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# Refinements of existence results for relaxed quasimonotone equilibrium problems

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**Abstract** We consider a general equilibrium problem in a normed vector space setting and we establish sufficient conditions for the existence of solutions in compact and non compact cases. Our approach is based on the concept of upper sign property for bifunctions, which turns out to be a very weak assumption for equilibrium problems. In the framework of variational inequalities, this notion coincides with the upper sign continuity for a set-valued operator introduced by Hadjisavvas. More in general, it allows to strengthen a number of existence results for the class of relaxed $\mu$-quasimonotone equilibrium problems.

**Keywords** Equilibrium problems · Relaxed $\mu$-quasimonotonicity · Upper sign property · Coercivity conditions

**Mathematics Subject Classification (2000)** 90C47 · 49J35

## 1 Introduction

In scientific contexts the term "equilibrium" has been widely used at least in physics, chemistry, engineering and economics within different frameworks, relying on different mathematical models such as optimization, variational inequalities and noncooperative games among others. In turn, these mathematical models share an underlying common structure which allows to conveniently formulate them in a unique format. Therefore, theoretical studies developed for one of these models can be generally modified to cope with the others through the common format in a unifying language. This format reads

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0 \quad \text{for all } y \in K, \qquad \text{EP}(f, K)$$

where $K \subseteq \mathbb{X}$ is a nonempty convex set of the normed vector space $\mathbb{X}$ and $f : K \times K \to \mathbb{R}$ is a bifunction.

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This general problem was named "equilibrium problem" by Blum and Oettli [9], who stressed this unifying feature and provided a thorough investigation of its theoretical properties. Until then, this general format was investigated in different directions by several authors. To the best of our knowledge, Nikaido and Isoda [25] were the first to characterize Nash equilibria as the solutions of EP($f$, $K$) for an appropriate auxiliary bifunction. Next, a lot of papers made a positive contribution to this field (see for instance [4,26,28] and references therein).

A large number of applications has been described successfully via the concept of equilibrium solution and therefore many researchers devoted their efforts to study EP($f$, $K$). In fact, nowadays there is a good theory for equilibria and a rapidly increasing number of algorithms for finding them (see [5] for a recent survey).

In this paper we aim at studying the existence of equilibria under mild assumptions of continuity and monotonicity. The equilibrium problem EP($f$, $K$) has been initially studied assuming that $f$ is lower semicontinuous in its first argument and quasiconvex in its second one. Under such assumptions Ky Fan [12] proved existence of solutions assuming compactness of $K$. Afterwards the same result was established in [10] replacing the compactness of $K$ with a suitable coerciveness of $f$.

Subsequently many efforts were concentrated for establishing existence results under weaker topological assumptions (continuity and coerciveness) and the monotonicity of $f$ plays a fundamental role. We recall that a bifunction $f$ is said *monotone* if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in K.$$

However the monotonicity seems too restrictive for many applied problems. In the last decade, several authors have relaxed this rather restrictive assumption to certain types of generalized monotonicity. For instance, under the assumption of pseudomonotonicity, Flores-Bazán [14] provided a characterization of the nonemptiness of the set of solutions by using an approach based on recession notions coming from minimization problems. His result was improved in [8] where a very weak coercivity condition was proposed. Moreover in the same paper, existence results for quasimonotone equilibrium problems with $f$ explicitly quasiconvex with respect to the second variable were presented. In [19] Iusem and Sosa, exploiting the relation between equilibrium problems and certain auxiliary convex feasibility problems (which will be called Minty equilibrium problems in the sequel), together with extensions to equilibrium problems of gap functions for variational inequalities, established necessary and sufficient conditions for pseudomonotone equilibrium problems. Their results were extended in [18] assuming the weaker condition of properly quasimonotonicity and a sort of pseudoconvexity of $f$ with respect to the second variable. Recently Farajzadeh and Zafarani [13] proposed different existence results which are based on an interesting technique due to Aussel and Hadjisavvas [1] in the framework of variational inequalities. These results are established under a technical condition which is weaker than the explicit quasiconvexity of $f(x, \cdot)$.

All these results are usually obtained exploiting concepts and methods introduced in the framework of variational inequalities. For instance one of the key tools for deriving existence results for EP($f$, $K$) is to establish a link between the set of solutions of EP($f$, $K$) and the set of solutions of a suitable equilibrium problem:

$$\text{find } \bar{x} \in K \quad \text{such that } f(y, \bar{x}) \leq 0 \quad \text{for all } y \in K. \qquad \text{MEP}(f, K)$$

In the framework of variational inequalities, MEP($f$, $K$) collapses into the so-called Minty variational inequality and for this reason we will call it *Minty equilibrium problem*. This problem is often called *dual equilibrium problem* since it is deeply related to the duality principle for saddle point problems. Unfortunately in the particular case of optimization, MEP$(f, K)$ does not recover any of the well-known dual problems: indeed, if $f(x, y) = \varphi(y) - \varphi(x)$ where $\varphi : \mathbb{X} \to \mathbb{R}$ then, both EP$(f, K)$ and MEP$(f, K)$ reduce to the same minimization problem. For this reason we prefer to use the term "Minty" instead of "dual". In the sequel we will denote by $S(f, K)$ the set of solutions of EP$(f, K)$, and by $M(f, K)$ the set of solutions of MEP$(f, K)$.

The Minty equilibrium problem was initially introduced for variational inequalities [11], and its relevance to applications was pointed out in [15]. A well-known result, formulated by Minty in [24], states the equivalence of the Minty and Stampacchia variational inequalities under continuity and monotonicity assumptions of the involved operator. This is a recurrent theme in the analysis of the existence of solutions of EP$(f, K)$: first showing the inclusion $M(f, K) \subseteq S(f, K)$, afterwards proving the nonemptiness of $M(f, K)$. Since $M(f, K)$ could be empty, applying to equilibrium problems a definition introduced in [1] for variational inequalities, Bianchi and Pini [8] considered a weaker concept of solution for MEP$(f, K)$: the set of the *local Minty solutions* is

$$M_L(f, K) = \{x \in K \ : \ \exists r > 0 \text{ s.t. } f(y, x) \leq 0, \quad \forall y \in K \cap B(x, r)\}$$

where $B(x, r)$ is the ball with center $x$ and radius $r > 0$. Clearly $M_L(f, K)$ is larger than $M(f, K)$ and therefore there are more possibilities that $M_L(f, K)$ can be nonempty. Conversely, it is a harder task to show the inclusion $M_L(f, K) \subseteq S(f, K)$.

