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Second-order conditions for nonsmooth multiobjective optimization problems with inclusion constraints

Ahmed Taa

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Abstract This paper investigates second-order optimality conditions for general multiobjective optimization problems with constraint set-valued mappings and an arbitrary constraint set in Banach spaces. Without differentiability nor convexity on the data and with a metric regularity assumption the second-order necessary conditions for weakly efficient solutions are given in the primal form. Under some additional assumptions and with the help of Robinson -Ursescu open mapping theorem we obtain dual second-order necessary optimality conditions in terms of Lagrange-Kuhn-Tucker multipliers. Also, the second-order sufficient conditions are established whenever the decision space is finite dimensional. To this aim, we use the second-order projective derivatives associated to the second-order projective tangent sets to the graphs introduced by Penot. From the results obtained in this paper, we deduce and extend, in the special case some known results in scalar optimization and improve substantially the few results known in vector case.

Keywords Vector optimization · Second-order Hadamard directional derivative · Lagrange multipliers · Second-order tangent sets · Tangent derivatives of multi functions

1 Introduction

Investigation on optimality conditions has been one of the most attracting topics in the theory of vector optimization problems. First-order optimality conditions both for differentiable and nondifferentiable problems have been studied by many authors; for example we cite [5,7,18,21,22]. For many optimization problems, notably in mathematical programming, the characterization of optimal with the help of second order conditions, is always of a great interest in order to refine first-order conditions. Second-order optimality have been examined substantially less than first optimality conditions, for example we cite the works concerning

A. Taa (✉)
Département de Mathématiques, Faculté des Sciences et Techniques, B.P. 549, Marrakech, Morocco
e-mail: taa@fstg-marrakech.ac.ma

the second-order case where C^1 programs are considered [8,11], and the papers [12,16,24], where $C^{1,1}$ and C^2 programs are investigated. In the most papers where problems with C^1 data are treated the results are obtained in terms of approximate Hessians and their variants. Cambini et al. [4] obtained second-order optimality conditions in a finite dimensional space, expressed by means of Lagrange-Kuhn-Tucker multipliers by using suitable separation theorems. Jiménez and Novo in [13], consider a vector optimization problem where objective and constraints are defined by vector valued mappings and established new second-order optimality conditions in the primal form and in the dual form whenever the data of the problem are twice Fréchet differentiables, by using the asymptotic second-order tangent cone introduced by Penot [19]. The results established in [13] extend those by Penot [19] and some of those by Jourani [14] to the vector case. Note that in [14], by using the directional derivative, the second order lower Dini directional derivative and the second-order set valued derivative, Jourani obtained second-order necessary optimality conditions in the primal form for a point that minimizes a real valued function over a constraint set defined by a multifunction and an arbitrary subset. When the objective function is twice differentiable he established second-order necessary optimality conditions in the dual form. However, he did not established such optimality conditions with asymptotic second-order tangent cones.

Inspired and motivated by the works of [13,14] and [19], the purpose of this paper is to investigate second-order optimality conditions for the following general multiobjective optimization problem with respect to a cone Y^+

$$(VP) : \begin{cases} \text{Minimize } f(x) \\ \text{Subject to } x \in C \text{ and } 0 \in F(x), \end{cases}$$

where f is a mapping from a Banach space X into a Banach space Y, C is a nonempty subset of X and F is a multifunction from X into a Banach space Z and $Y^+ \subset Y$. Without differentiability nor convexity on the data of the problem (VP) and with a metric regularity condition and by using the second order Hadamard directional derivative, the second-order tangent sets, and the second-order tangent derivatives of multifunctions, the second-order necessary conditions for weakly efficient solutions of (VP) are given in the primal form. Under some additional assumptions and with the help of Robinson-Ursescu open mapping theorem (see [20,23]), we derive second-order necessary optimality conditions for (VP) in the dual form. The second-order sufficient conditions for strictly efficient solution of (VP) are investigated whenever the decision space is finite dimensional.

From the results obtained in this paper, we deduce and extend, in the special case $F(x) = -g(x)+D$ (see Sect. 3) some known results in scalar optimization (see for instance [6,14,17], and [19]), and improve substantially the few results known in vector case (see for instance [11–13], and [24].

The rest of the paper is written as follows. Section 2, contains basic definitions and preliminary results. Section 3 is devoted to the second-order optimality conditions for the problem (VP). Section 4 discusses second-order optimality conditions for the problem (VP) with $F(x) = -g(x) + D$.

2 Preliminary results

Let X, Y and Z be three Banach spaces, A a subset of X and Y^+ be a pointed ($Y^+ \cap -Y^+ = \{0\}$) closed convex cone with nonempty interior introducing a partial order in Y. We denote by B_X the closed unit ball centred at the origin of X and by $Int A$, $cl A$, $bd A$, respectively, the interior, the closure, and the boundary of A. Throughout the paper, X^*, Y^* and Z^* will denote the continuous duals of X, Y and Z respectively, and we write $\langle ., . \rangle$ for the canonical bilinear form with respect to the duality (X^*, X).

Consider a multifunction F from X into Y. In the sequel we denote the domain ($dom(F)$), the graph ($Gr(F)$) and the epigraph ($epi(F)$) of F respectively by

$$dom(F) := \{x \in X : F(x) \neq \emptyset\},$$
$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\},$$
$$epi(F) := \left\{(x, y) \in X \times Y : y \in F(x) + Y^+\right\}.$$

For an arbitrary nonempty subset B of Y we denote by $F+B$ the multifunction from X into Y defined by $(F+B)(x) = F(x)+B$. If V is a nonempty subset of X then $F(V) = \bigcup_{x \in V} F(x)$.

Let H be a nonempty subset of Z. For all $z^* \in Z^*$ the support function of H is

$$s(z^*, H) = \sup \left\{\langle z^*, z\rangle : z \in H\right\}.$$

Let A be a nonempty subset of Y. The element $\overline{y} \in A$ is said to be a Pareto (respectively, a weak Pareto) minimal point of A with respect to Y^+ if

$$(A - \overline{y}) \cap (-Y^+) = \{0\} \quad (respectively, (A - \overline{y}) \cap (-IntY^+) = \emptyset).$$

We shall denote by $Min_{Y^+}(A)$ the set of all Pareto minimal points of A and by $W.Min_{Y^+}(A)$ the set of all weak Pareto minimal points of A with respect to Y^+. It is easy to see that

$$Min_{Y^+}(A) \subset W.Min_{Y^+}(A).$$

As it has mentioned in the introduction,we are concerned with following general vector optimization problem

$$(VP) : \begin{cases} Minimize\ f(x) \\ Subject\ to\ x \in C\ and\ 0 \in F(x), \end{cases}$$

where f is a mapping from X into Y, C is a nonempty subset of X and F is a multifunction from X into Z. Let E denotes the set of all feasible points for the problem (VP), i.e.,

$$E := \{x \in X : x \in C \text{ and } 0 \in F(x)\}.$$

A point $\overline{x} \in E$ is a local (respectively, a weak local) efficient solution with respect to Y^+ of the problem (VP) if there exists a neighborhood V of $\overline{x}$ such that

$$f(\overline{x}) \in Min\ f(V \cap E) \quad (respectively,\ f(\overline{x}) \in W.Min\ f(V \cap E)).$$

