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Global optimal solutions to a class of quadrinomial minimization problems with one quadratic constraint

Y.-B Yuan · S.-C. Fang · D. Y. Gao

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Abstract This paper studies the canonical duality theory for solving a class of quadrinomial minimization problems subject to one general quadratic constraint. It is shown that the nonconvex primal problem in $\mathbb{R}^n$ can be converted into a concave maximization dual problem over a convex set in $\mathbb{R}^2$, such that the problem can be solved more efficiently. The existence and uniqueness theorems of global minimizers are provided using the triality theory. Examples are given to illustrate the results obtained.

Keywords Nonconvex optimization · Canonical duality · Triality theory · NP-hard problem · Global optimization

1 Introduction

In this paper, we focus on the following quadrinomial minimization problem and call it a primal problem:

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Y.-B. Yuan
Institute of Metrology and Computational Science, China Jiliang University, Hangzhou 310018, China
e-mail: ybyuan@cjlu.edu.cn

S.-C. Fang
Department of Industrial and Systems Engineering, North Carolina State University, Raleigh, NC 27695, USA
e-mail: fang@ncsu.edu

D. Y. Gao (✉)
Graduate School of Information Technology and Mathematical Sciences, University of Ballarat, Ballarat, VIC 3350, Australia
e-mail: d.gao@ballarat.edu.au

$$(\mathcal{P}) : \min_{x\in\mathbb{R}^n} P(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \frac{1}{2}\alpha\left(\frac{1}{2}\|\mathbf{x}\|^2 - \beta\right)^2 - \mathbf{f}^T\mathbf{x}$$
$$s.t. \ \ \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x} + \mathbf{b}^T\mathbf{x} \leqslant c. \tag{1}$$

Where **A** and **B** are two symmetric matrices in $\mathbb{R}^{n\times n}$, **f** and **b** are two vectors in $\mathbb{R}^n$, $\alpha \geq 0$, $\beta \geq 0$ and c are real numbers, and $\|\cdot\|$ is the l_2 norm. Except for specially mentioned cases, the norm used in this manuscript is the l_2 norm.

Let us denote the feasible domain of Problem $(\mathcal{P})$ by

$$\mathcal{X} = \left\{\mathbf{x} \in \mathbb{R}^n | \ \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x} + \mathbf{b}^T x \leqslant c\right\} \tag{2}$$

and assume it is nonempty with c being large enough.

Notice that

$$W(\mathbf{x}) = \frac{1}{2}\alpha\left(\frac{1}{2}\|\mathbf{x}\|^2 - \beta\right)^2, \tag{3}$$

is a fourth-order *canonical polynomial* in $\mathbb{R}^n$ (see [11]). In a continuous system, this function is called the *double-well potential*, which was first studied by van der Waals in fluids mechanics in 1895. In the two dimensional space $\mathbb{R}^2$, $W(x_1, x_2)$ is called the *Mexican hat* function in cosmology and theoretical physics (see [11]). If we select $\alpha = 1$, $\beta = 2$, Fig. 1 shows the double-well function in one and two dimensional cases.

The Legendre transformation-based duality theory for nonconvex optimization and calculus of variations was studied by Ivar Ekeland ([1–3]). However, except for some special cases, using the Legendre dual to solve nonconvex systems remains an open territory ([2]). Canonical duality theory was originally developed by Gao and his colleagues for nonconvex analysis ([9,16]). This theory has shown its power in solving a large class of difficult global optimization problems ([4,5,7–16]).The goal of this paper is to apply the canonical duality theory for solving the proposed problem. We show that the constrained nonconvex primal problem $(\mathcal{P})$ in $\mathbb{R}^n$ can be reformulated as an unconstrained convex dual problems in $\mathbb{R}^2$ with no duality gap, which can be solved, under certain conditions, by well-developed nonlinear programming methods. In the next section, a "perfect dual problem" is formulated, which is equivalent to the primal problem in the sense that they share the same KKT points. The existence and uniqueness of a global minimizer is studied in Sect. 3. Some special cases and examples are given in Sect. 4 to illustrate the results obtained. Concluding remarks are given in the last section.

2 Canonical duality approach

Following the standard procedure of the canonical duality theory developed in ([9–11]), the geometrical operator for the primal problem $(\mathcal{P})$ can be defined as the following vector-valued mapping: $\Lambda(\mathbf{x}) = \{\rho(\mathbf{x}), y(\mathbf{x})\} : \mathbb{R}^n \to \mathbb{R}^2$, where

$$\rho(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x} + \mathbf{b}^T\mathbf{x},$$
$$y(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2 - \beta.$$

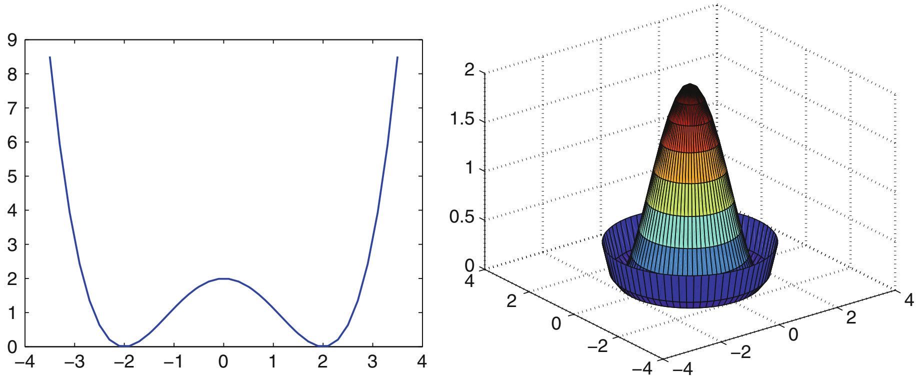

Fig. 1 Graphs of double-well potential $W(x)$

Also define that

$$\mathcal{D}_\rho = \{\rho \in \mathbb{R} |\ \ \rho \leqslant c\},$$
$$\mathcal{D}_y = \{y \in \mathbb{R} |\ \ y \geqslant -\beta\}.$$

The indicator function $\mathcal{I} : \mathbb{R} \to \{0\} \cup \{+\infty\}$ of $\mathcal{D}_\rho$ is defined by

$$\mathcal{I}(\rho) = \begin{cases} 0 & \text{if } \ \rho \in \mathcal{D}_\rho, \\ +\infty & \text{otherwise.} \end{cases}$$