However, this inclusion was established in Lemma 2.1 of [8] (and subsequently in Lemma 2.1 of [13]) where the authors exploited a weak concept of continuity introduced in [17] in the framework of variational inequalities: the upper sign continuity. We recall that a set-valued operator $T : K \rightrightarrows \mathbb{X}^*$ with $T(x) \neq \emptyset$ for every $x \in K$, where $\mathbb{X}^*$ is the dual space of $\mathbb{X}$, is called *upper sign continuous* at $x \in K$ if for every $y \in K$ the following implication holds:

$$\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle \geq 0, \quad \forall t \in (0, 1) \ \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0 \tag{1}$$

where $z_t = (1 - t)x + ty$. Upper sign continuity is a very weak notion of continuity. For instance, any upper hemicontinuous operator, i.e. whose restriction to line segments of $K$ is upper semicontinuous with respect to the weak* topology on $\mathbb{X}^*$, is upper sign continuous. Moreover any positive function on $\mathbb{R}$ is upper sign continuous. The upper sign continuity plays an important role for guaranteeing the $D$-maximal pseudomonotonicity of a pseudomonotone map $T$ [17] and it was used in [1] for proving the existence of strong solutions of a quasimonotone Stampacchia variational inequality.

The analogous concept of upper sign continuity for bifunctions introduced in [8] is a workable definition for showing the link between $S(f, K)$ and $M_L(f, K)$ but it does not coincide with the original one in the case of variational inequalities. This is one of the reasons that leads us to propose a more appropriate notion for bifunctions. This new concept has the advantage of unifying several known techniques and it provides existence results for a large class of EP$(f, K)$.

The paper is organized as follows. In Sect. 2 we recall the main definitions of generalized convexity and relaxed monotonicity used in literature. Section. 3 is devoted to analyze the concept of upper sign continuity introduced in [8]. As just written, we propose an alternative concept (which will be called upper sign property) and we provide certain conditions under which the upper sign property is weaker than the upper sign continuity. However this property will turn out to be sufficient to guarantee the inclusion of $M_L(f, K)$ in $S(f, K)$, which is one of the key stages for the nonemptiness of $S(f, K)$. In an analogous way we generalize the upper sign property in order to handle the relaxed $\mu$-Minty solutions. Finally, Sect. 4 is devoted to establish sufficient conditions for the existence of solutions of EP$(f, K)$. As usual we start considering the equilibrium problem with a compact feasible set, afterwards we introduce a coercivity condition for avoiding the compactness of $K$. Comparisons with other recent results end the paper.

For the sake of the reader, we conclude this section fixing the main notations which will be used in the rest of the paper.

### 1.1 Notation

Let $\mathbb{X}$ be a normed vector space with norm $\|\cdot\|$, $\mathbb{X}^*$ the dual space, and $\langle\cdot,\cdot\rangle$ the duality pairing between $\mathbb{X}$ and $\mathbb{X}^*$. For each $x \in \mathbb{X}$ and $r > 0$ we denote by $B(x, r)$ the closed ball with center $x$ and radius $r$. Given a set $A \subseteq \mathbb{X}$, conv $A$ is the convex hull of $A$.

From now on $K \subseteq \mathbb{X}$ will be a nonempty convex set and $f : K \times K \to \mathbb{R}$ an equilibrium bifunction, namely $f(x, x) = 0$ for every $x \in K$. Fixed $x \in K$, we will often use the following two level sets:

$$\text{lev}(f, x) = \{y \in K \;:\; f(x, y) \leq 0\}$$

and

$$\text{lev}_s(f, x) = \{y \in K \;:\; f(x, y) < 0\}.$$

If $T : K \rightrightarrows \mathbb{X}^*$ is a set-valued operator with nonempty and weak* compact convex values we denote by $f_T$ the associated bifunction

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle = \max_{x^* \in T(x)} \langle x^*, y - x\rangle.$$

Using this notation clearly MEP$(f_T, K)$ coincides with the Minty variational inequality associated to $T$. Instead, invoking the Sion's minimax theorem, EP$(f_T, K)$ is equivalent to the Stampacchia variational inequality. Actually, since $K$ and $T(x)$ are convex, $T(x)$ is nonempty and weak* compact, $x^* \in X^*$, $y - x \in X^{**}$, applying the Sion's minimax theorem we get

$$\max_{x^* \in T(x)} \inf_{y \in K} \langle x^*, y - x\rangle = \inf_{y \in K} \max_{x^* \in T(x)} \langle x^*, y - x\rangle.$$

A point $\bar{x} \in K$ solves EP$(f_T, K)$ if and only if

$$\inf_{y \in K} \max_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0,$$

and hence if and only if there exists $x^* \in T(x)$ such that $\langle x^*, y - x\rangle \geq 0$, for all $y \in K$.

## 2 Preliminaries: generalized convexity and monotonicity

This section is devoted to recall the main notions of convexity and monotonicity which will be used in the sequel. Even if most of all the concepts are well known, for the sake of the completeness, we underline their properties and we furnish some examples.

### 2.1 Generalized convexity

A function $\varphi : K \to \mathbb{R}$ is called *quasiconvex* if for all $x, y \in K$

$$\varphi(tx + (1-t)y) \le \max\{\varphi(x), \varphi(y)\}, \quad \forall t \in [0, 1].$$

It is well known that, from the geometrical point of view, the quasiconvexity of $\varphi$ coincides with the convexity either of the lower level set

$$\{x \in K \ : \ \varphi(x) \le \alpha\}, \quad \forall \alpha \in \mathbb{R}$$

or of the strict lower level set

$$\{x \in K \ : \ \varphi(x) < \alpha\}, \quad \forall \alpha \in \mathbb{R}.$$

The function $\varphi$ is said to be *semistrictly quasiconvex* if for all $x, y \in K$ such that $\varphi(x) \neq \varphi(y)$ it holds that

$$\varphi(tx + (1-t)y) < \max\{\varphi(x), \varphi(y)\}, \quad \forall t \in (0, 1).$$

This class of functions was introduced by Karamardian [20] under the name of "strict quasiconvexity" but we prefer to use this latter term for denoting the subclass of the quasiconvex functions such that

$$\varphi(tx + (1-t)y) < \max\{\varphi(x), \varphi(y)\}, \quad \forall t \in (0, 1)$$

even when $\varphi(x) = \varphi(y)$. Every strictly quasiconvex function is semistrictly quasiconvex but, without any continuity property, semistrictly quasiconvex functions need not to be quasiconvex. For instance the function $\varphi(x) = \min\{0, x\}$ defined on $\mathbb{R}$ is quasiconvex but not semistrictly quasiconvex; instead the function

$$\varphi(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

is semistrictly quasiconvex but not quasiconvex. This leads to the following definition: a function is said *explicitly quasiconvex* if it is quasiconvex and also semistrictly quasiconvex.

