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Strong convergence for maximal monotone operators, relatively quasi-nonexpansive mappings, variational inequalities and equilibrium problems

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Abstract In this paper, we introduce a new hybrid iterative scheme for finding a common element of the set of zeroes of a maximal monotone operator, the set of fixed points of a relatively quasi-nonexpansive mapping, the sets of solutions of an equilibrium problem and the variational inequality problem in Banach spaces. As applications, we apply our results to obtain strong convergence theorems for a maximal monotone operator and quasi-nonexpansive mappings in Hilbert spaces and we consider a problem of finding a minimizer of a convex function.

Keywords Hybrid projection method · Inverse-strongly monotone operator · Variational inequality · Equilibrium problem · Relatively quasi-nonexpansive mapping · Maximal monotone operator

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1 Introduction

The maximal monotone inclusion problem provides a powerful general framework for the study of many important optimization problems, such as convex programming problems and variational inequalities. One of the most interesting and important problems in the theory of maximal monotone operators is to find a zero point of maximal monotone operators. This problem contains the convex minimization problem and the variational inequality problem. A popular method for approximating this problem is called the proximal point algorithm introduced by Martinet [24] in a Hilbert space. In 1976, Rockafellar [33] extended the knowledge of Martinet [24] and proved the weak convergence of the proximal point algorithm. The proximal point algorithm of Rockafellar [33] is a successful algorithm for finding a zero point of maximal monotone operators. It gives an approximation to solutions of a variational inequality for monotone operators, and when the monotone operator be subdifferential of a proper, convex, and lower semicontinuous function, it gives an approximation to solutions of a minimization problem for the convex function.

Let E be a Banach space with the dual space E^* and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of E. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

The equilibrium problem (for short, EP) is as follows: Find $\hat{x} \in C$ such that

$$\theta(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\theta)$.

In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\theta)$. In other words, the $EP(\theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics and others. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\theta)$ (see, for example, [3,19] and references therein) and some solution methods have been proposed to solve the $EP(\theta)$ (see, for example, [3,7–10,13,16,17,21,27,29,31] and references therein).

Let $A : C \to E^*$ be an operator. *The classical variational inequality problem* for an operator A is as follows: Find $\hat{z} \in C$ such that

$$\langle A\hat{z}, y - \hat{z}\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solution of (1.2) is denote by $VI(A, C)$. This problems is interesting and have been studied by many mathematician because it includes various problems in many branches in mathematics and sciences, for example, linear programming, convex optimization problems, economics and physics. Let $A : C \to E^*$ be a mapping. Then A is said to be:

(1) *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(2) $\alpha-$*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C \quad \text{and} \quad x \neq y.$$

The class of inverse-strongly monotone mappings have been studied by many authors to approximating a common fixed point (see [17,21,37,44] for more details).

(3) An operator $B \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in B$. We denote the set $\{x \in E : 0 \in Bx\}$ by $B^{-1}0$.

(4) The monotone operator B is said to be *maximal* if its graph $G(B) = \{(x, y^*) : y^* \in Bx\}$ is not properly contained in the graph of any other monotone operator.

If B is maximal monotone, then the solution set $B^{-1}0$ is closed and convex. Let B be a monotone operator satisfying $D(B) \subset C \subset J^{-1}(\bigcap_{r>0} R(J + rB))$, where $D(B)$ is domain of B and $R(J + rB)$ is range of $J + rB$. Define the *resolvent* $J_r : C \to D(B)$ of B by $J_r x = x_r$. In other words, $J_r = (J + rB)^{-1} J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(B)$. For any $r > 0$, the *Yosida approximation* of B define by $B_r x = (Jx - JJ_r x)/r$. We know that $B_r x \in B(J_r x)$ for all $r > 0$ and $x \in E$.

Consider the problem: Find $v \in E$ such that

$$0 \in Bv, \tag{1.3}$$

where B is an operator from E into E^*. Such $v \in E$ is called a *zero point* of B. When B is a maximal monotone operator, a well-known method for solving (1.3) in a Hilbert space H is the *proximal point algorithm*: $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad \forall n \geq 1, \tag{1.4}$$

where $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n B)^{-1}$. Rockafellar [33] proved that the sequence $\{x_n\}$ converges weakly to an element of $B^{-1}0$.

Let C be a closed convex subset of E. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Recall that a mapping $T : C \to C$ is *closed* if, for each $\{x_n\}$ in C, $x_n \to x$ and $Tx_n \to y$ imply that $Tx = y$.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$.

The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x + y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.5}$$

where J is the normalized duality mapping. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.6}$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$.

Remark 1.1 If E is a reflexive, strictly convex and smooth Banach space, then, for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.5), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [12,38] for more details).

Definition 1.2 (1) A point p in C is called an *asymptotic fixed point* of T [34] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.
(2) A mapping T from C into itself is called *relatively nonexpansive* ([28,36,42]) if

(R1) $F(T)$ is nonempty;
(R2) $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;
(R3) $\widehat{F}(T) = F(T)$.

