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A complete characterization of strong duality in nonconvex optimization with a single constraint

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Abstract We first establish sufficient conditions ensuring strong duality for cone constrained nonconvex optimization problems under a generalized Slater-type condition. Such conditions allow us to cover situations where recent results cannot be applied. Afterwards, we provide a new complete characterization of strong duality for a problem with a single constraint: showing, in particular, that strong duality still holds without the standard Slater condition. This yields Lagrange multipliers characterizations of global optimality in case of (not necessarily convex) quadratic homogeneous functions after applying a generalized joint-range convexity result. Furthermore, a result which reduces a constrained minimization problem into one with a single constraint under generalized convexity assumptions, is also presented.

Keywords Strong duality · Nonconvex optimization

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1 Introduction and formulation of the problem

Let X be a real locally convex topological vector space; Y be a normed space; $P \subseteq Y$ be a closed convex cone with possibly empty interior, and C be a subset of X. Given $f : C \to \mathbb{R}$ and $g : C \to Y$, let us consider the cone constrained minimization problem

$$\mu \doteq \inf_{\substack{g(x)\in -P \\ x\in C}} f(x). \tag{P}$$

Thus, the constraint set may be described by inequality and equality constraints. The Lagrangian dual problem associated to (P) is

$$\nu \doteq \sup_{\lambda^*\in P^*} \inf_{x\in C}[f(x) + \langle \lambda^*, g(x)\rangle], \tag{D}$$

where P^* is the non negative polar cone of P. We say Problem (P) has a (Lagrangian) *zero duality gap* if the optimal values of (P) and (D) coincide, that is, $\mu = \nu$. The Problem (P) is said to have *strong duality* if it has a zero duality gap and Problem (D) admits a solution. To characterize this property is one of the most important problems in optimization, and certainly the lack of convexity makes the task an interesting challenge in mathematics. To that purpose, some constraints qualification (CQ) are needed, which may be of Slater-type, or interior-point condition, and in some other situation it requires a closed-cone CQ. Such CQ often restrict some applications.

More precisely, when $X = \mathbb{R}^n$ and $P = [0, +\infty[$ with g being a quadratic function that is not identically zero, the authors in [15] prove that, (P) has strong duality for each quadratic function f if, and only if there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$.

Similarly, when g is P-convex (see 3.8) and continuous, it is proven in [4] that (P) has strong duality for each $f \in X^*$ if, and only if a certain CQ holds. This CQ involves the epigraph of the support function of C and the epigraph of the conjugate of the function $x \mapsto \langle \lambda^*, g(x)\rangle$. This CQ is also equivalent to (P) has strong duality for each continuous and convex function f [14]. Stable zero duality gaps in convex programming (g is continuous, P-convex, and f is lower semicontinuous proper convex function), that is, strong duality for each linear perturbation of f, were characterized in terms of a similar CQ as above, see [16,18] for details.

Apart from these characterizations several sufficient conditions of the zero duality gap have been established in the literature, see [1,2,4,6,7,12,27].

Our goal in this paper is, firstly, to derive conditions for (P) to have strong duality under no convexity assumptions. Unlike some of the above results, which involve conditions on g and C that guarantee (P) has strong duality for every f in a certain class of functions, our approach allows us to derive conditions on the pair, f and g jointly, that ensure (P) has strong duality, under no convexity assumptions; this result can be used to situations where none of the results in [4–7,12,14,16], for instance, is applicable. Secondly, we provide a new characterization of strong duality in case we have a single constraint.

By assuming that x_0 is a solution to problem (P), the authors in [6, Corollary 3.1] prove that strong duality holds if and only if condition

$$T(\tilde{M}; (f(x_0), 0)) \cap (]-\infty, 0[\times\{0\}) = \emptyset \tag{S}$$

is satisfied, where $T(A; x)$ stands for the contingent cone to A at $x \in A$, and

$$\tilde{M} \doteq (f, g)(C \setminus K) + (\mathbb{R}_+ \times P), \ K \doteq \{x \in C : \ g(x) \in -P\},$$

with, $(f, g)(C \setminus K) \doteq \{(f(x), g(x)) \in \mathbb{R} \times Y : \ x \in C \setminus K\}$. Condition (S) has its origin in [9]. Since in most problems a solution to (P) is unknown, such an equivalent formulation, though interesting, has some disadvantages. Therefore, throughout this paper we do not assume that (P) has solution, no convexity assumption is imposed, and we use topological interior, allowing us to deal with cones possibly with empty interior.

The paper is organized as follows. In Sect. 2 some basic notations and preliminaries are collected. Section 3 establishes our first main theorems on strong duality for (P) via topological interior; such theorems cover situations where recent results cannot be applied, see Example 3.4; in addition, we present sufficient conditions allowing us to formulate (P) into one with a single inequality constraint; this result generalizes that due to Luenberger [20], valid when P is the usual non negative orthant, $\mathbb{R}^n_+$, and each component of g is convex. A new complete characterization of strong duality when the constrained set is determined by a single inequality, is established in Sect. 4; in particular, it is showed that strong duality holds without the standard Slater condition. Section 5 presents a Lagrange multipliers characterization for global optimality in case of quadratic homogeneous functions. Here, a generalized joint-range convexity result due to Jeyakumar, Lee and Li will play an important role.

2 Basic notations and preliminares

Throughout the paper, Y is a real normed vector space, its topological dual space is Y^*, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between Y and Y^*. Given $x, y \in Y$ we set $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$. The segments $]x, y]$ etc are defined analogously.

A set $P \subseteq Y$ is said to be a cone if $tP \subseteq P, \forall\, t \geq 0$; by P^* we mean the (non-negative) polar cone of P, i.e., $P^* = \{z \in Y^* : \ \langle z, p \rangle \geq 0, \ \ \forall\, p \in P\}$ Given $A \subseteq Y$, cone(A) stands for the smallest cone containing A, that is,

$$\text{cone}(A) = \bigcup_{t \geq 0} tA,$$

whereas $\overline{\text{cone}}(A)$ denotes the smallest closed cone containing A: obviously $\overline{\text{cone}}(A) = \overline{\text{cone}(A)}$, where $\overline{A}$ denotes the closure of A. Additionally, we set

$$\text{cone}_+(A) \doteq \bigcup_{t > 0} tA.$$

Evidently, $\text{cone}(A) = \text{cone}_+(A) \cup \{0\}$ and therefore $\overline{\text{cone}(A)} = \overline{\text{cone}_+(A)}$. Furthermore, a (not necessarily convex) cone $K \subseteq Y$ is called "pointed" (see for instance [23]) if $x_1 + \cdots + x_k = 0$ is impossible for $x_1, x_2, \ldots, x_k$ in K unless $x_1 = x_2 = \cdots = x_k = 0$.

It is easy to see that a cone K is pointed if, and only if $\text{co}(K) \cap (-\text{co}(K)) = \{0\}$ if, and only if 0 is a extremal point of $\text{co}(K)$.

In subsequent sections, the notations $\text{co}(A)$, int A, stand for the convex hull of A which is the smallest convex set containing A, and topological interior of A, respectively. We denote $\mathbb{R}_+ \doteq [0, +\infty[$, $\mathbb{R}_{++} \doteq \ ]0, +\infty[$, $\mathbb{R}_{--} = -\mathbb{R}_{++}$.

