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Robust solutions of quadratic optimization over single quadratic constraint under interval uncertainty

V. Jeyakumar · G. Y. Li

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Abstract In this paper we examine non-convex quadratic optimization problems over a quadratic constraint under unknown but bounded interval perturbation of problem data in the constraint and develop criteria for characterizing robust (i.e. uncertainty-immunized) global solutions of classes of non-convex quadratic problems. Firstly, we derive robust solvability results for quadratic inequality systems under parameter uncertainty. Consequently, we obtain characterizations of robust solutions for uncertain homogeneous quadratic problems, including uncertain concave quadratic minimization problems and weighted least squares. Using homogenization, we also derive characterizations of robust solutions for non-homogeneous quadratic problems.

Keywords Non-convex quadratic programming under uncertainty · Robust optimization · Single quadratic constraint · Robust solutions · Global optimality conditions

Mathematics Subject Classification (2000) 90C20 · 90C30 · 90C26 · 90C46

1 Introduction

Consider the non-convex quadratic optimization model problem with a quadratic inequality constraint

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V. Jeyakumar (✉) · G. Y. Li
Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia
e-mail: v.jeyakumar@unsw.edu.au

G. Y. Li
e-mail: g.li@unsw.edu.au

$$\begin{aligned} \min_{x\in\mathbb{R}^n}\ & \frac{1}{2}x^T Ax + a^T x \\ \text{s.t.}\ & \frac{1}{2}x^T Bx \leq \beta, \end{aligned} \tag{QP}$$

where the "data inputs" consist of the $n \times n$ matrices A, B and the vector a, and β is a scalar. Model problems of this form, in particular, appear in wireless communication and signal processing [19,26], and also arise as a subproblem in trust-region algorithms for unconstrained optimization and have been studied extensively in the literature [18,20–22,27,30]. These problems enjoy theoretically useful and computationally attractive features, including no duality gap [27,31], tight semi-definite programming relaxation [22,29] and dual characterization of the solution [13]. Unfortunately, these features can not, in general, be extended to quadratic problems with more than one quadratic constraint [1,2,16,22]. Moreover, in these models, it is assumed that data inputs are precisely known despite the reality of input data uncertainty in real-world models due to modeling or prediction errors.

Over the years, various approaches to addressing optimization under data uncertainty have been developed. These approaches can be classified into two broad categories: Stochastic and Deterministic. Stochastic approaches start by assuming the uncertainty has a probabilistic description. The best known technique is based on Stochastic Programming [24,25]. On the other hand, deterministic approaches are based on a description of uncertainty via bounded sets [4], as opposed to probability distribution. Robust optimization has emerged as a leading deterministic approach to address optimization under data uncertainty [3,5–9,14,17].

Following the framework of robust optimization [3,4], our approach, in this paper, to studying these quadratic model problems affected by data uncertainty is to treat uncertainty via bounded uncertainty sets, described by intervals.

Our main objective of this paper is to study quadratic model problems of type (QP) in the face of data uncertainty in the constraint, where the matrix B is uncertain and it belongs to an interval uncertainty set, $\mathcal{V} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\}$, B_1 and B_2 are given matrices and μ_1 and μ_2 are real numbers with $\mu_1 \leq \mu_2$. Our aim is to develop verifiable criteria for characterizing robust (i.e. uncertainty-immunized) solutions for classes of homogeneous and non-homogeneous quadratic problems under the interval uncertainty. As an illustration, consider the uncertain quadratic programming problem

$$\min_{x\in\mathbb{R}^2} \left\{ \frac{1}{2}x_1^2 - x_2^2 \ \mid\ \frac{1}{2}x^T Bx \leq 1 \right\},$$

where $B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$ is uncertain. Suppose that we know the exact value of b_1, b_3 which is $\overline{b}_1, \overline{b}_3$, and that the values of b_2 are uncertain in the sense that the nominal value of b_2 is $\overline{b}_2$ with the possible error of $\pm 10\%$. Then, the effect of the data uncertainty can be captured by considering the uncertainty set $\mathcal{V} = \{B_1 + \mu B_2 : \mu \in [-0.1, 0.1]\}$, where

$$B_1 = \begin{pmatrix} \overline{b}_1 & \overline{b}_2 \\ \overline{b}_2 & \overline{b}_3 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We make three key contributions to non-convex quadratic optimization under uncertainty.

First, we establish new robust theorems of the alternative for parameterized quadratic inequality systems, extending various corresponding powerful theorems of the alternative such as S-lemma [10,22,28]. Related theorems of the alternative may be found in [12,13,15, 22]. These theorems play a key role in deriving dual criteria characterizing robust solutions of uncertain quadratic programs later in the paper.

Second, we derive necessary and sufficient conditions for robust global optimality for classes of homogeneous problems (QP) under interval uncertainty. These results extend the useful features of non-convex quadratic optimization over single quadratic constraint to related problems under data uncertainty.

Third, using homogenization, we obtain characterizations of robust global solutions of uncertain non-homogeneous quadratic problems, where a more general quadratic constraint $\frac{1}{2}x^T Bx + b^T x + \beta \leq 0$ is uncertain and the uncertain input data $(B, b) \in \overline{\mathcal{V}} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}$.

The layout of the paper is as follows. Section 2 provides preliminary results on quadratic forms and inequalities that will be used and, in some cases, will be extended and applied later in the paper. Section 3 presents robust theorems of the alternative for quadratic inequalities. Section 4 characterizes robust global optimality for classes of homogeneous quadratic problems under uncertainty. Section 5 presents necessary and sufficient conditions for global optimality of non-homogeneous quadratic problems under uncertainty.

2 Preliminaries on quadratic systems

In this section we fix the notation and recall some basic facts on quadratic functions that will be used throughout this paper. The real line is denoted by $\mathbb{R}$ and the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The set of all non-negative vectors of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, and the interior of $\mathbb{R}^n_+$ is denoted by $\text{int}\mathbb{R}^n_+$. The space of all $(n \times n)$ symmetric matrices is denoted by S^n. The $(n \times n)$ identity matrix is denoted by I_n. The notation $A \succeq B$ means that the matrix $A - B$ is positive semidefinite. Moreover, the notation $A \succ B$ means the matrix $A - B$ is positive definite.

The basic and probably the most useful result on the joint-range convexity of homogeneous quadratic functions is given as follows.

Lemma 2.1 (Dine's Theorem [11,22]) *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = x^T A_1 x$ and $g(x) = x^T A_2 x$, where $A_1, A_2 \in S^n$. Then the set $\{(x^T A_1 x, x^T A_2 x) : x \in \mathbb{R}^n\}$ is convex.*

Dine's theorem is known to fail for more than two homogeneous quadratic functions. Polyak [23] established the following joint-range convexity result for three homogeneous quadratic functions under a positive definite condition on the matrices involved.

Lemma 2.2 (Polyak's Lemma [23, Theorem 2.1]) *Let $n \geq 3$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = x^T A_1 x$, $g(x) = x^T A_2 x$ and $h(x) = x^T A_3 x$, where $A_1, A_2, A_3 \in S^n$. Suppose that there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that*

$$\gamma_1 A_1 + \gamma_2 A_2 + \gamma_3 A_3 \succ 0. \tag{2.1}$$

Then the set $\{(x^T A_1 x, x^T A_2 x, x^T A_3 x) : x \in \mathbb{R}^n\}$ is convex.

Using Dine's Theorem, Yakubovich (cf [22]) obtained the following fundamental S-lemma which has played a key role in many areas of control and optimization.

