (x^*-\mu Bx^*)-y^*, j(x-y^*)\rangle \le 0, & \forall x\in C.\end{cases} \tag{2.2}$$

From Proposition 2.1, we deduce that (2.2) is equivalent to

$$\begin{cases} x^* = P_C(y^*-\lambda Ay^*),\\ y^* = P_C(x^*-\mu Bx^*).\end{cases}$$

This completes the proof. □

Lemma 2.12 *Let X be a Banach space having a single-valued normalized duality mapping j, assume that $A : C \to C$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Proof From (1.4), we know that $\|A\| = \sup_{\|x\|\leq 1} |\langle Ax, j(x)\rangle|$. Now for $x \in C$ with $\|x\| = 1$, we see that

$$\langle (I - \rho A)x, j(x)\rangle = 1 - \rho \langle Ax, j(x)\rangle \geq 1 - \rho \|A\| \geq 0.$$

(i.e., $I - \rho A$ is positive). It follows that

$$\begin{aligned}\|I - \rho A\| &= \sup \{\langle (I - \rho A)x, j(x)\rangle : x \in C, \|x\| = 1\}\\ &= \sup \{1 - \rho \langle Ax, j(x)\rangle : x \in C, \|x\| = 1\}\\ &\leq 1 - \rho\overline{\gamma}.\end{aligned}$$

This completes the proof. □

Lemma 2.13 *Let C be a closed convex subset of a uniformly smooth Banach space X, let $T, S : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Let $A : C \to C$ be a strongly positive linear bounded operator with efficient $\bar{\gamma} > 0$. Assume that $C \pm C \subset C$ and $0 < \gamma < \bar{\gamma}$. Then the sequence $\{x_t\}$ defined by $x_t = t\gamma Sx_t + (I - tA)Tx_t$ converges strongly to $z \in F(T)$ as $t \to 0$ which solves the following variational inequality*

$$\langle (A - \gamma S)z, j(z - p)\rangle \leq 0, \quad \forall p \in F(T). \tag{2.3}$$

Proof First, we show the uniqueness of the solution of the variational inequality (2.3). Suppose both $z_1 \in F(T)$ and $x_2 \in F(T)$ are solutions of (2.3), we have

$$\langle (A - \gamma S)z_1, j(z_1 - z_2)\rangle \leq 0,$$

and

$$\langle (A - \gamma S)z_2, j(z_2 - z_1)\rangle \leq 0.$$

Adding up the above two inequalities gets

$$\langle (A - \gamma S)z_1 - (A - \gamma S)z_2, j(z_1 - z_2)\rangle \leq 0.$$

Noting that

$$\begin{aligned}\langle (A - \gamma S)z_1 - (A - \gamma S)z_2, j(z_1 - z_2)\rangle &= \langle A(z_1 - z_2), j(z_1 - z_2)\rangle\\ &\quad - \gamma \langle Sz_1 - Sz_2, j(z_1 - z_2)\rangle\\ &\geq \bar{\gamma} \|z_1 - z_2\|^2 - \gamma \|z_1 - z_2\|^2\\ &= (\bar{\gamma} - \gamma) \|z_1 - z_2\|^2 \geq 0.\end{aligned}$$

Consequently we have $z_1 = z_2$ and the uniqueness is proved. We use $\tilde{z}$ to denote the unique solution of (2.3).

Next, we prove that $\{x_t\}$ is bounded. Indeed, we may assume, without loss of generality, $t \leq \|A\|^{-1}$, for $p \in F(T)$, it follows from Lemma 2.12 that

$$\begin{aligned}\|x_t - p\| &= \|t(\gamma Sx_t - Ap) + (I - tA)(Tx_t - p)\|\\ &\leq (1 - t\bar{\gamma}) \|x_t - p\| + t \|\gamma Sx_t - \gamma Sp\| + t \|\gamma Sp - Ap\|\\ &\leq (1 - t\bar{\gamma}) \|x_t - p\| + t\gamma \|x_t - p\| + t \|\gamma Sp - Ap\|\\ &= (1 - t(\bar{\gamma} - \gamma)) \|x_t - p\| + t \|\gamma Sp - Ap\|,\end{aligned}$$

which implies $\|x_t - p\| \leq \frac{\|\gamma Sp - Ap\|}{\bar{\gamma} - \gamma}$. This shows that $\{x_t\}$ is bounded.

Assume $t_n \to 0$ as $n \to \infty$. Set $x_n := x_{t_n}$ and define $\mu : C \to \mathbb{R}$ by

$$\mu(x) = \text{LIM} \|x_n - x\|^2, \quad x \in C,$$

where LIM is a Banach limit on l^∞. Let

$$K = \left\{ x \in C : \mu(x) = \min_{x \in C} \text{LIM} \|x_n - x\|^2 \right\}.$$

We see easily that K is a nonempty closed convex subset of X. Note

$$\|x_n - Tx_n\| = t_n \|\gamma Sx_n - ATx_n\| \to 0 \text{ as } n \to \infty.$$

It follows that

$$\mu(Tx) = \text{LIM} \|x_n - Tx\|^2 = \text{LIM} \|Tx_n - Tx\|^2 \leq \text{LIM} \|x_n - x\|^2 = \mu(x),$$

which implies that $T(K) \subset K$, that is , K is invariant under T. Since a uniformly smooth space has the fixed point property for nonexpansive mapping, T has a fixed point, say $z \in K$. Let $t \in (0, 1)$ and $x \in C$, then $z + t(x - Az) \in C$ by the assumption $C \pm C \subset C$. Since z is also a minimizer of μ over C, we have

$$\text{LIM} \|x_n - z\|^2 \leq \text{LIM} \|x_n - z - t(x - Az)\|^2 .$$

We observe

$$\begin{aligned} \|x_n - z - t(x - Az)\|^2 &= \langle x_n - z, j(x_n - z - t(x - Az))\rangle \\ &\quad - t \langle x - Az, j(x_n - z - t(x - Az))\rangle \\ &\leq \|x_n - z\| \|x_n - z - t(x - Az)\| \\ &\quad - t \langle x - Az, j(x_n - z - t(x - Az))\rangle \\ &\leq \frac{\|x_n - z\|^2 + \|x_n - z - t(x - Az)\|^2}{2} \\ &\quad - t \langle x - Az, j(x_n - z - t(x - Az))\rangle , \end{aligned}$$

which implies

$$\|x_n - z - t(x - Az)\|^2 \leq \|x_n - z\|^2 - 2t \langle x - Az, j(x_n - z - t(x - Az))\rangle .$$