3 Lagrangian strong duality and reducing to one single constraint

Given a real locally convex topological vector space X, a nonempty set $C \subseteq X$, a mapping $f : C \to \mathbb{R}$, let us consider the following cone constrained minimization problem

$$\mu \doteq \inf_{x \in K} f(x), \tag{3.1}$$

where $g : C \to Y$, with Y as before, $K \doteq \{x \in C : \ g(x) \in -P\}$ with $P \subseteq Y$ being a convex cone with possibly empty topological interior. This means that P may have the form $P = Q \times \{0\}$, in which case, the constraint set is described by inequality and equality constraints.
Let us introduce, as usual, the Lagrangian

$$L(\gamma^*, \lambda^*, x) = \gamma^* f(x) + \langle \lambda^*, g(x) \rangle.$$

Obviously,

$$\gamma^* \mu \geq \inf_{x \in C} L(\gamma^*, \lambda^*, x), \ \ \forall\, \lambda^* \in P^*, \ \ \forall\, \gamma^* \geq 0. \tag{3.2}$$

As pointed out in the introduction, our main concern is to find sufficient conditions ensuring strong duality for problem (3.1), that is, that there exists $\lambda_0^* \in P^*$ such that

$$\inf_{x \in K} f(x) = \inf_{x \in C} L(1, \lambda_0^*, x). \tag{3.3}$$

Throughout this section we do not assume that (3.1) has solution, and we will look for sufficient conditions implying strong duality, under no convexity assumption.

To that end, some constraint qualifications (CQ) are needed, which involve interior-point conditions, say Slater-type conditions. In addition, some regularity conditions will be also imposed. It is well known the standard Slater condition (SC) prevents to deal with inequality and equality constraints, since the cone involved has empty interior: for instance, Theorem 4.1 of [6] cannot be applied if the constraint set is determined by inequalities and equalities. The last part of this section establishes a result which reduces problem (3.1) into one with a single constraint, for a quasiconvex function f, a generalized convex mapping g, and a Slater-type condition. Such a result generalizes that due to Luenberger [20], valid when P is the usual non negative orthant and each component of g is convex. Set

$$F(C) \doteq (f, g)(C) = \{(f(x), g(x)) \in \mathbb{R} \times Y : \ x \in C\}.$$

3.1 Lagrangian strong duality: the general case

We obtain various equivalent formulations to have an equality in (3.2) for some (γ^*, λ^*). This preliminary result will allow us to get strong duality for problem (3.1) under a generalized Slater assumption.

Theorem 3.1 *Let us consider problem* (3.1). *Assume that μ is finite and*

$$\text{int}(\text{co}(F(C)) + (\mathbb{R}_+ \times P)) \neq \emptyset.$$

The following assertions are equivalent:

(a) there exist Lagrange multipliers $(\gamma_0^*, \lambda_0^*) \in \mathbb{R}_+ \times P^*$, $(\gamma_0^*, \lambda_0^*) \neq (0, 0)$, *such that*

$$\gamma_0^* \inf_{x \in K} f(x) = \inf_{x \in C} L(\gamma_0^*, \lambda_0^*, x);$$

(b) $\operatorname{cone}(\operatorname{int}(\operatorname{co}(F(C)) - \mu(1,0) + (\mathbb{R}_+ \times P)))$ *is pointed;*
(c) $(0,0) \notin \operatorname{int}(\operatorname{co}(F(C)) - \mu(1,0) + (\mathbb{R}_+ \times P)))$;
(d) $\operatorname{cone}(F(C) - \mu(1,0) + \operatorname{int}(\mathbb{R}_+ \times P))$ *is pointed, provided* $\operatorname{int} P \neq \emptyset$.

Proof $(b) \Longrightarrow (c)$: This is straightforward since otherwise, we had

$$\operatorname{cone}(\operatorname{int}(\operatorname{co}(F(C)) - \mu(1,0) + (\mathbb{R}_+ \times P))) = \mathbb{R} \times Y.$$

$(c) \Longrightarrow (a)$: We apply a standard convex separation theorem to obtain the existence of $\gamma_0^* \geq 0$ and $\lambda_0^* \in P^*$, not both zero, satisfying

$$\gamma_0^* f(x) + \langle \lambda_0^*, g(x) \rangle \geq \gamma_0^* \mu \quad \forall\, x \in C. \tag{3.4}$$

This implies

$$\inf_{x \in C} L(\gamma_0^*, \lambda_0^*, x) \geq \gamma_0^* \mu.$$

This together with (3.2) yield the desired result.
$(a) \Longrightarrow (b)$: From (a), (3.4) holds, and this amounts to writing

$$\langle (\gamma_0^*, \lambda_0^*), (f(x) - \mu, g(x)) \rangle \geq 0 \quad \forall\, x \in C. \tag{3.5}$$

Set $A \doteq F(C) - \mu(1,0)$. Since $\operatorname{cone}(\operatorname{int}(\operatorname{co}(A) + (\mathbb{R}_+ \times P)))$ is convex, we have to show that whenever $x, -x \in \operatorname{cone}(\operatorname{int}(\operatorname{co}(A) + (\mathbb{R}_+ \times P)))$, then $x = 0$. Assume that $x \neq 0$. Then, we can write $x = t_1\xi_1$, $-x = t_2\xi_2$, $t_i > 0$, $\xi_i \in \operatorname{int}(\operatorname{co}(A) + (\mathbb{R}_+ \times P))$, $i = 1, 2$. By (3.5), $\langle (\gamma_0^*, \lambda_0^*), \xi \rangle \geq 0 \ \ \forall\, \xi \in \operatorname{co}(A) + (\mathbb{R}_+ \times P)$. Given any $y \in \mathbb{R} \times Y$, we can choose $\delta > 0$ such that

$$\xi_i + \lambda y \in \operatorname{co}(A) + (\mathbb{R}_+ \times P), \ \ \forall\, |\lambda| < \delta, \ \forall\, i = 1, 2.$$

Then, by setting $p^* \doteq (\gamma_0^*, \lambda_0^*)$, we obtain

$$\langle p^*, \xi_i + \lambda y \rangle \geq 0, \ \ \forall\, |\lambda| < \delta, \ \forall\, i = 1, 2.$$

It follows that $\langle p^*, \lambda y(t_1 + t_2) \rangle \geq 0$, which implies that $\langle p^*, y \rangle = 0$ for all $y \in \mathbb{R} \times Y$. Hence $(\gamma_0^*, \lambda_0^*) = p^* = 0$, a contradiction.
$(b) \Longleftrightarrow (d)$: Since $K + \operatorname{int} Q = \operatorname{int}(K + Q)$ (see [8,25]) and $\operatorname{cone}(\operatorname{co}(K)) = \operatorname{co}(\operatorname{cone}(K))$, for every convex cone Q with nonempty interior and every set K, we obtain that $\operatorname{cone}(\operatorname{int}(\operatorname{co}(A) + P)) = \operatorname{co}(\operatorname{cone}(A + \operatorname{int} P))$. Taking into the account that pointedness of any cone is equivalent to pointednes of its convex hull, the result follows. □

In order to have strong duality, we need the non-verticality of the linear functional $(\gamma_0^*, \lambda_0^*)$, that is, we must have $\gamma_0^* > 0$. It holds whenever a Slater-type condition is imposed as the following corollary shows.