Lemma 2.3 (S-lemma [22]) *Let $A_1, A_2 \in S^n$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $x_0^T A_1 x_0 < 0$. Then the following statements are equivalent:*

(i) $x^T A_1 x \leq 0 \Rightarrow x^T A_2 x \geq 0$.
(ii) $(\exists \lambda \geq 0)\ (\forall x \in \mathbb{R}^n)\ x^T (A_1 + \lambda A_2) x \geq 0$.

The following alternative theorem of Yuan [31] for two strict inequalities played a key role in the study of eigenvalue problems and convergence analysis of trust-region algorithms.

Lemma 2.4 (Yuan's Alternative Theorem [31]) *Let $A_1, A_2 \in S^n$. Then, exactly one of the following two statements holds.*

(i) $(\exists x \in \mathbb{R}^n)\, x^T A_1 x < 0,\, x^T A_2 x < 0.$

(ii) $(\exists(\lambda_1, \lambda_2) \in \mathbb{R}^2_+ \backslash \{(0,0)\})\, (\forall x \in \mathbb{R}^n)\, x^T(\lambda_1 A_1 + \lambda_2 A_2)x \geq 0.$

3 Robust solvability of quadratic inequalities

In this Section, we examine robust theorems of the alternative for quadratic inequality systems under linear perturbations.

Theorem 3.1 *Let $A, B_1, B_2 \in S^n$ and let $\mu_1, \mu_2 \in \mathbb{R}$ with $\mu_1 \leq \mu_2$. Suppose that*

$$\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} \text{ is convex.}$$

Then, either one of the following two statements holds:

(1) $(\exists\, x \in \mathbb{R}^n)$, $\frac{1}{2}x^T Ax < \alpha$, $\frac{1}{2}x^T(B_1 + \mu B_2)x < \beta$, $\forall\, \mu \in [\mu_1, \mu_2]$
(2) $(\exists\, (\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+ \backslash \{0\})\, (\exists \mu \in [\mu_1, \mu_2])\, (\forall\, x \in \mathbb{R}^n)$,

$$\overline{\lambda}_1 \left(\frac{1}{2}x^T Ax - \alpha\right) + \overline{\lambda}_2 \left(\frac{1}{2}x^T(B_1 + \mu B_2)x - \beta\right) \geq 0.$$

Proof Clearly [(2) $\Rightarrow$ Not(1)] always holds. We show that [Not(1) $\Rightarrow$ (2)].

[Joint Range Convexity of the quadratic maps]. We first show that

$$\Omega_{\mathcal{V}} = \{(x^T Ax, \max_{B \in \mathcal{V}} x^T Bx) : x \in \mathbb{R}^n\} + \text{int}\mathbb{R}^2_+ \text{ is convex,}$$

where $\mathcal{V} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\}$. To see this, let $(a_1, b_1) \in \Omega_{\mathcal{V}}$ and $(a_2, b_2) \in \Omega_{\mathcal{V}}$, and let $\lambda \in [0, 1]$. Then, there exist $x_1, x_2 \in \mathbb{R}^n$ such that

$$x_1^T Ax_1 < a_1,\ \max_{B \in \mathcal{V}} x_1^T Bx_1 < b_1$$

and

$$x_2^T Ax_2 < a_2,\ \max_{B \in \mathcal{V}} x_2^T Bx_2 < b_2.$$

For each fixed $x \in \mathbb{R}^n$, as $B \mapsto x^T Bx$ is a linear map, we have $\max_{B \in \mathcal{V}} x^T Bx$ is attained at some extreme point of $\mathcal{V}$. Note that the extreme points of $\mathcal{V}$ are $B_1 + \mu_1 B_2$ and $B_1 + \mu_2 B_2$. We now see that, for each $x \in \mathbb{R}^n$, $\max_{B \in \mathcal{V}} x^T Bx = \max\{x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x\}$. So, we have

$$x_1^T Ax_1 < a_1,\ x_1^T(B_1 + \mu_1 B_2)x_1 < b_1,\ x_1^T(B_1 + \mu_2 B_2)x_1 < b_1,$$

and

$$x_2^T Ax_2 < a_2.\ x_2^T(B_1 + \mu_1 B_2)x_2 < b_2,\ x_2^T(B_1 + \mu_2 B_2)x_2 < b_2.$$

This implies that

$$(a_1, b_1, b_1) \in \{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} + \text{int}\mathbb{R}^3_+$$

and

$$(a_2, b_2, b_2) \in \{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} + \text{int}\mathbb{R}^3_+.$$

Now, from the assumption, we see that $\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} + \text{int}\mathbb{R}^3_+$ is convex. Then,

$$\lambda(a_1, b_1, b_1) + (1-\lambda)(a_2, b_2, b_2) \in \\ \times\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} + \text{int}\mathbb{R}^3_+.$$

So, we can find $x_3 \in \mathbb{R}^n$ such that

$$x_3^T Ax_3 < \lambda a_1 + (1-\lambda)a_2,\ x_3^T(B_1 + \mu_1 B_2)x_3 < \lambda b_1 + (1-\lambda)b_2$$

and

$$x_3^T(B_1 + \mu_2 B_2)x_3 < \lambda b_1 + (1-\lambda)b_2.$$

This gives us that

$$x_3^T Ax_3 < \lambda a_1 + (1-\lambda)a_2 \text{ and } \max_{B\in\mathcal{V}} x_3^T Bx_3 < \lambda b_1 + (1-\lambda)b_2.$$

So, $\lambda(a_1, b_1) + (1-\lambda)(a_2, b_2) \in \Omega_{\mathcal{V}}$, and hence $\Omega_{\mathcal{V}}$ is convex in this case.

[Dualization via separation.] Now, as (1) fails, we have $(2\alpha, 2\beta) \notin \Omega_{\mathcal{V}}$. Since $\Omega_{\mathcal{V}}$ is convex, the hyperplane separation theorem gives us that $(\exists\,(\lambda_1, \lambda_2) \in \mathbb{R}^2_+\backslash\{0\})\,(\forall x \in \mathbb{R}^n)$, $\lambda_1\left(\frac{1}{2}x^T Ax - \alpha\right) + \lambda_2\left(\max_{B\in\mathcal{V}} \frac{1}{2}x^T Bx - \beta\right) \geq 0$. As, for each x, $\max_{B\in\mathcal{V}} x^T Bx = \max\{x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x\}$, we have

$$\max\left\{\frac{1}{2}x^T(\lambda_1 A + \lambda_2(B_1 + \mu_1 B_2))x,\ \frac{1}{2}x^T(\lambda_1 A + \lambda_2(B_1 + \mu_2 B_2))x\right\} - (\lambda_1\alpha + \lambda_2\beta) \\ = \lambda_1\left(\frac{1}{2}x^T Ax - \alpha\right) + \lambda_2\left(\frac{1}{2}\max\{x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x\} - \beta\right) \geq 0.$$

[Simplification]. This shows that the following system has no solution

$$\frac{1}{2}x^T(\lambda_1 A + \lambda_2(B_1 + \mu_1 B_2))x < (\lambda_1\alpha + \lambda_2\beta) \text{ and } \frac{1}{2}x^T(\lambda_1 A + \lambda_2(B_1 + \mu_2 B_2))x \\ < (\lambda_1\alpha + \lambda_2\beta).$$