A mapping T is called *relatively quasi-nonexpansive* (or *quasi-ϕ-nonexpansive*) if the conditions $(R1)$ and $(R2)$ hold. Obviously, every relatively nonexpansive mapping is relatively quasi-nonexpansive mappings, but the converse is not true. The relatively quasi-nonexpansive mapping is sometimes called *hemirelatively nonexpansive mapping*. The asymptotic behavior of a relatively nonexpansive mapping was studied in [4,5,11]. The class of relatively quasi-nonexpansive mappings is more general than that of relatively nonexpansive mappings (see [4,5,11,25,35]) which requires the strong restriction: $F(T) = \widehat{F}(T)$.

On the author hand, the *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{1.7}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and the strict monotonicity of the mapping J (see, for example [1,2,12, 18,38]). If E is a Hilbert space, Π_C becomes the metric projection of E onto C.

Example 1.3 [29] Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

In 2004, Matsushita and Takahashi [26] introduced the following iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \quad \forall n \geq 0, \tag{1.8}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, $T : C \to C$ is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2005, Matsushita and Takahashi [25] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for a relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{1.9}$$

They proved that $\{x_n\}$ converges strongly to a point $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In 2008, Iiduka and Takahashi [14] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \geq 1, \tag{1.10}$$

where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.10) converges weakly to an element of $VI(A, C)$.

In 2009, Inoue et al. [15] proposed the hybrid method in a uniformly convex and uniformly smooth Banach space E for defined a sequence $\{x_n\}$ as follows:

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSJ_{r_n}x_n), \\ C_n = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n), \\ Q_n = \{z \in C_n : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 1, \end{cases} \tag{1.11}$$

and, under some control conditions, they proved that the sequence $\{x_n\}$ converge strongly to a point $\Pi_{F(S)\cap B^{-1}0}$.

In 2009, Klin-eam et al. [19] extended the result of Inoue et al. [15] for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space E by using a new hybrid method.

Recently, Takahashi and Zembayashi [39,40] studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces. Later, Qin et al. [30] introduced two kinds of iterative algorithms for the problem of finding zeros of maximal monotone operators. They proved weak and strong convergence theorems in a real Hilbert space. Also, they applied the results to a problem of finding a minimizer of a convex function.

In this paper, motivated and inspired by the work mentioned above of Inoue et al. [15], Klin-eam et al. [19], Matsushita and Takahashi [25] and Takahashi and Zembayashi [39,40], we introduce a new hybrid projection method for finding the fixed point set of relatively quasi-nonexpansive mappings, the set of variational inequality, the sets of solution of equilibrium problem and zeros of a maximal monotone operator in Banach spaces. As applications, we consider a problem of finding a minimizer of a convex function. The results presented in this paper improve and extend some recent results of Iiduka and Takahashi [14], Inoue et al. [15], Klin-eam et al. [19], Matsushita and Takahashi [25,26], Takahashi and Zembayashi [39,40] and given by some authors.

2 Preliminaries

We also need the following lemmas for the proof of our main results.

Let E be a Banach space with the dual space E^*. The *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Remark 2.1 Let E be a Banach space. Then the following are well known (see [12] for more details):

(1) If E is an arbitrary Banach space, then J is monotone and bounded.
(2) If E is a strictly convex, then J is strictly monotone.
(3) If E is a smooth, then J is single valued and semi-continuous.

(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(5) If E is reflexive, smooth and strictly convex, then the normalized duality mapping $J = J_2$ is single valued, one-to-one and onto.
(6) If E is reflexive, smooth and strictly convex, then J^{-1} is also single valued, one-to-one, onto and it is the duality mapping from E^* into E.
(7) If E is uniformly smooth, then E is smooth and reflexive.
(8) E is uniformly smooth if and only if E^* is uniformly convex.

Lemma 2.2 *[18] Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3 *[2] Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 *[2] Let E be a reflexive, strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.5 *[29] Let E be a real uniformly smooth and strictly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and relatively quasi-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

We make use of the following mapping $V : E^* \times E \to \mathbb{R}$ studied in Alber [2]:

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2, \quad \forall \varphi \in E^*, \ x \in E. \tag{2.1}$$

From the definition of the functional ϕ and V, we know that $V(Jy, x) = \phi(x, y)$ for all $x, y \in E$.

Definition 2.6 [2] (1) An operator $\Pi_C : E^* \to C$ is called the *generalized projection operator*—if it associates with an arbitrary fixed point $\varphi \in E^*$ to the minimum point of the functional $V(\varphi, x)$, i.e., a solution to the minimization problem:

$$V(\varphi, \Pi_C(\varphi)) = \inf_{y \in C} V(\varphi, y).$$

(2) $\Pi_C(\varphi) \in C \subset B$ is called the *generalized projection* of the point φ.

Remark 2.7 The following properties of V and Π_C hold (see [2,22] for more detail):

(1) $V(\varphi, x)$ is continuous.
(2) $V(\varphi, x)$ is convex with respect to φ when x is fixed and convex with respect to x when φ is fixed.
(3) $(\|\varphi\| - \|x\|)^2 \leq V(\varphi, x) \leq (\|\varphi\| + \|x\|)^2$.
(4) $V(\varphi, x) = 0$ if and only if $\varphi = Jx$.
(5) $V(J\Pi_C\varphi, x) \leq V(\varphi, x)$ for all $\varphi \in E^*$ and $x \in E$.
(6) $\Pi_C Jx = x$ for any $x \in C$.
(7) Π_C is monotone in E^*, i.e., for all $\varphi_1, \varphi_2 \in E^*$,

$$\langle \Pi_C\varphi_1 - \Pi_C\varphi_2, \varphi_1 - \varphi_2 \rangle \geq 0.$$

(8) If E is uniformly smooth, then $\varphi_1, \varphi_2 \in E^*$, we have

$$\|\Pi_C\varphi_1 - \Pi_C\varphi_2\| \leq 2R_1 g_E^{-1}(\|\varphi_1 - \varphi_2\|/R_1),$$

where $R_1 = (\|\Pi_C\varphi_1\|^2 + \|\Pi_C\varphi_2\|^2)^{1/2}$ and g_E^{-1} is the inverse function to g_E defined by the modulus of smoothness for a uniformly smooth Banach space E.