Corollary 3.2 *Let us consider problem* (3.1)*. Assume that* μ *is finite,*

$$\operatorname{int}(\operatorname{co}(F(C)) + (\mathbb{R}_+ \times P)) \neq \emptyset$$

and the generalized (SC) *that* $\overline{\operatorname{cone}}(g(C) + P) = Y$ *holds. The following assertions are equivalent:*

(a) *there exists a Lagrange multiplier* $\lambda_0^* \in P^*$*, such that*

$$\inf_{x \in K} f(x) = \inf_{x \in C} L(1, \lambda_0^*, x); \tag{3.6}$$

(b)

$$\inf_{x\in K} f(x) = \max_{\lambda^*\in P^*} \inf_{x\in C} L(1, \lambda^*, x);$$

(c) $\text{cone}(\text{int}(\text{co}(F(C)) - \mu(1,0) + (\mathbb{R}_+ \times P)))$ *is pointed;*

Proof $(a) \Longleftrightarrow (b)$: One implication is obvious. From (a) it follows that

$$\mu \le \max_{\lambda^*\in P^*} \inf_{x\in C} L(\lambda^*, x),$$

which together with (3.2) imply (b).

$(c) \Longleftrightarrow (a)$: One implication follows from the preceding theorem. If $\gamma_0^* = 0$, then $0 \neq \lambda_0^* \in P^*$ and $\langle \lambda_0^*, g(x)\rangle \ge 0$ for all $x \in C$. This implies that $\lambda_0^* = 0$, by the generalized Slater condition, which is a contradiction. Thus, we may suppose $\gamma_0^* = 1$ in (3.4), and therefore (a) holds. □

Some comments are in order. We compare our previous result with that given in [7, Theorem 4.4] where quasi relative interior is employed but at the expenses of requiring the convexity of $F(C) + (\mathbb{R}_+ \times P)$, which implies the convexity of $g(C) + P$. More precisely, with the same notations as in the mentioned paper, such a theorem is the following.

Theorem 3.3 [7, Theorem 4.4] *Suppose that* $F(C)+(\mathbb{R}_+\times P)$ *is convex,* $0 \in \text{qi}(g(C)+P)$ *and* $(0,0) \notin \text{qri}[\text{co}((F(C) - \mu(1,0) + \mathbb{R}_+ \times P) \cup \{(0,0)\})]$. *Then, there exists* $\lambda_0^* \in P^*$ *such that* (3.6) *holds.*

Here, given a convex set A, by $\text{qri}(A)$ and $\text{qi}(A)$ we mean the quasi relative interior and the quasi interior of A, see [3,7]. In order to prove the previous theorem, the authors show first that "Fenchel and Lagrange duality" are equivalent (so, some convexity assumptions are imposed) generalizing an earlier result due to Magnanti [21]. Then, from such an equivalence Theorem 3.3 is obtained.
As a by-product we observe that Theorems 4.2 and 4.4 in [7] are identical. Indeed, since $\overline{\text{cone}}\ A = \overline{\text{cone}\ A}$, and $\text{cone}(A - A) = \text{cone}\ A - \text{cone}\ A$ provided A is convex and $0 \in A$, we obtain

$$\overline{\text{cone}}\,((g(C) + P - (g(C) + P)) = \overline{\overline{\text{cone}}(g(C) + P) - \overline{\text{cone}}(g(C) + P)}.$$

From this, by assuming that $g(C) + P$ is convex, one immediately gets

$$0 \in \text{qi}(g(C) + P) \Longleftrightarrow 0 \in \text{qi}(g(C) + P - (g(C) + P)) \quad \text{and} \quad 0 \in \text{qri}(g(C) + P).$$

Our Corollary 3.2 may be applied to problems of minimizing a non quasiconvex function with equality and inequality constraints. It is illustrated in the following example. Notice that no result from [4,5,12,14,16], neither [6, Theorem 4.1], [13, Theorem 4.3], or [7, Theorem 4.4] can be applied, since we are dealing with an objective non-convex function and the mapping g is such that $g(C) + P$ is not convex.

Example 3.4 Notice this example shows our approach applies even if $\text{int}\ P = \emptyset$.
Take $C = \mathbb{R}$, $P = \mathbb{R}_+ \times \{0\}$,

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

$$g_1(x) = \begin{cases} x & \text{if } x \le -1, \\ -1 & \text{if } -1 < x < 0, \\ 0 & \text{if } x \ge 0. \end{cases} \qquad g_2(x) = \begin{cases} x+1 & \text{if } x < 0, \\ 0 & \text{if } x \ge 0, \end{cases}$$

and consider the problem

$$\mu \doteq \min\{f(x):\ g_1(x) \le 0,\ g_2(x) = 0, x \in \mathbb{R}\}.$$

Thus, $P^* = \mathbb{R}_+ \times \mathbb{R}$ and $\mu = 0$. Setting $F(x) = (f(x), g_1(x), g_2(x))$, $x \in C$, we obtain

$$F(x) = \begin{cases} (0, x, x+1) & \text{if} \quad x \le -1, \\ (0, -1, x+1) & \text{if} \quad -1 < x < 0, \\ (1, 0, 0) & \text{if} \quad x = 0, \\ (0, 0, 0) & \text{if} \quad x > 0. \end{cases}$$

It follows that

$$\begin{aligned} &F(C) - \mu(1, 0, 0) + (\mathbb{R}_+ \times P) \\ &\quad = \{(x, y, z) : x \ge 0, y \ge -1, 0 \le z < 1\} \cup \{(x, y, z) :\ x \ge 0, z \le y + 1, z \le 0\}. \end{aligned}$$

Then, $\text{int}(\text{co}(F(C)) - \mu(1, 0, 0) + (\mathbb{R}_+ \times P)) \ne \emptyset$ and

$$\text{cone}(\text{int}\,(\text{co}(F(C)) - \mu(1, 0, 0) + \mathbb{R}_+ \times P)) = \{(0, 0, 0)\} \cup \{(x, y, z) : x > 0, y, z \in \mathbb{R}\}$$

is pointed. Moreover,

$$(g_1, g_2)(x) = (g_1(x), g_2(x)) = \begin{cases} (x, x+1) & \text{if} \quad x \le -1, \\ (-1, x+1) & \text{if} \quad -1 < x < 0, \\ (0, 0) & \text{if} \quad x \ge 0. \end{cases}$$

Thus,

$$(g_1, g_2)(C) + P = \left\{(x, y) \in \mathbb{R}^2 : 0 \le y < 1,\ x \ge -1\right\} \cup \{(x, y) \in \mathbb{R}^2 : y \le 0, y \le x + 1\},$$

which is not convex. This yields

$$\overline{\text{cone}}((g_1, g_2)(C) + P) = \mathbb{R}^2,$$

that is, the generalized Slater condition is satisfied. On the other hand, given $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+ \times \mathbb{R}$, we obtain

$$L(\lambda, x) = \begin{cases} (\lambda_1 + \lambda_2)x + \lambda_2 & \text{if} \quad x \le -1, \\ \lambda_2 x + \lambda_2 - \lambda_1 & \text{if} \quad -1 < x < 0, \\ 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x > 0. \end{cases}$$

