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Existence results for proper efficient solutions of vector equilibrium problems and applications

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Abstract In this paper, we present sufficient conditions for the existence of Henig efficient solutions, superefficient solutions and Henig globally efficient solutions of a vector equilibrium problem in topological vector spaces, using a well-known separation theorem in infinite dimensional spaces. As an application, using a scalarization technique, existence results for proper efficient solutions of generalized vector variational inequalities are given.

Keywords Strong vector equilibrium problem · Henig efficient solution · Superefficient solution · Henig globally efficient solution · Generalized vector variational inequalities

1 Introduction

The study of equilibrium problems received a great attention, ever since the paper of Blum and Oettli appeared. They introduced the scalar equilibrium (EP), which consists in finding:

$$\bar{a} \in A \text{ such that } f(\bar{a}, b) \geq 0 \quad \text{for all } b \in B,$$

where A is a nonempty subset of a real topological vector space E, B a nonempty set and $f : A \times B \to \mathbb{R}$. This problem includes as particular cases optimization problems, saddle-point problems/minimax problems, variational inequalities, complementarity problems (see, for instance [5]).

In [3] and [4], the scalar equilibrium (EP) was extended to vector-valued bifunctions in the following way:

$$(VEP) \quad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \quad \text{for all } b \in B,$$

where $\varphi : A \times B \to Z$ is a given bifunction and C is a convex cone of a real topological vector space Z. We refer to this problem as the strong vector equilibrium problem. A point

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$\bar{a} \in A$ which satisfies the upper relation is called an efficient solution to (VEP). Denote by $V_{eff}(\varphi)$ the set of efficient solutions to (VEP). For existence results of (vector) equilibrium problems and their particular cases, as well as for properties of the set of solutions we refer the reader to [1,2,6–10,12,13,23,24] and [32].

Recently, Gong introduced in [18,19] and [21] different concepts of proper efficient solutions of the equilibrium problem (VEP) and stated existence results for proper efficient solutions, using scalarization techniques and Ky Fan's Lemma (see [14]). In this paper, we extend the existence results from [6], obtained for weak efficient solutions of (VEP) to existence results of proper efficient solutions of (VEP).

The paper is organized as follows. In the remaining part of the introduction, we recall some notions and properties considered in the past and necessary for our investigations.

In Sect. 2 we present existence theorems for Henig efficient solutions, superefficient solutions and Henig globally efficient solutions of (VEP), using the well-known Eidelheit's separation theorem in infinite dimensional spaces. After that, we give some corollaries which deal with stronger assumptions, and some of them are given for the ordering cone C. When we reduce the space Z to $\mathbb{R}$, and take C to be the set of positive real numbers, the considered proper solutions collapse into solutions of (EP), and we recover an earlier existence result of Kassay and Kolumban [25] for scalar equilibrium problems.

Motivated by the lack of results for the existence results of strong vector variational inequalities, as Chen and Hou mentioned in [12], in Sect. 3, using a scalarization technique, we present an application to generalized vector variational inequality problems, where we state existence results for proper efficient solutions. Whenever $C^\sharp \neq \emptyset$ and E is equipped with the weak topology, by Theorem 5 we recover Theorem 3.1 from [18].

Throughout this paper E and Z are considered to be real topological vector spaces, until something else is supposed, $A \subseteq E$ is a nonempty subset, B is a nonempty set, and $C \subseteq Z$ is a convex cone.

Recall that a subset $C \subseteq Z$ is called cone if $\lambda C \subseteq C$ for every $\lambda \geq 0$. The cone C is said to be:

(i) solid, if $\operatorname{int} C \neq \emptyset$;
(ii) pointed, if $C \cap (-C) = \{0\}$.

Let D be a nonempty subset of Z. The conic hull of D is defined as:

$$\operatorname{cone}(D) = \{td \mid t \geq 0,\ d \in D\}.$$

Let Z^* be the topological dual space of Z, and

$$C^* = \left\{z^* \in Z^* \mid z^*(c) \geq 0 \text{ for all } c \in C\right\}$$

be the positive dual cone of C. The quasi-interior of C^* is

$$C^\sharp = \left\{z^* \in C^* \mid z^*(c) > 0 \text{ for all } c \in C \setminus \{0\}\right\}.$$

We refer the reader to [16] for the fact that $C^\sharp \neq \emptyset$ if and only if C has a base, i.e. there is $\mathcal{B}$ a nonempty convex subset of the cone C such that $C = \operatorname{cone}(\mathcal{B})$ and $0 \notin \operatorname{cl}(\mathcal{B})$.

A neighborhood U of zero is said to be balanced, if $\lambda U \subseteq U$ for each scalar with $|\lambda| \leq 1$.

Lemma 1 *If $z^* \in C^*$ is a nonzero functional, then $z^*(z) > 0$ for all $z \in int\, C$.*

Let $\mathcal{B}$ be a base of C and let

$$C^\triangle = \left\{z^* \in C^\sharp \mid \text{ there is } t > 0 \text{ such that } z^*(b) \geq t \text{ for all } b \in \mathcal{B}\right\}.$$

The above notion, introduced by Zheng [33], satisfies the inclusion $C^{\triangle} \subseteq C^{\sharp}$. Since $\mathcal{B}$ is a base for C, we have $0 \notin \mathrm{cl}\,(\mathcal{B})$. So, by the Tukey's separation theorem (see for instance [27]) we get the existence of a nonzero functional $z^* \in Z^*$ such that

$$r = \inf\{z^*(b) \mid b \in \mathcal{B}\} > z^*(0) = 0.$$

Thus, $C^{\triangle} \neq \emptyset$.
Set

$$V_{\mathcal{B}} = \left\{z \in Z \mid |z^*(z)| < \frac{r}{2}\right\}.$$

Hence, $V_{\mathcal{B}}$ is a balanced neighborhood of zero in Z. For each convex neighborhood U of zero with the property $U \subseteq V_{\mathcal{B}}$, $\mathcal{B} + U$ is a convex set and $0 \notin \mathrm{cl}\,(\mathcal{B} + U)$. Therefore $C_U(\mathcal{B}) = \mathrm{cone}\,(U + \mathcal{B})$ is a pointed convex cone and $C\backslash\{0\} \subseteq \mathrm{int}\,C_U(\mathcal{B})$.

Gong showed in [17] and [18] that solutions of a (VEP) can be characterized and computed as solutions of an appropriate scalar equilibrium problem. Let us recall the next solution concepts.

