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Continuity of approximate solution mappings for parametric equilibrium problems

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Abstract In this paper, we obtain sufficient conditions for Hausdorff continuity and Berge continuity of an approximate solution mapping for a parametric scalar equilibrium problem. By using a scalarization method, we also discuss the Berge lower semicontinuity and Berge continuity of a approximate solution mapping for a parametric vector equilibrium problem.

Keywords Parametric equilibrium problems · Approximate solution mapping · Hausdorff upper semicontinuity and Hausdorff lower semicontinuity · Berge upper semicontinuity and Berge lower semicontinuity · Scalarization

1 Introduction

The equilibrium problem is a unified model of several problems, for example, variational inequalities and minmax problem. There are many papers ([6,12,14,20–22]) which intensively study different types of equilibrium problems and obtain many existence results. The stability analysis of solution mappings for equilibrium problems and variational inequalities is another important topic in optimization theory and applications. The semicontinuity of solution mappings for parametric equilibrium problems and parametric variational inequalities has been of increasing interest in the literature, such as [2,4,5,10,11,13,18,19]. Beside semicontinuity, the Hölder continuity of the solution mapping for parametric equilibrium problems has been also investigated intensively ([1,3,10,11,19]).

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On the other hand, exact solutions of the problems may not exist in many practical problems because the data of the problems are not sufficiently "regular". Moreover, these mathematical models are solved usually by numerical methods (iterative procedures or heuristic algorithms) which produce approximations to the exact solutions. So it is impossible to obtain an exact solution of many practical problems. Naturally, investigating approximate solutions of parametric equilibrium problems is of interest in both practical applications and computations. However, there are only a few results concerning the semicontinuity of approximate solution mappings for parametric variational inequality or parametric equilibrium problems. Kimura and Yao [17] have established the existence results for two types of approximate generalized vector equilibrium problems, and further obtained the semicontinuity of approximate solution mappings. Khanh and Luu [15] have discussed the semicontinuity of the approximate solution mappings of parametric multivalued quasivariational inequalities in topological vector spaces. Anh and Khanh [5] have considered two kinds of approximate solution mappings to parametric generalized vector quasiequilibrium problems and established the sufficient conditions for their Hausdorff semicontinuity (or Berge semicontinuity).

Motivated by the work reported in [5,15,17], the aim of this paper is to discuss the lower semicontinuity and continuity of the approximate solution mappings for a parametric scalar equilibrium problem (PSEP) and a parametric vector equilibrium problem (PVEP), respectively. Our main proof methods are different from the ones used in [5,17] and [15]. By using the monotonicity of the approximate solution mappings (with respect to the set-inclusion) for (PSEP), we establish the Hausdorff upper semicontinuity and lower semicontinuity of the approximate solution mappings for (PSEP). Then, the Berge semicontinuity of the approximate solution mapping for (PVEP) is derived by a scalarization method and a property involving the union of a family of Berge lower semicontinuity set-valued mappings. Moreover, we show that the sufficient condition which guarantees the Berge lower semicontinuity of the solution mapping for (PVEP) is also sufficient for Berge continuity. Our consequences are new and different from the corresponding ones in [5,15,17].

The rest of the paper is organized as follows. In Sect. 2, we recall semicontinuity and some of their properties. In Sect. 3, we discuss the Harsdorff continuity and Berge continuity of a solution mapping for (PSEP). In Sect. 4, by the results in Sect. 3, we establish the Berge lower semicontinuity and Berge continuity of a solution mapping for (PVEP).

2 Preliminaries

In this section, we recall some definitions and some properties needed in the following sections. Suppose that X and Y are two topological vector spaces, and $G : X \to 2^Y$ is a set-valued mapping.

Definition 2.1 (*see*[16])

(i) G is said to be Hausdorff upper semicontinuous (H-u.s.c) at $x_0 \in X$, if for every neighborhood B of the origin in Y, there is a neighborhood $N(x_0)$ of x_0 in X such that $G(x) \subseteq G(x_0) + B, \forall x \in N(x_0)$.

(ii) G is said to be Harsdorff lower semicontinuous (H-l.s.c) at $x_0 \in X$, if for every neighborhood B of the origin in Y, there is a neighborhood $N(x_0)$ of x_0 in X such that $G(x_0) \subseteq G(x) + B, \forall x \in N(x_0)$.