It is convex and lower semi-continuous on $\mathbb{R}$. Consequently, the primal problem $(\mathcal{P})$ takes the following unconstrained canonical form:

$$(\mathcal{P}) : \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \mathcal{I}(\rho(\mathbf{x})) + V(y(\mathbf{x})) + \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{f}^T\mathbf{x} \right\}, \tag{4}$$

where $V(y) = \frac{1}{2}\alpha y^2$. Since $\mathcal{I}(\rho)$ is lower semi-continuous and convex on $\mathbb{R}$ and $V(y)$ is a canonical (quadratic) function on $\mathcal{D}_y$, the corresponding canonical dual variables ρ^* and y^* satisfy the following duality relations:

$$\rho^* \in \partial^- \mathcal{I}(\rho) \Leftrightarrow \rho \in \partial^- \mathcal{I}^*(\rho^*) \Leftrightarrow \mathcal{I}(\rho) + \mathcal{I}^*(\rho^*) = \rho\rho^*, \tag{5}$$
$$y^* = DV(y) \Leftrightarrow y = DV^*(y^*) \Leftrightarrow V(y) + V^*(y^*) = yy^*, \tag{6}$$

where $\partial^-\mathcal{I}(\rho)$ is the sub-differential of $\mathcal{I}$, $DV(y)$ is the Gâteaux derivative of V with respect to y, $\mathcal{I}^*(\rho^*)$ is *Fenchel sup-conjugate* of $\mathcal{I}$ defined by

$$\mathcal{I}^*(\rho^*) = \sup_{\rho\in\mathbb{R}}\{\langle \rho, \rho^* \rangle - \mathcal{I}(\rho)\} = \begin{cases} c\rho^* & \text{if } \rho^* \geqslant 0, \\ +\infty & \text{otherwise,} \end{cases}$$

and $V^*(y^*)$ is the *Legendre conjugate* of V given by [11]

$$V^*(y^*) = \frac{1}{2\alpha}(y^*)^2.$$

Hence the so-called *total complementary function* can be written in the following form:

$$\Xi(\mathbf{x}, \rho^*, y^*) = \rho(\mathbf{x})\rho^* + y(\mathbf{x})y^* - \mathcal{I}^*(\rho^*) - V^*(y^*) + \frac{1}{2}\mathbf{x}^T\mathbf{A}x - \mathbf{f}^T\mathbf{x}, \tag{7}$$

which is well defined on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{D}_{y^*}$ with $\mathcal{D}_{y^*} = \{y^* \in \mathbb{R} | \ y^* \geqslant -\alpha\beta\}$. With this total complementary function, the canonical dual function

$$P^d(\rho^*, y^*) = -\frac{1}{2}(\mathbf{f} - \mathbf{b}\rho^*)^T \mathbf{G}(\rho^*, y^*)^{-1}(\mathbf{f} - \mathbf{b}\rho^*) - \frac{1}{2\alpha}(y^*)^2 - c\rho^* - \beta y^* \tag{8}$$

is defined on the dual feasible domain

$$\mathcal{Y}^* = \{(\rho^*, y^*) \in \mathbb{R} \times \mathcal{D}_{y^*} | \ \rho^* \geqslant 0, \ \det \mathbf{G}(\rho^*, y^*) \neq 0\}, \tag{9}$$

where $\mathbf{G}(\rho^*, y^*) = \mathbf{A} + \rho^*\mathbf{B} + y^*\mathbf{I}$.

The extremal values of $P^d(\rho^*, y^*)$ depend on the property of $\mathbf{G}(\rho^*, y^*)$. If $\mathbf{G}(\rho^*, y^*) \succ 0$, the unique supreme value can be checked out in the domain; If $\mathbf{G}(\rho^*, y^*) \prec 0$, both local min and local max can be identified by the triality theory under certain conditions [17]; If $\mathbf{G}(\rho^*, y^*)$ is indefinite, this becomes the worst case, but we can find boundary infimum values and all local optimal values corresponding to the stationary points.

In this paper, we restrict our attention on the dual feasible domain

$$\mathcal{Y}^*_+ = \{(\rho^*, y^*) \in \mathcal{Y}^* | \ \mathbf{G}(\rho^*, y^*) \succ 0\}. \tag{10}$$

Therefore, the canonical dual problem $(\mathcal{P}^d)$ associated with the problem $(\mathcal{P})$ is proposed as

$$(\mathcal{P}^d): \quad \max \quad \left\{ P^d(\rho^*, y^*) | \ (\rho^*, y^*) \in \mathcal{Y}^*_+ \right\}. \tag{11}$$

Theorem 1 *The problem $(\mathcal{P}^d)$ is canonically (or perfectly) dual to the primal problem $(\mathcal{P})$ in the sense that if $(\bar{\rho}^*, \bar{y}^*)$ is a KKT point of $(\mathcal{P}^d)$, then*

$$\bar{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\bar{\rho}^* + \bar{y}^*\mathbf{I})^{-1}(\mathbf{f} - \mathbf{b}\bar{\rho}^*) \tag{12}$$

is a feasible solution to the primal problem $(\mathcal{P})$ and

$$P(\bar{\mathbf{x}}) = P^d(\bar{\rho}^*, \bar{y}^*). \tag{13}$$

Proof If $(\bar{\rho}^*, \bar{y}^*)$ is a KKT point of $(\mathcal{P}^d)$, we have

$$\begin{aligned} \frac{1}{2}\|\bar{\mathbf{X}}\|^2 - \frac{1}{\alpha}\bar{y}^* - \beta = 0, \\ 0 \leqslant \bar{\rho}^* \perp \frac{1}{2}\bar{\mathbf{x}}^T\mathbf{B}\bar{\mathbf{x}} + \mathbf{b}^T\bar{\mathbf{x}} - c \leqslant 0, \end{aligned} \tag{14}$$

where $\bar{\mathbf{x}} = (\mathbf{A} + \bar{\rho}^*\mathbf{B} + \bar{y}^*\mathbf{I})^{-1}(\mathbf{f} - \mathbf{b}\bar{\rho}^*)$. The complementarity condition (14) means that $\bar{\mathbf{x}}$ is a KKT point of the problem $(\mathcal{P})$, and it is feasible. From the complementarity condition (14), we further have

$$c\bar{\rho}^* = \frac{1}{2}\bar{\rho}^*\bar{\mathbf{x}}^T\mathbf{B}\bar{\mathbf{x}} + \bar{\rho}^*\mathbf{b}^T\bar{\mathbf{x}}.$$