In [18] a subclass of the quasiconvex functions was considered. A function $\varphi : K \to \mathbb{R}$ is called *pseudoconvex* if for all $x, y \in K$ and all $t \in (0, 1)$ it holds that

$$\varphi(tx + (1-t)y) \ge \varphi(x) \quad \Rightarrow \quad \varphi(y) \ge \varphi(tx + (1-t)y).$$

This notion was introduced by García Ramos and Sosa in [16] and it must not be confused with the notion of pseudoconvexity introduced by Ortega and Rheinboldt in [27]. It is easy to check that pseudoconvexity implies quasiconvexity and semistrict quasiconvexity but the converse does not hold as the previous examples show. Nevertheless pseudoconvexity and semistrict quasiconvexity coincide when $\varphi$ is lower hemicontinuous, i.e. for every $x \in K$

$$\varphi(x) \le \liminf_{t \to 0^+} \varphi((1-t)x + ty), \quad \forall y \in K.$$

For the sake of completeness we prove this fact.

**Lemma 1** *Every lower hemicontinuous and semistrictly quasiconvex function $\varphi : K \to \mathbb{R}$ is pseudoconvex.*

*Proof* By contradiction, suppose there exist $x, y \in K$ and $t \in (0, 1)$ such that $\varphi(z) \geq \varphi(x)$ and $\varphi(z) > \varphi(y)$, where $z = (1-t)x + ty$. Since $\varphi$ is semistrictly quasiconvex it follows that $\varphi(x) = \varphi(y)$ and $\varphi((1-s)z + sx) < \varphi(z)$ for all $s \in (0, 1)$. Moreover the lower hemicontinuity of $\varphi$ at $z$ implies there exists a suitable $s \in (0, 1)$ such that $\varphi(x) < \varphi(w) < \varphi(z)$ with $w = (1-s)z + sx$. This leads to a contradiction considering the semistrict quasiconvexity of $\varphi$ on the interval with extreme points $w$ and $y$. □

As immediate consequence, a lower hemicontinuous and semistrictly quasiconvex function is always quasiconvex.

### 2.2 Generalized monotonicity

For proving the existence of solutions of variational inequalities, various generalized monotonicity assumptions have been assumed for the set-valued operator $T : K \rightrightarrows \mathbb{X}^*$. A different kind of generalization, namely relaxed monotonicity (or else weak monotonicity, or global hypomonotonicity) was considered by various authors in relation with algorithms for finding a solution of variational inequalities [22]. Recently, those different generalizations were combined and the existence solutions of a variational inequality was shown for the case of single valued, densely $\mu$-pseudomonotone operators [3]. Bai and Hadjisavvas [2] introduced a broader class of set-valued relaxed $\mu$-quasimonotone operators and established existence results. We start recalling the main definitions for set-valued operators proposed in [2].

Let $\mu \geq 0$ be fixed. A set-valued operator $T : K \rightrightarrows \mathbb{X}^*$ is called

– *relaxed $\mu$-pseudomonotone* if for all $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$ it holds that

$$\langle x^*, y - x \rangle \geq 0 \quad \Rightarrow \quad \langle y^*, x - y \rangle \leq \mu \|x - y\|^2,$$

– *relaxed $\mu$-quasimonotone* if for all $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$ it holds that

$$\langle x^*, y - x \rangle > 0 \quad \Rightarrow \quad \langle y^*, x - y \rangle \leq \mu \|x - y\|^2,$$

– *properly relaxed $\mu$-quasimonotone* if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in K$ and $\bar{x} \in \operatorname{conv}\{x_1, \ldots, x_n\}$ there exists $i$ such that

$$\langle x_i^*, \bar{x} - x_i \rangle \leq \mu \|x_i - \bar{x}\|^2, \quad \forall x_i^* \in T(x_i).$$

These concepts collapse into the usual ones of pseudomonotonicity, quasimonotonicity, and proper quasimonotonicity when $\mu = 0$.

Here we adapt these definitions to equilibrium bifunctions.

**Definition 1** Let $\mu \geq 0$ be fixed. The bifunction $f$ is called

– *relaxed $\mu$-pseudomonotone* if for all $x, y \in K$ it holds that

$$f(x, y) \geq 0 \quad \Rightarrow \quad f(y, x) \leq \mu \|x - y\|^2,$$

– *relaxed $\mu$-quasimonotone* if for all $x, y \in K$ it holds that

$$f(x, y) > 0 \quad \Rightarrow \quad f(y, x) \leq \mu \|x - y\|^2,$$

– *properly relaxed $\mu$-quasimonotone* if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in K$ and $\bar{x} \in \operatorname{conv}\{x_1, \ldots, x_n\}$ there exists $i$ such that

$$f(x_i, \bar{x}) \leq \mu \|x_i - \bar{x}\|^2.$$

In the case of set-valued operators, relaxed $\mu$-pseudomonotonicity implies proper relaxed $\mu$-quasimonotonicity which implies relaxed $\mu$-quasimonotonicity. In the general case, as considered here, relaxed $\mu$-pseudomonotonicity implies relaxed $\mu$-quasimonotonicity but proper relaxed $\mu$-quasimonotonicity neither implies, nor is implied by, relaxed $\mu$-pseudomonotonicity.