Hence, for $\lambda_1 \ge 0$ and $\lambda_2 \in \mathbb{R}$, we get

$$\inf_{x \in \mathbb{R}} L(\lambda, x) = \begin{cases} \lambda_2 - \lambda_1 & \text{if} \quad \lambda_1 + \lambda_2 \le 0, \\ -\infty & \text{if} \quad \lambda_1 + \lambda_2 > 0, \end{cases}$$

and therefore,

$$\max_{(\lambda_1,\lambda_2) \in P^*} \inf_{x \in \mathbb{R}} L(\lambda, x) = \max_{\substack{\lambda_1+\lambda_2 \le 0, \\ \lambda_1 \ge 0, \lambda_2 \in \mathbb{R}}} \inf_{x \in \mathbb{R}} L(\lambda, x) = \max_{\substack{\lambda_1+\lambda_2 \le 0, \\ \lambda_1 \ge 0, \lambda_2 \in \mathbb{R}}} (\lambda_2 - \lambda_1) = 0 = \mu.$$

$$\inf_{x \in \mathbb{R}} L(\lambda^*, x) = 0, \quad \lambda^* = (0, 0).$$

Example 3.5 This instance is discussed in [6, Example 3.1] for a different purpose and shows that a generalized SC is necessary to have strong duality.
Let $f(x_1,x_2)=-\sqrt{x_1}$, $g_1(x_1,x_2)=-x_1-x_2$, $g_2(x_1,x_2)=x_1$, $C=\mathbb{R}_+\times\mathbb{R}$, $P=\mathbb{R}_+\times\{0\}$. Thus, $P^*=\mathbb{R}_+\times\mathbb{R}$ and $\mu=0$. It is not difficult to see that $(g_1,g_2)(C)+(\mathbb{R}_+\times\{0\})=\mathbb{R}\times\mathbb{R}_+$, which implies that $\overline{\text{cone}}((g_1,g_2)(C)+P)\neq\mathbb{R}^2$. Moreover, setting $F(C)=\{(f(x),g_1(x),g_2(x)) : x\in C\}$, we obtain

$$F(C)+(\mathbb{R}^2_+\times\{0\})=\bigcup_{\substack{x_1>0\\ x_2\in\mathbb{R}}}\left\{(-\sqrt{x_1},-x_1-x_2,x_1)+(\mathbb{R}^2_+\times\{0\})\right\}$$
$$\cup\bigcup_{x_2\in\mathbb{R}}\left\{(0,-x_2,0)+(\mathbb{R}^2_+\times\{0\})\right\}.$$

Then, $\text{int}[\text{co}(F(C))+(\mathbb{R}^2_+\times\{0\})]\neq\emptyset$ and

$$\text{cone}\left(\text{int}[\text{co}(F(C))+(\mathbb{R}^2_+\times\{0\})]\right)\quad\text{is pointed.}$$

If there exists $(\lambda_1,\lambda_2)\in\mathbb{R}_+\times\mathbb{R}$ such that

$$-\sqrt{x_1}+\lambda_1(-x_1-x_2)+\lambda_2x_1\geq 0,\quad\forall\,(x_1,x_2)\in\mathbb{R}_+\times\mathbb{R},$$

then, setting $x_1=0$, we get $\lambda_1=0$. Thus, the previous inequality reduces

$$-\sqrt{x_1}+\lambda_2x_1\geq 0,\quad\forall\,x_1\geq 0,$$

which is impossible. Hence, strong duality does not hold.

3.2 Reducing a constrained problem into one with a single inequality constraint

We now establish an important result ensuring that problem (3.1) can be reformulated with a single constraint under generalized convexity assumptions and a Slater-type condition. Here, we restrict to the case int $P\neq\emptyset$. In addition, we consider the assumption

$$\forall p^*\in P^*,\quad\text{the restriction of } \langle p^*,g(\cdot)\rangle \text{ on any line segment of } C \text{ is lower semicontinuous.}\tag{3.7}$$

Next theorem extends and generalizes that of [20] valid for finite dimensional spaces and P-convex functions g. The latter means that, given a convex set C and $x,y\in C$, one has

$$g(tx+(1-t)y)\in tg(x)+(1-t)g(y)-P,\quad\forall\,t\in\ ]0,1[.\tag{3.8}$$

We recall a larger class of vector functions. According to [19] (where it was used to derive a Gordan-type alternative theorem), given a convex set $C\subseteq X$ with X as before, a mapping $g:C\to Y$ is called $*$-quasiconvex if $\langle p^*,g(\cdot)\rangle$ is quasiconvex for all $p^*\in P^*$. Independently, the author in [26] says that g is naturally P-quasiconvex if for all $x,y\in C$, $g([x,y])\subseteq[g(x),g(y)]-P$. Both classes coincide as shown in [10, Proposition 3.9], [11, Theorem 2.3] (it is still valid if P has empty interior) under assumption (3.7).

It is known from, Corollary 3.11 in [10], that every naturally P-quasiconvex function $g:C\to Y$ satisfying (3.7), is such that $g(C)+P$ is convex, so that $g(C')+P$ is also convex for every convex set $C'\subseteq C$.
A real-valued function $h:C\to\mathbb{R}$ is said to be semistrictly quasiconvex if

$$x,y\in C,\quad h(x)<h(y)\Longrightarrow h(\xi)<h(y),\quad\forall\ \xi\in\ ]x,y[.$$

Theorem 3.6 *Let us consider problem* (3.1) *with f being quasiconvex and upper semicontinuous along lines of C. Assume that μ is finite and g is naturally P-quasiconvex such that for all $p^* \in P^*\backslash\{0\}$, $x \in C \mapsto \langle p^*, g(x)\rangle$ is semistrictly quasiconvex and lsc along any line segment of C. If, in addition, the Slater-type condition that for some $\bar{x} \in C$, $\langle y^*, g(\bar{x})\rangle < 0$ for all $y^* \in P^*\backslash\{0\}$ holds, that is, $g(\bar{x}) \in -\text{int}\, P$, then, there exists $p^* \in P^*\backslash\{0\}$ such that*

$$\inf_{\substack{g(x)\in -P\\ x\in C}} f(x) = \inf_{\substack{\langle p^*, g(x)\rangle \le 0\\ x\in C}} f(x). \tag{3.9}$$

Hence, every solution to (3.1) *is also a solution to the problem of right hand-side of* (3.9).

Proof Let us consider

$$M \doteq g(C_0) + P, \quad C_0 \doteq \{x \in C : \ f(x) < \mu\}.$$

Since C_0 is convex and g is naturally P-quasiconvex on any convex subset C' of C, the set M is convex by Corollary 3.11 in [10]. We can assume that M is nonempty since otherwise any $p^* \in P^*$ verifies (3.9). Evidently, $M \cap (-P) = \emptyset$, for if not, there exists $z_0 \in -P$ such that $z_0 \in M$, that is, there is $x_0 \in C_0$ satisfying $z_0 - g(x_0) \in P$. It turns out that $g(x_0) \in -P, x_0 \in C, f(x_0) < \mu$, which cannot happen. We apply a convex separation theorem to obtain the existence of $p \in P^*$, $p^* \neq 0, \alpha \in \mathbb{R}$, such that

$$\langle p^*, z\rangle \geq \alpha \ \ \forall\, z \in M, \ \ \langle p^*, u\rangle \leq \alpha, \ \ \forall\, u \in -P.$$

Hence,

$$p^* \in P^* \ \text{ and } \ \langle p^*, g(x)\rangle \geq 0, \ \ \forall\, x \in C_0. \tag{3.10}$$