Definition 1 A vector $a \in A$ is said to be:

(i) a Henig efficient solution to (VEP) if there exists some neighborhoods U of zero with $U \subseteq V_{\mathcal{B}}$ such that

$$\varphi(a, B) \cap (-\mathrm{int}\,C_U(\mathcal{B})) = \emptyset.$$

(ii) a superefficient solution to (VEP) if, for each neighborhood V of zero, there exists some neighborhood U of zero such that

$$\mathrm{cone}\,(\varphi(a, B)) \cap (U - C) \subseteq V.$$

(iii) a Henig globally efficient solution to (VEP) if there exists a pointed convex cone $K \subseteq Z$, with $C\backslash\{0\} \subseteq \mathrm{int}\,K$, such that

$$\varphi(a, B) \cap (-K\backslash\{0\}) = \emptyset.$$

(iv) a weak efficient solution to (VEP) if the cone C is solid and

$$\varphi(a, B) \cap (-\mathrm{int}\,C) = \emptyset.$$

The sets of Henig efficient solutions, superefficient solutions, respectively Henig globally efficient solutions are denoted by $V_H(\varphi)$, $V_S(\varphi)$, respectively $V_G(\varphi)$.

For the solution sets we have that $V_G(\varphi) \subseteq V_{eff}(\varphi)$ and, if C has a base, then $V_S(\varphi) \subseteq V_H(\varphi) \subseteq V_{eff}(\varphi)$. To see that the set of Henig efficient solutions is greater than the set of superefficient solutions we give an example.

Example 1 Let $Z = \mathbb{R}^2, C = \mathbb{R}^2_+, A = [-2, -1], B = [1, 2]$ and let $f: [-2, -1] \times [1, 2] \to \mathbb{R}^2$ be defined by

$$f(x, y) = \begin{cases} (-2, 0), & \text{if } (x, y) = (-2, 2) \\ (x, y), & \text{otherwise.} \end{cases}$$

Take $z^* = (1, 1)$ and the base $\mathcal{B}$ to be the set

$$\{(x, y) \in \mathbb{R}^2_+ \mid x + y = 2\}.$$

We observe that the base is a closed convex subset of $\mathbb{R}^2$. Moreover, we find

$$r = \inf\{z^*(b) \mid b \in \mathcal{B}\} = 2.$$

For each balanced neighborhood $U = B(0, \epsilon)$ of zero, which has a radius $\epsilon \leq 1$, and for every $a \in [-2, -1]$ we obtain that

$$\varphi(a, B) \cap (-\text{int}\, C_U(\mathcal{B})) = \emptyset,$$

which means that all $a \in [-2, -1]$ are Henig efficient solutions of the vector equilibrium problem (VEP).

On the other part, each point $a \in (-2, -1]$ is a superefficient solution of (VEP). Hence, we have $V_S(f) = (-2, -1] \subseteq [-2, -1] = V_H(f)$.

Let $z^* \in C^* \backslash \{0\}$. A vector $a \in A$ is said to be a z^*-efficient solution to (VEP) if

$$z^*(\varphi(a, b)) \geq 0 \text{ for all } b \in B.$$

Denote by $V_{z^*}(\varphi)$ the set of all z^*-efficient solutions to (VEP).

The sets

$$\omega = \left\{ \bigcap_{i=1}^{n} \{z^* \in Z^* \mid \sup_{z \in D_i} \mid z^*(z) \mid < r\} \mid D_i \ (i \in \{1, \ldots, n\}) \text{ are bounded} \right.$$
$$\left. \text{subsets of } Z,\ r > 0,\ n \in \mathbb{N} \right\}$$

form a base of neighborhoods of zero with respect to the strong topology $\beta(Z^*, Z)$.

Lemma 2 (see [19,22]) *If the closed convex cone C has a bounded closed base $\mathcal{B}$, then*

$$int\ C^* = C^{\triangle}(\mathcal{B}),$$

where $int\ C^*$ is the interior of the dual cone C^* with respect to the strong topology $\beta(Z^*, Z)$.

In [21], the author gave a characterization of those proper efficient solutions in a particular framework. He considered E to be a real Hausdorff topological vector space, Z a real locally convex Hausdorff topological vector space and $A = B$. We say that a set $D \subseteq Z$ is a C-convex set, if $D + C$ is a convex set in Z.

Theorem 1 [21] *Assume that, for each $a \in A$, $\varphi(a, A)$ is a C-convex set. If C has a base $\mathcal{B}$, then:*

(i) $V_G(\varphi) = \bigcup_{z^* \in C^\sharp} V_{z^*}(\varphi)$;
(ii) $V_H(\varphi) = \bigcup_{z^* \in C^\triangle} V_{z^*}(\varphi)$;
(iii) *If C has a closed bounded base, then:*

$$V_S(\varphi) = \bigcup_{z^* \in int\ C^*} V_{z^*}(\varphi).$$

2 Sufficient conditions for proper efficient solutions of (VEP)

In this section we give sufficient conditions for the existence of proper solutions of (VEP) in a general framework. Let us begin with a definition.

Definition 2 [30] A function $f : E \to Z$ is said to be C-upper semicontinuous at $x \in E$ (C-usc in short) if it satisfies the following condition:

1° For any neighborhood $V_{f(x)} \subset Z$ of $f(x)$, there exists a neighborhood $U_x \subset E$ of x such that $f(u) \in V_{f(x)} - C$ for all $u \in U_x$.

The function f is said to be C-usc on E if, it is C-usc at every point $x \in E$

Remark 1 In [30] Tanaka characterized the above notion, in the hypothesis of a convex cone C with $\operatorname{int} C \neq \emptyset$. Thus, relation $1°$ is equivalent to:

$2°$ For any $k \in \operatorname{int} C$, there exists a neighborhood $U_x \subset E$ of x such that $f(u) \in f(x) + k - \operatorname{int} C$ for all $u \in U_x$.

Notice that in [26], this notion was termed $-C$-continuous function at x.

Theorem 1 *Let A be a compact set, C a convex cone with a base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ be a bifunction such that*

(i) *for each $b \in B$ and $U \subset V_{\mathcal{B}}$ the function $\varphi(\cdot, b) : A \to Z$ is $C_U(\mathcal{B})$-usc on A;*
(ii) *for each $a_1, \ldots, a_m \in A, \lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda m = 1, b_1, \ldots, b_n \in B$ and $U \subseteq V_{\mathcal{B}}$ there exists $u^* \in C_U^*(\mathcal{B}) \backslash \{0\}$ such that*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) \leq \sup_{a \in A} \min_{1 \leq j \leq n} u^* \left(\varphi(a, b_j)\right);$$

(iii) *there is $U_0 \subseteq V_{\mathcal{B}}$ such thatfor each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \in C_{U_0}^*(\mathcal{B})$ not all zero one has*

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) admits a Henig efficient solution.