For instance the bifunction $f(x, y) = e^{x^2}(x^2 - y^2)$ defined on $\mathbb{R} \times \mathbb{R}$ is pseudomonotone (and therefore relaxed $\mu$-pseudomonotone for each $\mu \geq 0$) but not properly relaxed $\mu$-quasimonotone for all $\mu \geq 0$ (take $x_1 = -\mu - 1$, $x_2 = \mu + 1$ and $\bar{x} = 0$). On the converse the bifunction

$$f(x, y) = \begin{cases} 1, & \text{if } xy < 0 \\ 0, & \text{if } xy \geq 0 \end{cases}$$

is properly quasimonotone (and therefore properly relaxed $\mu$-quasimonotone for each $\mu \geq 0$) but not relaxed $\mu$-quasimonotone for all $\mu \geq 0$ (take $x$ and $y$ such that $xy < 0$ and sufficiently close).

The following condition provides a simple sufficient criterion for the proper relaxed $\mu$-quasimonotonicity of an equilibrium bifunction.

**Lemma 2** *Let $f$ be a relaxed $\mu$-pseudomonotone equilibrium bifunction such that the strict level set* $\text{lev}_s(f, x)$ *is convex for every $x \in K$. Then $f$ is properly relaxed $\mu$-quasimonotone.*

*Proof* By contradiction assume there exist $x_1, \ldots, x_n \in K$ and $\bar{x} \in \text{conv}\{x_1, \ldots, x_n\}$ such that $f(x_i, \bar{x}) > \mu\|x_i - \bar{x}\|^2$ for all $i$. From the relaxed $\mu$-pseudomonotonicity of $f$ we deduce that, for every $i$, $f(\bar{x}, x_i) < 0$, that is $x_i \in \text{lev}_s(f, \bar{x})$. The convexity of $\text{lev}_s(f, \bar{x})$ implies that $\bar{x} \in \text{lev}_s(f, \bar{x})$ which is absurd since $f(\bar{x}, \bar{x}) = 0$. □

However, since the bifunction of the last example is quasiconvex with respect to the second variable, it follows that the quasiconvexity of $f(x, \cdot)$ is not enough to guarantee the relaxed $\mu$-quasimonotonicity of a properly relaxed $\mu$-quasimonotone function.

## 3 The upper sign property and its consequences

### 3.1 The definition

In order to extend to bifunctions the notion of upper sign continuity, in [8] Bianchi and Pini proposed the following definition.

**Definition 2** The equilibrium bifunction $f$ is *upper sign continuous* (with respect to the first variable) at $x \in K$ if for every $y \in K$ the following implication holds

$$f(z_t, y) \geq 0, \quad \forall t \in (0, 1) \;\Rightarrow\; f(x, y) \geq 0, \tag{2}$$

where $z_t = (1 - t)x + ty$.

The upper sign continuity is a very weak form of continuity. For instance if $f(\cdot, y)$ is upper hemicontinuous at $x$, i.e.

$$f(x, y) \geq \limsup_{t \to 0^+} f((1 - t)x + tz, y), \quad \forall z \in K$$

then it is upper sign continuous. Nevertheless in the particular case when the equilibrium bifunction is $f_T$ the definition of upper sign continuity for bifunctions does not coincide with the original definition for the set-valued operator $T$. Indeed for every $t \in (0, 1)$

$$f_T(z_t, y) = \sup_{z_t^* \in T(z_t)} \langle z_t^*, y - z_t \rangle = (1-t) \sup_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle.$$

Hence if $f_T$ is upper sign continuous then $T$ is upper sign continuous but the converse does not hold.

*Example 1* The set-valued operator $T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$T(x) = \begin{cases} \{1\}, & \text{if } x \geq 0 \\ [-1, 1], & \text{if } x < 0 \end{cases}$$

is upper sign continuous everywhere. The associated support bifunction

$$f_T(x, y) = \begin{cases} y - x, & \text{if } x \geq 0 \\ |y - x|, & \text{if } x < 0 \end{cases}$$

does not satisfy (2) at $x = 0$ (take $y < 0$).

For this reason we propose a slightly different notion which allows to embrace the definition of upper sign continuity for set-valued operators.

**Definition 3** The equilibrium bifunction $f$ is said to have the *upper sign property* (with respect to the first variable) at $x \in K$ if there exists $r > 0$ such that for every $y \in K \cap B(x, r)$ the following implication holds

$$f(z_t, x) \leq 0, \quad \forall t \in (0, 1) \;\Rightarrow\; f(x, y) \geq 0, \tag{3}$$

where $z_t = (1-t)x + ty$.

We prefer to use the term "property" instead of "continuity" for two reasons. First, the concept of upper sign continuity was introduced before and it has been used quite often (to our knowledge the notion has been used in [13]). In this way we do not mix up the two notions. Moreover our definition does not seem apparently related to some kind of continuity.

*Example 2* The bifunction $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x - y, & \text{if } x < y \\ 0, & \text{if } x \geq y \end{cases}$$

is continuous (and hence upper sign continuous) everywhere but it does not satisfy the upper sign property anywhere. Indeed take $x$ and $y = x + \varepsilon$ with arbitrary $\varepsilon > 0$. Since $z_t = x + t\varepsilon > x$ we have $f(z_t, x) = 0$ for all $t \in (0, 1)$ but $f(x, x + \varepsilon) < 0$.

However the two definitions are not comparable. We have just seen that there exist upper sign continuous bifunctions which have not the upper sign property. On the converse, the following example clarifies why we have used the term "upper sign" for our property.

*Example 3* The upper sign continuity for a set-valued operator $T$ can be equivalently reformulated as follows: there exists $r > 0$ such that for every $y \in K \cap B(x, r)$ the implication (1) follows. Indeed it is clear that this "local" condition is weaker than the "global" one. For the converse, fix $y \in K$ and assume that

$$\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle \geq 0, \quad \forall t \in (0, 1).$$

For a suitable $\bar{t}$ the point $z_{\bar{t}} \in K \cap B(x, r)$ and $z_{\bar{t}} - x = \bar{t}(y - x)$. From the "local" upper sign continuity we deduce that

$$0 \leq \sup_{x^* \in T(x)} \langle x^*, z_{\bar{t}} - x \rangle = \bar{t} \sup_{x^* \in T(x)} \langle x^*, y - x \rangle.$$

Hence the "local" definition of upper sign continuity is equivalent to the "global" one.