From Dine's theorem, we see that $\{(x^T(\lambda_1 A + \lambda_2(B_1 + \mu_1 B_2))x, x^T(\lambda_1 A + \lambda_2(B_1 + \mu_2 B_2))x) : x \in \mathbb{R}^n\}$ is convex, and so, the set

$$\Omega := \{(x^T(\lambda_1 A + \lambda_2(B_1 + \mu_1 B_2))x, x^T(\lambda_1 A + \lambda_2(B_1 + \mu_2 B_2))x) : x \in \mathbb{R}^n\} + \text{int}\mathbb{R}^2_+$$

is convex. So, $(2(\lambda_1\alpha + \lambda_2\beta), 2(\lambda_1\alpha + \lambda_2\beta)) \notin \Omega$. Then, the hyperplane separation theorem again gives us that there exists $(\delta_1, \delta_2) \in \mathbb{R}^2_+\backslash\{0\}$ such that for each $x \in \mathbb{R}^n$,

$$\delta_1\left(\frac{1}{2}x^T(\lambda_1 A + \lambda_2(B_1 + \mu_1 B_2))x - (\lambda_1\alpha + \lambda_2\beta)\right) \\ +\delta_2\left(\frac{1}{2}x^T(\lambda_1 A + \lambda_2(B_1 + \mu_2 B_2))x - (\lambda_1\alpha + \lambda_2\beta)\right) \geq 0.$$

Letting $\overline{\lambda}_1 = \lambda_1(\delta_1 + \delta_2), \overline{\lambda}_2 = \lambda_2(\delta_1 + \delta_2)$ and $\mu = \frac{\delta_1\mu_1+\delta_2\mu_2}{\delta_1+\delta_2}$, we see that $\mu \in [\mu_1, \mu_2]$, $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+\backslash\{0\}$ and, for each $x \in \mathbb{R}^n$,

$$\overline{\lambda}_1\left(\frac{1}{2}x^TAx - \alpha\right) + \overline{\lambda}_2\left(\frac{1}{2}x^T(B_1 + \mu B_2)x - \beta\right) \geq 0.$$

So, (2) holds. □

When $\mu_1 = \mu_2$, i.e, $\mathcal{V}$ is a singleton set, by Dine's theorem, we see that $\{(x^TAx, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_1 B_2)x) : x \in \mathbb{R}^n\}$ is always convex.

The following example illustrates that, the set $\mu_1 \neq \mu_2$, $\{(x^TAx, x^T(B_1+\mu_1 B_2)x, x^T(B_1 + \mu_1 B_2)x) : x \in \mathbb{R}^n\}$ is, in general, not convex and that the convexity requirement of Theorem 3.1 can not be dropped.

Example 3.1 Let $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $B_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mu_1 = -1, \mu_2 = 1$. Let

$$\begin{aligned}\Omega &= \{(x^TAx, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^2\}\\ &= \{(2x_1^2 - 2x_2^2, -2x_1^2 - 2x_1x_2, -2x_1^2 + 2x_1x_2) : (x_1, x_2) \in \mathbb{R}^2\}.\end{aligned}$$

We first show that Ω is not convex. Indeed, by considering $(x_1, x_2) = (1, 1)$ and $(x_1, x_2) = (1, -1)$, we see that

$$(0, -4, 0) \in \Omega \quad \text{and} \quad (0, 0, -4) \in \Omega.$$

On the other hand,

$$(0, -2, -2) = \frac{(0, -4, 0) + (0, -4, 0)}{2} \notin \Omega.$$

(Otherwise, there exists $z_1, z_2 \in \mathbb{R}$ such that

$$2z_1^2 - 2z_2^2 = 0, -2z_1^2 - 2z_1z_2 = -2 \text{ and } -2z_1^2 + 2z_1z_2 = -2.$$

Solving the last two equations gives us that $z_1^2 = 1$ and $z_2 = 0$ which clearly violates the first equation. This is a contradiction.)

Moreover, one can also verify that the statements (1) and (2) in Theorem 3.1 both fail. To see this, consider the following system

$$\left[x_1^2 - x_2^2 = \frac{1}{2}x^TAx < 0 \text{ and } \forall\, \mu \in [-1, 1],\ -x_1^2 + \mu x_1x_2 = \frac{1}{2}x^T(B_1 + \mu B_2)x < 0\right]. \tag{3.2}$$

From the first relation of (3.2), we see that $|x_1| < |x_2|$. On the other hand, the second relation of (3.2) entails that $-x_1^2 + |x_1x_2| < 0$, and so, $|x_2| < |x_1|$. So, the above system has no solution and hence (1) fails. Moreover, for any $(\lambda_1, \lambda_2) \in \mathbb{R}^2_+\backslash\{0\}$ and $\mu \in [-1, 1]$, we have

$$\lambda_1 A + \lambda_2(B_1 + \mu B_2) = \begin{pmatrix} 2(\lambda_1 - \lambda_2) & \mu\lambda_2 \\ \mu\lambda_2 & -2\lambda_1 \end{pmatrix}.$$

We note that, $\lambda_1 A + \lambda_2(B_1 + \mu B_2) \succeq 0$ implies that $2(\lambda_1 - \lambda_2) \geq 0$ and $-2\lambda_1 \geq 0$. This gives us that $\lambda_1 = \lambda_2 = 0$ which is impossible. So, (2) also fails.

Let us give certain easily verifiable conditions ensuring the convexity assumption in Theorem 3.1 when $\mu_1 < \mu_2$.

Proposition 3.1 *Let $A, B_1, B_2 \in S^n$ and $\mu_1 < \mu_2$. Suppose that A, B_1, B_2 are all diagonal matrices. Then,*

$$\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} \text{ is convex.}$$

Proof As B_1 and B_2 are both diagonal, $B_1 + \mu_1 B_2$ and $B_1 + \mu_2 B_2$ are also diagonal matrices. Let $A = \text{diag}(a_1, \ldots, a_n)$, $B_1 + \mu_1 B_2 = \text{diag}(b_1, \ldots, b_n)$ and $B_1 + \mu_2 B_2 = \text{diag}(c_1, \ldots, c_n)$. Then,

$$\begin{aligned}
&\left\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\right\}\\
&= \left\{\left(\sum_{i=1}^n a_i x_i^2, \sum_{i=1}^n b_i x_i^2, \sum_{i=1}^n c_i x_i^2\right) : (x_1, \ldots, x_n) \in \mathbb{R}^n\right\}\\
&= \left\{\left(\sum_{i=1}^n a_i y_i, \sum_{i=1}^n b_i y_i, \sum_{i=1}^n c_i y_i\right) : (y_1, \ldots, y_n) \in \mathbb{R}^n_+\right\},
\end{aligned}$$

and so, is convex. □

Proposition 3.2 *Let $A, B_1, B_2 \in S^n$ and $\mu_1 < \mu_2$. Suppose that $n \geq 3$ and that there exist $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$ such that*

$$\gamma_0 A + \gamma_1 B_1 + \gamma_2 B_2 \succ 0. \tag{3.3}$$

Then, $\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\}$ is convex.

Proof Let $\delta_1 = \frac{\gamma_1\mu_2 - \gamma_2}{\mu_2 - \mu_1}$ and $\delta_2 = \frac{\gamma_2 - \gamma_1\mu_1}{\mu_2 - \mu_1}$. Then, by (3.3), we have

$$\gamma_0 A + \delta_1(B_1 + \mu_1 B_2) + \delta_2(B_1 + \mu_2 B_2) = \gamma_0 A + \gamma_1 B_1 + \gamma_2 B_2 \succ 0.$$

So, the conclusion will follow from Polyak's lemma (Lemma 2.2). □

As a consequence of Theorem 3.1 we obtain a generalization of S-lemma for parameterized quadratic inequality systems.