For a closed cone S of Y, S^o will be the negative polar of S, that is

$$S^o := \left\{y^* \in Y^* :< y^*, y >\leq 0 \ \ \forall y \in S\right\}.$$

Let A be a nonempty convex subset of X the normal cone $N(A, a)$ to A at $a \in A$ is

$$N(A, a) = \left\{x^* \in X : \langle x^*, x - a\rangle \leq 0 \ for\ all\ x \in A\right\}.$$

Given a mapping $h : X \to Y$, the first-order Hadamard directional derivative $dh(\overline{x}; .)$ of h at $\overline{x} \in X$ in the direction v exists, if the following limit exists in Y

$$dh(\overline{x}; v) = \lim_{(t,w)\to(0^+,v)} t^{-1} \left[h(\overline{x} + tw) - h(\overline{x})\right].$$

When $dh(\overline{x}; v)$ exists for every $v \in X$, we say that h is first-order Hadamard directionally differentiable at $\overline{x}$. If $dh(\overline{x}; v)$ exists and if for a vector $w \in X$ there exists a point $d^2h(\overline{x}; v, w)$ of Y such that

$$d^2h(\overline{x}, v; w) = \lim_{(t,u)\to(0^+,w)} t^{-2}\left[h(\overline{x} + tv + t^2u) - h(\overline{x}) - tdh(\overline{x}; v)\right]$$

then we call $d^2h(\overline{x}; v, w)$ the second-order Hadamard directional derivative of h at $\overline{x}$ with respect to v in the direction w. When $d^2h(\overline{x}, v; w)$ exists, for every $w \in X$, we call h second-order Hadamard directionally differentiable at $\overline{x}$ with respect to v. When $d^2h(\overline{x}, v; w)$ exists, for every $(v, w) \in X \times X$ we call h second-order Hadamard directionally differentiable at $\overline{x}$. Moreover, if h is Lipschitzian at $\overline{x}$, then $dh(\overline{x}; v)$ and $d^2h(\overline{x}; v, w)$ respectively, can be reduced to

$$dh(\overline{x}; v) = \lim_{t\to0^+} t^{-1}\left[h(\overline{x} + tv) - h(\overline{x})\right],$$
$$d^2h(\overline{x}; v, w) = \lim_{t\to0^+} t^{-2}\left[h(\overline{x} + tv + t^2w) - h(\overline{x}) - tdh(\overline{x}; v)\right].$$

We say that a mapping $h : X \to Y$ is regular at $\overline{x}$ if the Hadamard directional derivative $dh(\overline{x}; v)$ exists at $\overline{x}$ in every direction v and satisfies

$$\lim_{(x,u,t)\to(\overline{x},v,0^+)} t^{-1}\left[h(x + tu) - h(x)\right] = dh(\overline{x}; v).$$

Moreover, if h is Lipschitzian at $\overline{x}$ then the latter can be reduced to

$$\lim_{(t,x)\to(0^+,\overline{x})} t^{-1}\left[h(x + tv) - h(x)\right] = dh(\overline{x}; v).$$

Proposition 2.1 *Let h be a mapping from X into Y and $\overline{x} \in X$. Suppose that h is Lipschitzian and regular at $\overline{x}$. If h is second-order Hadamard directionally differentiable at $\overline{x}$ with respect to $v \in X$. Then for all $w \in X$,*

$$d^2h(\overline{x}; v, w) = dh(\overline{x}; w) + d^2h(\overline{x}; v, 0).$$

Proof Let $w \in X$. By our assumption the mapping h is locally lipschitzian at $\overline{x}$, and so

$$d^2h(\overline{x}; v, w) = \lim_{t\to0^+} \left\{t^{-2}\left[h(\overline{x} + tv + t^2w) - h(\overline{x} + tv)\right] + t^{-2}\left[h(\overline{x} + tv) - h(\overline{x}) - tdh(\overline{x}; v)\right]\right\}.$$

By our assumptions one has

$$d^2h(\overline{x}; v, w) = dh(\overline{x}; w) + d^2h(\overline{x}; v, 0).$$

This completes the proof of Proposition 2.1. □

Let us recall the following concept. We say that a mapping $h : X \to Y$ is strictly differentiable at $\overline{x}$ if the Fréchet derivative $\nabla h(\overline{x})$ exists at $\overline{x}$ and satisfies

$$\lim_{(x,x')\to(\overline{x},\overline{x})} \|x - x'\|^{-1}\left[h(x) - h(x') - \nabla h(\overline{x})(x - x')\right] = 0.$$

If h is strictly differentiable at $\overline{x}$ then it is easy to see that h is Lipschitzian and regular at $\overline{x}$. Obviously, if h is continuously differentiable at $\overline{x}$ then h is strictly differentiable at $\overline{x}$.

Corollary 2.1 *Let h be a mapping from X into Y and $\overline{x} \in X$. Suppose that h is strictly differentiable at $\overline{x}$ and second-order Hadamard directionally differentiable at $\overline{x}$ with respect to $v \in X$. Then for all $w \in X$,*

$$d^2h(\overline{x}; v, w) = \nabla h(\overline{x})w + d^2h(\overline{x}; v, 0).$$

Proof The proof is a direct consequence of Proposition 2.1. □

The following Lemma will be used in Theorem 3.2.

Lemma 2.1 *Let $h : X \to Y$ be a mapping. Suppose that h is second-order Hadamard directionally differentiable at $\overline{x} \in X$ with respect to $v \in X$. Then the mapping $u \longmapsto d^2h(\overline{x}; v; u)$ is continuous.*

Proof Let $u_0 \in X$ and $\varepsilon > 0$. There exists $\alpha > 0$ and a neighborhood U_1 of zero in X such that

$$\|t^{-2}\left[h(\overline{x} + tv + t^2u) - h(\overline{x}) - tdh(\overline{x}; v)\right] - d^2h(\overline{x}; v; u_0)\| < \varepsilon$$

for all $t \in]0, \ \alpha]$ and $u \in u_0 + U_1$. Let U be any neighborhood of zero with $U + U \subset U_1$. Let $u \in u_0 + U$. Hence

$$\|t^{-2}\left[h(\overline{x} + tv + t^2u') - h(\overline{x}) - tdh(\overline{x}; v)\right] - d^2h(\overline{x}; v; u_0)\| < \varepsilon \tag{1}$$

for all $u' \in u + U$ and $t \in]0, \ \alpha]$. By taking the limit in (1) on t and u', we get

$$\|d^2h(\overline{x}; v; u) - d^2h(\overline{x}; v; u_0)\| \leq \varepsilon$$

for all $u \in u_0 + U$. This gives the result and the proof is complete. □

In this paper the following tangent sets will be used.

Definition 2.1 Let S be a nonempty subset of X, $\overline{x} \in S$ and $v \in X$.