Considering the Legendre-Young equality

$$\frac{1}{2\alpha}(\bar{y}^*)^2 + \beta\bar{y}^* = (\bar{y} + \beta)\bar{y}^* - \frac{1}{2}\alpha\bar{y}^2 = \frac{1}{2}\bar{\mathbf{x}}^T\bar{\mathbf{x}}\bar{y}^* - \frac{1}{2}\alpha\left(\frac{1}{2}\bar{\mathbf{x}}^T\bar{\mathbf{x}} - \beta\right)^2,$$

we have

$$\begin{aligned} P^d(\bar{\rho}^*, \bar{y}^*) &= -\frac{1}{2}(\mathbf{f} - \mathbf{b}\rho^*)^T \mathbf{G}(\bar{\rho}^*, \bar{y}^*)^{-1}\mathbf{G}(\bar{\rho}^*, \bar{y}^*) \\ &\quad \times \mathbf{G}(\bar{\rho}^*, \bar{y}^*)^{-1}(\mathbf{f} - \mathbf{b}\rho^*) - \frac{1}{2\alpha}(\bar{y}^*)^2 - \beta\bar{y}^* - \mathbf{c}\bar{\rho}^* \\ &= -\frac{1}{2}\bar{\mathbf{x}}^T \mathbf{G}(\bar{\rho}^*, \bar{y}^*)\bar{\mathbf{x}} - \left(\frac{1}{2\alpha}(\bar{y}^*)^2 + \beta\bar{y}^*\right) - \frac{1}{2}\bar{\mathbf{x}}^T \mathbf{B}\bar{\rho}^*\bar{\mathbf{x}} - \bar{\mathbf{x}}^T \mathbf{b}\bar{\rho}^* \\ &= -\frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}\bar{\mathbf{x}} + \frac{1}{2}\alpha\left(\frac{1}{2}\bar{\mathbf{x}}^T\bar{\mathbf{x}} - \beta\right)^2 - \bar{\mathbf{x}}^T\bar{\mathbf{x}}\bar{y}^* - \bar{\mathbf{x}}^T \mathbf{B}\bar{\rho}^*\bar{\mathbf{x}} - \bar{\mathbf{x}}^T \mathbf{b}\bar{\rho}^*, \\ &= -\frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}\bar{\mathbf{x}} + \frac{1}{2}\alpha\left(\frac{1}{2}\bar{\mathbf{x}}^T\bar{\mathbf{x}} - \beta\right)^2 + \bar{\mathbf{x}}^T(\mathbf{A}\bar{x} - f), \\ &= \frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}\bar{\mathbf{x}} + \frac{1}{2}\alpha\left(\frac{1}{2}\bar{\mathbf{x}}^T\bar{\mathbf{x}} - \beta\right)^2 - \mathbf{f}^T\bar{\mathbf{x}}, \\ &= P(\bar{\mathbf{x}}). \end{aligned}$$

This shows that there is no duality gap between the problems $(\mathcal{P})$ and $(\mathcal{P}^d)$. □

3 Global extrema

The triality theory of [9,11] charaterizes the global extrema of the primal problem $(\mathcal{P})$ as follows.

Theorem 2 (Global Extrema) *Assume that* $(\bar{\rho}*, \bar{y}*)$ *is a KKT point of* $(\mathcal{P}^d)$ *and* $\bar{\mathbf{x}} = \mathbf{G}(\bar{\rho}^*, \bar{y})^{-1}(\mathbf{f} - \mathbf{b}\bar{\rho}^*)$. *If* $(\bar{\rho}^*, \bar{y}^*) \in \mathcal{Y}_+^*$, *then* $(\bar{\rho}^*, \bar{y}^*)$ *is a global maximizer of* P^d *on* $\mathcal{Y}_+^*$, *while the vector* $\bar{\mathbf{x}}$ *is a global minimizer of* P *on* $\mathcal{X}$, *and*

$$P(\bar{\mathbf{x}}) = \min_{\mathbf{x}\in\mathcal{X}} P(\mathbf{x}) = \max_{(\rho^*, y^*)\in\mathcal{Y}_+^*} P^d(\rho^*, y^*) = P^d(\bar{\rho}^*, \bar{y}^*). \tag{15}$$

Proof The total complementarity function $\Xi(\mathbf{x}, \rho^*, y^*)$ can be written as

$$\begin{aligned} \Xi(\mathbf{x}, \rho^*, y^*) &= \frac{1}{2}\mathbf{x}^T\mathbf{B}\rho^*\mathbf{x} + \mathbf{b}^T\rho^*\mathbf{x} + \frac{1}{2}\mathbf{x}^T y^*\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{f}^T\mathbf{x} - \frac{1}{2\alpha}(y^*)^2 - c\rho^* - \beta y^*, \\ &= \frac{1}{2}\mathbf{x}^T\mathbf{G}(\rho^*, y^*)\mathbf{x} + (\mathbf{b}\rho^* - \mathbf{f})^T\mathbf{x} - \frac{1}{2\alpha}(y^*)^2 - c\rho^* - \beta y^*. \end{aligned} \tag{16}$$

If $(\bar{\rho}^*, \bar{y}^*) \in \mathcal{Y}_+^*$, then $\Xi(\mathbf{x}, \bar{\rho}^*, \bar{y}^*) : \mathbb{R}^n \mapsto \mathbb{R}$ is convex and $\Xi(\bar{\mathbf{x}}, \rho^*, y^*) : \mathbb{R}_+ \times \mathcal{D}_{y^*} \mapsto \mathbb{R}$ is concave. According to Theorem 1, $(\bar{\mathbf{x}}, \bar{\rho}^*, \bar{y}^*)$ is a saddle point of $\Xi(\mathbf{x}, \rho^*, y^*)$. Hence we have

$$\begin{aligned} P(\bar{\mathbf{x}}) &= \min_{\mathbf{x}\in\mathcal{X}} \max_{(\rho^*, y^*)\in\mathcal{Y}_+^*} \Xi(\mathbf{x}, \rho^*, y^*) = \max_{(\rho^*, y^*)\in\mathcal{Y}_+^*} \min_{\mathbf{x}\in\mathcal{X}} \Xi(\mathbf{x}, \rho^*, y^*) \\ &= P^d(\bar{\rho}^*, \bar{y}^*). \end{aligned}$$

□

Similar ideas can be traced back to [11–15].

3.1 Special case 1: quadratic programming over a sphere

Consider a special case of the primal problem. We select $\alpha = 0$, $\mathbf{B} = \mathbf{I}$, $\mathbf{b} = \mathbf{0}$, $c > 0$. Then the primal problem becomes

$$
\begin{aligned}
(\mathcal{P}): \min_{x\in\mathbb{R}^n} \; & P(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{f}^T\mathbf{x} \\
s.t. \; & \frac{1}{2}\|\mathbf{x}\|^2 \leqslant c,
\end{aligned} \tag{17}
$$

which is a quadratic programming problem with one quadratic (spherical) constraint [12]. Its canonical dual problem $(\mathcal{P}_s^d)$ becomes

$$
(\mathcal{P}_s^d): \max_{\rho^*\in\mathbb{R}} \left\{ P^d(\rho^*) = -\frac{1}{2}\mathbf{f}^T(\mathbf{A}+\rho^*\mathbf{I})^{-1}\mathbf{f} - c\rho^* \right\} \tag{18}
$$
$$
s.t. \;\; \rho^* \geqslant 0, \;\; \det(\mathbf{A}+\rho^*\mathbf{I}) \neq 0.
$$