(iii) G is said to be Berge upper semicontinuous (B-u.s.c) at $x_0 \in X$, if for every open set U with $G(x_0) \subseteq U$, there is a neighborhood $N(x_0)$ of x_0 in X such that $G(x) \subseteq U, \forall x \in N(x_0)$.

(iv) G is said to be Berge lower semicontinuous (B-l.s.c) at x_0, if for every open set U with $G(x_0) \cap U \neq \emptyset$, there is a neighborhood $N(x_0)$ of x_0 in X such that $G(x) \cap U \neq \emptyset, \forall x \in N(x_0)$.

We say that G is B-continuous (resp. H-continous) at x_0 if it is both B-l.s.c and B-u.s.c (resp. H-u.s.c) at x_0.

Lemma 2.1 *(*see [8]*)*

(i) If G is B-u.s.c at x_0, then G is H-u.s.c at x_0. Conversely if G is H-u.s.c at x_0 and $G(x_0)$ is compact, then G is B-u.s.c at x_0.
(ii) If G is H-l.s.c at x_0, then G is B-l.s.c at x_0. Conversely if G is B-l.s.c at x_0 and $cl(G(x_0))$ (i.e., the closure of $G(x_0)$) is compact , then G is H-l.s.c at x_0.

Lemma 2.2 *(*see [7]*) Let G be compact-valued on X. Then G is Berge upper semicontinuous at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for every $y_\alpha \in G(x_\alpha)$, there exist $y_0 \in G(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Now we recall the following lemma which plays an important role in Sect. 4.

Lemma 2.3 *(*see [9]*) The union $\Gamma = \bigcup_{i \in I} \Gamma_i$ of a family of B-l.s.c set-valued mappings Γ_i from a topological space X into a topological space Y is also a B-l.s.c set-valued mapping from X into Y, where I is an index set.*

3 Continuity of approximate solution mappings for (PSEP)

In this section, we deal with the following parametric scalar equilibrium problem (for short PSEP) of finding $\bar{x} \in E$ such that

$$f(\bar{x}, y, \mu) \geq 0, \quad \forall y \in E,$$

where $f : E \times E \times M \to R$, E is a nonempty convex compact subset of X and $M \subset Z$; X, Z are locally convex Hausdorff topological vector spaces.

Denote the approximate solution set of (PSEP) by

$$S_\varepsilon(\mu) := \{\bar{x} \in E : f(\bar{x}, y, \mu) + \varepsilon \geq 0, \quad \forall y \in E\},$$

where ε is a positive real number.

Fix $\mu_0 \in M$. First, we establish the compactness of $S_\varepsilon(\mu_0)$.

Lemma 3.1 *If for every $y \in E$, $f(\cdot, y, \mu_0)$ is continuous on E, then the approximate solution set $S_\varepsilon(\mu_0)$ of (PSEP) is compact.*

Proof Since the proof is trivial, we omit it. □

Lemma 3.2 *Assume that $S_\varepsilon(\mu) \neq \emptyset$ in a neighborhood of a fixed point (ε_0, μ_0). If for each $x, y \in E$, $f(x, y, \cdot)$ is continuous at μ_0, then there exists a neighborhood $N(\varepsilon_0) \times N(\mu_0)$ of μ_0 such that the approximate solution mapping $S.(\cdot)$ of (PSEP) satisfies the following condition: $\forall \mu_1, \mu_2 \in N(\mu_0), \forall \varepsilon_1, \varepsilon_2 \in N(\varepsilon_0) : \varepsilon_1 < \varepsilon_2$,*

$$S_{\varepsilon_1}(\mu_1) \subset S_{\varepsilon_2}(\mu_2).$$

Proof For any real number δ satisfying $0 < \delta < \varepsilon_0$, let $N(\varepsilon_0) := [\varepsilon_0 - \delta, \varepsilon_0 + \delta]$ be a given neighborhood of ε_0. Now, let $\varepsilon_1, \varepsilon_2$ be any two points from $[\varepsilon_0 - \delta, \varepsilon_0 + \delta]$ with $\varepsilon_1 < \varepsilon_2$. For any real number $\eta : 0 < \eta < \varepsilon_2 - \varepsilon_1$, by (ii), there exists a neighborhood $N(\mu_0)$ of μ_0 such that for each $x, y \in E$,

$$|f(x, y, \mu_1) - f(x, y, \mu_2)| \leq \eta, \quad \forall \mu_1, \mu_2 \in N(\mu_0). \tag{1}$$