(a) The first order contingent cone to S at $\overline{x}$ is

$$K(S, \overline{x}) = \left\{v \in X : \exists(t_n) \to 0^+, \exists(v_n) \to v \ such \ that \ \overline{x} + t_nv_n \in S \ \forall n \in \mathbb{N}\right\},$$

(b) The first order tangent cone to S at $\overline{x}$ is

$$k(S, \overline{x}) = \left\{v \in X : \forall(t_n) \to 0^+, \exists(v_n) \to v \ such \ that \ \overline{x} + t_nv_n \in S \ \forall n \in \mathbb{N}\right\},$$

(c) The second-order contingent set to S at $\overline{x}$ with respect to v is

$$K^2(S, \overline{x}, v) = \left\{\begin{array}{c} w \in X : \exists(t_n) \to 0^+, \exists(w_n) \to w \ such \ that \ \overline{x} + t_nv + t_n^2w_n \in S \\ \forall n \in \mathbb{N} \end{array}\right\},$$

(d) The second-order tangent set to S at $\overline{x}$ with respect to v is

$$k^2(S, \overline{x}, v) = \left\{\begin{array}{c} w \in X : \forall(t_n) \to 0^+, \exists(w_n) \to w \ such \ that \ \overline{x} + t_nv + t_n^2w_n \in S \\ \forall n \in \mathbb{N} \end{array}\right\},$$

(e) The asymptotic second-order contingent cone to S at $\overline{x}$ with respect to v is

$$K''(S, \overline{x}, v) = \left\{\begin{array}{c} w \in X : \exists(t_n, r_n) \to (0^+, 0^+) \ with \ t_nr_n^{-1} \to 0, \ \exists(w_n) \to w \\ such \ that \ \overline{x} + t_nv + r_nt_nw_n \in S \ \forall n \in \mathbb{N} \end{array}\right\}.$$

(f) The asymptotic second-order tangent cone to S at $\overline{x}$ with respect to v is

$$k''(S,\overline{x},v)=\left\{\begin{array}{c} w\in X:\forall(t_n,r_n)\to(0^+,0^+)\ with\ t_nr_n^{-1}\to 0,\exists(w_n)\to w \\ such\ that\ \overline{x}+t_nv+r_nt_nw_n\in S\ \forall n\in\mathbb{N}\end{array}\right\}.$$

(g) The interior tangent cone to S at $\overline{x}$ is

$$I(S,\overline{x})=\left\{\begin{array}{c} v\in X:\exists\delta>0\ such\ that\ x+tu\in S \\ \forall x\in(\overline{x}+\delta B_X)\cap S,\forall u\in v+\delta B_X,\forall t\in(0,\delta]\end{array}\right\}.$$

In fact the cones $K(S,\overline{x})$, $k(S,\overline{x})$, and $I(S,\overline{x})$, and the sets $K^2(S,\overline{x},v)$ and $k^2(S,\overline{x},v)$ are well-known. The cones $K''(S,\overline{x},v)$ and $k''(S,\overline{x},v)$ have been used by Penot [19] and Cambini et al. [3] in order to state optimality conditions in scalar optimization and $K''(S,\overline{x},v)$ used by Jiménez et al. [13] to establish second-order optimality conditions in vector optimization with twice differentiable data.

Remark 2.1 Let S be a nonempty subset of X, $\overline{x}\in S$ and $v\in X$.

(1) It is easy to see that
$k(S,\overline{x})\subset K(S,\overline{x}),\ k''(S,\overline{x},v)\subset K''(S,\overline{x},v)\ and\ k^2(S,\overline{x},v)\subset K^2(S,\overline{x},v)$.
(2) It is well known that the cone $I(S,\overline{x})$ is open and convex.
(3) The cones $K(S,\overline{x})$, $k(S,\overline{x})$, $K''(S,\overline{x},v)$ and $k''(S,\overline{x},v)$, and the sets $K^2(S,\overline{x},v)$ and $k^2(S,\overline{x},v)$ are closed. Moreover are convex whenever S is convex. In general the second-order contingent set (respectively, the second order tangent set) is not a cone.
(4) Note that if $v\notin K(S;\overline{x})$ (respectively, $v\notin k(S;\overline{x})$) then $K^2(S,\overline{x},v)=K''(S,\overline{x},v)=\emptyset$ (respectively, $k^2(S,\overline{x},v)=k''(S,\overline{x},v)=\emptyset$).
If $v=0$ then $K^2(S,\overline{x},v)=K''(S,\overline{x},v)=K(S;\overline{x})$ and $k^2(S,\overline{x},v)=k''(S,\overline{x},v)=k(S;\overline{x})$.

Definition 2.2 [19]. Let S be a nonempty subset of X, $\overline{x}\in S$ and $v\in X$. We denote by $\hat{K}^2(S,\overline{x},v)$ and $\hat{k}^2(S,\overline{x},v)$ respectively, the second-order projective contingent set and the second-order projective tangent set to S at $\overline{x}$ with respect to v defined by

$$\hat{K}^2(S,\overline{x},v)=\left\{\begin{array}{c}(w,r)\in X\times\mathbb{R}^+:\exists(t_n,r_n)\to(0^+,0^+),with\ t_nr_n^{-1}\to r,\exists(w_n)\to w \\ such\ that\ \overline{x}+t_nv+t_nr_nw_n\in S\ \forall n\in\mathbb{N}\end{array}\right\}.$$

$$\hat{k}^2(S,\overline{x},v)=\left\{\begin{array}{c}(w,r)\in X\times\mathbb{R}^+:\forall(t_n,r_n)\to(0^+,0^+),with\ t_nr_n^{-1}\to r,\exists(w_n)\to w \\ such\ that\ \overline{x}+t_nv+t_nr_nw_n\in S\ \forall n\in\mathbb{N}\end{array}\right\}.$$

Remark 2.2 One always has that

(i) $\hat{K}^2(S,\overline{x},v)=\mathbb{R}^+_*(K^2(S,\overline{x},v)\times\{1\})\cup K''(S,\overline{x},v)\times\{0\}$,
(ii) $\hat{k}^2(S,\overline{x},v)=\mathbb{R}^+_*(k^2(S,\overline{x},v)\times\{1\})\cup k''(S,\overline{x},v)\times\{0\}$,
where $\cup$ is the disjoint union and $\mathbb{R}^+_*:=\{x\in\mathbb{R}:x>0\}$.

Note that (i) of Remark 2.2 has been observed by Penot [19]. The proof of (ii)) is similar to that of (i).

Using Definitions 2.1 and 2.2 as point of departure, we introduce a similar concept for multifunctions.

Definition 2.3 Let G be a multifunction from X into Z with $(\overline{x},\overline{z})\in Gr(G)$ and let $x_1\in X$, $z_1\in Z$.