(9) By (8), we have Π_C continuous.

(10) If E is smooth, then, for any $\varphi \in E^*$ and $x \in C$,

$$x \in \Pi_C\varphi \Leftrightarrow \langle \varphi - Jx, x - y\rangle \geq 0, \quad \forall y \in C.$$

(11) If E is smooth, then, for any $\varphi \in E^*$ and $x \in \Pi_C\varphi$, the following inequality holds:

$$V(Jx, y) \leq V(\varphi, y) - V(\varphi, x), \quad \forall y \in C.$$

(12) The operator Π_C is single-valued if and only if E is strictly convex.

(13) If E is reflexive, then, for any $\varphi \in E^*$, $\Pi_C\varphi$ is a nonempty closed convex and bounded subset of C.

Lemma 2.8 *[23] Let E be a reflexive strictly convex and smooth Banach space, then* $\Pi_C = J^{-1}$.

Lemma 2.9 *[41] Let E be a uniformly convex Banach space. Then, for any* $r > 0$, *there exists a strictly increasing, continuous and convex function* $g : [0, 2r] \to R$ *with* $g(0) = 0$ *such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|), \quad \forall x, y \in B_r,\ t \in [0, 1],$$

where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.10 *[6] Let E be a uniformly convex and uniformly smooth Banach spaces. Then the following inequality hold*

$$\|\varphi + \Phi\|^2 \leq \|\varphi\|^2 + 2\langle \Phi, J^*(\varphi + \Phi)\rangle, \quad \forall \varphi, \Phi \in E^*.$$

Lemma 2.11 *[20] Let E be a smooth, strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and* $B : E \rightrightarrows E^*$ *be a monotone operator satisfying* $D(B) \subset C \subset J^{-1}(\cap_{r>0} R(J + rB))$. *Let* $r > 0$, J_r *and* B_r *be the resolvent and the Yosida approximation of B, respectively. Then the following hold:*

(i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$ *for all* $x \in C$ *and* $u \in B^{-1}0$;

(ii) $(J_r x, B_r x) \in B$ *for all* $x \in C$;

(iii) $F(J_r) = B^{-1}0$.

Lemma 2.12 *[32] Let E be a reflexive, strictly convex and smooth Banach space and* $B \subset E \times E^*$ *be a maximal monotone. Then* $R(J + rB) = E^*$ *for all* $r > 0$.

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, we assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for any $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1-t)x, y) \leq \theta(x, y);$$

(A4) for any $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

For example, let B be a continuous and monotone operator of C into E^* and define

$$\theta(x, y) = \langle Bx, y - x\rangle, \quad \forall x, y \in C.$$

Then θ satisfies the conditions $(A1)$–$(A4)$.

The following result is given in Blum and Oettli [3]:

Lemma 2.13 *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(A1)$-$(A4)$, $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$\theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.14 *[40] Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(A1) - (A4)$. For any $r > 0$ and $x \in E$, define a mapping $K_r : E \to C$ as follows:*

$$K_r x = \{z \in C : \theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C\}, \quad \forall x \in C.$$

Then the following hold:

(1) K_r is single-valued;
(2) K_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle;$$

(3) $F(K_r) = EP(\theta)$;
(4) $EP(\theta)$ is closed and convex.

Lemma 2.15 *[40] Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(A1)$-$(A4)$ and $r > 0$. Then, for any $x \in E$ and $q \in F(K_r)$,*

$$\phi(q, K_r x) + \phi(K_r x, x) \leq \phi(q, x).$$

Lemma 2.16 *[43] Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex real Banach space E and $A : C \to E^*$ be a continuous monotone mapping. For any $r > 0$, define a mapping $F_r : E \to C$ as follows:*

$$F_r x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C\}, \quad \forall x \in C.$$

Then the following hold:

(1) F_r is single-valued;
(2) $F(F_r) = VI(A, C)$;
(3) $VI(A, C)$ is closed and convex subset of C;
(4) $\phi(q, F_r x) + \phi(F_r x, x) \leq \phi(q, x)$ for all $q \in F(F_r)$.

3 Main results

In this section, we prove some new convergence theorems for finding a common solution of the set of common fixed points of relatively quasi nonexpansive mappings, the set of the variational inequality, the sets of solutions of the equilibrium problem and zeros of a maximal monotone operator in a real uniformly smooth and uniformly convex Banach space.