Taking any $\bar{x} \in S_{\varepsilon_1}(\mu_1)$, we have

$$\bar{x} \in E \quad \text{and} \quad f(\bar{x}, y, \mu_1) \geq -\varepsilon_1, \forall \quad y \in E.$$

Then, $f(\bar{x}, y, \mu_2) + f(\bar{x}, y, \mu_1) - f(\bar{x}, y, \mu_2) \geq -\varepsilon_1$, which along with (1) yields that

$$f(\bar{x}, y, \mu_2) \geq -\varepsilon_1 - \eta, \forall \quad y \in E.$$

Namely, $f(\bar{x}, y, \mu_2) \geq -\varepsilon_2$, which implies that $\bar{x} \in S_{\varepsilon_2}(\mu_2)$. Thus, it follows from the arbitrariness of $\bar{x}$ that $S_{\varepsilon_1}(\mu_1) \subset S_{\varepsilon_2}(\mu_2)$ and the proof is complete. □

Lemma 3.3 *Assume that* $S_\varepsilon(\mu) \neq \emptyset$ *in a neighborhood of a fixed point* (ε_0, μ_0). *If for each* $y \in E$, $f(\cdot, y, \mu_0)$ *is a concave function, then the approximate set* $S_\varepsilon(\mu_0)$ *is convex for each* $\varepsilon \in [\varepsilon_0 - \delta, \varepsilon_0 + \delta]$.

Proof Take any $x_1, x_2 \in S_\varepsilon(\mu_0)$ and any $\lambda \in [0, 1]$. Then, we have

$$f(x_1, y, \mu_0) + \varepsilon \geq 0, \quad \forall y \in E$$

and

$$f(x_2, y, \mu_0) + \varepsilon \geq 0, \quad \forall y \in E,$$

which along with (iii) yields that

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2, y, \mu_0) + \varepsilon &\geq \lambda(f(x_1, y, \mu_0) + \varepsilon) + (1-\lambda)(f(x_2, y, \mu_0) + \varepsilon) \\ &\geq 0, \quad \forall y \in E. \end{aligned}$$

Since E is a convex set, we have $\lambda x_1 + (1-\lambda)x_2 \in E$. Thus, $\lambda x_1 + (1-\lambda)x_2 \in S_\varepsilon(\mu_0)$ and the proof is complete. □

Now, we state our main result.

Theorem 3.1 *Assume that* $S_\varepsilon(\mu) \neq \emptyset$ *in a neighborhood of a fixed point* (ε_0, μ_0). *Furthermore, assume that the following conditions hold:*

(i) *For each* $y \in E$, $f(\cdot, y, \mu_0)$ *is continuous on* E*;*
(ii) *For each* $x, y \in E$, $f(x, y, \cdot)$ *is continuous at* μ_0*;*
(iii) *For each* $y \in E$, $f(\cdot, y, \mu_0)$ *is a concave function.*

Then, the approximate solution mapping $S_\cdot(\cdot)$ *of (PSEP) is H-continuous at* (ε_0, μ_0).

Proof Now, we verifies that the approximate solution mapping $S_\varepsilon(\mu)$ of (PSEP) is H-continuity at (ε_0, μ_0). Indeed, it follows from Lemma 3.2 that

$$S_{\varepsilon_0 - \delta}(\mu_0) \subset S_\varepsilon(\mu) \subset S_{\varepsilon_0 + \delta}(\mu_0), \forall \mu \in N(\mu_0), \quad \forall \varepsilon \in [\varepsilon_0 - \delta, \varepsilon_0 + \delta]. \tag{2}$$

On the other hand, for any $1 < \theta < \frac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1}$($\varepsilon_1, \varepsilon_2$ is defined as in Lemma 3.2), we have

$$\varepsilon_1 = \frac{1}{\theta}\gamma + \left(1 - \frac{1}{\theta}\right)\varepsilon_2 \quad \text{for} \quad \gamma := \varepsilon_2 + \theta(\varepsilon_1 - \varepsilon_2). \tag{3}$$

Obviously, $0 < \gamma < \varepsilon_1$. Then, we claim that

$$\frac{1}{\theta}S_\gamma(\mu_0) + \left(1 - \frac{1}{\theta}\right) S_{\varepsilon_2}(\mu_0) \subset S_{\varepsilon_1}(\mu_0). \tag{4}$$