Proof Suppose by contradiction that (VEP) has no Henig efficient solution, i.e. for each $a \in A$ and $U \subseteq V_{\mathcal{B}}$ there exists $b \in B$ with the property $\varphi(a, b) \in -\operatorname{int} C_U(\mathcal{B})$. This means that, for each $a \in A$ and $U \subseteq V_{\mathcal{B}}$ there exists $b \in B$ and $k \in \operatorname{int} C_U(\mathcal{B})$ such that

$$\varphi(a, b) + k \in -\operatorname{int} C_U(\mathcal{B}).$$

Consider the sets

$$U_{b,k} := \{a \in A \mid \varphi(a, b) + k \in -\operatorname{int} C_U(\mathcal{B})\},$$

where $b \in B$ and $k \in \operatorname{int} C_U(\mathcal{B})$. In what follows we show that the family of these sets forms an open covering of the compact set A.

Let $a_0 \in U_{b,k}$ and $k \in \operatorname{int} C_U(\mathcal{B})$. Since $a_0 \in U_{b,k}$ we have

$$\varphi(a_0, b) + k \in -\operatorname{int} C_U(\mathcal{B}) \quad \text{that is, } -\varphi(a_0, b) - k \in \operatorname{int} C_U(\mathcal{B}).$$

Denote $k^{'} := -\varphi(a_0, b) - k$, so $k^{'} \in \operatorname{int} C_U(\mathcal{B})$. Since the function $\varphi(\cdot, b)$ is $C_U(\mathcal{B})$-usc at $a_0 \in A$, we obtain for this $k^{'}$ that there exists a neighborhood $U_{a_0} \subset E$ of a_0 such that

$$\begin{aligned} \varphi(u, b) &\in \varphi(a_0, b) + k^{'} - \operatorname{int} C_U(\mathcal{B}) = \varphi(a_0, b) - \varphi(a_0, b) - k - \operatorname{int} C_U(\mathcal{B}) \\ &= -k - \operatorname{int} C_U(\mathcal{B}), \quad \text{for all } u \in U_{a_0}. \end{aligned}$$

Hence we obtained that $\varphi(u, b) + k \in -\operatorname{int} C_U(\mathcal{B})$ for all $u \in U_{a_0}$, which means that $U_{b,k}$ is an open set.

Since, for each $U \subseteq V_{\mathcal{B}}$ the family $\{U_{b,k}\}$ is an open covering of the compact set A, we can select a finite subfamily which covers the same set A, i. e. there exist $b_1, \ldots, b_n \in B$ and $k_1, \ldots, k_n \in \operatorname{int} C_U(\mathcal{B})$ such that

$$A \subseteq \bigcup_{j=1}^{n} U_{b_j,k_j}. \tag{1}$$

For these $k_1, \ldots, k_n \in \operatorname{int} C_U(\mathcal{B})$, we have that there exist $V_1, \ldots, V_n$ balanced neighborhoods of the origin of Z such that $k_j + V_j \subset C_U(\mathcal{B})$ for all $j \in \{1, \ldots, n\}$ (see e.g. [28]).

Define $V := V_1 \cap \cdots \cap V_n$, thus V is a balanced neighborhood of the origin of the space Z. Let $k_0 \in V \cap \operatorname{int} C_U(\mathcal{B})$, so we have $-k_0 \in V$. Hence,

$$k_j - k_0 \in k_j + V \subseteq k_j + V_j \subseteq C_U(\mathcal{B}), \text{ for all } j \in \{1, \ldots, n\},$$

which gives

$$k_j - k_0 \in C_U(\mathcal{B}), \quad \text{for all } j \in \{1, \ldots, n\}. \tag{2}$$

Now define the vector-valued function $F : A \to Z^n$ by

$$F(a) := (\varphi(a, b_1) + k_0, \ldots, \varphi(a, b_n) + k_0).$$

Assert that

$$\operatorname{co} F(A) \cap (\operatorname{int} C_U(B))^n = \emptyset, \tag{3}$$

where co $F(A)$ denotes the convex hull of the set $F(A)$. Supposing the contrary, there exist $a_1, \ldots, a_m \in A$ and $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that

$$\sum_{i=1}^{m} \lambda_i F(a_i) \in (\operatorname{int} \mathrm{C_U}(\mathcal{B}))^{\mathrm{n}}, \quad \text{or equivalently,}$$

$$\sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) + k_0 \in \operatorname{int} C_U(\mathcal{B}) \quad \text{for each } j \in \{1, \ldots, n\}. \tag{4}$$

Let $u^* \in C_U^*(\mathcal{B})$ be a nonzero functional for which (ii) holds. Applying u^* to the relation above and taking into account Lemma 1 we obtain that

$$\sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) + u^*(k_0) > 0.$$

Passing to the minimum over j we have

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) > -u^*(k_0), \tag{5}$$

thus, assumption (ii) and relation (5) imply that

$$\sup_{a \in A} \min_{1 \leq j \leq n} u^* \left(\varphi(a, b_j)\right) > -u^*(k_0). \tag{6}$$

For each $a \in A$, by relation (1) we have that there exists $j_0 \in \{1, \ldots, n\}$ such that $a \in U_{b_{j_0},k_{j_0}}$, i.e.$\varphi(a, b_{j_0}) + k_{j_0} \in -\operatorname{int} C_U(\mathcal{B})$. This, together with (2) imply that

$$\varphi(a, b_{j_0}) + k_0 \in -k_{j_0} + k_0 - \operatorname{int} C_U(\mathcal{B}) \subseteq -\operatorname{int} C_U(\mathcal{B}).$$

By Lemma 1 and using the fact that $u^* \in C_U^*(\mathcal{B})$ we obtain

$$u^*\left(\varphi(a, b_{j_0})\right) + u^*(k_0) < 0.$$

Thus for each $a \in A$

$$\min_{1 \le j \le n} u^*\left(\varphi(a, b_j)\right) < -u^*(k_0),$$

and passing to supremum over a we get a contradiction to (6).

By the separation theorem of convex sets of Eidelheit (see for instance [27]), we have that for each $U \subseteq V_{\mathcal{B}}$ there exists $z^* \in (Z^n)^*$ a nonzero functional such that

$$z^*(u) \le 0, \quad \text{for all } u \in \text{co}\, F(A) \text{ and} \tag{7}$$
$$z^*(c) \ge 0, \quad \text{for all } c \in (\text{int}\, C_U(\mathcal{B}))^n. \tag{8}$$

Using the representation $z^* = (z_1^*, \ldots, z_n^*)$, by a standard argument we deduce that $z_j^* \in C_U^*(\mathcal{B})$ for all $j \in \{1, \ldots, n\}$.