Theorem 3.1 *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $B : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(B) \subset C$ and $J_r = (J + rB)^{-1}J$ for all $r > 0$, where J is the duality mapping on E. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A1)–(A4). Let A be a continuous monotone mapping of C into E^* and $T : C \to C$ be a relatively quasi-nonexpansive mapping. Define two mappings $F_{r_n}, K_{r_n} : E \to C$ by*

$$F_{r_n}x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0, \ \ \forall y \in C\}$$

and

$$K_{r_n}x = \{z \in C : \theta(z, y) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0, \ \ \forall y \in C\}.$$

Assume that $\Theta := F(T) \cap B^{-1}0 \cap EP(\theta) \cap VI(A, C) \neq \emptyset$. For an initial point $x_1 \in E$ with $C_1 = C$, we define the iterative sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = F_{r_n}x_n, \\ y_n = \Pi_C(\alpha_n Jx_n + (1 - \alpha_n)JTJ_{r_n}z_n), \\ u_n = K_{r_n}y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}Jx_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a point $p \in \Theta$, where $p = \Pi_\Theta Jx_1$.

Proof We split the proof into seven steps as follows:

Step 1. We first show that C_{n+1} is closed and convex for each $n \geq 1$.

Clearly, $C_1 = C$ is closed and convex. From the definition of C_{n+1}, it is obvious that C_{n+1} is closed. Suppose that C_n is convex. Then for any $z \in C_n$, we know that $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2. \tag{3.2}$$

This inequality is affine in z and hence C_{n+1} is convex for each $n \geq 0$. Thus, C_{n+1} is closed and convex.

Step 2. We show that $\Theta \subset C_n$ for all $n \geq 1$ and $\{x_n\}$ is well defined.

We show by induction that $\Theta \subset C_n$ for all $n \geq 1$. Put $u_n = K_{r_n}y_n$ and $v_n = J_{r_n}z_n$ for all $n \geq 1$. From Lemma 2.14, it follows that K_{r_n} is a relatively quasi-nonexpansive mapping and $\Theta \subset C_1 = C$. Suppose that $\Theta \subset C_n$ for some $n \geq 1$. Let $q \in \Theta \subset C_n$. Since T is a relatively quasi-nonexpansive mapping, by nonexpansiveness of J_{r_n} (see [38, Theorem 4.6.3]), we have

$$\begin{aligned} \phi(q, u_n) &= \phi(q, K_{r_n}y_n) \\ &\leq \phi(q, y_n) \\ &= V(Jy_n, q) \\ &= V(J(\Pi_C(\alpha_n Jx_n + (1 - \alpha_n)JTJ_{r_n}z_n)), q) \\ &\leq V(\alpha_n Jx_n + (1 - \alpha_n)JTJ_{r_n}z_n, q) \\ &= V(\alpha_n Jx_n + (1 - \alpha_n)JTv_n, q) \\ &= \alpha_n V(Jx_n, q) + (1 - \alpha_n)V(JTv_n, q) \\ &= \alpha_n\phi(q, x_n) + (1 - \alpha_n)\phi(q, Tv_n) \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, v_n) \\
&= \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, J_{r_n} z_n) \\
&\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, z_n) \\
&= \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, F_{r_n} x_n) \\
&\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, x_n) \\
&\leq \phi(q, x_n). \qquad (3.3)
\end{aligned}
$$

This shows that $q \in C_{n+1}$ which implies that $\Theta \subset C_{n+1}$ and hence $\Theta \subset C_n$ for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We prove that $\{x_n\}$ is bounded.

From the definition of x_n that $x_n = \Pi_{C_n} Jx_1$, we have

$$V(Jx_1, x_n) \leq V(Jx_1, q), \quad \forall q \in \Theta. \qquad (3.4)$$

This implies that $\{V(Jx_1, x_n)\}$ is bounded. From the definition of V, it follows that $\{x_n\}$ is bounded and so $\{z_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{T v_n\}$ are also bounded.

Step 4. We show that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_n = \Pi_{C_n} Jx_1$ and $x_{n+1} = \Pi_{C_{n+1}} Jx_1$, we have

$$V(Jx_1, x_n) \leq V(Jx_1, x_{n+1}), \quad \forall n \geq 1, \qquad (3.5)$$

and hence $\{V(Jx_1, x_n)\}$ is nondecreasing. From (3.4) and (3.5), it follows that $\lim_{n\to\infty} V(Jx_1, x_n)$ exists. For any positive integers $m > n$, from $x_m = \Pi_{C_m} Jx_1 \in C_m \subset C_n$ and the property of V, we have

$$V(Jx_n, x_m) \leq V(Jx_1, x_m) - V(Jx_1, x_n), \quad \forall n \geq 1.$$

Taking $m, n \to \infty$, we have $\lim_{n\to\infty} V(Jx_n, x_m) = 0$ and also

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0. \qquad (3.6)$$

From Lemma 2.2, we get $\|x_n - x_m\| \to 0$ and so $\{x_n\}$ is a Cauchy sequence and, by the completeness of E and the closedness of C, we can assume that there exists $p \in C$ such that $x_n \to p \in C$ as $n \to \infty$.

Step 5. We show that $\|Ju_n - Jx_n\| \to 0$ as $n \to \infty$.