Taking the Banach limit over $n \geq 1$, we have

$$\text{LIM} \|x_n - z - t(x - Az)\|^2 \leq \text{LIM} \|x_n - z\|^2 - 2t\text{LIM} \langle x - Az, j(x_n - z - t(x - Az))\rangle ,$$

which implies

$$2t\text{LIM} \langle x - Az, j(x_n - z - t(x - Az))\rangle \leq \text{LIM} \|x_n - z\|^2 - \text{LIM} \|x_n - z - t(x - Az)\|^2 \leq 0.$$

Hence we obtain

$$\text{LIM} \langle x - Az, j(x_n - z - t(x - Az))\rangle \leq 0.$$

Since X is uniformly smooth, we have that the duality mapping j is norm-to-norm uniformly continuous on bounded set of X. Letting $t \to 0$, we have

$$\langle x - Ax, j(x_n - z)\rangle - \langle x - Az, j(x_n - z - t(x - Az))\rangle \to 0 \text{ uniformly}.$$

Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0, \delta)$ and for all $n \geq 1$,

$$\langle x - Ax, j(x_n - z)\rangle < \langle x - Az, j(x_n - z - t(x - Az))\rangle + \epsilon.$$

Consequently,

$$\text{LIM}\,\langle x - Ax,\, j(x_n - z)\rangle \leq \text{LIM}\,\langle x - Az,\, j(x_n - z - t(x - Az))\rangle + \epsilon \leq \epsilon.$$

Since ϵ is arbitrary, then

$$\text{LIM}\,\langle x - Az,\, j(x_n - z)\rangle \leq 0, \quad x \in C. \tag{2.4}$$

On the other hand, we have

$$x_n - z = t_n(\gamma S x_n - Az) + (I - t_n A)(T x_n - z).$$

It follows that

$$\begin{aligned}\|x_n - z\|^2 &= t_n\,\langle \gamma S x_n - Az,\, j(x_n - z)\rangle + \langle (I - t_n A)(T x_n - z),\, j(x_n - z)\rangle \\ &\leq t_n\,\langle \gamma S x_n - Az,\, j(x_n - z)\rangle + (1 - t_n\bar{\gamma})\,\|x_n - z\|^2,\end{aligned}$$

which implies

$$\begin{aligned}\|x_n - z\|^2 &\leq \frac{1}{\bar{\gamma}}\,\langle \gamma S x_n - Az,\, j(x_n - z)\rangle \\ &= \frac{1}{\bar{\gamma}}\,\langle \gamma S x_n - x,\, j(x_n - z)\rangle + \frac{1}{\bar{\gamma}}\,\langle x - Az,\, j(x_n - z)\rangle.\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\begin{aligned}\text{LIM}\,\|x_n - z\|^2 &\leq \frac{1}{\bar{\gamma}}\text{LIM}\,\langle \gamma S x_n - x,\, j(x_n - z)\rangle + \frac{1}{\bar{\gamma}}\text{LIM}\,\langle x - Az,\, j(x_n - z)\rangle \\ &\leq \frac{1}{\bar{\gamma}}\text{LIM}\,\langle \gamma S x_n - x,\, j(x_n - z)\rangle \\ &\leq \frac{1}{\bar{\gamma}}\text{LIM}\,\|\gamma S x_n - x\|\,\|x_n - z\|.\end{aligned}$$

In particular,

$$\bar{\gamma}\text{LIM}\,\|x_n - z\|^2 \leq \text{LIM}\,\|\gamma S x_n - \gamma S z\|\,\|x_n - z\| \leq \gamma\text{LIM}\,\|x_n - z\|^2.$$

Hence

$$(\bar{\gamma} - \gamma)\text{LIM}\,\|x_n - z\|^2 \leq 0.$$

Since $\bar{\gamma} > \gamma$, we have $\text{LIM}\,\|x_n - z\|^2 = 0$, and hence there exists a subsequence which is still denoted $\{x_n\}$ such that $x_n \to z$.

Next we prove that z solves the variational inequality (2.3). Since

$$x_t = t\gamma S x_t + (I - tA)T x_t.$$

We have

$$(A - \gamma S)x_t = -\frac{1}{t}(I - tA)(I - T)x_t.$$

On the other hand, note for all $x, y \in C$,

$$\begin{aligned}\langle (I - T)x - (I - T)y,\, j(x - y)\rangle &= \|x - y\|^2 - \langle Tx - Ty,\, j(x - y)\rangle \\ &\geq \|x - y\|^2 - \|Tx - Ty\|\,\|x - y\| \\ &\geq \|x - y\|^2 - \|x - y\|^2 = 0.\end{aligned}$$

For $p \in F(T)$, we have

$$\begin{aligned}\langle (A-\gamma S)x_t, j(x_t-p)\rangle &= -\frac{1}{t}\langle (I-tA)(I-T)x_t, j(x_t-p)\rangle \\ &= -\frac{1}{t}\langle (I-T)x_t-(I-T)p, j(x_t-p)\rangle \\ &\quad + \langle A(I-T)x_t, j(x_t-p)\rangle \\ &\le \langle A(I-T)x_t, j(x_t-p)\rangle .\end{aligned}$$

Replacing t with t_n and letting $n \to \infty$, note $(I-T)x_{t_n} \to (I-T)z = 0$, we have that

$$\langle (A-\gamma S)z, j(z-p)\rangle \le 0.$$

That is, $z \in F(T)$ is a solution of (2.3). Then $z = \tilde{z}$. In a summary, we have that each cluster point of $\{x_n\}$ converges strongly to z as $t_n \to 0$. This completes the proof. □