(a) The first-order contingent derivative of G at $(\overline{x},\overline{z})$ is the multifunction $D_KG(\overline{x},\overline{z})$ from X into Z defined by

$$Gr\ (D_KG(\overline{x},\overline{z}))=K(Gr(G),(\overline{x},\overline{z})).$$

(b) The second-order contingent derivative of G at $(\overline{x}, \overline{z})$ with respect to (x_1, z_1) is the multifunction $D^2_K G((\overline{x}, \overline{z}), (x_1, z_1))$ from X into Z defined by

$$Gr\left(D^2_K G((\overline{x}, \overline{z}), (x_1, z_1))\right) = K^2(Gr(G), (\overline{x}, \overline{z}), (x_1, z_1)).$$

(c) The asymptotic second-order contingent derivative of G at $(\overline{x}, \overline{z})$ with respect to (x_1, z_1) is the multifunction $D''_K G((\overline{x}, \overline{z}), (x_1, z_1))$ from X into Z defined by

$$Gr\left(D''_K G((\overline{x}, \overline{z}), (x_1, z_1))\right) = K''(Gr(G), (\overline{x}, \overline{z}), (x_1, z_1)).$$

(d) The second-order projective contingent derivative of G at $(\overline{x}, \overline{z})$ with respect to (x_1, z_1) is the multifunction $\hat{D}^2_K G((\overline{x}, \overline{z}), (x_1, z_1))$ from $X \times \mathbb{R}$ into Z defined by

$$z \in \hat{D}^2_K G((\overline{x}, \overline{z}), (x_1, z_1))(x, r) \Leftrightarrow ((x, y), r) \in \hat{K}^2(Gr(G), (\overline{x}, \overline{z}), (x_1, z_1)).$$

The proof of the following Lemma is a direct consequence of Definition 2.3

Lemma 2.2 *For all* $(x, r) \in X \times \mathbb{R}^+$

$$\hat{D}^2_K G((\overline{x}, \overline{z}), (x_1, z_1))(x, r) = \begin{cases} rD^2_K G((\overline{x}, \overline{z}), (x_1, z_1))(\frac{x}{r}) & if\ r \neq 0 \\ D''_K G((\overline{x}, \overline{z}), (x_1, z_1))(x) & if\ r = 0. \end{cases}$$

The following concept extends the one used in [13] *to the case of multifunctions: Let F be a multifunction from X into Y, $(\overline{x}, \overline{y}) \in Gr(F)$, and $v \in X$. One says that F is directionally metrically regular at $((\overline{x}, \overline{y}), v)$ with respect to a subset C of X, if there are two real numbers $a \geq 0$ and $r > 0$ such that*

$$(\mathbf{DMR}) \qquad d(\overline{x} + tu, F^{-1}(y) \cap C) \leq ad(y, F(\overline{x} + tu))$$

for all $t \in (0, r]$, $u \in v + rB_X$, with $\overline{x} + tu \in C$ and $y \in \overline{y} + rB_Y$, where $F^{-1}(y) = \{x \in X : y \in F(x)\}$. Here, we adopt the convention $d(x, \emptyset) = +\infty$.

Let us observe that condition (***DMR***) *is a consequence of the following well known metric regularity condition:* (***MR***) *there exist $a \geq 0$ and $r > 0$ such that*

$$d(x, F^{-1}(y) \cap C) \leq ad(y, F(x))$$

for all $x \in (\overline{x} + rB_X) \cap C$ and $y \in \overline{y} + rB_Y$. For more details about (***MR***) *(see* [1]*).*

The following concept is well known: We say that a multifunction H from X into Y is pseudo-Lipschitzian at $(\overline{x}, \overline{y}) \in Gr(H)$ if there are $k_H \geq 0$ and $r > 0$ such that

$$H(x) \cap (\overline{y} + rB_Y) \subset H(x') + k_H\|x - x'\|B_X, \ \forall x, x' \in \overline{x} + rB_X.$$

One can easily prove the following Proposition.

Proposition 2.2 *Let a multifunction F from X into Z, $\overline{x} \in E$ and $v \in X$. Then following conditions hold*

(a) $K^2(E, \overline{x}, v) \subset \left\{w \in K^2(C, \overline{x}, v) : 0 \in D^2_K \ F((\overline{x}, 0), (v, 0))(w)\right\}$,

(b) $K''(E, \overline{x}, v) \subset \left\{w \in K''(C, \overline{x}, v) : 0 \in D''_K F((\overline{x}, 0), (v, 0))(w)\right\}$,

(c) $\hat{K}^2(E, \overline{x}, v) \subset \left\{(w, r) \in \hat{K}(C, \overline{x}, v) : 0 \in \hat{D}^2_K F((\overline{x}, 0), (v, 0))(w, r)\right\}$.

The following proposition will be used in the next section.

Proposition 2.3 *Let $\overline{x} \in E$ and $v \in X$. Suppose that a multifunction F from X into Z is pseudo-Lipschitzian at $(\overline{x}, 0)$ and directionally metrically regular at $((\overline{x}, 0), v)$ with respect to C. Then the following conditions hold*

(a) $\left\{w \in k^2(C, \overline{x}, v) : 0 \in D^2_K F((\overline{x}, 0), (v, 0))(w)\right\} \subset K^2(E, \overline{x}, v)$,
(b) $\left\{w \in k''(C, \overline{x}, v) : 0 \in D''_K F((\overline{x}, 0), (v, 0))(w)\right\} \subset K''(E, \overline{x}, v)$,
(c) $\left\{(w, r) \in \hat{k}(C, \overline{x}, v) : 0 \in \hat{D}^2_K F((\overline{x}, 0), (v, 0))(w, r)\right\} \subset \hat{K}^2(E, \overline{x}, v)$,
(d) $\left\{w \in K^2(C, \overline{x}, v) : 0 \in D^2_k F((\overline{x}, 0), (v, 0))(w)\right\} \subset K^2(E, \overline{x}, v)$,
(e) $\left\{w \in K''(C, \overline{x}, v) : 0 \in D''_k F((\overline{x}, 0), (v, 0))(w)\right\} \subset K''(E, \overline{x}, v)$,
(f) $\left\{(w, r) \in \hat{K}(C, \overline{x}, v) : 0 \in \hat{D}^2_k F((\overline{x}, 0), (v, 0))(w, r)\right\} \subset \hat{K}^2(E, \overline{x}, v)$.

Proof (a) Let $w \in \left\{w \in k^2(C, \overline{x}, v) : 0 \in D^2_K F((\overline{x}, 0), (v, 0))(w)\right\}$.There exist sequences $(t_n) \to 0^+, (w_n) \to w, (w'_n) \to w$ and $(z_n) \to 0$ such that $\overline{x} + t_n v + t_n^2 w_n \in C$ and $t_n^2 z_n \in F(\overline{x} + t_n v + t_n^2 w'_n)$ for all $n \in \mathbb{N}$. From the directional metric regularity condition of F at $((\overline{x}, 0), v)$ with respect to C one can exhibit $a \geq 0$ such that for n large enough

$$d(\overline{x} + t_n v + t_n^2 w_n, E) \leq ad(0, F(\overline{x} + t_n v + t_n^2 w'_n).$$

Since F is pseudo-Lipschitzian at $(\overline{x}, 0)$, it follows that

$$d(\overline{x} + t_n v + t_n^2 w_n, E) \leq a(\|t_n^2 z_n\| + k_F t_n^2 \|w_n - w'_n\|) + t_n^3.$$

Therefore for n large enough there exists $u_n \in Z$ such that $\overline{x} + t_n v + t_n^2 u_n \in E$ and

$$\|w_n - u_n\| < a(\|z_n\| + k_F \|w_n - w'_n\|) + t_n,$$

so $(u_n) \to w \in K^2(E, \overline{x}, v)$. The proof of (b), (c), (d), (e) and (f) are similar to that of (a). □

3 Second-order optimality conditions in the general case

Inspired and motivated by the works of Jourani [14], Jimé nez and Novo [13], and Penot [19], the purpose of this section is to establish second-order conditions for local weak efficient solutions of the problem (VP) in the primal form and in the dual form. To this end we start with some lemmas. In what follows the multifunction (F, G) from X into $Y \times Z$ is defined by

$$(F, G)(x) = F(x) \times G(x) \quad for\ all\ x \in X.$$

and we adopt the convention $F(\emptyset) = \emptyset$.