From Step 4, taking $m = n + 1$, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \qquad (3.7)$$

Form Lemma 2.2, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.8)$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \qquad (3.9)$$

Since $x_{n+1} = \Pi_{C_{n+1}} Jx_1 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 1.$$

Thus, by (3.7), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \qquad (3.10)$$

Again, applying Lemma 2.2, we get

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.11}$$

From $\|u_n - x_n\| = \|u_n - x_{n+1} + x_{n+1} - x_n\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, it follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.12}$$

Thus, since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.13}$$

Step 6. We show that $p \in \Theta$, where

$$\Theta := F(T) \cap B^{-1}0 \cap EP(\theta) \cap VI(A, C)$$

(a) We show that $p \in F(T)$ as $n \to \infty$. From (3.3), for any $q \in \Theta$, it follows that $\lim_{n\to\infty} \phi(q, z_n) = \phi(q, p)$. Since $z_n = F_{r_n} x_n$

$$\phi(z_n, x_n) = \phi(F_{r_n} x_n, x_n) \leq \phi(q, x_n) - \phi(q, F_{r_n} x_n) = \phi(q, x_n) - \phi(q, z_n) \to 0 \text{ as } n \to \infty$$

applying Lemma 2.2, we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0. \tag{3.15}$$

From (3.3), we have $\phi(q, v_n) \geq \frac{1}{1-\alpha_n}(\phi(q, u_n) - \alpha_n \phi(q, x_n))$ and so it follows from Lemma 2.12 that

$$\begin{aligned}
\phi(v_n, z_n) &= \phi(J_{r_n} z_n, z_n) \\
&\leq \phi(q, z_n) - \phi(q, J_{r_n} z_n) \\
&= \phi(q, z_n) - \phi(q, v_n) \\
&\leq \phi(q, z_n) - \frac{1}{1-\alpha_n}(\phi(q, u_n) - \alpha_n \phi(q, x_n)) \\
&\leq \phi(q, x_n) - \frac{1}{1-\alpha_n}(\phi(q, u_n) - \alpha_n \phi(q, x_n)) \\
&= \frac{1}{1-\alpha_n}(\phi(q, x_n) - \phi(q, u_n)) \\
&= \frac{1}{1-\alpha_n}(\|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle) \\
&\leq \frac{1}{1-\alpha_n}(\|x_n\|^2 - \|u_n\|^2 + 2|\langle q, Jx_n - Ju_n\rangle|) \\
&\leq \frac{1}{1-\alpha_n}\big(\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|q\|\|Jx_n - Ju_n\|\big).
\end{aligned}$$

It follows from $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, (3.12) and (3.13) that

$$\lim_{n\to\infty} \phi(v_n, z_n) = 0.$$

Thus, from Lemma 2.2, we also have

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.16}$$

It follows from (3.3) that

$$\phi(q, v_n) \le \phi(q, x_n). \tag{3.17}$$

On the other hand, we note that

$$\begin{aligned}\phi(q, x_n) - \phi(q, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|q\|\|Jx_n - Ju_n\|.\end{aligned}$$

Thus it follows from $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$ that

$$\phi(q, x_n) - \phi(q, u_n) \to 0 \tag{3.18}$$

as $n \to \infty$. Since $\{x_n\}$ and $\{Tv_n\}$ are bounded, $\{Jx_n\}$ and $\{JTv_n\}$ are also bounded. From Lemma 2.9, if $r = \sup_{n\ge 0}\{\|Jx_n\}, \|JTv_n\|\}$, then there exists a continuous strictly increasing convex function g such that

$$\begin{aligned}\phi(q, u_n) &= \phi(q, K_{r_n} y_n)\\ &= \phi(q, y_n)\\ &= V(Jy_n, q)\\ &\le V(\alpha_n Jx_n + (1-\alpha_n)JTJ_{r_n}z_n, q)\\ &= V(\alpha_n Jx_n + (1-\alpha_n)JTv_n, q)\\ &= \|\alpha_n Jx_n + (1-\alpha_n)JTv_n\|^2 - 2\langle \alpha_n Jx_n + (1-\alpha_n)JTv_n, q\rangle + \|q\|^2\\ &\le \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|JTv_n\|^2 - 2\alpha_n\langle Jx_n, q\rangle - 2(1-\alpha_n)\langle JTv_n, q\rangle + \|q\|^2\\ &\quad -\alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|)\\ &\le \alpha_n V(Jx_n, q) + (1-\alpha_n)V(JTv_n, q) - \alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|)\\ &= \alpha_n\phi(q, x_n) + (1-\alpha_n)\phi(q, Tv_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|)\\ &\le \alpha_n\phi(q, x_n) + (1-\alpha_n)\phi(q, v_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|)\\ &\le \alpha_n\phi(q, x_n) + (1-\alpha_n)\phi(q, x_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|)\\ &\le \phi(q, x_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|).\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n)g(\|Jx_n - JTv_n\|) \le \phi(q, x_n) - \phi(q, u_n).$$

It follows from $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ and (3.18) that

$$\lim_{n\to\infty} g(\|Jx_n - JTv_n\|) = 0.$$

From the property of g such that $g(0) = 0$, it follow that

$$\lim_{n\to\infty}\|Jx_n - JTv_n\| = 0.$$

Since E be a uniformly smooth Banach spaces, E^* is a uniformly convex Banach spaces. Further, since J^{-1} is uniformly norm to norm continuous on bounded set, we get

$$\lim_{n\to\infty}\|x_n - Tv_n\| = 0. \tag{3.19}$$

By using the triangle inequality, we have $\|v_n - x_n\| \le \|v_n - z_n\| + \|z_n - x_n\|$. Thus, from (3.14) and (3.16), it follows that

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.20}$$

Again, by using the triangle inequality, we have $\|v_n - Tv_n\| \leq \|v_n - x_n\| + \|x_n - Tv_n\|$. Thus, from (3.19) and (3.20), it follows that

$$\liminf_{n\to\infty} \|v_n - Tv_n\| = 0. \tag{3.21}$$

Therefore, it follows from the clossedness of T and (3.20) that $p \in F(T)$.