Lemma 2.14 *Let C be a nonempty closed convex subset of a real Banach space X which has uniformly Gâteaux norm, and $T, S : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $C \pm C \subset C$ and $\{x_t\}$ converges strongly to $z \in F(T)$ as $t \to 0$, where x_t is defined by $x_t = t\gamma Sx_t + (I-tA)Tx_t$, where $\gamma > 0$ is a constant. Suppose $\{x_n\} \subset C$ is bounded and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then*

$$\limsup_{n\to\infty}\langle \gamma Sz - Az, j(x_n-z)\rangle \le 0.$$

Proof We note that

$$\begin{aligned}x_t - x_n &= t\gamma Sx_t + Tx_t - tATx_t - x_n \\ &= t(\gamma Sx_t - Ax_t) + (Tx_t - x_n) - t(ATx_t - Ax_t) \\ &= t(\gamma Sx_t - Ax_t) + (Tx_t - x_n) + t^2A(\gamma Sx_t - ATx_t).\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_t - x_n\|^2 \\ &\quad = t\langle \gamma Sx_t - Ax_t, j(x_t-x_n)\rangle + \langle Tx_t - x_n, j(x_t-x_n)\rangle \\ &\qquad + t^2\langle A(\gamma Sx_t - ATx_t), j(x_t-x_n)\rangle \\ &\quad = t\langle \gamma Sx_t - Ax_t, j(x_t-x_n)\rangle + \langle Tx_t - Tx_n, j(x_t-x_n)\rangle + \langle Tx_n - x_n, j(x_t-x_n)\rangle \\ &\qquad + t^2\langle A(\gamma Sx_t - ATx_t), j(x_t-x_n)\rangle \\ &\quad \le t\langle \gamma Sx_t - Ax_t, j(x_t-x_n)\rangle + \|x_t-x_n\|^2 + \|Tx_n - x_n\|\,\|x_t-x_n\| \\ &\qquad + t^2\|A(\gamma Sx_t - ATx_t)\|\,\|x_t-x_n\|,\end{aligned}$$

which implies

$$\langle \gamma Sx_t - Ax_t, j(x_n-x_t)\rangle \le \frac{\|Tx_n - x_n\|\,\|x_t-x_n\|}{t} + t\|A(\gamma Sx_t - ATx_t)\|\,\|x_t-x_n\|. \tag{2.6}$$

Since $\{x_t\}$, $\{x_n\}$ and $\{Tx_n\}$ are bounded and $x_n - Tx_n \to 0$, taking the upper limit as $n \to \infty$ in (2.6), we get that

$$\limsup_{n\to\infty}\langle \gamma Sx_t - Ax_t, j(x_n-x_t)\rangle \le t\|A(\gamma Sx_t - ATx_t)\|\limsup_{n\to\infty}\|x_t-x_n\|. \tag{2.7}$$

Taking the upper limit as $t \to 0$ in (2.7), we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma Sx_t - Ax_t, j(x_n - x_t)\rangle \le 0. \tag{2.8}$$

Since X has a uniformly Gâteaux norm, we obtain that j is single-valued and strong-weak* uniformly continuous on bounded set of X. We get that

$$\begin{aligned}
&|\langle \gamma Sz - Az, j(x_n - z)\rangle - \langle \gamma Sx_t - Ax_t, j(x_n - x_t)\rangle| \\
&\quad = |\langle \gamma Sz - Az, j(x_n - z) - j(x_n - x_t)\rangle + \langle \gamma Sz - \gamma Sx_t + Ax_t - Az, j(x_n - x_t)\rangle| \\
&\quad \le |\langle \gamma Sz - Az, j(x_n - z) - j(x_n - x_t)\rangle| + (\|\gamma Sz - \gamma Sx_t\| + \|Ax_t - Az\|)\,\|x_n - x_t\| \\
&\quad \to 0 \text{ as } t \to 0.
\end{aligned}$$

Hence, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall t \in (0, \delta)$, for all n, we have

$$\langle \gamma Sz - Az, j(x_n - z)\rangle \le \langle \gamma Sx_t - Ax_t, j(x_n - x_t)\rangle + \epsilon.$$

By (2.8), we get that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma Sz - Az, j(x_n - z)\rangle \\
&\quad = \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma Sz - Az, j(x_n - z)\rangle \\
&\quad \le \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma Sx_t - Ax_t, j(x_n - x_t)\rangle + \epsilon. \\
&\quad \le \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we get that

$$\limsup_{n\to\infty}\langle \gamma Sz - Az, j(x_n - z)\rangle \le 0.$$

This proof is complete. □

Lemma 2.15 (see [28], Lemma 3.1) *Let C be a nonempty subset of a Banach space X, and $\{T_n\}$ a sequence of mappings from C into X. Suppose that for any bounded subset B of C there exists a continuous increasing function h_B from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$ and* $\lim_{k,l\to\infty}\rho_l^k = 0$, *where* $\rho_l^k := \sup\{h_B(\|T_k z - T_l z\|) : z \in B\} < \infty$, *for all $k, l \in \mathbb{N}$. Then*

$$\lim_{n\to\infty}\sup\{h_B(\|Tz - T_n z\|) : z \in B\} = 0.$$

Remark 2.16 (see [28], Remark 3.2) If $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ and $h_B : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, increasing function such that $h_B(0) = 0$, then

$$\lim_{k,l\to\infty}\sup\{h_B(\|T_k z - T_l z\|) : z \in B\} = 0.$$

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space such that $C \pm C \subset C$. Let P_C be the sunny nonexpansive retraction from X to C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $\{T_i : C \to C\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings with $F := \cap_{i=0}^{\infty} F(T_i) \cap \Omega \neq \emptyset$. Let $S : C \to C$ be a nonexpansive mapping and $D : C \to C$ be a strongly positive linear bounded operator with the coefficient*

$\bar{\gamma}$ *such that* $0 < \gamma < \bar{\gamma}$. *For arbitrarily given* $x_0 \in C$, *let the sequence* $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ k_n = P_C(z_n - \lambda A z_n), \\ y_n = (1-\beta_n)x_n + \beta_n k_n, \\ x_{n+1} = \alpha_n \gamma S y_n + \gamma_n x_n + [(1-\gamma_n)I - \alpha_n D]T_n y_n, \end{cases} \tag{3.1}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ *and* $0 < \mu < \frac{\beta}{K^2}$. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$;
(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$, $\liminf_{n\to\infty} \beta_n > 0$.

Suppose that for any bounded subset D' *of* C *there exists an increasing, continuous and convex function* $h_{D'}$ *from* $\mathbb{R}^+$ *into* $\mathbb{R}^+$ *such that* $h_{D'}(0) = 0$ *and* $\lim_{k,l\to\infty} \sup\{h_{D'}(\|T_k z - T_l z\|) : z \in D'\} = 0$. *Let* T *be a mapping from* C *into* C *defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and suppose that* $F(T) = \cap_{i=0}^{\infty} F(T_i)$. *Then* $\{x_n\}$ *converges strongly to* $z \in F$, *which also solves the following variational inequality:*

$$\langle \gamma S z - D z, j(p - z)\rangle \le 0, \quad \forall\, p \in F. \tag{3.2}$$

Proof We divide the proof into four steps.