Corollary 3.1 (Robust S-Lemma) *Let $A, B_1, B_2 \in S^n$ and let $\mu_1, \mu_2 \in \mathbb{R}$ with $\mu_1 \leq \mu_2$. Suppose that the set*

$$\{(x^T Ax, x^T(B_1 + \mu_1 B_2)x, x^T(B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\}$$

is convex. If there exists $x_0 \in \mathbb{R}^n$ such that $x_0^T(B_1 + \mu B_2)x_0 < 0$, $\forall \mu \in [\mu_1, \mu_2]$ then, the following statements are equivalent:

(1) $[\forall \mu \in [\mu_1, \mu_2],\ x^T(B_1 + \mu B_2)x \leq 0] \Rightarrow x^T Ax \geq 0$
(2) $(\exists \overline{\lambda} \geq 0)\ (\exists \mu \in [\mu_1, \mu_2])\ (\forall x \in \mathbb{R}^n),\ x^T\big(A + \overline{\lambda}(B_1 + \mu B_2)\big)x \geq 0.$

Proof The conclusion will follow if we show [(1) ⇒ (2)] as the converse implication always holds. To see this, assume that (1) holds. Then the following system has no solution:

$$x^T Ax < 0,\ x^T(B_1 + \mu B_2)x \leq 0, \quad \forall \mu \in [\mu_1, \mu_2].$$

So, the system

$$x^T Ax < 0,\ x^T(B_1 + \mu B_2)x < 0, \quad \forall \mu \in [\mu_1, \mu_2]$$

has also no solution. Applying Theorem 3.1, there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+\backslash\{0\})$ and $\mu \in [\mu_1, \mu_2]$ such that for each $x \in \mathbb{R}^n$,

$$\overline{\lambda}_1 x^T Ax + \overline{\lambda}_2 x^T (B_1 + \mu B_2)x \geq 0.$$

But $\overline{\lambda}_1 \neq 0$ as $x_0^T (B_1 + \mu B_2)x_0 < 0,\ \forall \mu \in [\mu_1, \mu_2]$. Hence (2) holds. □

In passing, we note that the Robust S-lemma collapses to S-lemma [2] in the special case when $\mu_1 = \mu_2$. In this case, the convexity assumption always holds.

4 Uncertain homogeneous quadratic problems

Consider the uncertain quadratic programming problem

$$(HP) \quad \min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Ax \quad \text{s.t.} \quad \frac{1}{2}x^T Bx \leq \beta,$$

where $A \in S^n$, $\beta > 0$ and $B \in S^n$ is uncertain and it belongs to the interval uncertainty set $\mathcal{V}$, described by the bounded set $\mathcal{V} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\}$ where B_1, B_2 are given $n \times n$ matrices.

The robust counterpart of (HP) is

$$(HP_1) \quad \min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Ax \quad \text{s.t.} \quad \frac{1}{2}x^T Bx \leq \beta,\ \forall B \in \mathcal{V}.$$

Definition 4.1 We say that $\overline{x}$ is a (global) robust solution of (HP) in the sense that it is a (global) solution of its robust counterpart (HP_1).

Theorem 4.1 (Robust Solution Characterization) *For the problem* (HP)*, let* $\mathcal{V} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\}$ *with* $B_1, B_2 \in S^n$. *Suppose that*

$$\{(x^T Ax, x^T (B_1 + \mu_1 B_2)x, x^T (B_1 + \mu_2 B_2)x) : x \in \mathbb{R}^n\} \text{ is convex.}$$

Then, a robust feasible point $\overline{x}$ *is a robust solution of* (HP) *if and only if there exist* $\lambda \geq 0$ *and* $\mu \in [\mu_1, \mu_2]$ *with*

$$\begin{cases} \big(A + \lambda(B_1 + \mu B_2)\big)\overline{x} = 0 & \text{(First-order Condition)} \\ \lambda(\frac{1}{2}\overline{x}^T (B_1 + \mu B_2)\overline{x} - \beta) = 0 & \text{(Complementary Slackness)} \\ A + \lambda(B_1 + \mu B_2) \succeq 0 & \text{(Second-order Condition).} \end{cases} \tag{4.4}$$

Proof (*Necessary Part.*) Let $\overline{x}$ be a robust solution of problem (HP). Then,

$$\forall B \in \mathcal{V},\ \frac{1}{2}x^T Bx \leq \beta \Rightarrow \frac{1}{2}x^T Ax \geq \alpha := \frac{1}{2}\overline{x}^T A\overline{x}.$$

This implies that the following system has no solution

$$\frac{1}{2}x^T Ax < \alpha \text{ and } \max_{B\in\mathcal{V}} \frac{1}{2}x^T Bx < \beta.$$

So, Theorem 3.1 gives us that there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+ \backslash \{0\}$ and $\mu \in [\mu_1, \mu_2]$ such that

$$\overline{\lambda}_1 \left(\frac{1}{2} x^T A x - \alpha \right) + \overline{\lambda}_2 \left(\frac{1}{2} x^T (B_1 + \mu B_2) x - \beta \right) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

We now show that $\overline{\lambda}_1 > 0$. Otherwise, we have $\overline{\lambda}_1 = 0$ and so $\overline{\lambda}_2 \big(\frac{1}{2} x^T (B_1 + \mu B_2) x - \beta \big) \geq 0$ for all $x \in \mathbb{R}^n$. Letting $x = 0$ and noting that $\beta > 0$, we have $\overline{\lambda}_2 \leq 0$ and hence $\overline{\lambda}_2 = 0$ which contradicts $(\overline{\lambda}_1, \overline{\lambda}_2) \neq \{0\}$. Thus, $\overline{\lambda}_1 > 0$, and so,

$$\left(\frac{1}{2} x^T A x - \alpha \right) + \lambda \left(\frac{1}{2} x^T (B_1 + \mu B_2) x - \beta \right) \geq 0, \quad \forall x \in \mathbb{R}^n$$

where $\lambda = \frac{\overline{\lambda}_2}{\overline{\lambda}_1} \geq 0$. Letting $x = \overline{x}$, we see that $\lambda (\frac{1}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x} - \beta) = 0$. So, $h(x) = \big(\frac{1}{2} x^T A x - \alpha \big) + \lambda \big(\frac{1}{2} x^T (B_1 + \mu B_2) x - \beta \big)$ attains its global minimizer at $\overline{x}$, and hence $\nabla h(\overline{x}) = 0$ and $\nabla^2 h(\overline{x}) \succeq 0$. Thus,

$$\big(A + \lambda (B_1 + \mu B_2) \big) \overline{x} = 0 \text{ and } A + \lambda (B_1 + \mu B_2) \succeq 0.$$

[Sufficient Part.] Assume that there exist $\lambda \geq 0$ and $\mu \in [\mu_1, \mu_2]$ such that

$$\begin{cases} \big(A + \lambda (B_1 + \mu B_2) \big) \overline{x} = 0 & \text{(First-order Condition)} \\ \lambda (\frac{1}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x} - \beta) = 0 & \text{(Complementary Slackness)} \\ A + \lambda (B_1 + \mu B_2) \succeq 0 & \text{(Second-order Condition).} \end{cases}$$

Let x be any robust feasible point of (HP). Then $\forall\, B \in \mathcal{V}$, $\frac{1}{2} x^T B x \leq \beta$, and so,

$$\frac{1}{2} x^T (B_1 + \mu B_2) x \leq \beta.$$

By the complementary slackness condition, we have

$$\frac{\lambda}{2} x^T (B_1 + \mu B_2) x \leq \lambda \beta = \frac{\lambda}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x}.$$

Let $h(x) = \frac{1}{2} x^T A x + \lambda \big(\frac{1}{2} x^T (B_1 + \mu B_2) x - \beta \big)$. Then, the first-order condition and the second-order condition give us that h is a convex function with $\nabla h(\overline{x}) = 0$. So, $h(x) \geq h(\overline{x})$. That is to say,

$$\begin{aligned} \frac{1}{2} x^T A x &\geq \frac{1}{2} \overline{x}^T A \overline{x} + \lambda \left(\frac{1}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x} - \beta \right) - \lambda \left(\frac{1}{2} x^T (B_1 + \mu B_2) x - \beta \right) \\ &\geq \frac{1}{2} \overline{x}^T A \overline{x}. \end{aligned}$$

So, $\overline{x}$ is a robust solution of (HP). □

Remark 4.1 Note that the set of optimality conditions (4.4) means that $\overline{x}$ is a global solution of the tractable deterministic quadratic program

$$\begin{aligned} (P_\mu) \quad \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x^T A x \\ \text{s.t.} \quad & \frac{1}{2} x^T (B_1 + \mu B_2) x \leq \beta, \end{aligned}$$

for some $\mu \in [\mu_1, \ \mu_2]$.