Let $x \in C$, $\langle p^*, g(x)\rangle \leq 0$. In case $f(x) < \mu$, that is, $x \in C_0$, we get $g(x) \in M$ and thus $\langle p^*, g(x)\rangle = 0$. Set $x_t = t\bar{x} + (1-t)x$. By the upper semicontinuity of f, $f(x_t) < \mu$ for some $t \in \ ]0, 1[$, and therefore $x_t \in C_0$. Thus, by semistrict quasiconvexity, $0 \leq \langle p^*, g(x_t)\rangle < 0$, a contradiction. Whence $f(x) \geq \mu$. This implies

$$\inf_{\substack{\langle p^*, g(x)\rangle \le 0\\ x\in C}} f(x) \geq \inf_{\substack{g(x)\in -P\\ x\in C}} f(x).$$

The reverse inequality is trivial. □

The previous theorem allows us to reduce a problem with several quasiconvex constraints to a problem having a single quasiconvex constraint. If $\bar{x}$ solves the problem

$$\inf_{\substack{\langle p^*, g(x)\rangle \le 0\\ x\in C}} f(x), \tag{3.11}$$

we cannot assure that it necessarily solves (3.1) or satisfies $\langle p^*, g(\bar{x})\rangle = 0$. The latter is essentially due to the fact that relative minima need not be global minima.

Theorem 3.7 *Let us consider problem* (3.1). *Assume that x_0 solves problem* (3.1) ($\mu = f(x_0)$) *and that all the assumptions of Theorem* 3.6 *are fulfilled. Then, either*

(a) there is $\bar{x} \in C$, $g(\bar{x}) \in -\text{int}\, P$ and $f(\bar{x}) = \mu$, or
(b) there is $p^* \in P^*$ such that x_0 solves problem (3.11) *and $\langle p^*, g(x_0)\rangle = 0$.*

Proof Assume that (a) does not hold. Then, $0 \notin B + \text{int}\, P$, where $B \doteq g(C_1) + P$, $C_1 = \{x \in C : f(x) \leq \mu\}$. By assumption, B is convex and therefore, there exists $p^* \in Y^*$, $p^* \neq 0$, such that

$$0 \leq \langle p^*, \xi \rangle \quad \forall\, \xi \in B + \text{int}\, P.$$

this implies that $p^* \in P^*$ and $0 \leq \langle p^*, g(x) \rangle$ for all $x \in C_1$. Hence, $\langle p^*, g(x_0) \rangle = 0$, and by the previous theorem, x_0 solves problem (3.11). □

4 Characterizing strong duality: the case with a single inequality constraint

In this situation, we describe completely the pointedness of the cone appearing in Theorem 3.1; and as a consequence, a new characterization of strong duality is obtained, covering situations where a Slater-type condition may fail.
Here, $K = \{x \in C : g(x) \leq 0\}$, thus $K = S_g^-(0) \cup S_g^=(0)$, where

$$S_g^-(0) \doteq \{x \in C : g(x) < 0\}, \quad S_g^=(0) \doteq \{x \in C : g(x) = 0\},$$
$$S_g^+(0) \doteq \{x \in C : g(x) > 0\}.$$

Similarly, we define

$$S_f^-(\mu) \doteq \{x \in C : f(x) < \mu\}, \; S_f^+(\mu) \doteq \{x \in C : f(x) > \mu\},$$

$$S_f^=(\mu) \doteq \{x \in C : f(x) = \mu\}.$$

Set $\mathbb{R}^2_{++} \doteq \text{int}\, \mathbb{R}^2_+$, $F = (f, g)$ and $F(C) \doteq \{(f(x), g(x)) \in \mathbb{R}^2 : x \in C\}$. By writting $F(C) - \mu(1, 0) + \mathbb{R}^2_{++} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, it follows that

$$\text{cone}(F(C) - \mu(1, 0) + \mathbb{R}^2_{++}) = \text{cone}(\Omega_1) \cup \text{cone}(\Omega_2) \cup \text{cone}(\Omega_3), \tag{4.1}$$

where

$$\Omega_1 \doteq \bigcup_{x \in \text{argmin}_K f \cap S_g^=(0)} [(0, g(x)) + \mathbb{R}^2_{++}] \cup \bigcup_{x \in \text{argmin}_K f \cap S_g^-(0)} [(0, g(x)) + \mathbb{R}^2_{++}];$$
$$\Omega_2 \doteq \bigcup_{x \in K \setminus \text{argmin}_K f} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}];$$
$$\Omega_3 \doteq \bigcup_{x \in C \setminus K} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}] = \Omega_3^1 \cup \Omega_3^2 \cup \Omega_3^3,$$

with

$$\Omega_3^1 = \bigcup_{x \in S_g^+(0) \cap S_f^-(\mu)} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}]; \; \Omega_3^2 = \bigcup_{x \in S_g^+(0) \cap S_f^=(\mu)} [(0, g(x)) + \mathbb{R}^2_{++}],$$

$$\Omega_3^3 = \bigcup_{x \in S_g^+(0) \cap S_f^+(\mu)} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}].$$

On the other hand, whenever $S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$ and $S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$, we set

$$\alpha \doteq \inf_{x \in S_g^-(0) \cap S_f^+(\mu)} \frac{g(x)}{f(x) - \mu}, \quad \beta \doteq \sup_{x \in S_g^+(0) \cap S_f^-(\mu)} \frac{g(x)}{f(x) - \mu}.$$

Evidently, $-\infty \leq \alpha < 0$, $-\infty < \beta \leq 0$ and;

$$C = K \iff S_g^+(0) = \emptyset;$$

$$\operatorname{argmin}_K f \cap S_g^-(0) = \emptyset \quad \text{and} \quad S_g^-(0) \cap S_f^+(\mu) = \emptyset \iff S_g^-(0) = \emptyset;$$

$$S_g^-(0) \cap S_f^+(\mu) = \emptyset \iff S_g^-(0) \subseteq \operatorname{argmin}_K f.$$

The previous discussion along with (4.1) yield Figs. 1, 2 and 3. Such figures allow us to visualize the pointedness of $\operatorname{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$, which is expressed in the following theorem.

Theorem 4.1 *Let us consider problem (3.1) such that $K \neq \emptyset$ and μ is finite.*

(a) Assume that $\operatorname{argmin}_K f \neq \emptyset$. Then, $\operatorname{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$ is pointed if, and only if any of the following circumstances holds:
- *($a1$) $\operatorname{argmin}_K f \cap S_g^-(0) \neq \emptyset$ and, either $S_g^+(0) = \emptyset$ or $[S_g^+(0) \cap S_f^-(\mu) = \emptyset, S_g^+(0) \neq \emptyset]$;*
- *($a2$) $\operatorname{argmin}_K f \cap S_g^-(0) = \emptyset$, $\operatorname{argmin}_K f \cap S_g^=(0) \neq \emptyset$ and $K = \operatorname{argmin}_K f$;*
- *($a3$) $\operatorname{argmin}_K f \cap S_g^-(0) = \emptyset$, $\operatorname{argmin}_K f \cap S_g^=(0) \neq \emptyset$, $S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$, $-\infty < \alpha < 0$ and, either $S_g^+(0) = \emptyset$ or $[S_g^+(0) \cap S_f^-(\mu) \neq \emptyset, \beta \leq \alpha]$, or $[S_g^+(0) \cap S_f^-(\mu) = \emptyset, S_g^+(0) \neq \emptyset]$;*
- *($a4$) $\operatorname{argmin}_K f \cap S_g^-(0) = \emptyset$, $\operatorname{argmin}_K f \cap S_g^=(0) \neq \emptyset$, $S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$, $\alpha = -\infty$ and, either $S_g^+(0) = \emptyset$ or $[S_g^+(0) \cap S_f^-(\mu) = \emptyset, S_g^+(0) \neq \emptyset]$;*
- *($a5$) $\operatorname{argmin}_K f \cap S_g^-(0) = \emptyset$, $\operatorname{argmin}_K f \cap S_g^=(0) \neq \emptyset$, $S_g^-(0) \cap S_f^+(\mu) = \emptyset$ and $S_g^=(0) \cap S_f^+(\mu) \neq \emptyset$.*