The primal-dual relationship is given by

$$
\bar{\mathbf{x}} = (\mathbf{A}+\rho^*\mathbf{I})^{-1}\mathbf{f}.
$$

Following the ideas of reference [12], suppose that the symmetric matrix $\mathbf{A}$ has p (a positive integer less than or equal to n) distinct eigenvalues, and i_d (a positive integer $\leqslant p$) of them are negative such that

$$
a_1 < a_2 < \cdots < a_{i_d} < 0 \leqslant a_{i_d+1} < \cdots < a_p.
$$

Then for a given vector $\mathbf{f} \neq \mathbf{0} \in \mathbb{R}^n$ and a sufficiently large parameter $c > 0$, the canonical dual problem $(\mathcal{P}_s^d)$ has at most $2i_d + 1$ KKT points $\bar{\rho}_i^*$, $i = 1, 2, \ldots, 2i_d + 1$, satisfying the following distribution[1]

$$
\bar{\rho}_1^* > -a_1 > \bar{\rho}_2^* \geqslant \bar{\rho}_3^* > -a_2 > \cdots > -a_{i_d} > \bar{\rho}_{2i_d}^* \geqslant \bar{\rho}_{2i_d+1}^* > 0.
$$

For a given sufficiently large parameter $c > 0$, the problem $(\mathcal{P}_s^d)$ has at most $2i_d + 1$ KKT points on the boundary of the sphere. In practice, they can be obtained by solving the following algebraic equation

$$
\frac{1}{2}\mathbf{f}^T(\mathbf{A}+\rho^*\mathbf{I})^{-2}\mathbf{f} = c.
$$

Therefore, the global minimizer is

$$
\bar{\mathbf{x}}_1 = (\mathbf{A}+\bar{\rho}_1^*\mathbf{I})^{-1}\mathbf{f}.
$$

3.2 Special case 2: unconstrained double-well optimization

In this special case, $\mathbf{B} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, and $c > 0$. Then we have

$$
(\mathcal{P}): \min_{\mathbf{x}\in\mathbb{R}^n} P(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \frac{1}{2}\alpha\left(\frac{1}{2}\|\mathbf{x}\|^2 - \beta\right)^2 - \mathbf{f}^T\mathbf{x}.
$$

This problem is equivalent to finding the smallest stationary point of $P(\mathbf{x})$.

[1] This is a very special case. It is a challenging work that how to present a condition to make sure the canonical dual solution $\bar{\rho}_1^* > -a_1$.

Following [11], the canonical dual problem is to find the smallest stationary point of $P^d(y^*)$ over $\mathcal{D}_{y^*}$ such that

$$(\mathcal{P}^d): \quad \min \text{sta} \left\{ P^d(y^*) = -\frac{1}{2}\mathbf{f}^T(\mathbf{A} + y^*\mathbf{I})^{-1}\mathbf{f} - \frac{1}{2}\alpha^{-1}y^{*2} - \beta y^* \mid \ y^* \in \mathcal{D}_{y^*} \right\}.$$

The criticality condition associated with this dual problem is given by

$$\alpha^{-1}y^* + \beta = \frac{1}{2}\mathbf{f}^T(\mathbf{A} + y^*\mathbf{I})^{-2}\mathbf{f}.$$

For a given vector $\mathbf{f} \neq \mathbf{0} \in \mathbb{R}^n$ and $\alpha, \ \beta > 0$, if the symmetric matrix $\mathbf{A}$ has $m(\leq n)$ distinct eigenvalues $a_1 > a_2 > \cdots > a_m$, the above algebraic equation has at most $2m+1$ real roots $\bar{y}_1^* > \bar{y}_2^* \geq \bar{y}_3^* \geq \cdots \geq \bar{y}_{2m+1}^*$. These dual solutions lead to at most $2m+1$ critical points of $P(\mathbf{x})$ such that

$$\bar{x}_i = (\mathbf{A} + \bar{y}_i^*\mathbf{I})^{-1}\mathbf{f}, \quad i = 1, 2, \ldots, 2m+1.$$

The vector $\bar{x}_1 = (\mathbf{A} + \bar{y}_1^*\mathbf{I})^{-1}\mathbf{f}$ is a global minimizer of $P(\mathbf{x})$ and $\bar{x}_{2m+1} = (\mathbf{A} + \bar{y}_{2m+1}^*\mathbf{I})^{-1}\mathbf{f}$ is a local maximizer of $P(\mathbf{x})$.

4 Examples

In this section, we use three examples to illustrate the results obtained. In particular, we show that a global minimizer can be obtained by solving the canonical dual problem using MATLAB.

Example 1 Consider a one-dimensional double-well minimization problem. If we take $\mathbf{A} = a_1, f = f_1, \mathbf{B} = k_{11}, b = b_1$, the primal problem becomes

$$\begin{aligned} \min_{x \in \mathbb{R}} P(x) &= \frac{1}{2}a_1x^2 + \frac{1}{2}\alpha\left(\frac{1}{2}x^2 - \beta\right)^2 - f_1x \\ \textbf{s.t.} \quad & \frac{1}{2}k_{11}x^2 + b_1x \leq c. \end{aligned} \tag{19}$$

The canonical dual problem of (19) becomes

$$\begin{aligned} &\max_{(\rho^*, y^*) \in \mathbb{R}\times\mathbb{R}} P^d(\rho^*, y^*) = -\frac{1}{2}\frac{(f_1 - b_1\rho^*)^2}{k_{11}\rho^* + y^* + a_1} - \frac{1}{2\alpha}(y^*)^2 - \beta y^* - c\rho^* \\ &\textbf{s.t.} \quad \rho^* \geqslant 0, \quad y^* \geqslant -\alpha\beta, \quad k_{11}\rho^* + y^* \geqslant -a_1. \end{aligned} \tag{20}$$

The objective function in (20) is concave when $k_{11}\rho^* + y^* \geqslant -a_1$ and the dual problem has a unique solution in the feasible domain.