As a consequence of Theorem 4.1 we derive necessary and sufficient conditions for concave quadratic minimization problems (HP).

Corollary 4.1 (Robust Solution of Concave Minimization) *For the problem* (HP)*, let* $n \geq 3$ *and* $A \prec 0$*. Let* $\mathcal{V} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\}$ *with* $B_1, B_2 \in S^n$*. Then, a robust feasible point* $\overline{x}$ *is a robust solution of* (HP) *if and only if there exist* $\lambda \geq 0$ *and* $\mu \in [\mu_1, \mu_2]$ *such that* $(A + \lambda(B_1 + \mu B_2))\overline{x} = 0$, $\lambda(\frac{1}{2}\overline{x}^T(B_1 + \mu B_2)\overline{x} - \beta) = 0$ *and* $A + \lambda(B_1 + \mu B_2) \succeq 0$.

Proof The conclusion will follow from Theorem 4.1 and Proposition 3.2 as $A \prec 0$ ensures that (3.3) holds. □

Now, consider the following uncertain weighted least squares:

$$(WL) \quad \min \frac{1}{2}\sum_{i=1}^{n} v_i x_i^2 \quad \text{s.t.} \quad \frac{1}{2}\sum_{i=1}^{n} w_i x_i^2 \leq \beta, \qquad (WL)$$

where $\beta > 0$, $v_i \in \mathbb{R}$ and the data w_i, $i = 1, \ldots, n$ are uncertain and each w_i belongs to the uncertainty set $\mathcal{V}_i = [\underline{w}_i, \overline{w}_i]$ for some $\underline{w}_i, \overline{w}_i \in \mathbb{R}$ with $\underline{w}_i \leq \overline{w}_i$.

Theorem 4.2 (Robust Solution of Weighted Least Squares) *A feasible point* $\overline{x}$ *is a robust solution of* (WL) *if and only if there exist* $\lambda \geq 0$ *and* $w_i \in [\underline{w}_i, \overline{w}_i]$, $i = 1, \ldots, n$*, such that* $\sum_{i=1}^n (v_i + \lambda w_i)\overline{x}_i = 0$, $\lambda(\frac{1}{2}\sum_{i=1}^n w_i \overline{x}_i^2 - \beta) = 0$ *and* $v_i + \lambda w_i \geq 0$.

Proof The problem (WL) can be equivalently rewritten as

$$\min\left\{\frac{1}{2}x^T A x \ \mid\ \frac{1}{2}x^T B x \leq \beta\right\},$$

where $A = \operatorname{diag}(v_1, \ldots, v_n)$, $B \in \mathcal{V} := \{\mathcal{B}_1 + \mu\mathcal{B}_2 : \mu \in [\mu_1, \mu_2]\}$, $\mu_1 = -1$, $\mu_2 = 1$,

$$\mathcal{B}_1 = \operatorname{diag}\left(\frac{\overline{w}_1 + \underline{w}_1}{2}, \ldots, \frac{\overline{w}_n + \underline{w}_n}{2}\right) \text{ and } \mathcal{B}_2 = \operatorname{diag}\left(\frac{\overline{w}_1 - \underline{w}_1}{2}, \ldots, \frac{\overline{w}_n - \underline{w}_n}{2}\right)$$

[Necessary Part.] Let $\overline{x}$ be a robust solution of problem (WL). Then,

$$\forall\, B \in \mathcal{V},\ \frac{1}{2}x^T B x \leq \beta \Rightarrow \frac{1}{2}x^T A x \geq \alpha := \frac{1}{2}\overline{x}^T A \overline{x}.$$

This implies that the following system has no solution

$$\frac{1}{2}x^T A x < \alpha \text{ and } \max_{B \in \mathcal{V}} \frac{1}{2}x^T B x < \beta.$$

As A, B_1 and B_2 are all diagonal matrices, Proposition 3.1 gives us that there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+\backslash\{0\}$ and $\mu \in [\mu_1, \mu_2]$ such that

$$\overline{\lambda}_1\left(\frac{1}{2}x^T A x - \alpha\right) + \overline{\lambda}_2\left(\frac{1}{2}x^T(B_1 + \mu B_2)x - \beta\right) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Let $w_i = \frac{\overline{w}_i - \underline{w}_i}{2} + \mu\frac{\overline{w}_i - \underline{w}_i}{2}$. Then,

$$\overline{\lambda}_1\left(\frac{1}{2}\sum_{i=1}^{n} v_i x_i^2 - \alpha\right) + \overline{\lambda}_2\left(\frac{1}{2}\sum_{i=1}^{n} w_i x_i^2 - \beta\right) \geq 0.$$

We now show that $\overline{\lambda}_1 > 0$. Otherwise, we have $\overline{\lambda}_1 = 0$ and so $\overline{\lambda}_2\left(\frac{1}{2}\sum_{i=1}^n w_i x_i^2 - \beta\right) \geq 0$, for all $x \in \mathbb{R}^n$. Letting $x = 0$ and noting that $\beta > 0$, we have $\overline{\lambda}_2 \leq 0$ and hence $\overline{\lambda}_2 = 0$ which contradicts $(\overline{\lambda}_1, \overline{\lambda}_2) \neq \{0\}$. So,

$$\left(\frac{1}{2}\sum_{i=1}^n v_i x_i^2 - \alpha\right) + \lambda\left(\frac{1}{2}\sum_{i=1}^n w_i x_i^2 - \beta\right) \geq 0, \quad \forall x \in \mathbb{R}^n$$

where $\lambda = \frac{\overline{\lambda}_2}{\overline{\lambda}_1} \geq 0$. Now, letting $x = \overline{x}$, we see that $\lambda\left(\frac{1}{2}\sum_{i=1}^n w_i \overline{x}_i^2 - \beta\right) = 0$. Then, $h(x) = \left(\frac{1}{2}\sum_{i=1}^n v_i x_i^2 - \alpha\right) + \lambda\left(\frac{1}{2}\sum_{i=1}^n w_i x_i^2 - \beta\right)$ attains its global minimizer at $\overline{x}$, $\nabla h(\overline{x}) = 0$ and $\nabla^2 h(\overline{x}) \succeq 0$. These conditions yield

$$\sum_{i=1}^n (v_i + \lambda w_i)\overline{x}_i = 0 \text{ and } v_i + \lambda w_i \geq 0.$$