Step 1 We show that $\{x_n\}$ is bounded. By condition (i), we may assume, without loss of generality, that $\alpha_n \le (1-\gamma_n)\,\|D\|^{-1}$.

Since $D : C \to C$ is a strongly positive linear bounded operator, by (1.4), we have

$$\|D\| = \sup\{\,|\langle Du, j(u)\rangle| : u \in C,\ \|u\| = 1\}.$$

Observe that

$$\begin{aligned} \langle((1-\gamma_n)I - \alpha_n D)u, j(u)\rangle &= 1 - \gamma_n - \alpha_n \langle Du, j(u)\rangle \\ &\ge 1 - \gamma_n - \alpha_n \|D\| \\ &\ge 0. \end{aligned}$$

It follows that

$$\begin{aligned} \|(1-\gamma_n)I - \alpha_n D\| &= \sup\{\langle((1-\gamma_n)I - \alpha_n D)u, j(u)\rangle : u \in C,\ \|u\| = 1\} \\ &= \sup\{1 - \gamma_n - \alpha_n \langle Du, j(u)\rangle : u \in C,\ \|u\| = 1\} \\ &\le 1 - \gamma_n - \alpha_n \overline{\gamma}. \end{aligned}$$

Take $x^* \in F$, From Lemma 2.11, we have

$$x^* = P_C(P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)).$$

Put $y^* = P_C(x^* - \mu B x^*)$, then $x^* = P_C(y^* - \lambda A y^*)$. By Lemma 2.9, we obtain

$$\begin{aligned} \left\|k_n - x^*\right\| &= \left\|P_C(z_n - \lambda A z_n) - P_C(y^* - \lambda A y^*)\right\| \\ &\le \left\|(I - \lambda A)z_n - (I - \lambda A)y^*\right\| \\ &\le \left\|z_n - y^*\right\| \\ &= \left\|P_C(x_n - \mu B x_n) - P_C(x^* - \mu B x^*)\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\|(I-\mu B)x_n-(I-\mu B)x^*\right\| \\
&\leq \left\|x_n-x^*\right\| \qquad (3.3)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\|y_n-x^*\right\| &= \left\|(1-\beta_n)(x_n-x^*)+\beta_n(k_n-x^*)\right\| \\
&\leq (1-\beta_n)\left\|x_n-x^*\right\|+\beta_n\left\|k_n-x^*\right\| \\
&\leq (1-\beta_n)\left\|x_n-x^*\right\|+\beta_n\left\|x_n-x^*\right\| \\
&= \left\|x_n-x^*\right\|. \qquad (3.4)
\end{aligned}
$$

From (3.4), we have

$$
\begin{aligned}
&\left\|x_{n+1}-x^*\right\| \\
&= \left\|\alpha_n(\gamma S y_n-Dx^*)+\gamma_n(x_n-x^*)+((1-\gamma_n)I-\alpha_n D)(T_n y_n-x^*)\right\| \\
&\leq (1-\gamma_n-\alpha_n\bar{\gamma})\left\|T_n y_n-x^*\right\|+\gamma_n\left\|x_n-x^*\right\|+\alpha_n\left\|\gamma S y_n-\gamma S x^*\right\|+\alpha_n\left\|\gamma S x^*-Dx^*\right\| \\
&\leq (1-\gamma_n-\alpha_n\bar{\gamma})\left\|y_n-x^*\right\|+\gamma_n\left\|x_n-x^*\right\|+\alpha_n\gamma\left\|y_n-x^*\right\|+\alpha_n\left\|\gamma S x^*-Dx^*\right\| \\
&\leq (1-\gamma_n-\alpha_n\bar{\gamma})\left\|x_n-x^*\right\|+\gamma_n\left\|x_n-x^*\right\|+\alpha_n\gamma\left\|x_n-x^*\right\|+\alpha_n\left\|\gamma S x^*-Dx^*\right\| \\
&= (1-\alpha_n(\bar{\gamma}-\gamma))\left\|x_n-x^*\right\|+\alpha_n(\bar{\gamma}-\gamma)\frac{\|\gamma S x^*-Dx^*\|}{\bar{\gamma}-\gamma}.
\end{aligned}
$$

By induction, we have

$$
\left\|x_n-x^*\right\| \leq \max\left\{\left\|x_0-x^*\right\|, \frac{\|\gamma S x^*-Dx^*\|}{\bar{\gamma}-\gamma}\right\}, \quad \forall n \geq 1.
$$

Consequently $\{x_n\}$ is bounded. From (3.3) and (3.4), we know that $\{y_n\}$, $\{k_n\}$ and $\{z_n\}$ are also bounded.

Step 2 We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

We observe that

$$
\begin{aligned}
\|k_{n+1}-k_n\| &= \|P_C(z_{n+1}-\lambda A z_{n+1})-P_C(z_n-\lambda A z_n)\| \\
&\leq \|(I-\lambda A)z_{n+1}-(I-\lambda A)z_n\| \\
&\leq \|z_{n+1}-z_n\| \\
&= \|P_C(x_{n+1}-\mu B x_{n+1})-P_C(x_n-\mu B x_n)\| \\
&\leq \|(I-\mu B)x_{n+1}-(I-\mu B)x_n\| \\
&\leq \|x_{n+1}-x_n\|. \qquad (3.5)
\end{aligned}
$$

It follows from (3.5) that

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|(1-\beta_{n+1})x_{n+1}+\beta_{n+1}k_{n+1}-(1-\beta_n)x_n-\beta_n k_n\| \\
&= \|(1-\beta_{n+1})(x_{n+1}-x_n)+\beta_{n+1}(k_{n+1}-k_n)+(\beta_{n+1}-\beta_n)(k_n-x_n)\| \\
&\leq (1-\beta_{n+1})\|x_{n+1}-x_n\|+\beta_{n+1}\|k_{n+1}-k_n\|+|\beta_{n+1}-\beta_n|\,\|k_n-x_n\| \\
&\leq (1-\beta_{n+1})\|x_{n+1}-x_n\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,\|k_n-x_n\| \\
&= \|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,\|k_n-x_n\|. \qquad (3.6)
\end{aligned}
$$