Let $a_1 = -0.4, \alpha = 1, \beta = 0.45, f_1 = -0.3, k_{11} = 1, b_1 = 1, c = 1$, the minimization problem becomes

$$\begin{aligned} \min_{x \in \mathbb{R}} P(x) &= -0.2x^2 + \frac{1}{2}\left(\frac{1}{2}x^2 - 0.45\right)^2 + 0.3x \\ \textbf{s.t.} \quad & \frac{1}{2}x^2 + x \leq 1. \end{aligned} \tag{21}$$

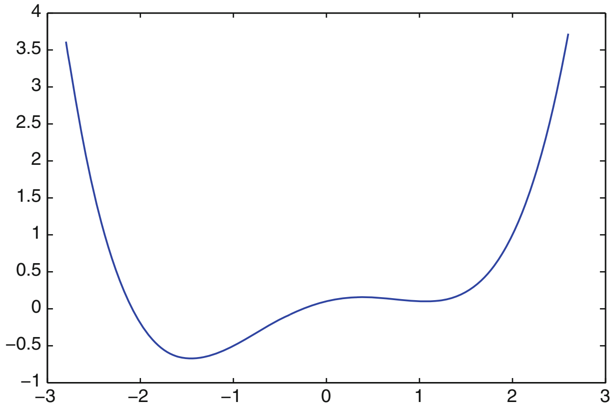

Fig. 2 Graphs of $P(x)$

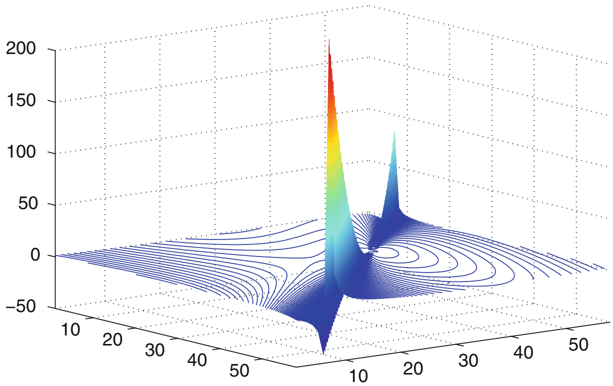

Fig. 3 Three dimensional contour of the canonical dual function $P^d(\rho^*, y^*)$

Its dual problem is

$$\max_{(\rho^*, y^*) \in \mathbb{R} \times \mathbb{R}} P^d(\rho^*, y^*) = -\frac{1}{2}\left(\frac{(-0.3-\rho^*)^2}{\rho^* + y^* - 0.4}\right) - \frac{1}{2}(y^*)^2 - 0.45y^* - \rho^*$$
$$\textbf{s.t.} \quad \rho^* \geqslant 0,\ y^* \geqslant -0.45,\ \rho^* + y^* \geqslant 0.4. \tag{22}$$

Figure 2 shows the graph of the primal objective function. The three dimensional contours of the canonical dual objective function is shown in Fig. 3.

Notice that the dual problem has a unique solution

$$\bar{\rho}^* = 0, \quad \bar{y}^* = 0.6064.$$

The optimal solution of the primal problem can be obtained by

$$\bar{x} = \frac{f_1 - b_1 \bar{\rho}^*}{\bar{\rho}^* + \bar{y}^* - 0.4} = \frac{-0.3}{0.6064 - 0.4} = -1.4535.$$

It is easy to verify that

$$P(\bar{x}) = -0.6748 = P^d(\bar{\rho}^*, \bar{y}^*).$$

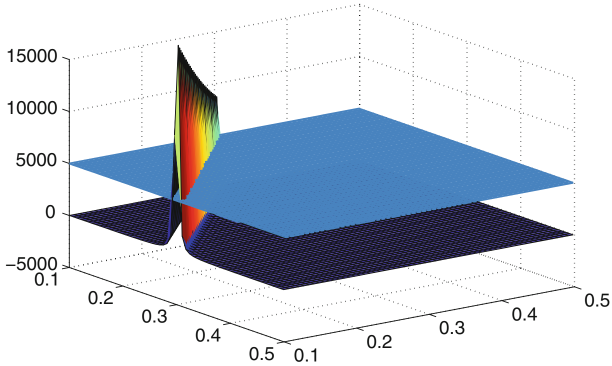

Fig. 4 Graph of $\Phi(\rho^*, y^*)$ and level surface $\Phi = c$

$\bar{\rho}^* = 0$ indicates that the solution $\bar{x}$ is in the interior of the feasible domain of the primal problem.

If we let $c = -0.45$ and keep other parameters the same, the dual solution becomes $\bar{\rho}^* = 0.8831$, $\bar{y}^* = 0.4159$. It indicates that the primal solution $\bar{x} = \frac{-0.3-0.8831}{0.8831+0.4159-0.4} = -1.3161$ is a boundary point of the interval defined by $\frac{1}{2}x^2 + x \leq -0.45$. Consequently, we can obtain a dual solution by solving the following algebraic equations

$$\begin{cases} \frac{1}{2}\frac{(-0.3-\bar{\rho}^*)^2}{(\bar{\rho}^*+\bar{y}^*-0.4)^2} + \frac{-0.3-\bar{\rho}^*}{\bar{\rho}^*+\bar{y}^*-0.4} = -0.45, \\ D_{\bar{y}^*}P^d(\bar{\rho}^*, \bar{y}^*) = \frac{1}{2}\frac{(-0.3-\bar{\rho}^*)^2}{(\bar{\rho}^*+\bar{y}^*-0.4)^2} - \bar{y}^* - 0.45 = 0. \end{cases} \tag{23}$$

We can select a solution pair such that $\bar{\rho}^* \geq 0$ and $\bar{y}^* \geq -0.45$.

If we let

$$\Phi(\bar{\rho}^*, \bar{y}^*) = \frac{1}{2}\frac{(-0.3-\bar{\rho}^*)^2}{(\bar{\rho}^*+\bar{y}^*-0.4)^2} + \frac{-0.3-\bar{\rho}^*}{\bar{\rho}^*+\bar{y}^*-0.4},$$

the graph of $\Phi(\rho^*, \bar{y}^*)$ and the surface of $\Phi = c$ are shown in Fig. 4.

Example 2 Consider a two-dimensional double-well potential minimization problem by taking

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The primal problem becomes

$$\begin{aligned} &\min_{(x_1,x_2)\in\mathbb{R}^2} P(x_1,x_2) = \frac{1}{2}\left(a_1x_1^2 + a_2x_2^2\right) + \frac{1}{2}\alpha\left(\frac{1}{2}\left(x_1^2+x_2^2\right) - \beta\right)^2 - f_1x_1 - f_2x_2 \\ &\textbf{s.t.} \quad \frac{1}{2}\left(k_{11}x_1^2 + k_{22}x_2^2\right) + b_1x_1 + b_2x_2 \leq c. \end{aligned} \tag{24}$$

The canonical dual problem of (24) is

$$\max_{(\rho^*, y^*)\in\mathbb{R}^2} P^d(\rho^*, y^*) = -\frac{1}{2}\left(\frac{(f_1 - b_1\rho^*)^2}{k_{11}\rho^* + y^* + a_1} + \frac{(f_2 - b_2\rho^*)^2}{k_{22}\rho^* + y^* + a_2}\right) - \frac{1}{2\alpha}(y^*)^2 - \beta y^* - c\rho^* \quad (25)$$
$$\mathbf{s.t.}\ \rho^* \geqslant 0,\ y^* \geqslant -\alpha\beta,$$
$$k_{11}\rho^* + y^* \geqslant -a_1, k_{22}\rho^* + y^* \geqslant -a_2.$$

The canonical dual objective function is concave when $k_{11}\rho^* + y^* \geqslant -a_1$, $k_{22}\rho^* + y^* \geqslant -a_2$, and the dual problem has a unique solution in the feasible domain.