In fact, take any $x_\gamma \in S_\gamma(\mu_0)$ and any $x_{\varepsilon_2} \in S_{\varepsilon_2}(\mu_0)$. Then,

$$f(x_\gamma, y, \mu_0) + \gamma \geq 0, \quad \forall y \in E \tag{5}$$

and

$$f\left(x_{\varepsilon_2}, y, \mu_0\right) + \varepsilon_2 \geq 0, \quad \forall y \in E. \tag{6}$$

Hence, (3), (5), (6) and (iii) together yields that

$$\begin{aligned} f\left(\frac{1}{\theta}x_\gamma + (1-\frac{1}{\theta})x_{\varepsilon_2}, y, \mu_0\right) + \varepsilon_1 &\geq \frac{1}{\theta}\left(f(x_\gamma, y, \mu_0) + \gamma\right) \\ &\quad + \left(1 - \frac{1}{\theta}\right)\left(f(x_{\varepsilon_2}, y, \mu_0) + \varepsilon_2\right) \\ &\geq 0, \quad \forall y \in E, \end{aligned}$$

which implies that $\frac{1}{\theta}x_\gamma + \left(1 - \frac{1}{\theta}\right)x_{\varepsilon_2} \in S_{\varepsilon_1}(\mu_0)$. It follows from the arbitrariness of $x_\gamma, x_{\varepsilon_2}$ that (4) holds.

By assumption i) and Lemma 3.1, $S_{2\varepsilon_0}(\mu_0)$ is compact. Then, it follows from Lemma 3.2 that $Q := \bigcup\{S_\varepsilon(\mu_0) : \varepsilon \leq 2\varepsilon_0\} \subset S_{2\varepsilon_0}(\mu_0)$ is bounded. Thus, for any closed convex neighborhood B of the origin, there exists a real number $\rho > 0$ such that

$$Q - Q \subset \rho B.$$

Notice that $\gamma < \varepsilon_1 < 2\varepsilon_0$. Hence, by Lemma 3.3 and (4), we have

$$\begin{aligned} S_{\varepsilon_2}(\mu_0) &\subset \left(1 - \frac{1}{\theta}\right)^{-1}\left[S_{\varepsilon_1}(\mu_0) - \frac{1}{\theta}S_\gamma(\mu_0)\right] \\ &= S_{\varepsilon_1}(\mu_0) + \left(\frac{1}{\theta - 1}\right)\left[S_{\varepsilon_1}(\mu_0) - S_\gamma(\mu_0)\right] \\ &\subset S_{\varepsilon_1}(\mu_0) + \left(\frac{\rho}{\theta - 1}\right)B. \end{aligned} \tag{7}$$

Let $\delta_0 < \frac{\varepsilon_0}{\rho+1}$ and consider the interval $[\varepsilon_0 - \delta_0, \varepsilon_0 + \delta_0]$. Let in (7) $\varepsilon_2 = \varepsilon_0$ and $\varepsilon_1 = \varepsilon_0 - \delta_0$. Choose $\theta = \rho + 1$. Then, $1 < \theta < \frac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1} = \frac{\varepsilon_0}{\delta_0}$ and it follows from (2) and (7) that

$$\begin{aligned} S_{\varepsilon_0}(\mu_0) &\subset S_{\varepsilon_0 - \delta}(\mu_0) + B \\ &\subset S_\varepsilon(\mu) + B, \quad \forall \mu \in N(\mu_0), \quad \forall \varepsilon \in [\varepsilon_0 - \delta_0, \varepsilon_0 + \delta_0], \end{aligned}$$

which implies that $S_\cdot(\cdot)$ is H-l.s.c at (ε_0, μ_0).

On the other hand, let in (7) $\varepsilon_2 = \varepsilon_0 + \delta_0$ and $\varepsilon_1 = \varepsilon_0$. Choose $\theta = \rho + 1$. Then, it follows from (2) and (7) that

$$\begin{aligned} S_\varepsilon(\mu) &\subset S_{\varepsilon_0 + \delta}(\mu_0) \\ &\subset S_{\varepsilon_0}(\mu_0) + B, \quad \forall \mu \in N(\mu_0), \quad \forall \varepsilon \in [\varepsilon_0 - \delta_0, \varepsilon_0 + \delta_0], \end{aligned}$$

which means that $S_\cdot(\cdot)$ is H-u.s.c at (ε_0, μ_0). This completes the proof. □