Now, let $f_T$ be the equilibrium bifunction associated to $T$ and $y \in K \cap B(x, r)$. Choosing $z_t = (1 - t)x + ty$ with $t \in (0, 1)$ we have

$$f_T(z_t, x) = \sup_{z_t^* \in T(z_t)} \langle z_t^*, x - z_t \rangle = -t \inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle,$$

and the inequality $f_T(z_t, x) \leq 0$ coincides with $\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle \geq 0$. Hence, in the case of variational inequalities, the upper sign property of $f_T$ is the upper sign continuity of $T$. For this reason the bifunction $f_T$ introduced in Example 1 has the upper sign property but it is not upper sign continuous.

Even if upper sign continuity and upper sign property are not comparable in general, the previous example shows that in the case of variational inequalities the upper sign property is a weaker condition than the upper sign continuity proposed in [8]. The following result highlights a large class of bifunctions for which the previous implication holds.

**Lemma 3** *Let $f$ be an equilibrium bifunction such that for every $x, y_1, y_2 \in K$ it holds that*

$$f(x, y_1) \leq 0 \textit{ and } f(x, y_2) < 0 \Rightarrow f(x, z_t) < 0, \quad \forall t \in (0, 1) \tag{4}$$

*where $z_t = (1 - t)y_1 + ty_2$. If $f$ is upper sign continuous then it has the upper sign property. Moreover, under the above assumption, every upper hemicontinuous bifunction with respect to the first variable has the upper sign property.*

*Proof* Actually, we show that (3) holds for all $y \in K$. Take $x, y \in K$ such that $f(z_t, x) \leq 0$ for all $t \in (0, 1)$ and assume that $f$ is upper sign continuous. By contradiction, suppose that $f(x, y) < 0$. From (2), there exists $s \in (0, 1)$ such that $f(z_s, y) < 0$. But $f(z_s, x) \leq 0$ and $f(z_s, z_s) = 0$, and the contradiction descends from (4). □

*Remark 1* Condition (4) is a technical assumption introduced in [13] in order to show the inclusion of $M_L(f, K)$ in $S(f, K)$. It can be easily shown that (4) is equivalent to assume that $\text{lev}_s(f, x)$ is convex for every $x \in K$ and moreover the implication

$$f(x, y_1) = 0 \quad \text{and} \quad f(x, y_2) < 0 \Rightarrow f(x, z_t) < 0, \quad \forall t \in (0, 1) \tag{5}$$

holds for every $x, y_1, y_2 \in K$. The convexity of $\text{lev}_s(f, x)$ is guaranteed by the quasiconvexity of $f(x, \cdot)$. Instead condition (5), which was introduced and named *sign preserving property* in [7], is clearly satisfied if $f$ is semistrictly quasiconvex with respect to the second variable. Hence if $f(x, \cdot)$ is explicitly quasiconvex (and in particular when it is pseudoconvex), the upper sign continuity is stronger than the upper sign property.

### 3.2 Applications of the upper sign property

We have stressed the fact that, for a quite reasonable class of equilibrium problems, the upper sign property is a very weak form of continuity, weaker than the upper sign continuity.

The main purpose of this subsection is to highlight links between $S(f, K)$ and suitable sets of solutions related to the MEP$(f, K)$. The first result is a generalization of Lemma 2.1

in [8] and Lemma 2.1 in [13] and we prove that $M_L(f, K)$ is a subset of $S(f, K)$ under the upper sign property of the involved bifunction.

**Theorem 1** *Let $f$ be an equilibrium bifunction with the upper sign property such that for every $x, y \in K$ it holds that*

$$f(x, y) < 0 \Rightarrow f(x, z_t) < 0, \quad \forall t \in (0, 1) \tag{6}$$

*where $z_t = (1-t)x + ty$. Then $M_L(f, K) \subseteq S(f, K)$.*

*Proof* Assume that $x \in M_L(f, K)$ and take $y \in K$. From the definitions of upper sign property and $M_L(f, K)$ there exists $r > 0$ such that (3) holds and $f(z, x) \leq 0$ for all $z \in K \cap B(x, r)$. In particular take any $z \in K \cap B(x, r)$ belonging to the interval with extreme points $x$ and $y$. Therefore $f(z_t, x) \leq 0$ where $z_t = (1-t)x + tz$ with $t \in (0, 1)$, and the upper sign property implies that $f(x, z) \geq 0$. From (6) we deduce that $f(x, y)$ must be necessarily non negative and the arbitrariness of $y$ concludes the proof. □

Some remarks are needed about the proof and the comparison between our result and the others already presented in literature.

*Remark 2*
- From the proof we notice that the upper sign property of $f$ allows to deduce the inclusion of $M_L(f, K)$ in the set of the local solutions of EP$(f, K)$. Instead condition (6) is a technical assumption which ensures that every local solution of EP$(f, K)$ belongs to $S(f, K)$.
- Clearly condition (6) is a particular case of conditions (4) and (5): it is enough to choose $y_1 = x$ and $y_2 = y$.
- Now we compare our result with Lemma 2.1 in [8] and Lemma 2.1 in [13] where the same conclusion is reached. In [8] the authors assume that $f$ is upper sign continuous and quasiconvex with respect to the second variable. Moreover they require that (5) holds. As just written, (6) is weaker than (5). Besides, Lemma 3 guarantees that our assumptions are globally weaker.
  Instead in [13] the authors do not require the quasiconvexity of $f(x, \cdot)$ but, in addition to the upper sign continuity, they assume the stronger technical condition (4). All these assumptions imply ours by Lemma 3.
- An analogous result in [18] was proved under the assumptions of lower semicontinuity and pseudoconvexity of $f$ with respect to the second variable and upper hemicontinuity with respect to the first variable. Clearly all these assumptions are stronger than ours.

*Example 4* Let $K = [-1, 1] \subseteq \mathbb{R}$ and $f$ be defined as follows:

$$f(x, y) = \begin{cases} y - x, & \text{if } x \geq 0 \quad \text{or } x < 0 \text{ and } y \geq x \\ 0, & \text{if } x < 0 \quad \text{and } y < x. \end{cases}$$

The bifunction $f$ has the upper sign property and the convexity of $f(x, \cdot)$ guarantees that the condition (6) holds. Hence Theorem 1 implies that $M_L(f, K) \subseteq S(f, K)$. Indeed $M_L(f, K) = \{-1\}$ and $S(f, K) = [-1, 0)$. Notice that this inclusion cannot be deduced by Lemma 2.1 in [8] or Lemma 2.1 in [13] because $f$ is not upper sign continuous at $x = 0$ (take $y < 0$).