In the next of the paper, we say that $v \in X$ is a critical direction for (VP) at $\overline{x} \in X$ if $v \in K(E, \overline{x})$ and $df(\overline{x}; v) \in bd(-Y^+)$; and we write $v \in K(\overline{x})$.

The proof of the following lemma uses some ideas of [13].

Lemma 3.1 *Let $\overline{x}$ be a local weak efficient solution of (VP) with respect to Y^+ and f be second-order Hadamard directionally differentiable at $\overline{x}$. Then*

(a) $cl\left[(df(\overline{x}; .) + IntY^+)(K(E, \overline{x}))\right] \cap -IntY^+ = \emptyset$,
(b) $cl\left[(d^2 f(\overline{x}; v, .) + IntY^+)(K^2(E, \overline{x}, v))\right] \cap IntK(-Y^+, df(\overline{x}; v)) = \emptyset$, *for all* $v \in K(\overline{x})$.
Moreover, if f is regular and Lipschitzian at $\overline{x}$ then the following statement hold
(c) $cl\left[(df(\overline{x}; .) + IntY^+)(K''(E, \overline{x}, v))\right] \cap IntK(-Y^+, df(\overline{x}; v)) = \emptyset$, *for all* $v \in K(\overline{x})$.

Proof Since (a) follows from (b), it suffices to prove (b) and (c). (b) Suppose on the contrary that there is $y \in cl\left[(d^2 f(\overline{x}; v, .) + IntY^+)(K^2(E, \overline{x}, v))\right]$ with $y \in IntK(-Y^+, df(\overline{x}; v))$. Choose sequences $(x_n) \subset K^2(E, \overline{x}, v)$ and $(y_n) \to y$ such that

$$y_n - d^2 f(\overline{x}; v, x_n) \in IntY^+, \quad for\ all\ n \in \mathbb{N}. \tag{2}$$

By definition of $K^2(E, \overline{x}, v)$, for each $n \in \mathbb{N}$ there exist sequences $(t_{n,m})_{m\in\mathbb{N}} \to 0^+$ and $(x_{n,m})_{m\in\mathbb{N}} \to x_n$ such that $\overline{x} + t_{n,m}v + t_{n,m}^2 x_{n,m} \in E$ for all $m \in \mathbb{N}$. By (2), for each $n \in \mathbb{N}$ there exists an integer $m_0(n) \geq 0$ such that for all $m \geq m_0(n)$

$$f(\overline{x} + t_{n,m}v + t_{n,m}^2 x_{n,m}) - f(\overline{x}) - t_{n,m}df(\overline{x}; v) - t_{n;m}^2 y_n \in -IntY^+. \tag{3}$$

Since $IntY^+ \neq \emptyset$, then by Proposition 2.4 of [13] one has

$$IntK(-Y^+, df(\overline{x}; v)) = I(-IntY^+, df(\overline{x}; v)).$$

Hence there exists $\delta > 0$ such that

$$df(\overline{x}; v) + tz \in -IntY^+, \quad for\ all\ t \in (0, \delta]\ and\ z \in y + \delta B_Y.$$

consequently there exists an integer $n_0 \geq 0$ such that for all $n \geq n_0$ one has $y_n \in y + \delta B_Y$, and hence for each $n \geq n_0$ there exists an integer $m_1(n)$ such that for all $m \geq m_1(n)$ we obtain $t_{n,m} \in (0, \delta]$. Therefore for each $n \geq n_0$ one has $t_{n,m}df(\overline{x}, v) + t_{n,m}^2 y_n \in -IntY$ for all $m \geq \max(m_1(n), m_0(n))$. Thus (3) can be written as

$$f(\overline{x} + t_{n,m}v + t_{n,m}^2 x_{n,m}) - f(\overline{x}) \in -IntY^+ \quad for\ all\ m \geq \max(m_1(n), m_0(n)),$$

which contradicts that $\overline{x}$ is a local weak efficient solution of (VP) with respect to Y^+.

(c) Suppose on the contrary that there is $y \in cl\left[(df(\overline{x}; .) + IntY^+)(K''(E, \overline{x}, v))\right]$ with $y \in IntK(-Y^+, df(\overline{x}; v))$. There exist sequences $(x_n) \subset K''(E, \overline{x}, v)$ and $(y_n) \to y$ such that

$$y_n - df(\overline{x}; x_n) \in IntY^+ \quad for\ all\ n \in \mathbb{N}. \tag{4}$$

By definition of $K''(E, \overline{x}, v)$, for each $n \in \mathbb{N}$ there exist sequences $(t_{n,m}, r_{n,m})_{m\in\mathbb{N}} \to 0^+$ and $(x_{n,m})_{m\in\mathbb{N}} \to x_n$, with $\left(\frac{t_{n,m}}{r_{n,m}}\right) \to 0^+$ such that $\overline{x} + t_{n,m}v + t_{n,m}r_{n,m}x_{n,m} \in E$ for all $m \in \mathbb{N}$. Since f is regular at $\overline{x}$ one has

$$\lim_{m\to\infty}(t_{n,m}r_{n,m})^{-1}\left[f(\overline{x} + t_{n,m}v + t_{n,m}r_{n,m}x_{n,m}) - f(\overline{x} + t_{n,m}v)\right] = df(\overline{x}; x_n). \tag{5}$$

Since $\left(\frac{t_{n,m}}{r_{n,m}}\right) \to 0^+$, it follows with the Lipschitzity and the second-order Hadamard directional differentiability of f at $\overline{x}$ that

$$\lim_{m\to\infty}\left(t_{n,m}r_{n,m}\right)^{-1}\left[f(\overline{x} + t_{n,m}v) - f(\overline{x}) - t_{n,m}df(\overline{x}; v)\right] = 0. \tag{6}$$

By (4), (5) and (6), then for each $n \in \mathbb{N}$, we conclude for m large enough that

$$f(\overline{x} + t_{n,m}v + t_{n,m}r_{n,m}x_{n,m}) - f(\overline{x}) - \left(t_{n,m}df(\overline{x}; v) - t_{n,m}r_{n,m}y\right) \in -IntY^+.$$

Since $IntK(-Y^+, df(\overline{x}; v)) = I(-IntY^+, df(\overline{x}; v))$, then by the arguments similar to that of (b) we obtain a contradiction. □

Remark 3.1 (1) Part (b) and (c) of Lemma 3.1 are valid for all $v \in X$, but is only meaningful for $v \in K(\overline{x})$, since if $v \notin K(E, \overline{x})$ then $K^2(E, \overline{x}, v) = K''(E, \overline{x}, v) = \emptyset$ (see Remark 2.1) and if $df(\overline{x}, v) \notin -Y^+$ then $K(-Y^+, df(\overline{x}, v)) = \emptyset$. Finally, if $df(\overline{x}, v) \in -IntY^+$ then by (i) one has $v \notin K(E, \overline{x})$.