Put $x_{n+1}=\gamma_n x_n+(1-\gamma_n)l_n$. Then we obtain

$$
\begin{aligned}
&l_{n+1}-l_n \\
&= \frac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}}-\frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha_{n+1}\gamma S y_{n+1} + ((1-\gamma_{n+1})I - \alpha_{n+1}D)T_{n+1}y_{n+1}}{1-\gamma_{n+1}} \\
&\quad - \frac{\alpha_n \gamma S y_n + ((1-\gamma_n)I - \alpha_n D)T_n y_n}{1-\gamma_n} \\
&= \frac{\alpha_{n+1}}{1-\gamma_{n+1}}(\gamma S y_{n+1} - DT_{n+1}y_{n+1}) - \frac{\alpha_n}{1-\gamma_n}(\gamma S y_n - DT_n y_n) \\
&\quad + T_{n+1}y_{n+1} - T_{n+1}y_n + T_{n+1}y_n - T_n y_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|l_{n+1} - l_n\| \le \frac{\alpha_{n+1}}{1-\gamma_{n+1}} \|\gamma S y_{n+1} - DT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\gamma_n} \|\gamma S y_n - DT_n y_n\| \\
+ \|y_{n+1} - y_n\| + \|T_{n+1}y_n - T_n y_n\|. \qquad (3.7)
\end{aligned}
$$

Substituting (3.6) into (3.7), we have that

$$
\begin{aligned}
&\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\gamma_{n+1}} \|\gamma S y_{n+1} - DT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\gamma_n} \|\gamma S y_n - DT_n y_n\| \\
&+ |\beta_{n+1} - \beta_n| \, \|k_n - x_n\| + \|T_{n+1}y_n - T_n y_n\|. \qquad (3.8)
\end{aligned}
$$

Let D' be a subset of C containing $\{y_n\}$ and $\{x_n\}$, since $\lim_{k,l\to\infty} \sup\{h_{D'}(\|T_k z - T_l z\|) : z \in D'\} = 0$, we have

$$
\begin{aligned}
h_{D'}(\|T_{n+1}y_n - T_n y_n\|) &\le \sup\left\{h_{D'}(\|T_{n+1}x - T_n x\|) : x \in D'\right\} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

It follows from the property of $h_{D'}$, we have

$$
\lim_{n\to\infty} \|T_{n+1}y_n - T_n y_n\| = 0.
$$

From (3.8) and conditions (i), (ii) and (iii), we obtain

$$
\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0.
$$

By Lemma 2.4, we have $\lim_{n\to\infty} \|l_n - x_n\| = 0$. Consequently, we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\gamma_n) \|l_n - x_n\| = 0. \qquad (3.9)
$$

We observe that

$$
x_{n+1} - x_n = \alpha_n(\gamma S y_n - DT_n y_n) + (1-\gamma_n)(T_n y_n - x_n).
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(\gamma S y_n - DT_n y_n) + (1-\gamma_n)(T_n y_n - x_n)\| \\
&\ge (1-\gamma_n) \|T_n y_n - x_n\| - \alpha_n \|\gamma S y_n - DT_n y_n\|,
\end{aligned}
$$

which implies that

$$
\|T_n y_n - x_n\| \le \frac{1}{1-\gamma_n}(\|x_{n+1} - x_n\| + \alpha_n \|\gamma S y_n - DT_n y_n\|).
$$

Noticing conditions (i) and (ii) and (3.9), we have

$$
\lim_{n\to\infty} \|T_n y_n - x_n\| = 0. \qquad (3.10)
$$

Step 3 We prove that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$.

From Lemma 2.9, we have

$$\begin{aligned}\left\|z_n - y^*\right\|^2 &= \left\|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\right\|^2\\ &\le \left\|x_n - x^* - \mu(Bx_n - Bx^*)\right\|^2\\ &\le \left\|x_n - x^*\right\|^2 - 2\mu(\beta - K^2\mu)\left\|Bx_n - Bx^*\right\|^2,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\left\|k_n - x^*\right\|^2 &= \left\|P_C(z_n - \lambda Az_n) - P_C(y^* - \lambda Ay^*)\right\|^2\\ &\le \left\|z_n - y^* - \lambda(Az_n - Ay^*)\right\|^2\\ &\le \left\|z_n - y^*\right\|^2 - 2\lambda(\alpha - K^2\lambda)\left\|Az_n - Ay^*\right\|^2.\end{aligned} \tag{3.12}$$

Substituting (3.11) into (3.12), we have

$$\left\|k_n - x^*\right\|^2 \le \left\|x_n - x^*\right\|^2 - 2\mu(\beta - K^2\mu)\left\|Bx_n - Bx^*\right\|^2 - 2\lambda(\alpha - K^2\lambda)\left\|Az_n - Ay^*\right\|^2. \tag{3.13}$$

It follows that

$$\begin{aligned}&\left\|y_n - x^*\right\|^2\\ &= \left\|(1-\beta_n)(x_n - x^*) + \beta_n(k_n - x^*)\right\|^2\\ &\le (1-\beta_n)\left\|x_n - x^*\right\|^2 + \beta_n\left\|k_n - x^*\right\|^2\\ &\le (1-\beta_n)\left\|x_n - x^*\right\|^2 + \beta_n(\left\|x_n - x^*\right\|^2 - 2\mu(\beta - K^2\mu)\left\|Bx_n - Bx^*\right\|^2)\\ &\quad - 2\beta_n\lambda(\alpha - K^2\lambda)\left\|Az_n - Ay^*\right\|^2)\\ &= \left\|x_n - x^*\right\|^2 - 2\beta_n\mu(\beta - K^2\mu)\left\|Bx_n - Bx^*\right\|^2 - 2\beta_n\lambda(\alpha - K^2\lambda)\left\|Az_n - Ay^*\right\|^2.\end{aligned} \tag{3.14}$$

By the convexity of $\|.\|^2$ and Lemma 2.7, we obtain

$$\begin{aligned}&\left\|x_{n+1} - x^*\right\|^2\\ &= \left\|\alpha_n(\gamma Sy_n - DT_ny_n) + \gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*)\right\|^2\\ &\le \left\|\gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*)\right\|^2 + 2\alpha_n\left\langle\gamma Sy_n - DT_ny_n, j(x_{n+1} - x^*)\right\rangle\\ &\le \gamma_n\left\|x_n - x^*\right\|^2 + (1-\gamma_n)\left\|T_ny_n - x^*\right\|^2 + 2\alpha_n\|\gamma Sy_n - DT_ny_n\|\left\|x_{n+1} - x^*\right\|.\\ &\le \gamma_n\left\|x_n - x^*\right\|^2 + (1-\gamma_n)\left\|y_n - x^*\right\|^2 + 2\alpha_n M_1,\end{aligned} \tag{3.15}$$

where $M_1 = \sup_{n\ge 0}\{\|\gamma Sy_n - DT_ny_n\|\,\|x_{n+1} - x^*\|\}$.