Remark 3.1 If ε is a fixed positive real number and $f(x, y, \mu) = f'(x, y, \mu) + \varepsilon$, then the parametric scalar equilibrium problem (PSEP) was investigated by Ahn and Khanh in [5]. They defined another approximate solution mapping $\widetilde{S_\varepsilon}(\cdot)$ for (PSEP) and discussed H-l.s.c (or B-l.s.c) of the solution mapping $\widetilde{S_\varepsilon}(\cdot)$, see Theorems 2.1, 2.3, 2.6 and 2.8 of [5]. They also studied H-u.s.c (or B-u.s.c) of the approximate solution mapping $S_\varepsilon(\cdot)$ for (PSEP) in Theorems 3.1, 3.3 and 3.5 of [5]. However, the assumptions and our proof methods in Theorem 3.1 are very different from the corresponding ones in [5].

Corollary 3.1 *Assume that all assumptions of Theorem 3.1 are satisfied. Then, the approximate solution mapping $S_{\cdot}(\cdot)$ of (PSEP) is B-continuous at at (ε_0, μ_0).*

Proof From Lemma 3.1, $S_{\varepsilon_0}(\mu_0)$ is compact. Thus, by Theorem 3.1 and Lemma 2.1, the approximate solution mapping $S_{\cdot}(\cdot)$ of (PSEP) is B-continuous at (ε_0, μ_0) and the proof is complete. □

The following example is given to illustrate that assumption (iii) in Theorem 3.1 is essential.

Example 3.1 Let $X = Z = R$, $E = [0, 1]$, $M = [-1, 1]$. Furthermore, let $\mu_0 = 0 \in M = [-1, 1]$, $\varepsilon_0 = \frac{1}{4}$ and $f(x, y, \mu) = \mu x(x - y) - \frac{1}{4}$. Obviously, all conditions of Theorem 4.1 except for (iii) are satisfied. The direct computation shows that

$$S_{\varepsilon_0}(\mu) = \begin{cases} [0, 1], & \text{if} \quad \mu \in [0, 1], \\ \{0\}, & \text{if} \quad \mu \in [-1, 0). \end{cases}$$

Clearly, we see that $S_{\varepsilon_0}(\cdot)$ is even not B-l.s.c at $\mu_0 = 0$. Hence assumption (iii) in Theorem 3.1 is essential.

4 Continuity of approximate solution mappings for (PVEP)

Now, we consider the following parametric vector approximate equilibrium problem (PVEP): find $\bar{x} \in E$ such that

$$f(\bar{x}, y, \mu) \notin -\text{int}C, \quad \forall y \in E,$$

where $f : E \times E \times M \to Y$ is a vector-valued function, C is a pointed closed convex cone in Y with $\text{int}C \neq \emptyset$, $e \in \text{int}C$ and $E \subset X$ is a nonempty compact convex subset and $M \subset Z$; X, Y, Z are locally convex Hausdorff topological vector spaces.

For each $\varepsilon > 0$, $\mu \in M$, by $S_{\varepsilon e}(\mu)$ we denote the approximate solution set of (PVEP), i.e.,

$$S_{\varepsilon e}(\mu) := \{\bar{x} \in E : f(\bar{x}, y, \mu) + \varepsilon e \notin -\text{int}C, \forall y \in E\},$$

where ε is a positive real number.

Let $C^* := \{\xi \in Y^* : \xi(y) \geq 0, \forall y \in C\}$ be the dual cone of C. Letting $e \in \text{int}C$ be given, we have that $B_e^* := \{\xi \in C^* : \xi(e) = 1\}$ is a weak* compact base of C^*. For each $\xi \in B_e^*$, by S_ε^ξ we denote the ξ-approximate solution set of (PVEP), i.e.,

$$S_\varepsilon^\xi(\mu) := \{\bar{x} \in E : \xi(f(\bar{x}, y, \mu)) + \varepsilon \geq 0, \forall y \in E\}.$$

A vector valued function $g : X \to Y$ is said to be C-convex on X if, for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $\lambda g(x_1) + (1 - \lambda)g(x_2) \in g(\lambda x_1 + (1 - \lambda)x_2) + C$; f is C-concave if $-f$ is C-convex.