(2) Suppose that f is regular, second-order Hadamard directionally differentiable and Lipschitzian at $\overline{x}$. Using Remark 2.2 and Proposition 2.1, it is easy to prove that if the conditions (b) and (c) of Lemma 3.1 hold then the following hold and vice versa

$$cl\left[(A+IntY^{+})(\hat{K}^{2}(E,\overline{x},v))\right]\cap IntK(-Y^{+},df(\overline{x};v))=\emptyset,$$

where $A(w,r):=df(\overline{x},w)+rd^{2}f(\overline{x};v,0)$ for all $(w,r)\in X\times\mathbb{R}^{+}$.

Now, we are able to state the primal second-order necessary optimality conditions for (VP). The following result extends Theorem 4.2 in [13] to the case of nondifferentiable vector optimization problems with inclusion constraints, and that of [14] to the vector case.

Theorem 3.1 *Let $\overline{x}$ be a local weak efficient solution of (VP) with respect to Y^{+} and $v\in K(\overline{x})$. Suppose that*

(1) *f is second-order Hadamard directionally differentiable at $\overline{x}$.*
(2) *the multifunction F is pseudo-Lipschitzian at $(\overline{x},0)$ and directionally metrically regular at $((\overline{x},0),v)$ with respect to C.*
Then the following system has no solution $x\in X$

$$\text{(a)}\quad\begin{cases} d^{2}f(\overline{x};v;x)\in IntK(-Y^{+},df(\overline{x};v))\\ x\in k^{2}(C,\overline{x},v),\ 0\in D_{K}^{2}F(\overline{x},0)(v,0)(x).\end{cases}$$

Moreover, suppose that
(3) *f is regular and Lipschitzian at $\overline{x}$.*
Then the following system has no solution $x\in X$

$$\text{(b)}\quad\begin{cases} df(\overline{x};x)\in IntK(-Y^{+},df(\overline{x};v)\\ x\in k''(C,\overline{x},v),\ \ 0\in D_{K}''F(\overline{x},0)(v,0)(x).\end{cases}$$

Proof (a) Suppose on the contrary that there is $x\in k^{2}(C,\overline{x},v)$ with $0\in D_{K}^{2}F((\overline{x},0),(v,0))(x)$ such that $d^{2}f(\overline{x};v,x)\in IntK(-Y^{+},df(\overline{x},v))$. By Proposition 2.3 and Lemma 3.1 we have a contradiction since $d^{2}f(\overline{x};v,x)\in cl\left[(d^{2}f(\overline{x};v,.)+IntY^{+})(K^{2}(E,\overline{x},v))\right]$, where E is the feasible set of the problem (VP) defined in Sect. 2. The proof of (b) is similar to that of (a). □

Remark 3.2 Let the assumptions (1) and (3) of Theorem 3.1 be satisfied. Using Remark 2.2, Proposition 2.1 and Lemma 2.2, it is easy to see that if the systems (a) and (b) of Theorem 3.1 in $x\in X$ are incompatible then the following system in $(x,r)\in X\times\mathbb{R}^{+}$ is incompatible and vice versa

$$\begin{cases} df(\overline{x};x)+rd^{2}f(\overline{x};v;0)\in IntK(-Y^{+},df(\overline{x};v))\\ (x,r)\in\hat{k}^{2}(C,\overline{x},v),0\in\hat{D}_{K}^{2}F((\overline{x},0),(v,0))(x,r),\end{cases}$$

where $\mathbb{R}^{+}=\{x\in\mathbb{R}:x\geq 0\}$.

Let us turn to sufficient optimality conditions of the problem (VP). To this end we start by the following recalling appeared in [13]: Let $m\geq 1$ be an integer. The point $\overline{x}\in E$ is said to be a strict local efficient solution of order m for problem (VP), if there exist $\alpha>0$ and a neighborhood U of $\overline{x}$ such that

$$(f(x)+Y^{+})\cap\left(f(\overline{x})+\alpha\|x-\overline{x}\|^{m}IntB_{Y}\right)=\emptyset,\qquad\forall x\in(E\cap U)\backslash\{\overline{x}\}.$$

It is easy to see that every strict local efficient solution of order m is a local efficient solution.

In what follows for $v\in X$ we denote by $v^{\perp}$ the set defined by $v^{\perp}:=\{x\in X:\langle v,x\rangle=0\}$.

Lemma 3.2 *Let $\overline{x} \in E$, X is finite dimensional and S_X denotes the unit sphere of X. Let the assumptions* (1) *and* (3) *of Theorem* 3.1 *be satisfied. Suppose that for each $v \in K(E, \overline{x}) \setminus \{0\}$ satisfying $df(\overline{x}; v) \in -Y^+$ and that two following conditions hold*

(a) $cl\left[(df(\overline{x}; .) + d^2 f(\overline{x}; v; 0) + IntY^+)(K^2(E, \overline{x}, v) \cap v^\perp)\right] \cap K(-Y^+, df(\overline{x}; v)) = \emptyset,$
(b) $cl\left[(df(\overline{x}; .) + IntY^+)(K''(E, \overline{x}, v) \cap v^\perp \setminus \{0\})\right] \cap \left(K(-Y^+, df(\overline{x}; v)\right) = \emptyset.$
Then $\overline{x}$ is a strict local efficient solution of order two for (VP).

Proof Suppose on the contrary that for all $n \in \mathbb{N}$, there exist $x_n \in E \cap (\overline{x} + 1/(n+1) B_X) \setminus \{\overline{x}\}$ and $b_n \in IntB$ such that

$$f(x_n) - f(\overline{x}) - 1/(n+1) \|x_n - \overline{x}\|^2 b_n \in -Y^+, \forall n \in \mathbb{N}. \tag{7}$$

Let $t_n = \|x_n - \overline{x}\|$, $v_n = t_n^{-1}(x_n - \overline{x})$. Since X is finite dimensional, by taking a subsequence if necessary we may suppose that (v_n) has a limit $v \in K(E, \overline{x})$ with $\|v\| = 1$. By (7), it follows that $df(\overline{x}, v) \in -Y^+$. Consider now, the sequence $(w_n)_{n \in \mathbb{N}}$ such that $x_n = \overline{x} + t_n v + t_n^2 w_n$ for all $n \in \mathbb{N}$. We have two cases. The case (a): The sequence (w_n) is bounded. Observe that $v + t_n w_n \in S_X$ for all $n \in \mathbb{N}$. Since S_X is compact, passing to a subsequence, we may assume that $(w_n) \to w \in K^2(E, \overline{x}, v) \cap K(S_X, v)$. By (7) one has

$$\begin{aligned}&\lim_{n \to +\infty} (t_n^{-2}\left[f(\overline{x} + t_n v + t_n^2 w_n) - f(\overline{x} + t_n v)\right] + t_n^{-2}\left[f(\overline{x} + t_n v) - f(\overline{x}) - t_n df(\overline{x}; v)\right]\\ &\quad - (1/(n+1))\, b_n)\\ &\quad \in K(-Y^+, df(\overline{x}; v)).\end{aligned}$$