In particular, for each $U \subseteq V_{\mathcal{B}}$ there are $z_1^*, \ldots, z_n^* \in C_U^*(\mathcal{B})$ not all zero such that $z^*(u) \le 0$ for all $u \in F(A)$. This means that for any $a \in A$, $z^*(F(a)) \le 0$, or equivalently,

$$\sum_{j=1}^{n} z_j^*\left(\varphi(a, b_j) + k_0\right) \le 0.$$

Taking into account the linearity of $z_j^* \in C_U^*(\mathcal{B})$ for all $j \in \{1, \ldots, n\}$, Lemma 1 and the fact that not all z_j^* are zero we obtain

$$\sum_{j=1}^{n} z_j^*\left(\varphi(a, b_j)\right) \le -\sum_{j=1}^{n} z_j^*(k_0) < 0.$$

Passing to supremum over $a \in A$ in the upper relation we deduce that

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^*\left(\varphi(a, b_j)\right) < 0,$$

which is a contradiction to assumption (iii). This completes the proof. □

Assumption (ii) of Theorem 1 is a kind of generalized concavity of the bifunction φ in its first variable. To see that, we recall the notions of C-subconcavelikeness, respectively C-concavelikeness of a bifunction is its first variable, which originates from [6], respectively [20].

Definition 3 Let $\varphi : A \times B \to Z$ be a bifunction and C a convex cone with $\text{int}\, C \neq \emptyset$. The bifunction φ is said to be:

(i) C-subconcavelike in its first variable if for each $l \in \text{int}\, C$, $a_1, a_2 \in A$ and $\lambda \in [0, 1]$ there exists $\bar{a} \in A$ such that

$$\varphi(\bar{a}, b) \ge_C \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) - l \text{ for all } b \in B.$$

(ii) C-concavelike in its first variable if for all $a_1, a_2 \in A$ and $\lambda \in [0, 1]$ there exists $\bar{a} \in A$ such that

$$\varphi(\bar{a}, b) \ge_C \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) \text{ for all } b \in B.$$

We observe that, if the cone C is solid, then each C-concavelike bifunction in its first variable is C-subconcavelike in its first variable. The next corollaries deals with stronger assumptions than those of Theorem 1.

Corollary 1 *Let A be a compact set, C a convex cone with a base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ be a bifunction such that*

(i) *for each $b \in B$ and $U \subset V_{\mathcal{B}}$ the function $\varphi(\cdot, b) : A \to Z$ is $C_U(\mathcal{B})$-usc on A;*
(ii) *for each $U \subseteq V_{\mathcal{B}}$, φ is $C_U(\mathcal{B})$-subconcavelike in its first variable;*
(iii) *there is $U_0 \subseteq V_{\mathcal{B}}$ such that for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \in C_{U_0}^*(\mathcal{B})$ not all zero one has*

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) admits a Henig efficient solution.

Proof It is enough to show that assumption (ii) of Theorem 1 is satisfied. Let us prove that the $C_U(\mathcal{B})$-subconcavelikeness of the bifunction φ implies assumption (ii) of the above theorem.

For $U \subseteq V_{\mathcal{B}}$ arbitrarily chosen, take $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_n \in B$, $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and $u^* \in C_U^*(\mathcal{B}) \backslash \{0\}$.

Since φ is $C_U(\mathcal{B})$-subconcavelike in its first variable, for each $l \in \operatorname{int} C_U(\mathcal{B})$ there exists $\bar{a} \in A$ such that

$$\sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \leq_{C_U(\mathcal{B})} \varphi(\bar{a}, b_j) + l \quad \text{for each } j \in \{1, \ldots, n\}. \tag{9}$$

Applying u^* to relation (9), this becomes

$$\sum_{i=1}^{m} \lambda_i u^* \varphi(a_i, b_j) \leq u^* \left(\varphi(\bar{a}, b_j)\right) + u^*(l) \quad \text{for each } j \in \{1, \ldots, n\}.$$

Passing to minimum over j yields:

$$\begin{aligned} \min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) &\leq \min_{1 \leq j \leq n} u^* \left(\varphi(\bar{a}, b_j)\right) + u^*(l) \\ &\leq \sup_{a \in A} \min_{1 \leq j \leq n} u^* \left(\varphi(a, b_j)\right) + u^*(l). \end{aligned}$$

Since this relation holds for each $l \in \operatorname{int} C_U(\mathcal{B})$ we obtain assumption (ii) of Theorem 1 satisfied. Hence (VEP) admits a Henig efficient solution. □

Corollary 2 *Let A be a compact set, C a convex cone with a base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ be a bifunction such that*

(i) *for each $b \in B$ and $U \subset V_{\mathcal{B}}$ the function $\varphi(\cdot, b) : A \to Z$ is C-usc on A;*
(ii) *it is C-concavelike in its first variable;*
(iii) *there is $U_0 \subseteq V_{\mathcal{B}}$ such that for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \in C_{U_0}^*(\mathcal{B})$ not all zero one has*

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) admits a Henig efficient solution.

Proof For the proof of this corollary we have to verify if the assumptions of Corollary 1 are satisfied.

In order to prove this, let $U \subseteq V_{\mathcal{B}}$ be arbitrarily chosen. Since $\varphi(\cdot, b)$ is C-usc on A, we have that for every $a \in A$ and each neighborhood V of $\varphi(a, b)$, there exists a neighborhood U of a such that

$$\varphi(u, b) \in V - C \subseteq V - C_U(\mathcal{B}) \quad \text{for all } u \in U.$$

Hence, $\varphi(\cdot, b)$ is $C_U(\mathcal{B})$-usc for all $b \in B$.

Furthermore, take $a_1, a_2 \in A$ and $\lambda \in [0, 1]$. By the C-concavelikeness of the bifunction φ in its first variable we have the existence of an element $\bar{a} \in A$ such that

$$\varphi(\bar{a}, b) \geq_C \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) \quad \text{for all } b \in B. \tag{10}$$

We know, that for each balanced neighborhood U of the origin, with the property $U \subseteq V_{\mathcal{B}}$, $C\backslash\{0\} \subseteq \operatorname{int} C_U(\mathcal{B})$. Hence, by(10) we obtain

$$\varphi(\bar{a}, b) \geq_{C_U(\mathcal{B})} \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) \quad \text{for all } b \in B. \tag{11}$$

Moreover, for each $l \in \operatorname{int} C_U(\mathcal{B})$, by inequality (11) we get

$$\varphi(\bar{a}, b) \geq_{C_U(\mathcal{B})} \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) - l \quad \text{for all } b \in B.$$

So, assumption (ii) of Corollary 1 is satisfied and the proof is completed. □

In what follows, we state existence results for superefficient solutions and Henig globally efficient solutions.

If C has a closed bounded base $\mathcal{B}$, in view of Lemma 2, we have $\operatorname{int} C^* = C^{\triangle}$. Moreover, by Proposition 2 of [17], $a \in A$ is a superefficient solution of (VEP) if and only if $a \in A$ is a Henig efficient solution.