(b) Assume that $\operatorname{argmin}_K f = \emptyset$. Then, $\operatorname{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$ is pointed if, and only if any of the following instances holds:
- *($b1$) $S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$, $-\infty < \alpha < 0$ and, either $S_g^+(0) = \emptyset$ or $[S_g^+(0) \cap S_f^-(\mu) \neq$*

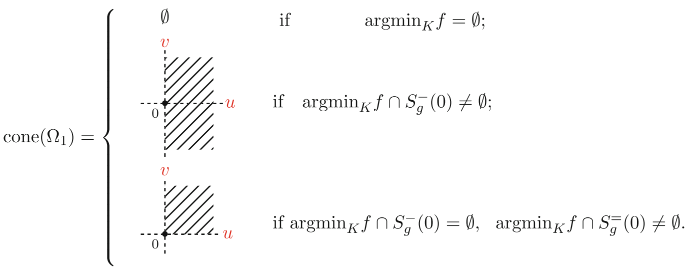


Fig. 1 Visualizing Theorem 4.1

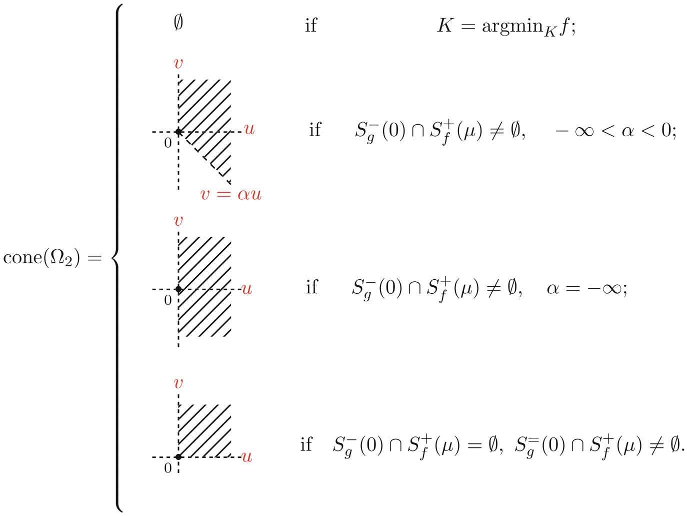


Fig. 2 Visualizing Theorem 4.1

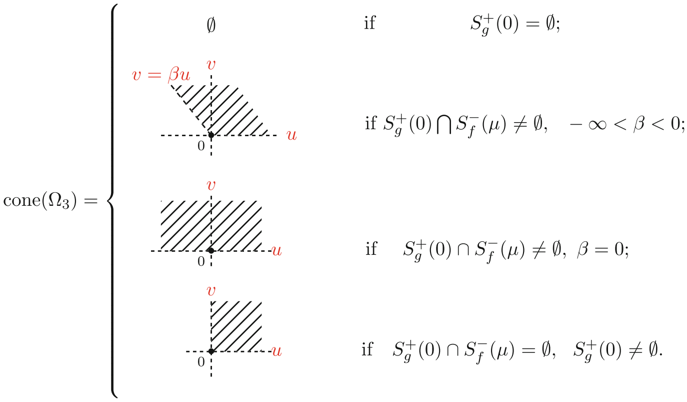


Fig. 3 Visualizing Theorem 4.1

$\emptyset$, $\beta \leq \alpha$] or [$S_g^+(0) \cap S_f^-(\mu) = \emptyset$, $S_g^+(0) \neq \emptyset$];

(b2) $S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$, $\alpha = -\infty$ and, either $S_g^+(0) = \emptyset$ or [$S_g^+(0) \cap S_f^-(\mu) = \emptyset$, $S_g^+(0) \neq \emptyset$];

(b3) $S_g^-(0) \cap S_f^+(\mu) = \emptyset$, $S_g^=(0) \cap S_f^+(\mu) \neq \emptyset$.

Proof We omit the long but easy proof, once we get Figs. 1, 2, and 3. □

Looking at (4.1) and the expressions for cone(Ω_i), $i = 1, 2, 3$ (see, Figs. 1, 2, and 3), we get the following corollary which establishes a complete description concerning the validity of strong duality for nonconvex minimization problems with a single constraint. In particular, we observe that strong duality holds even if the standard Slater condition is not satisfied.

Corollary 4.2 *Let K be non-empty and μ finite.*

(a) *If either* $\operatorname{argmin}_K f \cap S_g^-(0) \neq \emptyset$ *or* $[S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$ *with* $\alpha = -\infty]$*, then*

$$\lambda^* \geq 0,\ f(x) + \lambda^* g(x) \geq \mu \quad \forall\, x \in C \implies \lambda^* = 0;$$

consequently,

$$\inf_{x\in K} f(x) = \inf_{x \in C} f(x).$$

(b) *Assume that* $S_g^+(0) = \emptyset$ *or* $[S_g^+(0) \cap S_f^-(\mu) = \emptyset,\ S_g^+(0) \neq \emptyset]$ *are satisfied; if either* $(a3)$ *or* $(a5)$ *with* $\operatorname{argmin}_K f \neq \emptyset$*, or* $[(b3)$ *with* $\operatorname{argmin}_K f = \emptyset]$ *holds, then, any* $(\gamma^*, \lambda^*) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$*, verifies*

$$\gamma^*(f(x) - \mu) + \lambda^* g(x) \geq 0 \quad \forall\, x \in C,$$

consequently,

$$\gamma^* \inf_{x \in K} f(x) = \inf_{x \in C} L(\gamma^*, \lambda^*, x).$$

(c) *Assume that* $\beta \leq \alpha$*; if* $[(a3)$ *with* $\operatorname{argmin}_K f \neq \emptyset]$*, or* $[(b1)$ *with* $\operatorname{argmin}_K f = \emptyset]$ *hold, then any* λ^* *such that* $-\frac{1}{\beta} \leq \lambda^* \leq -\frac{1}{\alpha}$ *satisfies*

$$f(x) + \lambda^* g(x) \geq \mu \quad \forall\, x \in C. \tag{4.2}$$

(d) *Assume that* $-\infty < \beta < 0$ *and* $S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$ *hold; if either* $(a2)$ *or* $(a5)$ *with* $\operatorname{argmin}_K f \neq \emptyset$*, or* $[(b3)$ *with* $\operatorname{argmin}_K f = \emptyset]$*, then any* λ^* *such that* $-\frac{1}{\beta} \leq \lambda^*$ *verifies* (4.2).