[Sufficient Part.] This part is similar to the proof in Theorem 4.1. □

5 Uncertain non-homogeneous problems

In this Section, we derive necessary and sufficient conditions for robust global optimality for classes nonhomogeneous quadratic problems with a single uncertain quadratic inequality constraint. Let $A \in S^n$, $a, b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, and consider the following nonhomogeneous quadratic problem with constraint uncertainty:

$$\begin{aligned} (QP) \quad & \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x + a^T x \\ & \text{s.t.} \quad \frac{1}{2} x^T B x + b^T x + \beta \leq 0, \end{aligned}$$

where $(B, b) \in S^n \times \mathbb{R}^n$ is uncertain and

$$(B, b) \in \overline{\mathcal{V}} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}.$$

The robust counterpart of (QP) is

$$\begin{aligned} (QP_1) \quad & \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x + a^T x \\ & \text{s.t.} \quad \frac{1}{2} x^T B x + b^T x + \beta \leq 0, \quad \forall (B, b) \in \overline{\mathcal{V}}. \end{aligned}$$

For a given point $\overline{x}$, let $\alpha = -(\frac{1}{2}\overline{x}^T A \overline{x} + a^T \overline{x})$ and denote

$$H_0 = \begin{pmatrix} A & a \\ a^T & 2\alpha \end{pmatrix}, H_1 = \begin{pmatrix} B_1 + \mu_1 B_2 & b_1 + \delta_1 b_2 \\ (b_1 + \delta_1 b_2)^T & 2\beta \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} B_1 + \mu_2 B_2 & b_1 + \delta_2 b_2 \\ (b_1 + \delta_2 b_2)^T & 2\beta \end{pmatrix}.$$

We shall say the problem (QP) is **regular** with respect to $\overline{x}$ if

$$\{(x^T H_0 x, x^T H_1 x, x^T H_2 x) : x \in \mathbb{R}^n\} \text{ is convex.}$$

Note again that in the case where $\mu_1 = \mu_2$ and $\delta_1 = \delta_2$ (i.e. $H_1 = H_2$) the problem (QP) is always regular for any given point $\overline{x}$.

Theorem 5.1 (Robust Solution Characterization) *Let $\overline{x}$ be a robust feasible point of (QP). Suppose that (QP) is regular with respect to $\overline{x}$ and that there exists $x_0 \in \mathbb{R}^n$ such that*

$$\frac{1}{2}x_0^T Bx_0 + b^T x_0 + \beta < 0, \ \forall\, (B, b) \in \overline{\mathcal{V}}.$$

Then, $\overline{x}$ is a robust solution of (QP) if and only if there exist $\lambda \geq 0, \mu \in [\mu_1, \mu_2]$ and $\delta \in [\delta_1, \delta_2]$ such that

$$\begin{cases} (A + \lambda(B_1 + \mu B_2))\,\overline{x} = -\,(a + \lambda(b_1 + \delta b_2)) & \text{(First-order Condition)} \\ \lambda(\frac{1}{2}\overline{x}^T (B_1 + \mu B_2)\overline{x} + (b_1 + \delta b_2)^T \overline{x} + \beta) = 0 & \text{(Complementary-Slackness)} \\ A + \lambda(B_1 + \mu B_2) \succeq 0 & \text{(Second-order Condition).} \end{cases} \tag{5.5}$$

Proof (Necessary Part.) Let $\overline{x}$ be a robust solution of problem (QP) and let $\alpha := -\frac{1}{2}\overline{x}^T A\overline{x} - a^T\overline{x}$. Then,

$$\forall\, (B, b) \in \overline{\mathcal{V}}, \ \frac{1}{2}x^T Bx + b^T x + \beta \leq 0 \Rightarrow \frac{1}{2}x^T Ax + a^T x + \alpha \geq 0.$$

This implies that the following system has no solution:

$$\left[\frac{1}{2}x^T Ax + a^T x + \alpha < 0 \text{ and } \forall\, (B, b) \in \overline{\mathcal{V}}, \ \frac{1}{2}x^T Bx + b^T x + \beta < 0\right]. \tag{5.6}$$

[Homogenization.] We first show that the homogeneous system in $\mathbb{R}^{n+1}$

$$\left[(x, t) \in \mathbb{R}^n \times \mathbb{R}, \max\left\{\frac{1}{2}x^T Bx + tb^T x + \beta t^2 : (B, b) \in \overline{\mathcal{V}}\right\} < 0 \ \text{ and } \ \frac{1}{2}x^T Ax + ta^T x + \alpha t^2 < 0\right]$$

also has no solution. Otherwise, there exists $(\tilde{x}, \tilde{t}) \in \mathbb{R}^{n+1}$ such that

$$\frac{1}{2}\tilde{x}^T A\tilde{x} + \tilde{t}a^T \tilde{x} + \alpha\tilde{t}^2 < 0 \text{ and } \max\left\{\frac{1}{2}\tilde{x}^T B\tilde{x} + \tilde{t}b^T \tilde{x} + \beta\tilde{t}^2 : (B, b) \in \overline{\mathcal{V}}\right\} < 0.$$

If $\tilde{t} \neq 0$, then, we have

$$\frac{1}{2}\left(\frac{\tilde{x}}{\tilde{t}}\right)^T A\left(\frac{\tilde{x}}{\tilde{t}}\right) + a^T\left(\frac{\tilde{x}}{\tilde{t}}\right) + \alpha < 0 \text{ and } \max\left\{\frac{1}{2}\left(\frac{\tilde{x}}{\tilde{t}}\right)^T B\left(\frac{\tilde{x}}{\tilde{t}}\right) + b^T\left(\frac{\tilde{x}}{\tilde{t}}\right) + \beta : (B, b) \in \overline{\mathcal{V}}\right\} < 0$$

which contradicts the fact that the system (5.6) has no solution. On the other hand, if $\tilde{t} = 0$, then we have

$$\frac{1}{2}\tilde{x}^T A\tilde{x} < 0 \text{ and } \max\left\{\frac{1}{2}\tilde{x}^T B\tilde{x} : B = B_1 + \mu B_2, \mu \in [\mu_1, \mu_2]\right\} < 0.$$

Let $x_s = s\tilde{x}$. Then, as $s \to +\infty$, $\frac{1}{2}x_s^T Ax_s + a^T x_s + \alpha \to -\infty$ and

$$\max\left\{\frac{1}{2}x_s^T Bx_s + b^T x_s + \beta : (B, b) \in \overline{\mathcal{V}}\right\} = \max\left\{s^2\frac{1}{2}\tilde{x}^T B\tilde{x} + sb^T \tilde{x} + \beta : (B, b) \in \overline{\mathcal{V}}\right\} \to -\infty$$

This shows us that for large enough s, x_s satisfies (5.6) which is a contradiction. So, we see that the following homogeneous system in $\mathbb{R}^{n+1}$ has no solution

$$\left[\frac{1}{2}x^T Ax + ta^T x + \alpha t^2 < 0 \text{ and } \max\left\{\frac{1}{2}x^T Bx + tb^T x + \beta t^2 : (B, b) \in \overline{\mathcal{V}}\right\} < 0\right]. \tag{5.7}$$

[Dualization]. Now, we show that there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+\backslash\{0\}$, $\mu \in [\mu_1, \mu_2]$ and $\delta \in [\delta_1, \delta_2]$ such that

$$\overline{\lambda}_1\left(\frac{1}{2}x^T Ax + a^T x + \alpha\right) + \overline{\lambda}_2\left(\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x + \beta\right) \geq 0,\ \forall x \in \mathbb{R}^n.$$

We now split the proof into two cases: Case 1, $\mu_1 < \mu_2$; Case 2, $\mu_1 = \mu_2$.