By Theorem 1, Corollary 1 and Corollary 2, we have the following results.

Theorem 2 *Let E be a real Hausdorff topological vector space, Z a real locally convex Hausdorff topological vector space, $A = B$ a compact set, C a convex cone with a closed and bounded base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ be a bi function such that*

(i) *for each $b \in A$ and $U \subset V_{\mathcal{B}}$ the function $\varphi(\cdot, b) : A \to Z$ is $C_U(\mathcal{B})$-usc on A;*
(ii) *for each $a_1, \ldots, a_m \in A$, $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, $b_1, \ldots, b_n \in A$ and $U \subseteq V_{\mathcal{B}}$ there exists $u^* \in C_U^*(\mathcal{B})\backslash\{0\}$ such that*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) \leq \sup_{a\in A} \min_{1\leq j\leq n} u^* \left(\varphi(a, b_j)\right);$$

(iii) *there is $U_0 \subseteq V_{\mathcal{B}}$, such that for each $b_1, \ldots, b_n \in A$, $z_1^*, \ldots, z_n^* \in C_{U_0}^*(\mathcal{B})$ not all zero one has*

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) admits a superefficient solution.

Corollary 3 *Let E be a real Hausdorff topological vector space, Z a real locally convex Hausdorff topological vector space, $A = B$ a compact set, C a convex cone with a closed and bounded base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ be a bifunction such that*

(i) *for each* $b \in A$ *and* $U \subset V_{\mathcal{B}}$ *the function* $\varphi(\cdot, b) : A \to Z$ *is* $C_U(\mathcal{B})$*-usc on* A*;*
(ii) *for each* $U \subseteq V_{\mathcal{B}}$*,* φ *is* $C_U(\mathcal{B})$*-subconcavelike in its first variable;*
(iii) *there is* $U_0 \subseteq V_{\mathcal{B}}$*, such that for each* $b_1, \ldots, b_n \in A$*,* $z_1^*, \ldots, z_n^* \in C_{U_0}^*(\mathcal{B})$ *not all zero one has*

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) *admits a superefficient solution.*

Corollary 4 *Let* E *be a real Hausdorff topological vector space,* Z *a real locally convex Hausdorff topological vector space,* $A = B$ *a compact set,* C *a convex cone with a closed and bounded base* $\mathcal{B}$*, and let* $\varphi : A \times B \to Z$ *be a bifunction such that*

(i) *for each* $b \in A$ *and* $U \subset V_{\mathcal{B}}$ *the function* $\varphi(\cdot, b):A \to Z$ *is* C*-usc on* A*;*
(ii) *it is* C*-concavelike in its first variable;*
(iii) *there is* $U_0 \subseteq V_{\mathcal{B}}$*, such that for each* $b_1, \ldots, b_n \in A$*,* $z_1^*, \ldots, z_n^* \in C_{U_0}^*(\mathcal{B})$ *not all zero one has*

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) *admits a superefficient solution.*

In the finally part we present sufficient conditions for the existence of Henig globally efficient solutions of the vector equilibrium problem (VEP).

Theorem 3 *Let* A *be a compact set,* K *a pointed convex cone with the property* $C\setminus\{0\} \subseteq int\, K$*, and let* $\varphi : A \times B \to Z$ *be a bifunction such that*

(i) *for each* $b \in B$*, the function* $\varphi(\cdot, b) : A \to Z$ *is* K*-usc on* A*;*
(ii) *for each* $a_1, \ldots, a_m \in A$*,* $\lambda_1, \ldots, \lambda_m \geq 0$ *with* $\lambda_1 + \cdots + \lambda_m = 1, b_1, \ldots, b_n \in B$ *there exists* $u^* \in K^*\setminus\{0\}$ *such that*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) \leq \sup_{a\in A} \min_{1\leq j\leq n} u^* \left(\varphi(a, b_j)\right);$$

(iii) *for each* $b_1, \ldots, b_n \in B, z_1^*, \ldots, z_n^* \in K^*$ *not all zero one has*

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) *admits a Henig globally efficient solution.*

Proof Suppose by contradiction that (VEP) has no Henig globally efficient solution, i.e. for each pointed and convex cone $H \subseteq Z$ with the property $C\setminus\{0\} \subseteq \operatorname{int} H$, and for each $a \in A$ there exists $b \in B$ such that

$$\varphi(a, b) \in -H\setminus\{0\}.$$

In particular, this relation holds for the pointed convex cone $\operatorname{int} K \cup \{0\}$, where K is the cone from the hypothesis.

Hence, for each $a \in A$ there are $b \in B$ and $k \in \text{int } K$ such that

$$\varphi(a, b) + k \in -\text{int } K.$$

Consider the sets

$$U_{b,k} := \{a \in A \mid \varphi(a, b) + k \in -\text{int } K\},$$

where $b \in B$ and $k \in \text{int } K$. In what follows we show that the family of these sets forms an open covering of the compact set A.

Let $a_0 \in U_{b,k}$ and $k \in \text{int } K$. Since $a_0 \in U_{b,k}$ we have that

$$\varphi(a_0, b) + k \in -\text{int } K \text{ that is, } -\varphi(a_0, b) - k \in \text{int } K.$$

Denote $k' := -\varphi(a_0, b) - k$, so $k' \in \text{int } K$.Since the function $\varphi(\cdot, b)$ is K-usc at $a_0 \in A$, we obtain for k' that there exists a neighborhood $U_{a_0} \subset E$ of a_0 such that

$$\begin{aligned} \varphi(u, b) &\in \varphi(a_0, b) + k' - \text{int } K \\ &= \varphi(a_0, b) - \varphi(a_0, b) - k - \text{int } K \\ &= -k - \text{int } K \quad \text{for all } u \in U_{a_0}. \end{aligned}$$

Hence we have that $\varphi(u, b) + k \in -\text{int } K$ for all $u \in U_{a_0}$, which means that $U_{b,k}$ is an open set.

Since the family $\{U_{b,k}\}$ is an open covering of the compact set A, we can select a finite subfamily which covers the same set A, i.e., there exist $b_1, \ldots, b_n \in B$ and $k_1, \ldots, k_n \in \text{int } K$ such that

$$A \subseteq \bigcup_{j=1}^{n} U_{b_j,k_j}. \tag{12}$$

For these $k_1, \ldots, k_n \in \text{int } K$, we have that there exist $V_1, \ldots, V_n$ balanced neighborhoods of the origin of Z such that $k_j + V_j \subset K$ for all $j \in \{1, \ldots, n\}$.