(b) We show that $p \in EP(\theta)$. From (3.3), we get $\phi(q, y_n) \leq \phi(q, x_n)$. From Lemma 2.14 and $u_n = K_{r_n} y_n$, we observe that

$$\begin{aligned}\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n)\\ &\leq \phi(q, y_n) - \phi(q, K_{r_n} y_n)\\ &\leq \phi(q, x_n) - \phi(q, K_{r_n} y_n)\\ &= \phi(q, x_n) - \phi(q, u_n).\end{aligned} \tag{3.22}$$

From (3.18) and Lemma 2.2, we get

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.23}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the condition $\{r_n\} \subset [d, \infty)$ for some $d > 0$, we have $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ as $n \to \infty$ and

$$\theta(u_n, y) + \tfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

By (A2), we have

$$\begin{aligned}\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle\\ &\geq -\theta(u_n, y)\\ &\geq \theta(y, u_n), \quad \forall y \in C,\end{aligned}$$

and $u_n \to p$ and so $\theta(y, p) \leq 0$ for all $y \in C$. For any $0 < t < 1$, define $y_t = ty + (1-t)p$. Then $y_t \in C$, which imply that $\theta(y_t, p) \leq 0$. From (A1), it follows that

$$0 = \theta(y_t, y_t) \leq t\theta(y_t, y) + (1-t)\theta(y_t, p) \leq t\theta(y_t, y).$$

Thus $\theta(y_t, y) \geq 0$. From (A3), we have $\theta(p, y) \geq 0$ for all $y \in C$ and so $p \in EP(\theta)$.

(c) We show that $p \in VI(A, C)$. From $F_{r_n} x_n = z_n \in C$, we have

$$\langle v - z_n, Az_n\rangle + \frac{1}{r_n}\langle v - z_n, Jz_n - Jx_n\rangle \geq 0,$$

that is,

$$\langle v - z_n, Az_n\rangle + \langle v - z_n, \tfrac{Jz_n - Jx_n}{r_n}\rangle \geq 0, \quad \forall v \in C. \tag{3.24}$$

For any $0 < t < 1$, define $v_t = tv + (1-t)p$. Then $v_t \in C$. It follows from (3.24) that

$$\langle v_t - z_n, Az_n\rangle + \langle v_t - z_n, \frac{Jz_n - Jx_n}{r_n}\rangle \geq 0, \quad \forall v_t \in C,$$

that is,

$$\langle v_t - z_n, Av_t\rangle \geq \langle v_t - z_n, Av_t\rangle - \langle v_t - z_n, Az_n\rangle - \langle v_t - z_n, \frac{Jz_n - Jx_n}{r_n}\rangle \geq 0, \quad \forall v \in C. \tag{3.25}$$

Thus, by the condition $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and (3.25), we have $\frac{Jz_n - Jx_n}{r_n} = 0$. Since A is monotone, we have

$$\langle v_t - z_n, Av_t\rangle \geq \langle v_t - z_n, Av_t - Az_n\rangle \geq 0$$

and so

$$\lim_{n\to\infty} \langle v_t - z_n, Av_t\rangle = \langle v_t - p, Av_t\rangle \geq 0,$$

since $z_n \to p$, that

$$\langle v - p, Av_t\rangle \geq 0, \quad \forall v \in C.$$

Again, taking $t \to 0$ in the inequality above,

$$\langle v - p, Ap\rangle \geq 0, \quad \forall v \in C.$$

This implies that $p \in VI(A, C)$.

(d) We show that $p \in B^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, it follows from (3.16) that

$$\lim_{n\to\infty} \|Jz_n - Jv_n\| = 0.$$

From the condition $\{r_n\} \subset [d, \infty)$ for some $d > 0$, we have

$$\lim_{n\to\infty} \frac{1}{r_n}\|Jz_n - Jv_n\| = 0.$$

thus, since $J_{r_n} z_n = v_n$, we have

$$\lim_{n\to\infty} \|B_{r_n} z_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|Jz_n - JJ_{r_n} z_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|Jz_n - Jv_n\| = 0.$$

From the monotonicity of B, for any $(w, w^*) \in G(B)$, we have $\langle w - v_n, w^* - B_{r_n} z_n\rangle \geq 0$ for all $n \geq 0$ and so, letting $n \to \infty$, we get $\langle w - p, w^*\rangle \geq 0$. So, from the maximality of B, we have $p \in B^{-1}0$. Therefore, it follows from (a), (b), (c) and (d) that $p \in \Theta$.

Step 7. we show that $p = \Pi_\Theta Jx_1$.