(e) *Assume that either* $S_g^+(0) = \emptyset$ *or* $[S_g^+(0) \cap S_f^-(\mu) = \emptyset,\ S_g^+(0) \neq \emptyset]$ *are satisfied; if* $[(a3)$ *with* $\operatorname{argmin}_K f \neq \emptyset]$ *or* $[(b1)$ *with* $\operatorname{argmin}_K f = \emptyset]$*, hold, then any* λ^* *such that* $-\frac{1}{\alpha} \geq \lambda^* > 0$ *verifies* (4.2).

(f) *If* $S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$ *with* $\beta = 0$*, then*

$$\gamma^* \geq 0, \quad \gamma^*(f(x) - \mu) + g(x) \geq 0 \quad \forall\, x \in C \implies \gamma^* = 0.$$

Figures 4 and 5 provide the geometry of different situations occuring in Corollary 4.2.

The preceding corollary applies to situations where strong duality still holds even if the Slater condition fails. This is illustrated in the following example.

Example 4.3 Take $C = \{(x_1, x_2) :\ x_2 \leq x_1, x_1 \geq 0, x_2 \geq -1\}$, $P = \mathbb{R}_+$. Let

$$f(x_1, x_2) = x_1^2 + x_1 x_2 - 2x_2^2, \quad g(x_1, x_2) = x_1 - x_2.$$

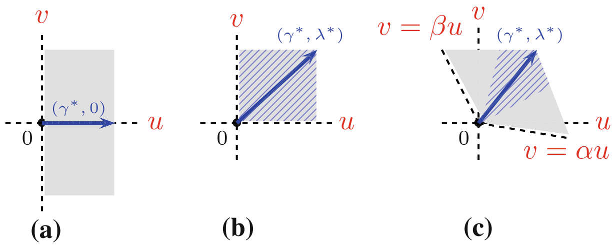
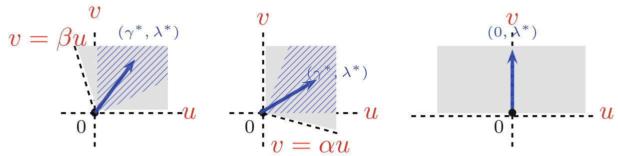


Fig. 4 Corollary 4.2(*a*), (*b*), (*c*)



(d) **(e)** **(f)**

Fig. 5 Corollary 4.2(d), (e), (f)

Thus, $K = \{(x_1, x_2) : x_1 = x_2,\ x_2 \geq 0\}$, and there is no $x \in C$ such that $g(x) < 0$, i.e., $S_g^-(0) = \emptyset$. In this case,

$$\text{cone}(F(C) + \mathbb{R}^2_{++}) = \left\{(u, v) \in \mathbb{R}^2 :\ v > -\frac{1}{2}u,\ v > 0\right\} \cup \{(0, 0)\}$$

is pointed; $\mu = 0$, $\text{argmin}_K f = K$, $S_g^+(0) = \{(x_1, x_2) \in C : x_1 > x_2\}$ and

$$S_f^-(\mu) = \left\{(x_1, x_2) : x_2 < -\frac{1}{2}x_1, x_2 \geq -1, x_1 \geq 0\right\}.$$

Thus, $S_g^+(0) \cap S_f^-(\mu) = S_f^-(\mu)$, and so $\beta = -1/2$. Hence, according to Corollary 4.2(d), any $\lambda^* \geq 2$ satisfies $f(x) + \lambda^* g(x) \geq 0\ \ \forall\, x \in C$, i.e.,

$$\min_{\substack{g(x)\leq 0 \\ x\in C}} f(x) = \min_{x\in C}(f(x) + \lambda^* g(x)), \quad \forall\, \lambda^* \geq 2.$$

Next theorem provides a complete characterization of strong duality for our problem with a single constraint.

Theorem 4.4 *Let K be non-empty and μ finite. The following assertions are equivalent:*

(a) *strong duality holds;*
(b) $\text{cone}(F(C) - \mu(1, 0) + \mathbb{R}^2_{++})$ *is pointed, and either* $S_g^+(0) \cap S_f^-(\mu) = \emptyset$ *or* $[S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$ *with* $\beta < 0]$ *holds.*

Proof $(a) \implies (b)$: The pointedness follows from Theorem 3.1. Suppose that $S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$. By assumption, there exists $\lambda_0^* \geq 0$ such that $f(x) + \lambda_0^* g(x) \geq \mu$ for all $x \in C$. This implies that $\lambda_0^* > 0$. Indeed, if $\lambda_0^* = 0$, the previous inequality gives $f(x) - \mu \geq 0$ for all $x \in C$, which is impossible if $S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$.

Now, suppose that $\beta = 0$. Then, there exists $\bar{x} \in S_g^+(0) \cap S_f^-(\mu) \neq \emptyset$ such that

$$\frac{g(\bar{x})}{f(\bar{x}) - \mu} > -\frac{1}{\lambda_0^*}.$$

It follows that $f(\bar{x}) + \lambda_0^* g(\bar{x}) < \mu$, yielding a contradiction; this proves that $\beta < 0$.

$(b) \implies (a)$: This is a consequence of Theorem 4.1 and Corollary 4.2. □

We observe the condition $S_g^+(0) \cap S_f^-(\mu) = \emptyset$ can be split into the following two expressions:

(1) $S_g^+(0) = \emptyset$;
(2) $S_g^+(0) \cap S_f^-(\mu) = \emptyset$ and $S_g^+(0) \neq \emptyset$.

The first gives immediately

$$\inf_{x\in K} f(x) = \inf_{x\in C} f(x).$$

Furthermore, since $(F(C) - \mu(1,0)) \cap (-\mathbb{R}^2_{++}) = \emptyset$, it is not difficult to see the pointedness of $\text{cone}(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$ is equivalent to the convexity of $\text{cone}(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$, or $\overline{\text{cone}}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$, or $\text{cone}(F(C) - \mu(1,0)) + \mathbb{R}^2_{++}$, see for instance Theorem 4.1 in [10].

As we will see in the next section, the convexity of $F(C)$, and so of $\text{cone}(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$, is guaranteed by an important class of quadratic functions, as a consequence of Dine's theorem, see [17].

5 A concrete application: the (non convex) quadratic homogeneous case

We now derive, from Theorem 4.4, a necessary and sufficient optimality condition for a class of homogeneous programming problems, arising in telecommunications and robust control, see [22,24].

Let us consider the following homogeneous optimization problem:

$$\mu \doteq \inf \left\{ \frac{1}{2}x^\top Ax : \ \frac{1}{2}x^\top Bx \leq 1, x \in C \right\}, \tag{5.1}$$

where C is a regular cone [17, Definition 3.1], that is, $C \cup (-C)$ is a linear subspace. Setting

$$f(x) = \frac{1}{2}x^\top Ax, \quad g(x) = \frac{1}{2}x^\top Bx - 1,$$

where A, B are symmetric matrices, the generalized Dine's theorem ([17, Theorem 3.2]) ensures that

$$F(C) - \mu(1,0) = \left\{ \left(\frac{1}{2}x^\top Ax, \frac{1}{2}x^\top Bx \right) : \ x \in C \right\} - (\mu, 1) \ \text{ is convex,}$$

and therefore so is $\text{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$, which is equivalent to the pointedness of $\text{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$. We notice the Slater condition is also satisfied. As a consequence, strong duality always holds for problem (5.1) by Corollary 3.2. We recall that A is copositive (on C) if $x^\top Ax \geq 0$ for all $x \in C$. Notice that copositivity on C is equivalent to copositivity on $C \cup (-C)$. By $H^\perp$ we mean the orthogonal subspace of $H \subseteq \mathbb{R}^m$, that is, $H^\perp = \{\xi \in \mathbb{R}^m : \ \langle \xi, x\rangle = 0 \ \ \forall\, x \in H\}$. Next theorem, which is new in the literature, considers non-convex situations.