In the framework of variational inequalities, Bai and Hadjisavvas [2] introduced the concept of relaxed $\mu$-quasimonotone set-valued operator. By using this property, they established new existence results. The crucial point of their paper lies in the relation between the solution set of the variational inequality and the solution set of a suitable relaxed $\mu$-Minty problem. Adapting their definition to the case of equilibrium problems we introduce the following concept.

**Definition 4** Let $\mu \geq 0$ be fixed. A point $x \in K$ is called a *local relaxed $\mu$-Minty solution* if there exists $r > 0$ such that

$$f(y, x) \leq \mu \|y - x\|^2, \quad \forall y \in K \cap B(x, r). \tag{7}$$

We denote by $M_L^\mu(f, K)$ the set of $x \in K$ which satisfy (7).

Clearly every local relaxed $\mu$-Minty solution is a local Minty solution of the problem associated to the bifunction $f(x, y) - \mu\|x - y\|^2$ and $M_L^0(f, K) = M_L(f, K)$. Moreover if $\mu' < \mu$ then $M_L^{\mu'}(f, K) \subseteq M_L^\mu(f, K)$. The next goal is to prove the inclusion $M_L^\mu(f, K) \subseteq S(f, K)$. For this reason we need to link the definition of upper sign property to the parameter $\mu$.

**Definition 5** Let $\mu \geq 0$ be fixed. A bifunction $f$ is said to have the *$\mu$-upper sign property* (with respect to the first variable) at $x \in K$ if there exists $r > 0$ such that for every $y \in K \cap B(x, r)$ the following implication holds

$$f(z_t, x) \leq \mu \|z_t - x\|^2, \quad \forall t \in (0, 1) \;\Rightarrow\; f(x, y) \geq 0, \tag{8}$$

where $z_t = (1 - t)x + ty$.

If $\mu > \mu'$, every bifunction with the $\mu$-upper sign property has the $\mu'$-upper sign property and, as before, the upper sign property coincides with the 0-upper sign property.

**Lemma 4** *Let $f$ be an equilibrium bifunction such that for every $x \in K$ there exists $r > 0$ such that for every $y \in K \cap B(x, r)$*

$$(1 - t) f(z_t, x) + t f(z_t, y) \geq 0, \quad \forall t \in (0, 1). \tag{9}$$

*If $f$ is upper hemicontinuous with respect to the first variable then it has the $\mu$-upper sign property for every $\mu \geq 0$.*

*Proof* By contradiction, suppose there exists $\mu \geq 0$ such that $f$ has not the $\mu$-upper sign property at $x$. Therefore for every $r' < r$ there exists $y' \in K \cap B(x, r')$ such that $f(z'_t, x) \leq \mu \|z'_t - x\|^2$ for all $t \in (0, 1)$ and $f(x, y') < 0$. Since $f(\cdot, y')$ is upper hemicontinuous at $x$, there are $\varepsilon > 0$ and $\delta \in (0, 1)$ such that $f(z'_t, y') < -\varepsilon$ for every $t \in (0, \delta)$. From (9) we deduce

$$0 \leq (1 - t) f(z'_t, x) + t f(z'_t, y') \leq \mu(1 - t)t^2 \|x - y'\|^2 - \varepsilon t$$

and the contradiction is achieved choosing $t$ sufficiently small. □

It is easy to show that the technical assumption (9) holds if $f(x, \cdot)$ is locally convex at $x$, uniformly with respect to $x$, that is there exists $r > 0$, independent on $x$, such that the function $f(x, \cdot)$ restricted to $B(x, r)$ is convex. Hence under a mild assumption of convexity, also the $\mu$-upper sign property is a kind of weak continuity.

Moreover, arguing as in Theorem 1, it is possible to get the following result.

**Theorem 2** *Let $f$ be an equilibrium bifunction with the $\mu$-upper sign property such that (6) holds. Then $M_L^\mu(f, K) \subseteq S(f, K)$*

We conclude this section dealing with all these last results in the case of variational inequalities. Let us consider the bifunction $f_T$ associated to the set-valued operator $T$. The $\mu$-upper sign property of $f_T$ assumes the following equivalent form

$$\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x\rangle \geq -\mu t \|y - x\|^2, \quad \forall t \in (0,1) \quad \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0 \tag{10}$$

where $x, y \in K$ and $z_t = (1-t)x + ty$. Since $f_T$ is convex with respect to $y$, from Lemma 4 we deduce that every upper hemicontinuous set-valued operator $T$ with nonempty weakly* compact values verifies (10) for each $\mu \geq 0$. The converse does not hold. For instance the set-valued operator $T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$T(x) = \begin{cases} [0,1], & \text{if } x > 0 \\ \{0\}, & \text{if } x = 0 \\ [-1, 0], & \text{if } x < 0 \end{cases}$$

is not upper hemicontinuous at $x = 0$ but property (10) is verified for every $\mu \geq 0$ since

$$\inf_{z_t^* \in T(z_t)} \langle z_t^*, y\rangle = 0 \quad \text{and} \quad \sup_{x^* \in T(0)} \langle x^*, y\rangle = 0$$

for each $y \in \mathbb{R}$. For all these considerations Theorem 2 not only generalizes to equilibrium problems the fundamental Lemma 2.1 in [2] stated for variational inequalities, but also allows to improve it in the variational inequalities' context.

## 4 Existence results

Theorem 1 and Theorem 2 pave the way towards a simple approach for deriving the existence of solutions of EP$(f, K)$: any assumption which guarantees the nonemptiness of $M_L^\mu(f, K)$ for a suitable $\mu \geq 0$ is a sufficient condition for the nonemptiness of $S(f, K)$.