Suppose that Case 1 holds. Let

$$W_1 = \begin{pmatrix} B_1 & b_1 + \frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_2 - \mu_1}b_2 \\ (b_1 + \frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_2 - \mu_1}b_2)^T & 2\beta \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} B_2 & \frac{\delta_2 - \delta_1}{\mu_2 - \mu_1}b_2 \\ \frac{\delta_2 - \delta_1}{\mu_2 - \mu_1}b_2^T & 0 \end{pmatrix}.$$

Then,

$$W_1 + \mu_1 W_2 = \begin{pmatrix} B_1 + \mu_1 B_2 & b_1 + \delta_1 b_2 \\ (b_1 + \delta_1 b_2)^T & 2\beta \end{pmatrix} \text{ and } W_1 + \mu_2 W_2 = \begin{pmatrix} B_1 + \mu_2 B_2 & b_1 + \delta_2 b_2 \\ (b_1 + \delta_2 b_2)^T & 2\beta \end{pmatrix}.$$

It follows from (5.7) that

$$\frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T H_0 \begin{pmatrix} x \\ t \end{pmatrix} < 0 \text{ and } \forall\, \mu \in [\mu_1, \mu_2],\ \frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (W_1 + \mu W_2)\begin{pmatrix} x \\ t \end{pmatrix} < 0$$

has no solution. From the assumption, we see that

$$\{(z^T H_0 z, z^T(W_1 + \mu_1 W_2)z, z^T(W_1 + \mu_2 W_2)z) : z \in \mathbb{R}^{n+1}\} \text{ is convex.}$$

So, Theorem 3.1 implies that there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+\backslash\{0\}$, $\mu \in [\mu_1, \mu_2]$, and for each $(x, t)^T \in \mathbb{R}^{n+1}$,

$$\overline{\lambda}_1\left(\frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T H_0 \begin{pmatrix} x \\ t \end{pmatrix}\right) + \overline{\lambda}_2\left(\frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (W_1 + \mu W_2)\begin{pmatrix} x \\ t \end{pmatrix}\right) \geq 0.$$

In particular, letting $t = 1$, we see that, for each $x \in \mathbb{R}^n$

$$\overline{\lambda}_1\left(\frac{1}{2}x^T Ax + a^T x + \alpha\right)$$
$$+\overline{\lambda}_2\left(\frac{1}{2}x^T(B_1 + \mu B_2)x + \left(b_1 + \left(\frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_2 - \mu_1} + \mu\frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_2 - \mu_1}\right)b_2\right)^T x + \beta\right) \geq 0.$$

Let $\delta := \frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_2 - \mu_1} + \mu\frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_2 - \mu_1}$. As $\mu \in [\mu_1, \mu_2]$, we see that $\delta \in [\delta_1, \delta_2]$ and

$$\overline{\lambda}_1\left(\frac{1}{2}x^T Ax + a^T x + \alpha\right) + \overline{\lambda}_2\left(\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x + \beta\right) \geq 0.$$

Suppose that Case 2 holds and let $\mu_1 = \mu_2 = \mu$ and let

$$W_1 = \begin{pmatrix} B_1 + \mu B_2 & b_1 \\ b_1^T & 2\beta \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} 0_{n\times n} & b_2 \\ b_2^T & 0 \end{pmatrix}.$$

Then,

$$W_1 + \delta_1 W_2 = \begin{pmatrix} B_1 + \mu B_2 & b_1 + \delta_1 b_2 \\ (b_1 + \delta_1 b_2)^T & 2\beta \end{pmatrix} \text{ and } W_1 + \delta_2 W_2 = \begin{pmatrix} B_1 + \mu B_2 & b_1 + \delta_2 b_2 \\ (b_1 + \delta_2 b_2)^T & 2\beta \end{pmatrix}.$$

It follows from (5.7) that

$$\frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T H_0 \begin{pmatrix} x \\ t \end{pmatrix} < 0 \text{ and } \forall\, \delta \in [\delta_1, \delta_2],\ \frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (W_1 + \delta W_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0$$

has no solution. From the assumption, we see that

$$\{(z^T H_0 z, z^T (W_1 + \delta_1 W_2)z, z^T (W_1 + \delta_2 W_2)z) : z \in \mathbb{R}^{n+1}\} \text{ is convex.}$$

So, Theorem 3.1 gives us that there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+ \backslash \{0\}$, $\delta \in [\delta_1, \delta_2]$ such that for each $(x, t)^T \in \mathbb{R}^{n+1})$,

$$\overline{\lambda}_1 \left(\frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T H_0 \begin{pmatrix} x \\ t \end{pmatrix} \right) + \overline{\lambda}_2 \left(\frac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (W_1 + \delta W_2) \begin{pmatrix} x \\ t \end{pmatrix} \right) \geq 0.$$

Letting $t = 1$, we see that, for each $x \in \mathbb{R}^n$

$$\overline{\lambda}_1 \left(\frac{1}{2} x^T A x + a^T x + \alpha \right) + \overline{\lambda}_2 \left(\frac{1}{2} x^T (B_1 + \mu B_2) x + (b_1 + \delta b_2)^T x + \beta \right) \geq 0.$$

Therefore, there exist $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}^2_+ \backslash \{0\}$, $\mu \in [\mu_1, \mu_2]$ and $\delta \in [\delta_1, \delta_2]$ such that

$$\overline{\lambda}_1 \left(\frac{1}{2} x^T A x + a^T x + \alpha \right) + \overline{\lambda}_2 \left(\frac{1}{2} x^T (B_1 + \mu B_2) x + (b_1 + \delta b_2)^T x + \beta \right) \geq 0,\ \ \forall x \in \mathbb{R}^n.$$

[Simplification]. Clearly, $\overline{\lambda}_1 \neq 0$, Otherwise, we have $\overline{\lambda}_2 \big(x^T (B_1 + \mu B_2) x + (b_1 + \delta b_2)^T x + \beta \big) \geq 0$ for all $x \in \mathbb{R}^n$. Letting $x = x_0$, we see that $\overline{\lambda}_2 \leq 0$. So, $\overline{\lambda}_2 = 0$ which contradicts $(\overline{\lambda}_1, \overline{\lambda}_2) \neq \{0\}$. Thus, $\overline{\lambda}_1 > 0$, and so,

$$\left(\frac{1}{2} x^T A x + a^T x + \alpha \right) + \lambda \left(\frac{1}{2} x^T (B_1 + \mu B_2) x + (b_1 + \delta b_2)^T x + \beta \right) \geq 0,\ \ \forall x \in \mathbb{R}^n,$$

where $\lambda = \frac{\overline{\lambda}_2}{\overline{\lambda}_1} \geq 0$. Now, letting $x = \overline{x}$, we see that

$$\lambda \left(\frac{1}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x} + (b_1 + \delta b_2)^T \overline{x} + \beta \right) \geq 0.$$

This together with the fact that $\lambda \geq 0$ and $\overline{x}$ is robust feasible yields that

$$\lambda \left(\frac{1}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x} + (b_1 + \delta b_2)^T \overline{x} + \beta \right) = 0.$$

Note that $\frac{1}{2} \overline{x}^T A \overline{x} + a^T \overline{x} + \alpha = 0$. It follows that, for each $x \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{1}{2} x^T A x + a^T x + \alpha + \lambda \left(\frac{1}{2} x^T (B_1 + \mu B_2) x + (b_1 + \delta b_2)^T x + \beta \right) \\ &\quad \geq \frac{1}{2} \overline{x}^T A \overline{x} + a^T \overline{x} + \alpha + \lambda \left(\frac{1}{2} \overline{x}^T (B_1 + \mu B_2) \overline{x} + (b_1 + \delta b_2)^T \overline{x} + \beta \right). \end{aligned}$$