Theorem 5.1 *Let μ finite and $\bar{x}$ feasible for* (5.1)*. The following assertions are equivalent:*

(a) *$\bar{x}$ is a solution to* (5.1)*;*
(b) *$\exists\, \lambda^* \geq 0$ such that $\nabla f(\bar{x}) + \lambda^*\nabla g(\bar{x}) \in (C \cup (-C))^\perp$, $\lambda^* g(\bar{x}) = 0$, $A + \lambda^* B$ is copositive (on C)*

Proof $(a) \Longrightarrow (b)$: By the remark above, strong duality holds, thus, there exists $\lambda^* \geq 0$ such that

$$f(\bar{x}) + \lambda^* g(\bar{x}) \leq f(\bar{x}) = \inf_{x\in C} (f(x) + \lambda^* g(x)).$$

This implies that $\lambda^* g(\bar{x}) = 0$ and $\bar{x}$ is a minimum for $L(x) = f(x) + \lambda^* g(x)$ on C, and so also on $C \cup (-C)$ since

$$\min\{L(x) : x \in C\} = \min\{L(x) : x \in C \cup (-C)\}.$$

The necessary optimality condition yields

$$\langle \nabla f(\bar{x}) + \lambda^* \nabla g(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in C \cup (-C).$$

We also have $f(x) + \lambda^* g(x) \geq f(\bar{x})$ for all $x \in C$, which gives $x^\top (A + \lambda^* B) x \geq 0$ for all $x \in C$.
$(b) \Longrightarrow (a)$: Setting $L(x) = f(x) + \lambda^* g(x)$, $x \in C \cup (-C)$, we write

$$L(x) - L(\bar{x}) = \langle \nabla f(\bar{x}) + \lambda^* \nabla g(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \langle (A + \lambda^* B)(x - \bar{x}), x - \bar{x} \rangle.$$

Since $A + \lambda^* B$ is also copositive on $C \cup (-C)$, and this is a subspace, the previous equality reduces to

$$f(x) \geq L(x) \geq L(\bar{x}) = f(\bar{x}) + \lambda^* g(\bar{x}) = f(\bar{x}),$$

implying $f(x) \geq f(\bar{x})$. This proves that $\bar{x}$ is a solution to (5.1). □

References

1. Auslender, A.: Existence of optimal solutions and duality results under weak conditions. Math. Program. Ser. A **88**, 45–59 (2000)
2. Auslender, A., Teboulle, M.: Asymptotic Cones and Functions in Optimization and Variational Inequalities. Springer Monograph. Springer, New york (2003)
3. Borwein, J.M., Lewis, A.S.: Partially finite convex programming, part I: quasi relative interiors and duality theory. Math. Program. Ser. A **57**, 15–48 (1992)
4. Bot, R.I., Wanka, G.: An alternative formulation for a new closed cone constraint qualification. Nonlinear Anal. **64**, 1367–1381 (2006)
5. Bot, R.I., Grad, S.-M., Wanka, G.: New regularity conditions for strong and total Fenchel-Lagrange duality in infinite dimensional spaces. Nonlinear Anal. **69**, 323–336 (2008)
6. Bot, R.I., Csetnek, E.R., Moldovan, A.: Revisiting some duality theorems via the quasirelative interior in convex optimization. J. Optim. Theory Appl. **139**, 67–84 (2008)
7. Bot, R.I., Csetnek, E.R., Wanka, G.: Regularity conditions via quasi-relative interior in convex programming. SIAM J. Optim. **19**, 217–233 (2008)
8. Brecker, W.W., Kassay, G.: A systematization of convexity concepts for sets and functions. J. Convex Anal. **4**, 109–127 (1997)
9. Daniele, P., Giuffrè, S.: General infinite dimensional duality and applications to evolutionary networks equilibrium problems. Optim. Lett. **1**, 227–243 (2007)
10. Flores-Bazán, F., Hadjisavvas, N., Vera, C.: An optimal alternative theorem and applications to mathematical programming. J. Global Optim. **37**, 229–243 (2007)
11. Flores-Bazán, F., Vera, C.: Unifying efficiency and weak efficiency in generalized quasiconvex vector minimization on the real-line. Int. J. Optim. Theory Methods Appl. **1**, 247–265 (2009)
12. Frenk, J.B.G., Kassay, G.: On classes of generalized convex functions. Gordan-Farkas type theorems, and Lagrangian duality. J. Optim. Theory Appl. **102**, 315–343 (1999)
13. Giannessi, F., Mastroeni, G.: Separation of sets and Wolfe duality. J. Global Optim. **42**, 401–412 (2008)
14. Jeyakumar, V.: Constraint qualifications characterizing Lagrangian duality in convex optimization. J. Optim. Theory Appl. **136**, 31–41 (2008)
15. Jeyakumar, V., Huy, N.Q., Li, G.Y.: Necessary and sufficient conditions for S-lemma and nonconvex quadratic optimization. Optim. Eng. **10**, 491–503 (2009)
16. Jeyakumar, V., Lee, G.M.: Complete characterizations of stable Farkas' lemma and cone-convex programming duality. Math. Prog. A **114**, 335–347 (2008)
17. Jeyakumar, V., Lee, G.M., Li, G.Y.: Alternative theorems for quadratic inequality systems and global quadratic optimization. SIAM J. Optim **20**, 983–1001 (2009)

18. Jeyakumar, V., Li, G.Y.: Stable zero duality gaps in convex programming: complete dual characterizations with applications to semidefinite programs. J. Math. Anal. Appl. **360**, 156–167 (2009)
19. Jeyakumar, V., Oettli, W., Natividad, M.: A solvability theorem for a class of quasiconvex mappings with applications to optimization. J. Math. Anal. Appl. **179**, 537–546 (1993)
20. Luenberger, D.G.: Quasi-convex programming. SIAM J. Appl. Math. **16**(5), 1090–1095 (1968)
21. Magnanti, T.L.: Fenchel and Lagrange duality are equivalent. Math. Prog. A **7**, 253–258 (1974)
22. Matskani, E., Sidiropoulos, N.D., Luo, Z.-Q., Tassiulas, L.: Convex approximations techniques for joint multiuser downlink beamforming and admission control. IEEE Trans. Wireless Commun. **7**, 2682–2693 (2008)
23. Rockafellar, R.T., Wets, R.J.-B.: Variational analysis. Springer, Berlin, Hedelberg (1998)
24. Sidiropoulos, N.D., Davidson, T.N., Luo, Z.-Q.: Transmit beamforming for physical layer multicasting. IEEE Trans. Signal Process **54**, 2239–2251 (2006)
25. Song, W.: Duality in set-valued optimization. Dissert. Math. **375**, 1–67 (1998)
26. Tanaka, T.: General quasiconvexities, cones saddle points and minimax theorem for vector-valued functions. J. Optim.Theory Appl. **81**, 355–377 (1994)
27. Zalinescu, C.: Convex analysis in general vector spaces. World Scientific, Singapore (2002)