When $f$ is not properly relaxed $\mu$-quasimonotone, we follow the idea which was originally expressed in [1] for variational inequalities and adapted in [8] and [13] for equilibrium problems. In the sequel, for each $x \in K$ we denote by $F_\mu$ the set

$$F_\mu(x) = \{y \in K : f(x, y) \leq \mu \|x - y\|^2\}.$$

**Theorem 3** *Let $f$ be a relaxed $\mu$-quasimonotone equilibrium bifunction which is not properly relaxed $\mu$-quasimonotone. Suppose that the set $F_\mu(x)$ is closed and* lev$(f, x)$ *is convex for every $x \in K$. Then $M_L(f, K)$ is nonempty.*

*Proof* Since $f$ is not properly relaxed $\mu$-quasimonotone, there exist $x_1, \ldots, x_n \in K$ and $\bar{x} \in \text{conv}\{x_1, \ldots, x_n\}$ such that

$$f(x_i, \bar{x}) > \mu \|x_i - \bar{x}\|^2, \quad \forall i = 1, \ldots, n.$$

Since $F_\mu(x_i)$ are closed, there exists $r > 0$ such that for every fixed $y \in K \cap B(\bar{x}, r)$

$$f(x_i, y) > \mu \|x_i - y\|^2, \quad \forall i = 1, \ldots, n.$$

From the relaxed $\mu$-quasimonotonicity of $f$ we deduce

$$f(y, x_i) \leq 0, \quad \forall i = 1, \ldots, n.$$

Lastly, the convexity of lev$(f, y)$ concludes the proof. □

Now the nonemptiness of $S(f, K)$ descends immediately from Theorem 1 and Theorem 3.

**Corollary 1** *Let $f$ be a relaxed $\mu$-quasimonotone equilibrium bifunction which is not properly relaxed $\mu$-quasimonotone. Suppose that the set $F_\mu(x)$ is closed and $\operatorname{lev}(f, x)$ is convex for every $x \in K$. Moreover assume that $f$ has the upper sign property and condition (6) holds. Then $S(f, K)$ is nonempty.*

The requirement that $f$ must not be properly relaxed $\mu$-quasimonotone is fundamental. For instance the linear bifunction $f(x, y) = x - y$ is clearly properly relaxed $\mu$-quasimonotone and relaxed $\mu$-quasimonotone for each $\mu$. All the others assumptions of Corollary 1 are satisfied but EP$(f, K)$ with feasible region $K = [0, +\infty)$ has not solution. Notice that $K$ is not compact.

If we assume the compactness of $K$ it is possible to invoke the so-called Three Polish Theorem [21]. This result was originally stated in $\mathbb{R}^n$ by Knaster, Kuratowski and Mazurkiewicz and later it was extended to the case of an infinite-dimensional topological vector space by Ky Fan [12]. We recall this fundamental result.

**Theorem 4** *Let $\mathbb{Y}$ be a real topological Hausdorff vector space, $C$ be a subset of $\mathbb{Y}$ and $T : C \rightrightarrows \mathbb{Y}$ be a set-valued operator satisfying the following conditions:*

(a) *$T$ is a KKM-map, that is for any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in C$*

$$\operatorname{conv}\{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i),$$

(b) *$T(x)$ is closed for each $x \in C$,*
(c) *there exists $x \in C$ such that $T(x)$ is compact.*

*Then*

$$\bigcap_{x \in C} T(x) \neq \emptyset.$$

Notice that the proper relaxed $\mu$-quasimonotonicity of $f$ is equivalent to affirm that $F_\mu$ is a KKM-map. Since

$$\bigcap_{x \in K} F_\mu(x) \subseteq M_L^\mu(f, K)$$

the following result is an immediate consequence of Theorem 4.

**Corollary 2** *Let $K$ be weakly compact and $f$ be a properly relaxed $\mu$-quasimonotone equilibrium bifunction such that $F_\mu(x)$ is weakly closed for every $x \in K$. Then $M_L^\mu(f, K)$ is nonempty.*

The next step is to replace the assumption of compactness of $K$ with weaker conditions, which clearly have to involve some suitable form of coercivity on $f$.

**Theorem 5** *Assume that $\mathbb{X}$ is a reflexive Banach space and $K$ is closed. Let $f$ be a $\mu$-quasimonotone equilibrium bifunction with the $\mu$-upper sign property and satisfying (5). Suppose that $F_\mu(x)$ is closed and $\operatorname{lev}(f, x)$ is convex for every $x \in K$. Assume that the following coercivity condition holds:*

(C) *for any sequence* $\{x_n\} \subseteq K$ *such that* $\|x_n\| \to \infty$, *there exist* $n_0 \in \mathbb{N}$ *and* $y_{n_0} \in K$ *such that* $\|y_{n_0}\| < \|x_{n_0}\|$ *and* $f(x_{n_0}, y_{n_0}) \leq 0$.

*Then* $S(f, K)$ *is nonempty.*

*Proof* For each fixed $n \in \mathbb{N}$, we denote by $K_n$ the intersection of $K$ with the closed ball centered at the origin with radius $n$, i.e.

$$K_n = \{x \in K : \|x\| \leq n\}.$$

Since $K_n$ is nonempty for sufficiently large $n$, in what follows, for the sake of simplicity, we suppose without loss of generality that $K_n$ is nonempty for all $n \in \mathbb{N}$. Moreover the reflexivity of $\mathbb{X}$ and the closedness of $K$ imply that $K_n$ is weakly compact for every $n$. From Theorem 3 and Corollary 2, for every $n$ there exists $x_n \in M_L^{\mu}(f, K_n)$. We distinguish two cases.

(a) There exists $n_0 \in \mathbb{N}$ such that $\|x_{n_0}\| < n_0$. Therefore $x_{n_0} \in M_L^{\mu}(f, K)$ and the nonemptiness of $S(f, K)$ descends from Theorem 2.
(b) Suppose that $\|x_n\| = n$ for all $n \in \mathbb{N}$. In this case the coercivity condition guarantees the existence of $n_0 \in \mathbb{N}$ and $y_{n_0} \in K$ such that $\|y_{n_0}\| < \|x_{n_0}\| = n_0$ and $f(x_{n_0}, y_{n_0}) \leq 0$. From Theorem 2, $f(x_{n_0}, y) \geq 0$ for every $y \in K_{n_0}$; in particular $f(x_{n_0}, y_{n_0}) = 0$. Take any $y \in K \setminus K_{n_0}$. Since $\|y_{n_0}\| < n_0$, there exists $t \in (0, 1)$ such that $z_t = ty + (1-t)y_{n_0}$ belongs to $K_{n_0}$. It follows that $f(x_{n_0}, z_t) \geq 0$. Since $f(x_{n_0}, y_{n_0}) = 0$, we conclude from (5) that $f(x_{n_0}, y) \geq 0$ and $x_{n_0} \in S(f, K)$. □

We give an application of Theorem 5 when $\mu = 0$.