From the property of Π_C and $p \in \Theta$, we have

$$V(J\Pi_\Theta Jx_1, p) + V(Jx_1, \Pi_\Theta Jx_1) \leq V(Jx_1, p). \tag{3.26}$$

Since $x_{n+1} = \Pi_{C+1} Jx_1 \in C_{n+1}$ and $\Pi_\Theta \in C_{n+1}$ for all $n \geq 1$, it follows that

$$V(Jx_{n+1}, \Pi_\Theta Jx_1) + V(Jx_1, x_{n+1}) \leq V(Jx_1, \Pi_\Theta Jx_1). \tag{3.27}$$

By Remark 2.7(1), that V is continuous and $\lim_{n\to\infty} x_n = p$. Then we get

$$\lim_{n\to\infty} V(Jx_1, x_{n+1}) = V(Jx_1, p). \tag{3.28}$$

Thus, from (3.26), (3.27) and (3.28), we can conclude that

$$V(Jx_1, p) = V(Jx_1, \Pi_\Theta Jx_1),$$

that is, $p = \Pi_\Theta Jx_1$. The proof is completed. □

Setting $A \equiv 0$ in Theorem 3.1, then we obtain the following corollary:

Corollary 3.2 *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $B : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(B) \subset C$ and $J_r = (J + rB)^{-1} J$ for all $r > 0$, where J is the duality mapping on E. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A1)–(A4) and $T : C \to C$ be a relatively quasi-nonexpansive mapping. Define a mappings $K_{r_n} : E \to C$ by*

$$K_{r_n}x = \{z \in C : \theta(z, y) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\}, \quad \forall x \in E.$$

Assume that $\Theta := F(T) \cap B^{-1}0 \cap EP(\theta) \neq \emptyset$. For an initial point $x_1 \in E$ with $C_1 = C$, we define the iterative sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = \Pi_C(\alpha_n Jx_n + (1 - \alpha_n)JTJ_{r_n}x_n), \\ u_n = K_{r_n}y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}Jx_1, \quad \forall n \geq 1, \end{cases} \tag{3.29}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $p \in \Theta$, where $p = \Pi_\Theta Jx_1$.

Let E be a Banach space and $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the *subdifferential* of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^*\rangle + f(x),\ \forall y \in E\}, \quad \forall x \in E.$$

Then ∂f is a maximal monotone operator (see [38] for more details).

Corollary 3.3 *Let C be a nonempty closed, convex subset of a uniformly convex and uniformly smooth Banach space E. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A1)–(A4), A be a continuous monotone mapping of C into E^* and $T : C \to C$ be a relatively quasi-nonexpansive mapping. Define mappings $F_{r_n}, K_{r_n} : E \to C$ by*

$$F_{r_n}x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\}, \quad \forall x \in E,$$

and

$$K_{r_n}x = \{z \in C : \theta(z, y) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\}, \quad \forall x \in E.$$

Assume that $\Theta := F(T) \cap EP(\theta) \cap VI(A, C) \neq \emptyset$. For an initial point $x_1 \in E$ with $C_1 = C$, define the iterative sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = F_{r_n}x_n, \\ y_n = \Pi_C(\alpha_n Jx_n + (1 - \alpha_n)JTz_n), \\ u_n = K_{r_n}y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}Jx_1, \quad \forall n \geq 1, \end{cases} \tag{3.30}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $p \in \Theta$, where $p = \Pi_\Theta Jx_1$.

Proof Let $B = \partial i_C$ as in Theorem 3.1, where i_C is the indicator function, that is,

$$i_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & \text{otherwise}. \end{cases} \tag{3.31}$$

For any $x \in E$ and $r > 0$, it follows that

$$\begin{aligned} p = J_r x &\Longleftrightarrow Jp + r\partial i_C(p) \ni Jx \\ &\Longleftrightarrow Jx - Jp \in r\partial i_C(p) \\ &\Longleftrightarrow i_C(y) \geq \left\langle y - p, \frac{Jx - Jp}{r} \right\rangle + i_C(p), \ \forall y \in E \\ &\Longleftrightarrow 0 \geq \langle y - p, Jx - Jp \rangle, \ \forall y \in C \\ &\Longleftrightarrow p = argmin_{y \in C} \phi(y, x) \\ &\Longleftrightarrow p = \Pi_C x. \end{aligned}$$

Then we know that B is a maximal monotone operator and $J_r = \Pi_C$ for any $r > 0$. Thus, by Theorem 3.1, we obtain the conclusion. The proof is completed. □

Remark 3.4 Theorem 3.1 and Corollary 3.2 extend and improve the result of Inoue et al. [15] and Matsushita and Takahashi [25] in the following aspect:

1. from the viewpoint of computation, we remove Q_n (from the CQ-method to the shrinking projection method);
2. from the viewpoint of mappings, from relatively nonexpansive mapping to quasi -ϕ-nonexpansive mappings;
3. from the viewpoint of method, we modify and improve the result's Matsushita and Takahashi [25,26] and Iiduka and Takahashi [14] to the new method by using the generalized projection method, also we obtain a strong convergence theorem.