Define $V := V_1 \cap \cdots \cap V_n$, thus V is a balanced neighborhood of the origin of the space Z. Let $k_0 \in V \cap \text{int } K$, so we have $-k_0 \in V$. Hence,

$$k_j - k_0 \in k_j + V \subseteq k_j + V_j \subseteq K \quad \text{for all } j \in \{1, \ldots, n\},$$

which gives

$$k_j - k_0 \in K \quad \text{for all } j \in \{1, \ldots, n\}. \tag{13}$$

Now define the vector-valued function $F : A \to Z^n$ by

$$F(a) := (\varphi(a, b_1) + k_0, \ldots, \varphi(a, b_n) + k_0).$$

Assert that

$$\text{co } F(A) \cap (\text{int } K)^n = \emptyset, \tag{14}$$

where co $F(A)$ denotes the convex hull of the set $F(A)$. Supposing the contrary, there exist $a_1, \ldots, a_m \in A$ and $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that $\sum_{i=1}^{m} \lambda_i F(a_i) \in (\text{int } K)^n$, or equivalently,

$$\sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) + k_0 \in \text{int } K \quad \text{for each } j \in \{1, \ldots, n\}. \tag{15}$$

Let $u^* \in K^*$ be a nonzero functional for which (ii) holds. Applying u^* to (12), we obtain

$$\sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) + u^*(k_0) > 0.$$

Passing to the minimum over j we have

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) > -u^*(k_0), \tag{16}$$

thus, assumption (ii) and relation (16) imply

$$\sup_{a\in A} \min_{1\leq j\leq n} u^* \left(\varphi(a, b_j)\right) > -u^*(k_0). \tag{17}$$

For each $a \in A$, by relation (12) we have that there exists $j_0 \in \{1, \ldots, n\}$ such that $a \in U_{b_{j_0}, k_{j_0}}$, i.e. $\varphi(a, b_{j_0}) + k_{j_0} \in -\text{int}\, K$. This, together with (13) imply that

$$\varphi(a, b_{j_0}) + k_0 \in -k_{j_0} + k_0 - \text{int}\, K \subseteq -\text{int}\, K.$$

By Lemma 1 we have

$$u^* \left(\varphi(a, b_{j_0})\right) + u^*(k_0) < 0.$$

Thus for each $a \in A$

$$\min_{1\leq j\leq n} u^* \left(\varphi(a, b_j)\right) < -u^*(k_0),$$

and passing to supremum over a we obtain a contradiction to (17).

By the separation theorem of convex sets of Eidelheit (see for instance [28]), we have that there exists $z^* \in (Z^n)^*$ a nonzero functional such that

$$z^*(u) \leq 0 \quad \text{for all}\ u \in \text{co}\, F(A)\ \text{and} \tag{18}$$

$$z^*(c) \geq 0 \quad \text{for all}\ c \in (\text{int}\, K)^n. \tag{19}$$

Using the representation $z^* = (z_1^*, \ldots, z_n^*)$, by a standard argument we deduce that $z_j^* \in K^*$ for all $j \in \{1, \ldots, n\}$.

In particular, by (18), we have $z^*(u) \leq 0$ for all $u \in F(A)$. This means that for any $a \in A$, $z^*(F(a)) \leq 0$, or equivalently,

$$\sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j) + k_0\right) \leq 0.$$

Taking into account the linearity of $z_j^* \in K^*$ for all $j \in \{1, \ldots, n\}$ and the fact that not all z_j^* are zero we obtain

$$\sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \leq -\sum_{j=1}^{n} z_j^*(k_0) < 0.$$

Passing to supremum over $a \in A$ in the upper relation we deduce that

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) < 0,$$

which is a contradiction to assumption (iii). This completes the proof. □

The second assumption of Theorem 3, can be replaced by a stronger one, namely a generalized concavity notion.

Corollary 5 *Let A be a compact set, K a pointed convex cone with the property $C\backslash\{0\} \subseteq \mathrm{int}\, K$, and let $\varphi : A \times B \to Z$ be a bifunction such that*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to Z$ is K-usc on A;*
(ii) *it is K-subconcavelike in its first variable;*
(iii) *for each $b_1, \dots, b_n \in B$, $z_1^*, \dots, z_n^* \in K^*$ not all zero one has*

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) admits a Henig globally efficient solution.

Proof We show that assumption (ii) of Theorem 3 is satisfied. Let us prove that the K-subconcavelikeness of the bifunction φ implies assumption (ii) of the theorem.

Consider $a_1, \dots, a_m \in A$, $b_1, \dots, b_n \in B$, $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and $u^* \in K^*\backslash\{0\}$.

Since φ is K-subconcavelike in its first variable, for each $l \in \mathrm{int}\, K$ there exists $\bar{a} \in A$ such that

$$\sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \leq_K \varphi(\bar{a}, b_j) + l \quad \text{for each } j \in \{1, \dots, n\}. \tag{20}$$

Applying u^* to relation (20), this becomes

$$\sum_{i=1}^{m} \lambda_i u^* \varphi(a_i, b_j) \leq u^* \left(\varphi(\bar{a}, b_j)\right) + u^*(l) \quad \text{for each } j \in \{1, \dots, n\}.$$

Passing to minimum over j yields:

$$\begin{aligned}\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i u^* \left(\varphi(a_i, b_j)\right) &\leq \min_{1\leq j\leq n} u^* \left(\varphi(\bar{a}, b_j)\right) + u^*(l)\\ &\leq \sup_{a\in A} \min_{1\leq j\leq n} u^* \left(\varphi(a, b_j)\right) + u^*(l).\end{aligned}$$

Since this relation holds for each $l \in \mathrm{int}\, K$ we obtain assumption (ii) of Theorem 1 satisfied. Hence, (VEP) admits a Henig globally efficient solution. □

Corollary 6 *Let A be a compact set, K a pointed convex cone with the property $C\backslash\{0\} \subseteq \mathrm{int}\, K$, and let $\varphi : A \times B \to Z$ be a bi function such that*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to Z$ is C-usc on A;*
(ii) *it is C-concavelike in its first variable;*
(iii) *for each $b_1, \dots, b_n \in B$, $z_1^*, \dots, z_n^* \in K^*$ not all zero one has*

$$\sup_{a\in A} \sum_{j=1}^{n} z_j^* \left(\varphi(a, b_j)\right) \geq 0.$$

Then the equilibrium problem (VEP) admits a Henig globally efficient solution.

Proof We show that the assumptions (i), respectively (ii) Corollary 5 are satisfied.Is easy to see, that the C-upper semicontinuity assumption implies the K-upper semicontinuity of the function $\varphi(\cdot, b)$, for all $b \in B$.