*Example 5* Let $K = (-\infty, 1] \subseteq \mathbb{R}$ and $f$ be defined by

$$f(x, y) = \begin{cases} y^2 - (1 + x)y + x, & \text{if } x \neq 0 \\ -y, & \text{if } x = 0. \end{cases}$$

Obviously $f(x, \cdot)$ is convex and continuous, for every fixed $x \in K$. Hence $f$ satisfies the condition (5) and $\text{lev}(f, x)$ is closed and convex. It can be verified that $f$ is quasimonotone but not pseudomonotone and that $f$ has the upper sign property. The feasible set $K$ is unbounded but the coercivity condition (C) holds. In fact, if $\{x_n\} \subseteq K$ and $x_n \to -\infty$, there exists a suitable $n_0$ such that $x_{n_0} < -1$. Then $y_{n_0} = x_{n_0} + 1 \in K$ verifies $|y_{n_0}| < |x_{n_0}|$ and $f(x_{n_0}, y_{n_0}) = x_{n_0} < 0$. Theorem 5 guarantees that $S(f, K)$ is nonempty. However, $f$ is not an upper sign continuous bifunction. Indeed, for $x = 0$, $y = 1$ and $z_t = x + t(y - x) = t$, we have $f(z_t, 1) = 0$, for all $t \neq 0$ but $f(0, 1) = -1$.

We end with a brief comparison with other results from the literature. The coercivity condition (C) was initially considered in the framework of variational inequality problems by Luc [23] when the operator is single-valued. In [8] the authors adapted to equilibrium problems a minimal coercivity condition stated in [6] for variational inequalities:

(C1) there exists $n_0 \in \mathbb{N}$ such that for every $x \in K \setminus K_{n_0}$ there exists $y \in K$ such that $\|y\| < \|x\|$ and $f(x, y) \leq 0$.

It is not hard to verify that that (C) and (C1) are equivalent. Hence Theorem 5 improves Theorem 4.2 in [8]. Moreover as underlined at the end of Sect. 3, the $\mu$-upper sign property of the bifunction $f_T$ associated to a set-valued operator $T$ is weaker than the upper hemicontinuity of $T$. Therefore adapting Theorem 5 to the particular case of variational inequalities, an improvement of the existence result in [2] can be derived.

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## References

1. Aussel, D., Hadjisavvas, N.: On quasimonotone variational inequalities. J. Optim. Theory Appl. **121**, 445–450 (2004)
2. Bai, M.R., Hadjisavvas, N.: Relaxed quasimonotone operators and relaxed quasiconvex functions. J. Optim. Theory Appl. **138**, 329–339 (2008)
3. Bai, M.R., Zhou, S.Z., Ni, G.Y.: On generalized monotonicity of variational inequalities. Comput. Math. Appl. **53**, 910–917 (2007)
4. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities. Applications to Free Boundary Problems. Wiley, New York (1984)
5. Bigi, G., Castellani, M., Pappalardo, M., Passacantando, M.: Existence and solution methods for equilibria. Eur. J. Oper. Res., doi:10.1016/j.ejor.2012.11.037
6. Bianchi, M., Hadjisavvas, N., Schaible, S.: Minimal coercivity conditions for variational inequalities. Application to exceptional families of elements. J. Optim. Theory Appl. **122**, 1–7 (2004)
7. Bianchi, M., Pini, R.: A note on stability for parametric equilibrium problems. Oper. Res. Lett. **31**, 445–450 (2003)
8. Bianchi, M., Pini, R.: Coercivity conditions for equilibrium problems. J. Optim. Theory Appl. **124**, 79–92 (2005)
9. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
10. Brézis, H., Nirenberg, L., Stampacchia, G.: A remark on Ky Fan's minimax principle. Boll. Un. Mat. Ital. **6**, 293–300 (1972)
11. Debrunner, H., Flor, P.: Ein Erweiterungssatz für monotone Mengen. Arch. Math. **15**, 445–447 (1964)
12. Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequalities III, pp. 103–113. Academic Press, New York (1972)
13. Farajzadeh, A.P., Zafarani, J.: Equilibrium problems and variational inequalities in topological vector spaces. Optimization **59**, 485–499 (2010)
14. Flores-Bazán, F.: Existence theorems for generalized noncoercive equilibrium problems: the quasiconvex case. SIAM J. Optim. **11**, 675–690 (2000)
15. Giannessi, F.: On Minty variational principle. In: Giannessi, F., Komlósi, S., Rapcsák, T. (eds.) New Trends in Mathematical Programming, pp. 93–99. Kluwer, Dordrecht (1998)
16. García, Ramos Y., Sosa, W.: El Lema de Ky Fan y sus aplicaciones. Revista TECNIA **13**(2), 75–86 (2003)
17. Hadjisavvas, N.: Continuity and maximality properties of pseudomonotone operators. J. Convex Anal. **10**, 459–469 (2003)
18. Iusem, A.N., Kassay, G., Sosa, W.: On certain conditions for the existence of solutions of equilibrium problems. Math. Program. **116**, 259–273 (2009)
19. Iusem, A.N., Sosa, W.: New existence results for equilibrium problems. Nonlinear Anal. **52**, 621–635 (2003)
20. Karamardian, S.: Strictly quasi-convex (concave) functions and duality in mathematical programming. J. Math. Anal. Appl. **20**, 344–358 (1967)
21. Knaster, B., Kuratowski, C., Mazurkiewicz, S.: Ein Beweies des Fixpunktsatzes für $N$ Dimensionale Simplexe. Fund. Math. **14**, 132–137 (1929)
22. Konnov, I.V.: Partial proximal point method for nonmonotone equilibrium problems. Optim. Methods Softw. **21**, 373–384 (2006)
23. Luc, D.T.: Existence results for densely pseudomonotone variational inequalities. J. Math. Anal. Appl. **254**, 291–308 (2001)
24. Minty, G.J.: On the generalization of a direct method of the calculus of variations. Bull. Am. Math. Soc. **73**, 315–321 (1967)
25. Nikaido, H., Isoda, K.: Note on noncooperative convex games. Pac. J. Math. **5**, 807–815 (1955)
26. Nirenberg, L.: Topics in Nonlinear Functional Analysis. New York University Press, New York (1974)
27. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970)
28. Zukhovitskii, S.I., Polyak, R.A., Primak, M.E.: On an $n$-person concave game and a production model. Sov. Math. Dokl. **11**, 522–526 (1970)