With the proof similar to Lemma 3.1 in [13], the following result can be proved:

Lemma 4.1 *If for each $x \in E$, $\mu \in M$, $f(x, \cdot, \mu)$ is a C-convex function, then*

$$S_{\varepsilon e}(\mu) = \bigcup_{\xi \in B_e^*} S_\varepsilon^\xi(\mu).$$

Theorem 4.1 *Assume that, for each $\xi \in B_e^*$, the ξ-efficient solution of (PVEP) exists in a neighborhood of the considered point (ε_0, μ_0). Assume furthermore that the following conditions hold:*

(i) *For each $y \in E$, $f(\cdot, y, \mu_0)$ is continuous on E;*
(ii) *For each $x, y \in E$, $f(x, y, \cdot)$ is continuous at μ_0;*
(iii) *For each $x \in E$, $f(x, \cdot, \mu_0)$ is a C-convex function;*
(iv) *For each $y \in E$, $f(\cdot, y, \mu_0)$ is a C-concave function.*

Then, the approximate solution mapping $S_{\cdot e}(\cdot)$ of (PVEP) is B-continuous at (ε_0, μ_0).

Proof (a) We first show that $S_{\cdot e}(\cdot)$ for (PVEP) is B-l.s.c at (ε_0, μ_0). Indeed, set $g(x, y, \mu) := \xi(f(x, y, \mu))$ for each $x, y \in E, \mu \in M$ and $\xi \in B_e^*$. Applying Corollary 3.1 to g, we have that the ξ-approximate solution mapping $S_{\cdot}^\xi(\cdot)$ of (PVEP) is B-l.s.c at (ε_0, μ_0). By (iii) and Lemma 4.1, we have

$$S_{\varepsilon e}(\mu) = \bigcup_{\xi \in B_e^*} S_\varepsilon^\xi(\mu),$$

which along with Lemma 2.3 yields that $S_{\cdot e}(\cdot)$ is B-l.s.c at (ε_0, μ_0).

(b) By (a), it suffices to show that the approximate solution mapping $S_{\cdot e}(\cdot)$ for (PVEP) is B-u.s.c at (ε_0, μ_0). Suppose that the solution mapping $S_{\cdot e}(\cdot)$ of (PVEP) is not B-u.s.c at (ε_0, μ_0). Then there exists a open neighborhood U satisfying $S_{\varepsilon_0 e}(\mu_0) \subset U$, and sequences $\varepsilon_n \to \varepsilon_0$ and $\mu_n \to \mu_0$ with $x_n \in S_{\varepsilon_n e}(\mu_n)$ such that $x_n \notin U, \forall n$. Since E is compact, $x_n \to x_0 \in E$. If $x_0 \notin S_{\varepsilon_0 e}(\mu_0)$, then there exists $y_0 \in E$ such that $f(x_0, y_0, \mu_0) + \varepsilon_0 e \in -\text{int}C$. By assumptions (i) and (ii), there exists a index $\bar{n}$ such that $f(x_{\bar{n}}, y_0, \mu_{\bar{n}}) + \varepsilon_{\bar{n}} e \in -\text{int}C$, which is impossible as $x_{\bar{n}} \in S_{\varepsilon_{\bar{n}} e}(\mu_{\bar{n}})$. Thus, $x_0 \in S_{\varepsilon_0 e}(\mu_0) \subset U$, which contradicts $x_n \notin U, \forall n$. This completes the proof. □

Remark 4.1 When is replaced by any $\gamma \in \text{int}C$ and $f(x, y, \mu) = f(x, y) + \gamma$, the (PVEP) reduces to the problem in [17]. However, the assumptions and proof method of Theorem 4.1 are very different from the corresponding ones in [17].

The following example is given to illustrate that the assumption (iii) in Theorem 4.1 is essential.

Example 4.1 Let $X = Z = R$, $E = [0, 1]$, and $Y = R^2$, $e = (1, 1) \in \text{int}R_+^2$, $M = (0, 1)$ and $\varepsilon, \mu \in M$. Suppose that

$$f(x, y, \varepsilon) = ((2 + \log_2^\varepsilon) y(x - y) - \varepsilon, y(x - y) - \varepsilon) \text{ and } \varepsilon_0 = \mu_0 = \frac{1}{4}.$$

Obviously, all assumptions of Theorem 4.1 except for (iii) are satisfied. Moreover, the direct computation shows that

$$S_{\varepsilon e}(\varepsilon) = \begin{cases} [0, 1], & \text{if } \varepsilon \in [\frac{1}{4}, 1), \\ \{0\}, & \text{if } \varepsilon \in (0, \frac{1}{4}), \end{cases}$$

which implies that $S_{\cdot e}(\cdot)$ is not B-l.s.c at $\mu_0 = \frac{1}{4}$. Thus the assumption (iii) in Theorem 4.1 is essential.

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