Substituting (3.14) into (3.15), we have

$$\begin{aligned}&\left\|x_{n+1} - x^*\right\|^2\\ &\le \gamma_n\left\|x_n - x^*\right\|^2 + (1-\gamma_n)\left\|x_n - x^*\right\|^2 - 2(1-\gamma_n)\beta_n\mu(\beta - K^2\mu)\left\|Bx_n - Bx^*\right\|^2\\ &\quad - 2(1-\gamma_n)\beta_n\lambda(\alpha - K^2\lambda)\left\|Az_n - Ay^*\right\|^2 + 2\alpha_n M_1\\ &= \left\|x_n - x^*\right\|^2 - 2(1-\gamma_n)\beta_n\mu(\beta - K^2\mu)\left\|Bx_n - Bx^*\right\|^2\\ &\quad - 2(1-\gamma_n)\beta_n\lambda(\alpha - K^2\lambda)\left\|Az_n - Ay^*\right\|^2 + 2\alpha_n M_1,\end{aligned}$$

which implies that

$$\begin{aligned}&2(1-\gamma_n)\beta_n\mu(\beta-K^2\mu)\left\|Bx_n-Bx^*\right\|^2+2(1-\gamma_n)\beta_n\lambda(\alpha-K^2\lambda)\left\|Az_n-Ay^*\right\|^2\\&\quad\le\left\|x_n-x^*\right\|^2-\left\|x_{n+1}-x^*\right\|^2+2\alpha_nM_1\\&\quad\le\|x_n-x_{n+1}\|\left(\left\|x_n-x^*\right\|+\left\|x_{n+1}-x^*\right\|\right)+2\alpha_nM_1.\end{aligned}$$

Since $0<\lambda<\frac{\alpha}{K^2}, 0<\mu<\frac{\beta}{K^2}, \liminf_{n\to\infty}\beta_n>0, \liminf_{n\to\infty}(1-\gamma_n)=1-\limsup_{n\to\infty}\gamma_n>0, \lim_{n\to\infty}\alpha_n=0$, and (3.9), we have

$$\lim_{n\to\infty}\left\|Bx_n-Bx^*\right\|=0,\ \lim_{n\to\infty}\left\|Az_n-Ay^*\right\|=0. \tag{3.16}$$

Let $r_1=\sup_{n\ge0}\{\|z_n-y^*\|,\|x_n-x^*\|\}$. By Proposition 2.1 and Lemma 2.6, we have

$$\begin{aligned}&\left\|z_n-y^*\right\|^2\\&=\left\|P_C(x_n-\mu Bx_n)-P_C(x^*-\mu Bx^*)\right\|^2\\&\le\left\langle x_n-\mu Bx_n-(x^*-\mu Bx^*),\,j(z_n-y^*)\right\rangle\\&=\left\langle x_n-x^*,\,j(z_n-y^*)\right\rangle+\mu\left\langle Bx^*-Bx_n,\,j(z_n-y^*)\right\rangle\\&\le\frac{1}{2}(\left\|x_n-x^*\right\|^2+\left\|z_n-y^*\right\|^2-g_1(\left\|x_n-z_n-(x^*-y^*)\right\|)\\&\quad+\mu\left\langle Bx^*-Bx_n,\,j(z_n-y^*)\right\rangle,\end{aligned}$$

where $g_1:[0,\infty)\to[0,\infty)$ is a continuous, strictly increasing and convex function such that $g_1(0)=0$. Hence we have

$$\begin{aligned}\left\|z_n-y^*\right\|^2&\le\left\|x_n-x^*\right\|^2-g_1(\left\|x_n-z_n-(x^*-y^*)\right\|)+2\mu\left\langle Bx^*-Bx_n,\,j(z_n-y^*)\right\rangle\\&\le\left\|x_n-x^*\right\|^2-g_1(\left\|x_n-z_n-(x^*-y^*)\right\|)+2\mu\left\|Bx_n-Bx^*\right\|\left\|z_n-y^*\right\|.\end{aligned} \tag{3.17}$$

Let $r_2=\sup_{n\ge0}\{\|z_n-y^*\|,\|k_n-x^*\|\}$. By Proposition 2.1 and Lemma 2.6, we have

$$\begin{aligned}&\left\|k_n-x^*\right\|^2\\&=\left\|P_C(z_n-\lambda Az_n)-P_C(y^*-\lambda Ay^*)\right\|^2\\&\le\left\langle z_n-\lambda Az_n-(y^*-\lambda Ay^*),\,j(k_n-x^*)\right\rangle\\&=\left\langle z_n-y^*,\,j(k_n-x^*)\right\rangle+\lambda\left\langle Ay^*-Az_n,\,j(k_n-x^*)\right\rangle\\&\le\frac{1}{2}(\left\|z_n-y^*\right\|^2+\left\|k_n-x^*\right\|^2-g_2(\left\|z_n-k_n+(x^*-y^*)\right\|)\\&\quad+\lambda\left\langle Ay^*-Az_n,\,j(k_n-x^*)\right\rangle,\end{aligned}$$

where $g_2:[0,\infty)\to[0,\infty)$ is a continuous, strictly increasing and convex function such that $g_2(0)=0$. Therefore we get

$$\begin{aligned}\left\|k_n-x^*\right\|^2&\le\left\|z_n-y^*\right\|^2-g_2(\left\|z_n-k_n+(x^*-y^*)\right\|)+2\lambda\left\langle Ay^*-Az_n,\,j(k_n-x^*)\right\rangle\\&\le\left\|z_n-y^*\right\|^2-g_2(\left\|z_n-k_n+(x^*-y^*)\right\|)+2\lambda\left\|Az_n-Ay^*\right\|\left\|k_n-x^*\right\|.\end{aligned} \tag{3.18}$$