Then, $h(x) = \left(\frac{1}{2}x^T Ax + a^T x + \alpha\right) + \lambda\left(\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x + \beta\right)$ attains its global minimizer at $\overline{x}$, and hence $\nabla h(\overline{x}) = 0$ and $\nabla^2 h(\overline{x}) \succeq 0$. They give us that

$$\left(A + \lambda(B_1 + \mu B_2)\right)\overline{x} = -\left(a + \lambda(b_1 + \delta b_2)\right) \text{ and } A + \lambda(B_1 + \mu B_2) \succeq 0.$$

[Sufficient Part.] Suppose that there exist $\lambda \geq 0$, $\mu \in [\mu_1, \mu_2]$ and $\delta \in [\delta_1, \delta_2]$ such that $\left(A+\lambda(B_1+\mu B_2)\right)\overline{x} = -\left(a+\lambda(b_1+\delta b_2)\right)$, $\lambda(\frac{1}{2}\overline{x}^T(B_1+\mu B_2)\overline{x}+(b_1+\delta b_2)^T\overline{x}+\beta) = 0$ and $A + \lambda(B_1 + \mu B_2) \succeq 0$. Then, for any robust feasible point x, we have $\forall (B, b) \in \overline{\mathcal{V}}$, $\frac{1}{2}x^T Bx + b^T x + \beta \leq 0$, and so,

$$\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x + \beta \leq 0.$$

By the complementary slackness condition, we have

$$\lambda\left(\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x\right) \leq -\lambda\beta = \lambda\left(\frac{1}{2}\overline{x}^T(B_1 + \mu B_2)\overline{x} + (b_1 + \delta b_2)^T\overline{x}\right).$$

Let $h(x) = \left(\frac{1}{2}x^T Ax + a^T x\right) + \lambda(x^T(B_1+\mu B_2)x+(b_1+\delta b_2)^T x+\beta)$. Then, the first-order condition and the second-order condition give us that h is a convex function with $\nabla h(\overline{x}) = 0$. So, $h(x) \geq h(\overline{x})$. So,

$$\begin{aligned}\frac{1}{2}x^T Ax + a^T x &\geq \frac{1}{2}\overline{x}^T A\overline{x} + a^T\overline{x} + \lambda\left(\frac{1}{2}\overline{x}^T(B_1 + \mu B_2)\overline{x} + (b_1 + \delta b_2)^T\overline{x} + \beta\right)\\ &\quad -\lambda\left(\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x + \beta\right)\\ &\geq \frac{1}{2}\overline{x}^T A\overline{x} + a^T\overline{x}.\end{aligned}$$

So, $\overline{x}$ is a robust solution of (QP). □

Corollary 5.1 *Let $n \geq 2$ and $\overline{x}$ be a robust feasible point. Suppose that there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 H_1 + \gamma_2 H_2 \succ 0$. and that there exists $x_0 \in \mathbb{R}^n$ such that*

$$\frac{1}{2}x^T Bx + b^T x + \beta < 0,\ \forall (B, b) \in \overline{\mathcal{V}}.$$

Then, $\overline{x}$ is a robust solution of (QP) if and only if there exist $\lambda \geq 0$, $\mu \in [\mu_1, \mu_2]$ and $\delta \in [\delta_1, \delta_2]$ such that $\left(A + \lambda(B_1 + \mu B_2)\right)\overline{x} = -\left(a + \lambda(b_1 + \delta b_2)\right)$, $\lambda(\frac{1}{2}\overline{x}^T(B_1 + \mu B_2)\overline{x} + (b_1 + \delta b_2)^T\overline{x} + \beta) = 0$ and $A + \lambda(B_1 + \mu B_2) \succeq 0$.

Proof As $\gamma_1 H_1 + \gamma_2 H_2 \succ 0$, Proposition 3.2 implies that (QP) is regular for any feasible point $\overline{x}$. So, the conclusion follows from the preceding theorem. □

We finish this Section by providing an example to illustrate our robust solution characterization.

Example 5.1 Consider the following nonconvex quadratic problem with data uncertainty

$$\begin{aligned}&\min\ x_1^2 + x_2^2 - \frac{3}{2}x_3^2 + x_3\\ &\text{s.t}\ \ a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_3 \leq 1,\end{aligned}$$

where the data $a_1, a_2, a_3, a_4 \in \mathbb{R}$ are uncertain. Suppose that we know that the nominal value of a_1, a_3, a_4 are 1, 1, −0.5 with the possible error of ±0.5, and the nominal value of a_2 is 1 with the possible error of ± 1. Then, this problem can be captured by the following uncertain (QP):

$$(QP) \quad \min_{x\in\mathbb{R}^3} \frac{1}{2}x^T Ax + a^T x$$
$$\text{s.t.} \quad \frac{1}{2}x^T Bx + b^T x + \beta \le 0,$$

where $\beta = -1$,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix},\ a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and $(B, b) \in S^3 \times \mathbb{R}^3$ is uncertain and

$$(B, b) \in \overline{\mathcal{V}} = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}$$

with $\mu_1 = -1, \mu_2 = 1, \delta_1 = -1, \delta_2 = 1$,

$$B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 0 \\ 0 \\ -0.5 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 0 \\ -0.5 \end{pmatrix}.$$

Then,

$$H_1 = \begin{pmatrix} B_1 + \mu_1 B_2 & b_1 + \delta_1 b_2 \\ (b_1 + \delta_1 b_2)^T & 2\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} B_1 + \mu_2 B_2 & b_1 + \delta_2 b_2 \\ (b_1 + \delta_2 b_2)^T & 2\beta \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}.$$

So, this problem is regular as

$$(-2)H_1 + H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \succ 0.$$

Let $\overline{x} = (0, 0, \frac{1-\sqrt{7}}{3})^T$, $\mu = \delta = 1$ and $\lambda = 1$. It can be verified that $\overline{x}$ is robust feasible. Moreover, we see that

$$\begin{cases} (A + \lambda(B_1 + \mu B_2))\,\overline{x} = (0, 0, 0)^T = -\big(a + \lambda(b_1 + \delta b_2)\big) & \text{(First-order Condition)} \\ \lambda\left(\frac{1}{2}\overline{x}^T(B_1 + \mu B_2)\overline{x} + (b_1 + \delta b_2)^T\overline{x} + \beta\right) = 0 & \text{(Complementary Slackness)} \\ A + \lambda(B_1 + \mu B_2) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \succeq 0 & \text{(Second-order Condition).} \end{cases} \tag{5.8}$$

So, our robust solution characterization is satisfied at $x = \overline{x}$. Moreover, we can directly verify that $\overline{x}$ is a robust solution. To see this, we note that for any robust feasible point

$x = (x_1, x_2, x_3)^T$, we have $\frac{1}{2}x^T(B_1 + \mu B_2)x + (b_1 + \delta b_2)^T x + \beta \leq 0$ with $\mu = 1$ and $\delta = 1$, i.e.,

$$1.5x_1^2 + 2x_2^2 + 1.5x_3^2 - x_3 \leq 1.$$

This gives us that, for any robust feasible point $x = (x_1, x_2, x_3)^T$, we must have $x_3 \geq -1 + 1.5x_1^2 + 2x_2^2 + 1.5x_3^2$ and hence the objective value

$$\frac{1}{2}x^T Ax + a^T x = x_1^2 + x_2^2 - 1.5x_3^2 + x_3 \geq -1 + 2.5x_1^2 + 3x_2^2 \geq -1.$$

As $\frac{1}{2}\overline{x}^T A\overline{x} + a^T\overline{x} = -1$, we see that $\overline{x}$ is a robust solution.

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