4 Applications

4.1 Application to Hilbert spaces

If $E = H$, a Hilbert space, then H is a uniformly smooth and uniformly convex Banach space E and every closed relatively quasi-nonexpansive mapping reduces to a closed quasi-nonexpansive mapping. Moreover, $J = I$ (: the identity operator on H) and $\Pi_C = P_C$ (: the projection mapping from H into C). Thus the following corollaries hold:

Theorem 4.1 *Let C be a nonempty closed and convex subset of a Hilbert space H. Let $B : H \rightrightarrows H$ be a maximal monotone operator satisfying $D(B) \subset C$ and $J_r = (I + rB)^{-1}$ for all $r > 0$. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(A1)$–$(A4)$, let A be a continuous monotone mapping of C into H and $T : C \to C$ be a quasi-nonexpansive mapping. Define mappings $F_{r_n}, K_{r_n} : H \to C$ by*

$$F_{r_n} x = \{z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}, \quad \forall x \in H,$$

and

$$K_{r_n} x = \{z \in C : \theta(z, y) + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}, \quad \forall x \in H.$$

Assume that $\Theta := F(T) \cap B^{-1}0 \cap EP(\theta) \cap VI(A, C) \neq \emptyset$. *For an initial point* $x_1 \in H$ *with* $C_1 = C$, *define the iterative sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = F_{r_n} x_n, \\ y_n = P_C(\alpha_n x_n + (1-\alpha_n) T J_{r_n} z_n), \\ u_n = K_{r_n} y_n, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $p \in \Theta$, *where* $p = P_\Theta x_1$.

Corollary 4.2 *Let* C *be a nonempty closed and convex subset of a Hilbert space* H. *Let* θ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying the conditions* $(A1)$–$(A4)$, *let* A *be a continuous monotone mapping of* C *into* H *and* $T : C \to C$ *be a quasi-nonexpansive mapping. Define mappings* $F_{r_n}, K_{r_n} : H \to C$ *by*

$$F_{r_n} x = \{z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}, \quad \forall x \in H,$$

and

$$K_{r_n} x = \{z \in C : \theta(z, y) + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}, \quad \forall x \in H.$$

Assume that $\Theta := F(T) \cap EP(\theta) \cap VI(A, C) \neq \emptyset$. *For an initial point* $x_1 \in H$ *with* $C_1 = C$, *define the iterative sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = F_{r_n} x_n, \\ y_n = P_C(\alpha_n x_n + (1-\alpha_n) T z_n), \\ u_n = K_{r_n} y_n, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $p \in \Theta$, *where* $p = P_\Theta x_1$.

Proof Let $B = \partial i_C$ as in Theorem 3.1, where i_C is the indicator function. For any $x \in H$ and $r > 0$, we have

$$\begin{aligned} p = J_r x &\iff p + r\partial i_C(p) \ni x \\ &\iff x - p \in r\partial i_C(p) \\ &\iff i_C(y) \geq \left\langle y - p, \frac{x - p}{r} \right\rangle + i_C(p), \ \forall y \in H \\ &\iff 0 \geq \langle y - p, x - p \rangle, \ \forall y \in C \\ &\iff p = P_C x. \end{aligned}$$

Then we know that B is a maximal monotone operator and $J_r = P_C$ for any $r > 0$. Thus, by Theorem 4.1, we obtain the conclusion. □

4.2 Application to a proper lower semi-continuous convex function

In this section, by using Theorem 3.1, we can consider the problem of finding a minimizer of a proper lower semi-continuous convex function f in a Banach space.

Theorem 4.3 *Let C be a nonempty closed and convex subset of a Banach space E. Let f be a proper lower semi-continuous convex function. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A1)-(A4), A be a continuous monotone mapping of C into E and $T : C \to C$ be a quasi-nonexpansive mapping. Define mappings $F_{r_n}, K_{r_n} : E \to C$ by Theorem 3.1.*

Assume that $\Theta := F(T) \cap \partial f^{-1}0 \cap EP(\theta) \cap VI(A, C) \neq \emptyset$. For an initial point $x_1 \in E$ with $C_1 = C$, define the iterative sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = F_{r_n} x_n, \\ v_n = \operatorname{argmin}_{w \in E}\{f(w) + \frac{1}{2r_n}\|w\|^2 + \frac{1}{r_n}\langle w, z_n\rangle\}, \\ y_n = \Pi_C(\alpha_n x_n + (1-\alpha_n)T v_n), \\ u_n = K_{r_n} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $p \in \Theta$, where $p = P_\Theta x_1$.

Proof Since $f : E \to (-\infty, +\infty]$ is a proper convex lower semi-continuous function, then, we obtain that the subdifferential ∂f of f is maximal monotone (see Rockafellar [33]). For $r > 0$ and $x \in E$, denote J_r be the resolvent of ∂f. Then we notice that

$$Jx \in JJ_r x + r\partial f(J_r x)$$

and hence

$$\begin{aligned} 0 &\in \partial f(J_r x) + \frac{1}{r} JJ_r x - \frac{1}{r} x \\ &= \partial\Big(f + \frac{1}{2r}\|\cdot\|^2 - \frac{1}{r} Jx\Big)(J_r x). \end{aligned} \tag{4.4}$$

This implies that

$$J_r x = \operatorname{argmin}_{w \in E}\Big\{f(w) + \frac{1}{2r}\|w\|^2 + \frac{1}{r}\langle w, Jx\rangle\Big\}, \tag{4.5}$$

that is, for $z_n \in C \subset E$, we have $v_n = \operatorname{argmin}_{w\in E}\{f(w) + \frac{1}{2r_n}\|w\|^2 + \frac{1}{r_n}\langle w, Jz_n\rangle\} = J_r z_n$. Thus, from Theorem 3.1, we can get the conclusion. The proof is completed. □

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