Substituting (3.17) into (3.18), we obtain

$$\|k_n - x^*\|^2 \le \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu \|Bx_n - Bx^*\| \|z_n - y^*\| \\ -g_2(\|z_n - k_n + (x^* - y^*)\|) + 2\lambda \|Az_n - Ay^*\| \|k_n - x^*\| . \tag{3.19}$$

From (3.19), we get

$$\begin{aligned} &\|y_n - x^*\|^2 \\ &= \|(1-\beta_n)(x_n - x^*) + \beta_n(k_n - x^*)\|^2 \\ &\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|k_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \beta_n g_1(\|x_n - z_n - (x^* - y^*)\|) - \beta_n g_2(\|z_n - k_n + (x^* - y^*)\|) \\ &\quad +2\beta_n\mu\|Bx_n - Bx^*\|\|z_n - y^*\| + 2\beta_n\lambda\|Az_n - Ay^*\|\|k_n - x^*\| . \end{aligned} \tag{3.20}$$

Combining (3.15) and (3.20), we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\le \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\|x_n - x^*\|^2 - (1-\gamma_n)\beta_n g_1(\|x_n - z_n - (x^* - y^*)\|) \\ &\quad -(1-\gamma_n)\beta_n g_2(\|z_n - k_n + (x^* - y^*)\|) + 2(1-\gamma_n)\beta_n\mu\|Bx_n - Bx^*\|\|z_n - y^*\| \\ &\quad +2(1-\gamma_n)\beta_n\lambda\|Az_n - Ay^*\|\|k_n - x^*\| + 2\alpha_n M_1 \\ &\le \|x_n - x^*\|^2 - (1-\gamma_n)\beta_n g_1(\|x_n - z_n - (x^* - y^*)\|) \\ &\quad -(1-\gamma_n)\beta_n g_2(\|z_n - k_n + (x^* - y^*)\|) \\ &\quad +2\mu\|Bx_n - Bx^*\|\|z_n - y^*\| + 2\lambda\|Az_n - Ay^*\|\|k_n - x^*\| + 2\alpha_n M_1, \end{aligned}$$

which implies

$$\begin{aligned} &(1-\gamma_n)\beta_n g_1(\|x_n - z_n - (x^* - y^*)\|) + (1-\gamma_n)\beta_n g_2(\|z_n - k_n + (x^* - y^*)\|) \\ &\quad\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\| \\ &\qquad +2\lambda\|Az_n - Ay^*\|\|k_n - x^*\| + 2\alpha_n M_1 \\ &\quad\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\| \\ &\qquad +2\lambda\|Az_n - Ay^*\|\|k_n - x^*\| + 2\alpha_n M_1. \end{aligned}$$

Since $\liminf_{n\to\infty}\beta_n > 0$, $\liminf_{n\to\infty}(1-\gamma_n) > 0$, $\lim_{n\to\infty}\alpha_n = 0$, (3.9) and (3.16), we have

$$\lim_{n\to\infty} g_1(\|x_n - z_n - (x^* - y^*)\|) = 0, \ \lim_{n\to\infty} g_2(\|z_n - k_n + (x^* - y^*)\|) = 0.$$

It follows from the properties of g_1 and g_2, we obtain

$$\lim_{n\to\infty}\|x_n - z_n - (x^* - y^*)\| = 0, \ \lim_{n\to\infty}\|z_n - k_n + (x^* - y^*)\| = 0. \tag{3.21}$$

From (3.21), we have

$$\begin{aligned} \|x_n - k_n\| &\le \|x_n - z_n - (x^* - y^*)\| + \|z_n - k_n + (x^* - y^*)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.22}$$

Consequently, we obtain

$$\|y_n - x_n\| = \beta_n\|x_n - k_n\| \to 0 \text{ as } n \to \infty. \tag{3.23}$$

From (3.10) and (3.23), we have

$$\|T_n y_n - y_n\| \leq \|T_n y_n - x_n\| + \|x_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.24}$$

It follows that

$$\begin{aligned}\|T_n x_n - x_n\| &\leq \|T_n x_n - T_n y_n\| + \|T_n y_n - y_n\| + \|y_n - x_n\| \\ &\leq 2\|x_n - y_n\| + \|T_n y_n - y_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{3.25}$$

Since

$$h_{D'}(\|T_n x_n - T x_n\|) \leq \sup\left\{h_{D'}(\|T_n x - T x\| : x \in D')\right\}.$$

By Lemma 2.15 and the continuity of $h_{D'}$, we have $\lim_{n\to\infty} h_{D'}(\|T_n x_n - T x_n\|) = 0$. And the properties of $h_{D'}$ yield

$$\lim_{n\to\infty} \|T_n x_n - T x_n\| = 0.$$

It follows that

$$\|T x_n - x_n\| \leq \|T x_n - T_n x_n\| + \|T_n x_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.26}$$

Define a mapping $U : C \to C$ as $Ux = \delta T x + (1-\delta) G x$, where G is defined by Lemma 2.10 and $\delta \in (0, 1)$ is a constant. Then by Lemma 2.8, we know that U is nonexpansive and $F(U) = F(T) \cap F(G) = \cap_{n=1}^{\infty} F(T_n) \cap \Omega = F$. We define $x_t = t\gamma S x_t + (I - tD)U x_t$, it follows from Lemma 2.13 that $\{x_t\}$ converges strongly to $z \in F(U) = F$. From (3.22) and (3.26), we have

$$\begin{aligned}\|x_n - U x_n\| &= \|\delta(x_n - T x_n) + (1-\delta)(x_n - G x_n)\| \\ &= \|\delta(x_n - T x_n) + (1-\delta)(x_n - k_n)\| \\ &\leq \delta\|x_n - T x_n\| + (1-\delta)\|x_n - k_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{3.27}$$

By Lemma 2.14, we have

$$\limsup_{n\to\infty} \langle \gamma S z - D z, j(x_n - z)\rangle \leq 0. \tag{3.28}$$