By our assumptions we obtain

$$df(\overline{x}; w) + d^2 f(\overline{x}; v, 0) \in K(-Y^+, df(\overline{x}; v)),$$

which is a contradiction with our assumption since $K(S_X, v) = v^\perp$ and

$$\begin{aligned}&df(\overline{x}; w) + d^2 f(\overline{x}; v, 0) \in cl\\ &\left[(df(\overline{x}; .) + d^2 f(\overline{x}; v, 0) + IntY^+)(K^2(E, \overline{x}, v) \cap K(S_X, v))\right].\end{aligned}$$

The case (b): The sequence (w_n) is not bounded. We may suppose that $(\|w_n\|) \to +\infty$ (taking a subsequence if necessary). Then the sequence (x_n) can be written as

$$x_n = \overline{x} + t_n v + t_n^2 \|w_n\| (w_n \|w_n\|^{-1}). \tag{8}$$

Observe that, $t_n \|w_n\| = \|v_n - v\| \to 0^+$ and $v + t_n^2 \|w_n\| (w_n \|w_n\|^{-1}) \in S$ for all $n \in \mathbb{N}$. Since S_X is compact, without loss of generality we may suppose that $(w_n \|w_n\|^{-1}) \to w$. Then $w \in K''(E, \overline{x}, v) \cap v^\perp \setminus \{0\}$. By (7) and (8) we have

$$\begin{aligned}&\lim_{n \to +\infty} (t_n^{-2} \|w_n\|^{-1}\left[f(\overline{x} + t_n v + t_n^2 \|w_n\| (w_n \|w_n\|^{-1})) - f(\overline{x} + t_n v)\right]\\ &\quad + t_n^{-2} \|w_n\|^{-1}\left[f(\overline{x} + t_n v) - f(\overline{x}) - t_n df(\overline{x}; v)\right] - (1/(n+1))\, b_n \|w_n\|^{-1})\\ &\quad \in K(-Y^+, df(\overline{x}; v)).\end{aligned}$$

By our assumptions we get $df(\overline{x}; w) \in K(-Y^+, df(\overline{x}; v))$, which contradicts our assumption since

$$df(\overline{x}; w) \in cl\left[(df(\overline{x}; .) + IntY^+)(K''(E, \overline{x}, v) \cap v^\perp\right].$$

□

Now, we can state second-order sufficient optimality conditions of the problem (VP) in the primal form. The following result extends Theorem 4.8 of [13] to the case of nondifferentiable vector optimization problems with inclusion constraints.

Proposition 3.1 *Let the assumptions of Lemma 3.2 be satisfied. If for any* $v \in K(E, \overline{x}) \setminus \{0\}$ *satisfying* $df(\overline{x}, v) \in -Y^+$ *and if the following systems in* $x \in X$ *are incompatible*

$$\text{(a)} \quad \begin{cases} df(\overline{x}; x) + d^2 f(\overline{x}; v; 0) \in K(-Y^+, df(\overline{x}; v)) \\ x \in K^2(C, \overline{x}, v) \cap v^\perp,\ 0 \in D^2_K F((\overline{x}, 0), (v, 0))(x), \end{cases}$$

$$\text{(b)} \quad \begin{cases} df(\overline{x}; x) \in K(-Y^+, df(\overline{x}; v)), \\ x \in K''(C, \overline{x}, v) \cap v^\perp \setminus \{(0\},\ 0 \in D''_K F((\overline{x}, 0), (v, 0))(x) \end{cases}$$

then $\overline{x}$ is a strict local efficient solution of order two for (VP).

Proof Observe that $df(\overline{x}; x) + d^2 f(\overline{x}; v; 0) \in cl\left[df(\overline{x}; x) + d^2 f(\overline{x}; v; 0) + IntY^+\right]$ and that $df(\overline{x}; x) \in cl(df(\overline{x}; x) + IntY^+)$. As $K^2(E, \overline{x}, v) \subset \{x \in K^2(C, \overline{x}, v) : 0 \in D^2_K F((\overline{x}, 0), (v, 0))(x)\}$ and $K''(E, \overline{x}, v) \subset \left\{x \in K''(C, \overline{x}, v) : 0 \in D''_K F((\overline{x}, 0), (v, 0))(x)\right\}$. The proof follows by Lemma 3.2. □

Remark 3.3 Let the assumptions of Proposition 3.2 be satisfied. Using Remark 2.2 and Lemma 2.2 it is easy to shake that if the systems (a) and (b) of Proposition 3.1 are incompatible in $x \in X$ then the following system in $(x, r) \in X \times \mathbb{R}^+$ is incompatible and vice versa

$$\begin{cases} df(\overline{x}; x) + rd^2 f(\overline{x}; v; 0) \in K(-Y^+, df(\overline{x}; v)) \\ (x, r) \in \hat{K}^2(C, \overline{x}, v) \cap v^\perp \times \mathbb{R}^+ \setminus \{(0, 0)\},\ 0 \in \hat{D}^2_K F(\overline{x}, 0), (v, 0))(x, r). \end{cases}$$

The following lemmas prepare dual second-order necessary optimality conditions of (VP).

Lemma 3.3 *Let* $\overline{x}$ *be a local weak efficient solution of* (VP) *with respect to* Y^+ *and* $v \in K(\overline{x})$. *Let the assumptions* (1) *and* (2) *of Theorem 3.1 be satisfied. Then*

(a) $(d^2 f(\overline{x}; v, .) + IntY^+, D^2_K F((\overline{x}, 0), (v, 0)))(k^2(C, \overline{x}, v)) \cap IntK(-Y^+, df(\overline{x}; v)) \times \{0\} = \emptyset$.
Moreover, suppose that the assumption (3) of Theorem 3.1 is satisfied. Then
(b) $\left(df(\overline{x}; .) + IntY^+, D''_K F((\overline{x}, 0), (v, 0))(k''(C, \overline{x}, v)\right) \cap \left(IntK(-Y^+, df(\overline{x}; v)) \times \{0\}\right) = \emptyset$.

Proof (a) Let $v \in K(\overline{x})$. Suppose on the contrary that there are $y \in Y$ and $x \in k^2(C, \overline{x}, v)$ such that

$$y \in (d^2 f(\overline{x}; v, .) + IntY^+)(x) \cap IntK(-Y^+, df(\overline{x}; v)) \text{ and } 0 \in D^2_K F((\overline{x}, 0), (v, 0))(x).$$

This implies with Proposition 2.3 that $x \in K^2(E, \overline{x}, v)$. Thus the condition (a) follows by Lemma 3.1. The proof of (b) is similar to that of (a). □

Remark 3.4 Let the assumptions (1) and (3) of Theorem 3.1 be satisfied. Using Remark 2.1, Proposition 2.1 and Lemma 2.2, it is easy to prove that if (a) and (b) of Lemma 3.3 hold, then the following holds and vice versa

$$(A + IntY^+, \hat{D}^2_K F((\overline{x}, 0), (v, 0)))(\hat{k}^2(C, \overline{x}, v)) \cap \left(IntK(-Y^+, df(\overline{x}; v)) \times \{0\}\right) = \emptyset$$

where $A(x, r) := df(\overline{x}; x) + rd^2 f(\overline{x}; v, 0)$ for all $(x, r) \in X \times \mathbb{R}^+$.