If we let $a_1 = -0.4, a_2 = -0.2, \alpha = 1, \beta = 0.45, f_1 = f_2 = -0.3, k_{11} = 2, k_{22} = 4, b_1 = 2, b_2 = 2.4, c = -0.72$, the minimization problem becomes

$$\min_{(x_1,x_2)\in\mathbb{R}^2} P(x_1, x_2) = -0.2x_1^2 - 0.1x_2^2 + \frac{1}{2}\left(\frac{1}{2}(x_1^2 + x_2^2) - 0.45\right)^2 + 0.3x_1 + 0.3x_2$$
$$\mathbf{s.t.}\ x_1^2 + 2x_2^2 + 2x_1 + 2.4x_2 \leq -1.65. \quad (26)$$

The corresponding dual problem is

$$\max_{(\rho^*, y^*)\in\mathbb{R}^2} P^d(\rho^*, y^*) = -\frac{1}{2}\left(\frac{(-0.3 - 2\rho^*)^2}{2\rho^* + y^* - 0.4} + \frac{(-0.3 - 2.4\rho^*)^2}{4\rho^* + y^* - 0.2}\right) - \frac{1}{2}(y^*)^2 - 0.45y^* + 1.65\rho^* \quad (27)$$
$$\mathbf{s.t.}\ \rho^* \geqslant 0,\ y^* \geqslant -0.45,$$
$$2\rho^* + y^* \geqslant 0.4, 4\rho^* + y^* \geqslant 0.2.$$

The dual problem has a unique solution

$$\bar{\rho}^* = 0.2314, \quad \bar{y}^* = 0.5499.$$

The solution of the primal problem can be obtained as

$$\bar{x}_1 = \frac{f_1 - b_1\bar{\rho}^*}{k_{11}\bar{\rho}^* + \bar{y}^* + a_1} = \frac{-0.3 - 2 * 0.2314}{2 * 0.2314 + 0.5499 - 0.4} = -1.2450,$$
$$\bar{x}_2 = \frac{f_2 - b_2\bar{\rho}^*}{k_{22}\bar{\rho}^* + \bar{y}^* + a_2} = \frac{-0.3 - 2.4 * 0.2314}{4 * 0.2314 + 0.5499 - 0.2} = -0.6706.$$

It is easy to verify that

$$P(\bar{x}) = -0.7785 = P^d(\bar{\rho}^*, \bar{y}^*).$$

Figure 5 shows the graph of the primal objective function and Fig. 6 shows its three-dimensional contour.

If $c = -0.72$, the dual solution becomes $\bar{\rho}^* = 0,\ \bar{y}^* = 0.6315$, and the primal solution is $\bar{x}_1 = -1.2959, \bar{x}_2 = -0.6952$. The global minimizer becomes

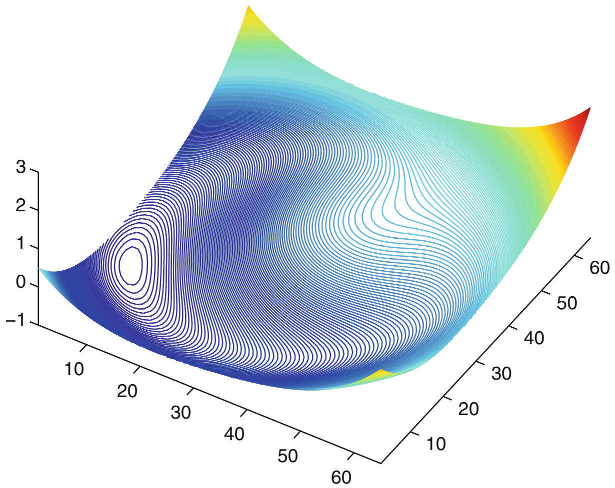

Fig. 5 Graph of $P(x_1, x_2)$

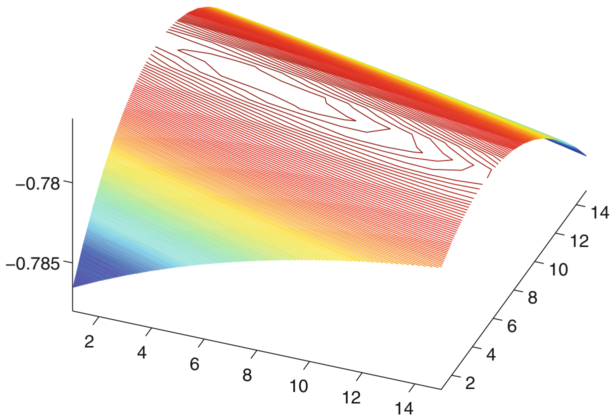

Fig. 6 Canonical dual figure

$$P(-1.2959, -0.6952) = -0.7822 = P^d(0, 0.6315).$$

In fact, $c = -1.6143$ is a critical value such that $\bar{\rho}^* = 0$ when $c \geq -1.6143$ and $\bar{\rho}^* \neq 0$ when $c \leq -1.6143$. The canonical dual solution can be obtained by solving the following algebraic equations

$$\begin{cases} \frac{(2\rho^*+0.3)^2}{(2\rho^*+y^*-0.4)^2} + \frac{2(2.4\rho^*+0.3)^2}{(4\rho^*+y^*-0.2)^2} + \frac{2(-2\rho^*-0.3)}{2\rho^*+y^*-0.4} + \frac{2.4(-2.4\rho^*-0.3)}{4\rho^*+y^*-0.2} = c, \\ \frac{1}{2}\left(\frac{(2\rho^*+0.3)^2}{(2\rho^*+y^*-0.4)^2} + \frac{2(2.4\rho^*+0.3)^2}{(4\rho^*+y^*-0.2)^2}\right) - y^* - 0.45 = 0. \end{cases}$$

Graph of $\Phi(\bar{\rho}^*, \bar{y}^*)$ and the level surface $\Phi = c$ are shown in Fig. 7.