Step 4 Finally we prove that $x_n \to z \in F$ as $n \to \infty$.
From (3.1) and (3.4), we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &= \langle((1-\gamma_n)I - \alpha_n D)(T_n y_n - z), j(x_{n+1} - z)\rangle + \gamma_n \langle x_n - z, j(x_{n+1} - z)\rangle \\ &\quad + \alpha_n \langle \gamma S y_n - D z, j(x_{n+1} - z)\rangle \\ &\leq (1 - \gamma_n - \alpha_n \bar{\gamma})\|T_n y_n - z\|\|x_{n+1} - z\| + \gamma_n \|x_n - z\|\|x_{n+1} - z\| \\ &\quad + \alpha_n \langle \gamma S y_n - \gamma S z, j(x_{n+1} - z)\rangle + \alpha_n \langle \gamma S z - D z, j(x_{n+1} - z)\rangle \\ &\leq (1 - \gamma_n - \alpha_n \bar{\gamma})\|y_n - z\|\|x_{n+1} - z\| + \gamma_n \|x_n - z\|\|x_{n+1} - z\| \\ &\quad + \alpha_n \gamma \|y_n - z\|\|x_{n+1} - z\| \\ &\quad + \alpha_n \langle \gamma S z - D z, j(x_{n+1} - z)\rangle \\ &\leq (1 - \gamma_n - \alpha_n \bar{\gamma})\|x_n - z\|\|x_{n+1} - z\| + \gamma_n \|x_n - z\|\|x_{n+1} - z\| \\ &\quad + \alpha_n \gamma \|x_n - z\|\|x_{n+1} - z\|\end{aligned}$$

$$
\begin{aligned}
&+\alpha_n \langle \gamma Sz - Dz, j(x_{n+1} - z)\rangle \\
&= (1-\alpha_n(\bar{\gamma}-\gamma))\,\|x_n - z\|\,\|x_{n+1} - z\| + \alpha_n \langle \gamma Sz - Dz, j(x_{n+1} - z)\rangle \\
&\le \frac{1-\alpha_n(\bar{\gamma}-\gamma)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle \gamma Sz - Dz, j(x_{n+1} - z)\rangle \\
&\le \frac{1-\alpha_n(\bar{\gamma}-\gamma)}{2}\,\|x_n - z\|^2 + \frac{1}{2}\,\|x_{n+1} - z\|^2 + \alpha_n \langle \gamma Sz - Dz, j(x_{n+1} - z)\rangle,
\end{aligned}
$$

which implies

$$
\|x_{n+1} - z\|^2 \le (1-\alpha_n(\bar{\gamma}-\gamma))\,\|x_n - z\|^2 + \alpha_n(\bar{\gamma}-\gamma)\frac{2\langle \gamma Sz - Dz, j(x_{n+1} - z)\rangle}{\bar{\gamma}-\gamma}. \tag{3.29}
$$

Apply Lemma 2.3 to (3.29), we obtain $x_n \to z \in F$ as $n \to \infty$.This completes the proof. □

Next we give two simple examples about the control parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ of Theorem 3.1.

Example 3.2 Put $\alpha_n = \frac{1}{3n}, \gamma_n = \frac{1}{3} + \frac{1}{6n}, \beta_n = \frac{1}{3} + \frac{1}{3n}$. Then the conditions (i)–(iii) are satisfied in theorem 3.1.

Example 3.3 Put $\alpha_n = \frac{1}{4n^{\frac{1}{2}}}, \gamma_n = \frac{1}{10}e^{\frac{1}{n}}, \beta_n = \frac{1}{2\pi}\arctan n + \frac{1}{3}$. Then the conditions (i)–(iii) are satisfied in theorem 3.1.

Theorem 3.4 *Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space such that $C \pm C \subset C$. Let P_C be the sunny nonexpansive retraction from X to C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $\{T_i : C \to C\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings with $F := \cap_{i=0}^{\infty} F(T_i) \cap \Omega \neq \emptyset$. Let $f : C \to C$ be a contraction with a constant $\theta \in (0,1)$ and $D : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma' < \frac{\bar{\gamma}}{\theta}$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$
\begin{cases}
z_n = P_C(x_n - \mu B x_n),\\
k_n = P_C(z_n - \lambda A z_n),\\
y_n = (1-\beta_n)x_n + \beta_n k_n,\\
x_{n+1} = \alpha_n \gamma' f(y_n) + \gamma_n x_n + [(1-\gamma_n)I - \alpha_n D]T_n y_n,
\end{cases} \tag{3.30}
$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$;
(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0, \liminf_{n\to\infty} \beta_n > 0$.

Suppose that for any bounded subset D' of C there exists an increasing, continuous and convex function $h_{D'}$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_{D'}(0) = 0$ and $\lim_{k,l\to\infty} \sup\{h_{D'}(\|T_k z - T_l z\|) : z \in D'\} = 0$. Let T be a mapping from C into C defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \cap_{i=0}^{\infty} F(T_i)$.Then $\{x_n\}$ converges strongly to $z \in F$, which also solves the following variational inequality:

$$
\langle \gamma f(z) - Dz, j(p - z)\rangle \le 0, \quad \forall\, p \in F. \tag{3.31}
$$

Proof Take $S = \frac{f}{\theta}, \gamma = \gamma'\theta$ in Theorem 3.1, we note that $\frac{f}{\theta}$ is nonexpansive. So we obtain the desired result by Theorem 3.1. □

Theorem 3.5 *Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space such that $C \pm C \subset C$. Let P_C be the sunny nonexpansive retraction from X to C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $\{T_i : C \to C\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings with $F := \cap_{i=0}^{\infty} F(T_i) \cap \Omega \neq \emptyset$. Let $\phi : C \to C$ be a Meir-Keeler contraction and $D : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma < \bar{\gamma}$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ k_n = P_C(z_n - \lambda A z_n), \\ y_n = (1 - \beta_n)x_n + \beta_n k_n, \\ x_{n+1} = \alpha_n \gamma \phi(y_n) + \gamma_n x_n + [(1 - \gamma_n)I - \alpha_n D]T_n y_n, \end{cases} \tag{3.32}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;
(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0, \liminf_{n\to\infty} \beta_n > 0$.

Suppose that for any bounded subset D' of C there exists an increasing, continuous and convex function $h_{D'}$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_{D'}(0) = 0$ and $\lim_{k,l\to\infty} \sup \{h_{D'}(\|T_k z - T_l z\|) : z \in D'\} = 0$. Let T be a mapping from C into C defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \cap_{i=0}^{\infty} F(T_i)$. Then $\{x_n\}$ converges strongly to $z \in F$, which also solves the following variational inequality:

$$\langle \gamma\phi(z) - Dz, j(p - z)\rangle \leq 0, \quad \forall p \in F. \tag{3.33}$$

Proof We note that ϕ is nonexpansive, so the conclusion of Theorem 3.5 can be obtained from Theorem 3.1 immediately. □

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