Take $a_1, a_2 \in A$ and $\lambda \in [0, 1]$. By the C-concavelikeness of the bifunction φ in its first variable we have the existence of an element $\bar{a} \in A$ such that

$$\varphi(\bar{a}, b) \geq_C \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) \quad \text{for all } b \in B. \tag{21}$$

Since $C \setminus \{0\} \subseteq \text{int}\, K$, by (21) we obtain

$$\varphi(\bar{a}, b) \geq_K \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) \quad \text{for all } b \in B. \tag{22}$$

Moreover, for each $l \in \text{int}\, K$, by inequality (22) we get

$$\varphi(\bar{a}, b) \geq_K \lambda\varphi(a_1, b) + (1 - \lambda)\varphi(a_2, b) - l \quad \text{for all } b \in B.$$

So, assumption (ii) of Corollary 6 is satisfied. □

Whenever $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, Theorem1 and Theorem 3 permit us to reobtain an earlier result of existence of solutions for (EP).

Corollary 7 [25] *Let A be a compact set, let B be a nonempty set and let* $\varphi : A \times B \to \mathbb{R}$ *be a bi function such that*

(i) *for each* $b \in B$, *the function* $\varphi(\cdot, b) : A \to \mathbb{R}$ *is usc on A;*
(ii) *for each* $a_1, \ldots, a_m \in A, \lambda_1, \ldots, \lambda_m \geq 0$ *with* $\lambda_1 + \cdots + \lambda_m = 1$, *and* $b_1, \ldots, b_n \in B$

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \sup_{a \in A} \min_{1 \leq j \leq n} \varphi(a, b_j);$$

(iii) *for each* $b_1, \ldots, b_n \in B, \mu_1, \ldots, \mu_n \geq 0$ *with* $\mu_1 + \cdots + \mu_n = 1$

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j \varphi(a, b_j) \geq 0.$$

Then the equilibrium problem (EP) *admits a solution.*

Proof We show that the assumptions of Theorem 1, respectively Theorem 3 are satisfied. Since for each $U \subseteq V_{\mathcal{B}}$ the set cone $(U + \mathcal{B}) = \mathbb{R}_+$ and choosing $K = \mathbb{R}_+$ in Theorem 3, it is obvious that conditions (i) and (ii) of Theorem 1, respectively Theorem 3 are satisfied (the latter with $u^* = 1$).

Let $b_1, \ldots, b_n \in B, z_1^*, \ldots, z_n^* \in \mathbb{R}_+$ not all zero and denote $\mu_j := \dfrac{z_j^*}{\nu}$ for all $j \in \{1, \ldots, n\}$, where $\nu = \sum_{j=1}^{n} z_j^*$. Thus each $\mu_j \geq 0$ and $\mu_1 + \cdots + \mu_n = 1$. Hence by assumption (iii) we obtain

$$\frac{1}{\nu} \sup_{a \in A} \sum_{j=1}^{n} z_j^* \varphi(a, b_j) \geq 0,$$

i.e. assumption (iii) of Theorem 1, respectively Theorem 3 is satisfied. □

3 Applications to vector variational inequalities

The domain of vector variational inequalities received a great attention ever since the paper of Giannessi [15] appeared and the first existence results for vector variational inequalities were published in [11]. Most of the research results in this area deal with a weak version of vector variational inequalities and their generalizations. Hence, the authors of [12] suggested to study the existence of solutions for strong vector variational inequalities.

In this section we consider Minty and Stampacchia generalized vector variational inequalities, and we state existence results for proper solutions.

Let $F : A \to LC(A, Z)$ and $q : A \to Z$ be given mappings, where $LC(A, Z)$ denotes the set of all linear and continuous functionals from A to Z. Taking $A = B$, we study the following vector variational inequalities:

$$(MVI) \text{ find } \bar{a} \in A \text{ such that } \langle F(b), b-a\rangle + q(b) - q(a) \notin -C \setminus \{0\} \quad \text{for all } b \in A$$

and

$$(SVI) \text{ find } \bar{a} \in A \text{ such that } \langle F(a), b-a\rangle + q(b) - q(a) \notin -C\setminus\{0\} \quad \text{for all } b \in A.$$

By $\langle F(a), b-a\rangle$ we understand the value of $F(b)$ at $b-a$, for all $a, b \in A$.

First, let us recall some definitions concerning the study of vector variational inequalities (see [18,29,31]).

Definition 4 Let $F : A \to LC(A, Z)$ be a given mapping.

(i) F is said to be C-monotone if for each $a, b \in A$, we have

$$\langle F(b) - F(a), b-a\rangle \geq_C 0.$$

(ii) Let $e^* \in C^* \setminus \{0\}$. F is said to be e^*-monotone if for each $a, b \in A$, we have

$$e^*(\langle F(b) - F(a), b-a\rangle) \geq 0.$$

(iii) F is said to be v-hemicontinuous if for each $a, b \in A$ and $t \in [0, 1]$, the mapping $t \mapsto \langle F(tb + (1-t)a), b-a\rangle$ is continuous at 0^+.

(iv) Let $e^* \in C^* \setminus \{0\}$. F is said to be e^*-upper hemicontinuous if for each $a, b \in A$ and $t \in [0, 1]$, the mapping $t \mapsto e^*(\langle F(tb + (1-t)a), b-a\rangle)$ is upper semicontinuous at 0^+.

It is clear that, if F is C-monotone and v-hemicontinuous on A, then for any $e^* \in C^* \setminus \{0\}$, F is e^*-monotone, respectively e^*-upper-hemicontinuous.

Definition 5 A vector $a \in A$ is said to be:

(i) a Henig globally efficient solution to (MVI) if there exists a pointed convex cone $K \subseteq Z$, with $C\setminus\{0\} \subseteq \operatorname{int} K$, such that

$$\langle F(b), b-a\rangle + q(b) - q(a) \notin -K \setminus \{0\} \quad \text{for all } b \in A.$$

(ii) a Henig globally efficient solution to (SVI) if there exists a pointed convex cone $K \subseteq Z$, with $C\setminus\{0\} \subseteq \operatorname{int} K$, such that

$$\langle F(a), b-a\rangle + q(b) - q(a) \notin -K \setminus \{0\} \quad \text{for all } b \in A.$$

(iii) a Henig efficient solution to (SVI) if there exists some neighborhoods U of zero, with $U \subseteq V_{\mathcal{B}}$, such that

$$\langle F(a), b-a\rangle + q(b) - q(a) \notin -\text{int}\, C_U(\mathcal{B}) \quad \text{for all } b \in A.$$

Theorem 4 *Let $K \subseteq Z$ be a pointed convex cone, with $C\backslash\{0\} \subseteq int\, K$, $e^* \in K^\sharp$, A a compact and convex set, and the following assumptions satisfied:*

(i) *$e^* \circ q$ is lower semicontinuous;*
(ii) *q is K-convex;*
(iii) *F is e^*-monotone.*

Then, (MVI) admits a Henig globally efficient solution.