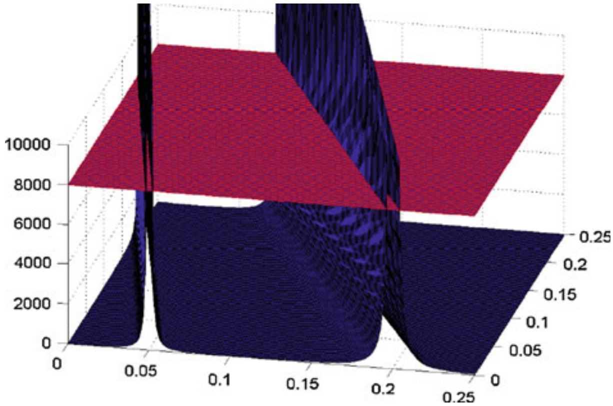

Fig. 7 Graph of $\Phi(\rho^*, y^*)$ and level surface of $\Phi = c$

Example 3 Consider a three-dimensional double-well potential minimization problem by taking

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The primal problem becomes

$$\min_{(x_1,x_2,x_3)\in\mathbb{R}^3} P(x_1,x_2,x_3) = \frac{1}{2}\left(\sum_{i=1}^{3} a_i x_i^2\right) + \frac{1}{2}\alpha\left(\frac{1}{2}\left(\sum_{i=1}^{3} x_i^2\right) - \beta\right)^2 - \sum_{i=1}^{3} f_i x_i$$
$$\textbf{s.t.}\, \frac{1}{2}\left(\sum_{i=1}^{3} k_{ii} x_i^2\right) + \sum_{i=1}^{3} b_i x_i \le c. \tag{28}$$

The canonical dual problem of (28) is

$$\max_{(\rho^*,y^*)\in\mathbb{R}^2} P^d(\rho^*, y^*) = -\frac{1}{2}\left(\sum_{i=1}^{3} \frac{(f_i - b_i\rho^*)^2}{k_{ii}\rho^* + y^* + a_i}\right) - \frac{1}{2\alpha}(y^*)^2 - \beta y^* - c\rho^*$$
$$\textbf{s.t.}\, \rho^* \geqslant 0,\ y^* \geqslant -\alpha\beta,\ k_{ii}\rho^* + y^* \ge -a_i,\ i = 1, 2, 3. \tag{29}$$

Note that the canonical objective function is concave, if $k_{ii}\rho^* + y^* \ge -a_i$ for $i = 1, 2, 3$ and the dual problem has a unique solution in the feasible domain.

If we let $a_1 = -0.6, a_2 = -0.4, a_3 = -0.2, \alpha = 1, \beta = 0.45, f_1 = f_2 = f_3 = -0.3, k_{11} = 2, k_{22} = 4, k_{33} = 0, b_1 = 2, b_2 = 2, b_3 = -1, c = 1$, the primal problem becomes

$$\min_{(x_1,x_2,x_3)\in\mathbb{R}^3} P(x_1,x_2,x_3) = -(0.3x_1^2 + 0.2x_2^2 + 0.1x_3^2) + \frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{3} x_i^2 - 0.45\right)^2$$
$$+0.3\sum_{i=1}^{3} x_i$$
$$\textbf{s.t.}\, x_1^2 + 2x_2^2 + 2x_1 + 2x_2 - x_3 \le 1. \tag{30}$$

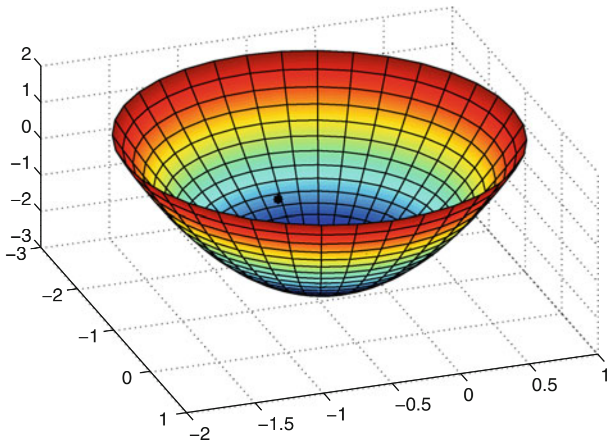

Fig. 8 Graph of the quadratic constrain $x_1^2 + 2x_2^2 + 2x_1 + 2x_2 - x_3 \leq 1$. The black dot denotes the global minimal point (−1.3459,−0.7094,−0.4816) of $P(x)$

The feasible set determined by the constraint is the inner part of paraboloid as shown in Fig. 8.

The canonical dual problem becomes

$$\max_{(\rho^*, y^*)\in\mathbb{R}^2} P^d(\rho^*, y^*) = -0.5\left(\frac{(-0.3-2\rho^*)^2}{2\rho^* + y^* - 0.6} + \frac{(-0.3-2\rho^*)^2}{4\rho^* + y^* - 0.4} + \frac{(-0.3+\rho^*)^2}{y^* - 0.2}\right)$$
$$-0.5(y^*)^2 - 0.45y^* - \rho^* \tag{31}$$
$$\textbf{s.t.}\,\rho^* \geqslant 0,\ y^* \geqslant -0.45,$$
$$2\rho^* + y^* > 0.6,\ 4\rho^* + y^* > 0.4,\ y^* > 0.2.$$

The unique dual solution is given by

$$\bar{\rho}^* = 0, \quad \bar{y}^* = 0.8229.$$

The corresponding solution of the primal problem can be obtained as

$$\begin{cases} \bar{x}_1 = \frac{f_1}{\bar{y}^* + a_1} = \frac{-0.3}{0.8229-0.6} = -1.3459; \\ \bar{x}_2 = \frac{f_2}{\bar{y}^* + a_2} = \frac{-0.3}{0.8229-0.4} = -0.7094; \\ \bar{x}_3 = \frac{f_3}{\bar{y}^* + a_3} = \frac{-0.3}{0.8229-0.2} = -0.4816. \end{cases}$$

It is easy to verify that the global minimizer

$$P(\bar{x}) = -1.0894 = P^d(\bar{\rho}^*, \bar{y}^*).$$

The equation of $\bar{\rho}^* = 0$ indicates that the primal solution (−1.3459,−0.7094,−0.4816) is in the interior of the feasible domain defined by $x_1^2 + 2x_2^2 + 2x_1 + 2x_2 - x_3 \leq 1$. Actually, $(-1.3459)^2 + 2(-0.7094)^2 + 2(-1.3459) + 2(-0.7094) + 0.4816 = -0.8113 < 1$, and $c = -0.8113$ is a critical value.

If we select $c = -0.9$ in (28), the corresponding dual solution is

$$\bar{\rho}^* = 0.0587, \quad \bar{y}^* = 0.7886.$$

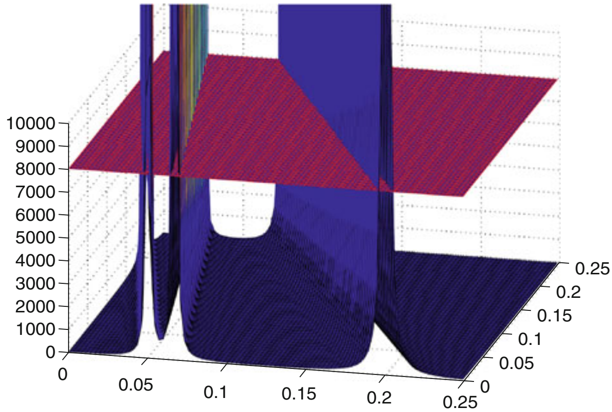

Fig. 9 The graphes of $\Phi(\rho^*, y^*)$ and surface $\Phi = c$

The primal solution is obtained as

$$\begin{cases} \bar{x}_1 = \frac{f_1 - b_1\rho^*}{k_{11}\rho^* + y^* + a_1} = \frac{-0.3-2*0.0587}{2*0.0587+0.8229-0.6} = -1.3641; \\ \bar{x}_2 = \frac{f_2 - b_2\rho^*}{k_{22}\rho^* + y^* + a_2} = \frac{-0.3-2*0.0587}{4*0.0587+0.8229-0.4} = -0.6696; \\ \bar{x}_3 = \frac{f_3 - b_3\rho^*}{k_{33}\rho^* + y^* + a_3} = \frac{-0.3+0.0587}{0.8229-0.2} = -0.4100. \end{cases}$$

It locates on the boundary of $x_1^2 + 2x_2^2 + 2x_1 + 2x_2 - x_3 \le -0.9$. Actually, we have $(-1.3641)^2 + 2(-0.6696)^2 + 2(-1.3641) + 2(-0.6696) - 0.4100 = 0.9000$.

The global minimizer is

$$P(\bar{x}) = -1.0869 = P^d(\bar{\rho}^*, \bar{y}^*).$$

We see $\bar{\rho}^* = 0$, when $c > -0.8113$, and $\bar{\rho}^* \neq 0$, when $c \le -0.8113$. The canonical dual solution can be obtained by the solving the following algebraic equations

$$\begin{cases} \frac{(-0.3-2\rho^*)^2}{(2\rho^*+y^*-0.6)^2} + \frac{2(-0.3-2\rho^*)^2}{(4\rho^*+y^*-0.4)^2} + \frac{2(-0.3-2\rho^*)}{2\rho^*+y^*-0.6} + \frac{2(-0.3-2\rho^*)}{4\rho^*+y^*-0.4} - \frac{2(-0.3+\rho^*)}{y^*-0.2} = c, \\ \frac{1}{2}\left(\frac{(-0.3-2\rho^*)^2}{(2\rho^*+y^*-0.6)^2} + \frac{(-0.3-2\rho^*)^2}{(4\rho^*+y^*-0.4)^2} + \frac{(-0.3+\rho^*)^2}{(y^*-0.2)^2}\right) - y^* - 0.45 = 0. \end{cases}$$

Same as previous examples, the graphs of $\Phi(\bar{\rho}^*, \bar{y}^*)$ and $\Phi = c$ are shown in Fig. 9.

Notice that our examples show that there is no duality gap between the primal problem and its canonical dual. Higher dimensional examples can also be illustrated using the same procedure. Over $\mathcal{Y}_+^*$, when c is large enough, the canonical dual is strongly concave. After solving the canonical dual problem by MATLAB, a global minimizer can be found.

5 Concluding remark

In this paper, we have further extended the canonical duality theory to handle a class of quadrinomial minimization problems with one general quadratic constraint. Since the double-well function has its special meanings in physics and mechanics, the study of our problem becomes useful. Our results show how a global minimizer can be found by solving the canonical dual problem. Although the existence theorem has not been presented completely, this work begins to reveal some insights of this type of difficult nonconvex optimization problems. The triality theory can also be used to identify both local minimizer and local maximizers. Interested readers are refereed to the recent work [17].

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References

1. Ekeland, I.: Legendre duality in nonconvex optimization and calculus of variations. SIAM J. Control Optim. **15**, 905–934 (1977)
2. Ekeland, I.: Nonconvex duality. In: Gao, D.Y. (ed.) Proceedings of IUTAM Symposium on Duality, Complementarity and Symmetry in Nonlinear Mechanics. Kluwer Academic, Dordrecht/Boston/London (2003)
3. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. North-Holland, Amsterdam (1976)
4. Fang, S.-C., Gao, D.Y., Sheu, R.L., Wu, S.Y.: Canonical dual approach for solving 0–1 quadratic programming problems. J. Ind. Manage. Optim. **4**(1), 125–142 (2008)
5. Fang, S.-C., Gao, D.Y., Sheu, R.L., Xing, W.X.: Global optimization for a class of fractional programming problems. J. Glob. Optim. **45**, 337–353 (2009)
6. Floudas, C.A., Visweswaran, V.: Quadratic optimization. In: Horst, R., Pardalos, P.M. (eds.) Handbook of Global Optimization, pp. 217–270. Kluwer Academic Publishers, Dordrecht/Boston/London (1995)
7. Gao, D.Y.: Dual extremum principles in finite deformation theory with applications to post-buckling analysis of extended nonlinear beam theory. Appl. Mech. Rev. **50**(11), S64–S71 (1997)
8. Gao, D.Y.: Duality-mathematics. Wiley Encyclopedia Electronical Electronical Eng. **6**, 68–77 (1999)
9. Gao, D.Y.: Duality Principles in Nonconvex Systems: Theory, Methods and Applications. Kluwer Academic Publishers, Dordrecht/Boston/London (2000a)
10. Gao, D.Y.: Canonical dual transformation method and generalized triality theory in nonsmooth global optimization. J. Glob. Optim. **17**(1/4), 127–160 (2000b)
11. Gao, D.Y.: Perfect duality theory and complete set of, solutions to a class of global optimization. Optimization **52**(4–5), 467–493 (2003)
12. Gao, D.Y.: Complete solutions to constrained quadratic optimization problems. J. Glob. Optim. **29**, 377–399 (2004)
13. Gao, D.Y.: Sufficient conditions and canonical duality in nonconvex minimization with inequality constraints. J. Ind. Manage. Optim. **1**(1), 53–63 (2005)
14. Gao, D.Y.: Complete solutions and extremality criteria to polynomial optimization problems. J. Glob. Optim. **35**, 131–143 (2006)
15. Gao, D.Y., Sherali, H.D. : Canonical duality theory: connections between nonconvex mechanics and global optimization. In: Gao, D.Y., Sherali, H. (eds.) Advances in Applied Mathematics and Global Optimization, pp. 257–326. Springer, USA (2009)
16. Gao, D.Y., Strang, G.: Geometric nonlinearity: potential energy, complementary energy, and the gap function. Q. Appl. Math. **47**(3), 487–504 (1989)
17. Gao, D.Y., Wu, C.Z.: On the triality theory in global optimization (2010)