Proof For the proof of this theorem, we show that the assumption of Corollary 7 are satisfied.
Define a real-valued bifunction $g : A \times A \to \mathbb{R}$, by

$$g(a,b) = e^*(\langle F(b), b-a\rangle + q(b) - q(a)) \quad \text{for all } a, b \in A.$$

The linearity of $F(b)$ in the variable a and assumption (i) assure the upper semicontinuity of the function g in its first variable.

We observe that g is concave in its first variable, due to assumption (ii). So, assumption (ii) of Corollary 7 is satisfied.

Now, take $b_1, \ldots, b_n \in A$, $\mu_1, \ldots, \mu_n \geq 0$ with $\mu_1 + \cdots + \mu_n = 1$. Since, F is e^*-monotone we have

$$e^*(\langle F(b_j), b_j - a\rangle + q(b_j) - q(a)) \geq e^*(\langle F(a), b_j - a\rangle + q(b_j) - q(a)) \tag{23}$$

for all $j \in \{1, \ldots, n\}$ and $a \in A$. □

Summing over j and taking into account assumption (ii), by (23) we obtain

$$\sup_{a\in A} \sum_{j=1}^{n} \mu_j g(a, b_j) \geq 0.$$

So, the assumptions of Corollary 7 are satisfied and, by this, we have the existence of an $\bar{a} \in A$ such that

$$g(\bar{a}, b) \geq 0 \quad \text{for all } b \in A.$$

Since $e^* \in K^\sharp$ we get

$$\langle F(b), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -K\backslash\{0\} \quad \text{for all } b \in A,$$

i.e. (MVI) admits a Henig globally efficient solution.

Under the e^*-upper hemicontinuity of the operator F and the convexity assumption on A, we state existence results for (SVI).

The next theorem gives existence results for Henig globally efficient solutions of (SVI), under additional assumptions than those of Theorem 3.1 from [18], where the author states existence results for efficient solution of (SVI), namely that there exists a pointed convex cone K with the property $C\backslash\{0\} \subseteq \text{int}\, K$ and $K^\sharp \neq \emptyset$. Such an hypothesis is not very demanding, since such a cone always exists if we suppose the cone C to admit a base.

Theorem 5 *Let $K \subseteq Z$ be a pointed convex cone, with $C \setminus \{0\} \subseteq int\, K$, $e^* \in K^\sharp$, A a compact and convex set, and the following assumptions satisfied:*

(i) *$e^* \circ q$ is lower semicontinuous;*
(ii) *q is K-convex;*
(iii) *F is e^*-monotone;*
(iv) *F is e^*-upper hemicontinuous.*

Then, (SVI) admits a Henig globally efficient solution.

Proof By Theorem 4, we have the existence of a point $\bar{a} \in A$ such that

$$e^*(\langle F(b), b - \bar{a}\rangle + q(b) - q(\bar{a})) \geq 0 \quad \text{for all } b \in A. \tag{24}$$

For all $t \in [0, 1]$, let $b(t) = tb + (1-t)\bar{a}$ which belongs to A, by the convexity assumption on A. Thus, by (24), for all $t \in [0, 1]$ we have

$$e^*(\langle F(b(t)), b(t) - \bar{a}\rangle + q(b(t)) - q(\bar{a})) \geq 0. \tag{25}$$

By assumption (ii) and relation (25), for each $t \in (0, 1]$

$$e^*(\langle F(b(t)), b - \bar{a}\rangle + q(b) - q(\bar{a})) \geq 0. \tag{26}$$

Let $\{t_n\}_{n\geq 1}$ be a sequence of positive numbers, such that $t_n \to 0$, whenever $n \to \infty$. Assumption (iv) guarantees for each $\epsilon > 0$, the existence of $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ holds:

$$e^*(\langle F(b(t_n)), b - \bar{a}\rangle + q(b) - q(\bar{a})) < e^*(\langle F(\bar{a}), b - \bar{a}\rangle + q(b) - q(\bar{a})) + \epsilon^{'},$$

where $\epsilon^{'} = \min\{\epsilon, 1\}$. By this and (26) we have

$$0 \leq e^*(\langle F(\bar{a}), b - \bar{a}\rangle + q(b) - q(\bar{a})) + \epsilon^{'}.$$

Since the inequality holds for each $\epsilon^{'}$ we deduce

$$e^*(\langle F(\bar{a}), b - \bar{a}\rangle + q(b) - q(\bar{a})) \geq 0.$$

By this, the vector variational inequality (SVI) admits a Henig globally efficient solution. □

In the final part of this section, we give existence results for the Stampacchia vector variational inequality problem, under stronger assumptions than those of Theorem 5.

Corollary 8 *Let $K \subseteq Z$ be a pointed convex cone, with $K^\sharp \neq \emptyset$ such that $C \setminus \{0\} \subseteq int\, K$, A a compact and convex set,and the following assumptions satisfied:*

(i) *q is K-lower semicontinuous;*
(ii) *q is K-convex;*
(iii) *F is K-monotone;*
(iv) *F is v-hemicontinuous.*

Then, (SVI) admits a Henig globally efficient solution.

Proof It is an easy exercise to verify that for each $e^* \in K^\sharp$, the assumptions of Theorem 5 are satisfied. □

Assumptions (i), (ii) and (iii) of Corollary 8 are satisfied, if we consider the function q to be C-lsc, C-convex and the operator F to be C-monotone on A.

Corollary 9 *Let C be a cone with a base, A a compact and convex set, and the following assumptions satisfied:*

(i) *q is C-lower semicontinuous;*
(ii) *q is C-convex;*
(iii) *F is C-monotone;*
(vi) *F is v-hemicontinuous.*

Then, (SVI) admits a Henig efficient solution.

Proof Since C admits a base $\mathcal{B}$, then there exists a pointed convex cone $C_U(\mathcal{B})$ such that $C\backslash\{0\} \subseteq \text{int}\, C_U(\mathcal{B})$ and $C_U^\sharp(\mathcal{B}) \neq \emptyset$ (see the first section). By the hypothesis, the assumptions of Corollary 8 are satisfied. So, there exists an element $\bar{a} \in A$ such that

$$\langle F(\bar{a}), b - \bar{a}\rangle + q(b) - q(\bar{a}) \notin -C_U(\mathcal{B})\backslash\{0\} \quad \text{for all } b \in A.$$

By this, we deduce

$$\langle F(\bar{a}), b - \bar{a}\rangle + q(b) - q(\bar{a}) \notin -\text{int}\, C_U(\mathcal{B}) \quad \text{for all } b \in A,$$

i.e. the strong vector variational inequality (SVI) admits a Henig efficient solution. □

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