The following concept will be used: For a convex subset B of Y, we recall that the $Core$ or algebraic interior of B, denoted by $CoreB$, is the subset of B denoted by

$$CoreB := \{b \in B : \forall y \in Y, \exists \varepsilon > 0, b + [-\varepsilon, \varepsilon]\, y \subseteq B\}.$$

It satisfies the following relations

$$IntB \neq \emptyset \Rightarrow IntB = CoreB \text{ and } 0 \in CoreB \Leftrightarrow \mathbb{R}^+ B = Y.$$

For more details about this notion see for instance [10] and [20].

Lemma 3.4 *Let $h : X \to Y$ be a continuous mapping and $H : X \to Z$ be a multifunction with closed convex graph of $X \times Z$ and S be a nonempty closed convex subset of X. If $0 \in coreH(S)$ then the set $(h + IntY^+, H)(S)$ has nonempty interior.*

Proof Let $d_0 \in IntY^+$. Then there exists $\varepsilon > 0$ such that for all $d_1, d_2 \in \varepsilon IntB_Y$

$$d_0 + d_1 + d_2 \in IntY^+. \tag{9}$$

Let $x_0 \in S$ with $0 \in H(x_0)$. By the continuity of h at x_0 there exists a neighborhood U_X of zero such that for all $x \in x_0 + U_X$

$$h(x) \in h(x_0) + \varepsilon IntB_Y.$$

Using relation (9), we obtain for all $x \in x_0 + U_X$ and $d_2 \in \varepsilon IntB_Y$ that

$$d_0 + h(x_0) + d_2 - h(x) \in IntY^+. \tag{10}$$

By the Robinson -Ursescu open mapping theorem (see [20,23]) there exists $r > 0$ such that

$$rB_Z \subset H\left((x_0 + U_X) \cap S\right).$$

Let $z \in rB_Z$. There exists $x_1 \in (x_0 + U_X) \cap S$ such that $z \in H(x_1)$, and by (10) we get

$$(d_0 + h(x_0) + d_2, z) \in (h(x_1) + IntY^+, H)(x_1),$$

and this completes the proof of Lemma 3.4. □

Remark 3.5 Let $\gamma > 1$. Lemma 3.4 remains valid if, instead of convexity of H we consider that H^{-1} is γ -paraconvex (for definition and properties of this notion see Jourani [15]). Indeed, by Jourani [15], the set-valued mapping H is open at any point $(x_0, 0) \in Gr(H)$ with $x_0 \in S$.

We can now state dual second-order necessary optimality conditions of (VP) in terms of Lagrange-Kuhn-Tucker multipliers. The following Theorem 3.2, improves and extends for instance the results in [6,14,17] to the vector case and those in [13,24] to the case of nondifferentiable vector optimization problems with inclusion constraints. The proof used by [13] and [24] is different and depends heavily on the use of Farkas lemma.

Theorem 3.2 *Let $\overline{x}$ be a local weak efficient solution of (VP) with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions (1) and (2) of Theorem 3.1 be satisfied. Moreover suppose that*

(i) *The subset C is convex,*
(ii) *The set valued map $D_K^2 F((\overline{x}, 0), (v, 0))$ has a convex graph on $X \times Z$,*
(iii) $0 \in coreD_K^2 F((\overline{x}, 0), (v, 0))(k^2(C, \overline{x}, v))$,
(iv) $(d^2 f(\overline{x}; v, .) + IntY^+, D_K^2 F((\overline{x}, 0), (v, 0)))(k^2(C, \overline{x}, v))$ *is convex.*

Then there exists $(y^*, z^*) \in (-Y^+)^o \times Z^*$ with $y^* \neq 0$ such that $\langle y^*, df(\overline{x}; v)\rangle = 0$ and

$$\langle y^*, d^2 f(\overline{x}; v, x)\rangle \geq \langle z^*, z\rangle, \qquad for\ all\ x \in k^2(C, \overline{x}, v)\ and\ z \in D_K^2 F((\overline{x}, 0), (v, 0))(x).$$

Proof From Lemmas 2.1, 3.3, 3.4 and the standard Hahn-Banach separation theorem there exists $(y^*, u^*) \in Y^* \times Z^* \setminus \{(0,0)\}$ such that

$$\langle y^*, d^2 f(\overline{x}; v, x) + d\rangle + \langle u^*, z\rangle \geq \langle y^*, y\rangle \tag{11}$$

for all $x \in k^2(C, \overline{x}, v), d \in IntY^+, z \in D^2_K F((\overline{x}, 0), (v, 0))(x)$ and $y \in IntK(-Y^+, df(\overline{x}; v))$. Hence

$$y^* \in (-Y^+)^o \cap N(-Y^+, df(\overline{x}, v)).$$

As $df(\overline{x}; v) \in -Y^+$, we get $\langle y^*, df(\overline{x}; v)\rangle = 0$. Since $0 \in K(-Y^+, df(\overline{x}; v)) \cap Y^+$ we conclude from (11) with $z^* = -u^*$ that

$$\langle y^*, d^2 f(\overline{x}; v, x)\rangle \geq \langle z^*, z\rangle \tag{12}$$

for all $x \in k^2(E, \overline{x}, v)$ and $z \in D^2_K F((\overline{x}, 0), (v, 0))(x)$. Now, suppose that $y^* = 0$. From (12) one has $\langle z^*, z\rangle \leq 0$ for all $x \in k^2(C, \overline{x}, v)$ and $z \in D^2_K F((\overline{x}, 0), (v, 0))(x)$. By the Robinson-Ursescu open mapping theorem (see [20] and [23]) for a certain $x_0 \in k^2(C; \overline{x}, v)$ with $0 \in D^2_K F((\overline{x}, 0), (v, 0))(x_0)$ there exists $r > 0$ such that

$$rB_Z \subset D^2_k F((\overline{x}, 0), (v, 0)) \left((x_0 + B_X) \cap k^2(C; \overline{x}, v)\right).$$

Then $\langle z^*, z\rangle \leq 0$ for all $z \in rB_Z$. Thus $z^* = 0$, which is a contradiction. □

The following Theorem 3.3 extends for instance Theorem 5.3 of [13] to the case of non-differentiable vector optimization problems with inclusion constraints. The proof used by [13] is different and depends heavily on the use of Farkas lemma.

Theorem 3.3 *Let $\overline{x}$ be a local weak efficient solution of (VP) with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions* (1), (2) *and* (3) *of Theorem 3.1 be satisfied. Moreover, suppose that*

(i) *The subset C is convex,*
(ii) *The set-valued map $D''_K F((\overline{x}, 0), (v, 0))$ has a convex graph on $X \times Z$,*
(iii) $0 \in core D''_K F((\overline{x}, 0), (v, 0))(k''(C, \overline{x}, v))$,
(iv) $(df(\overline{x}; .) + IntY^+, D''_K F((\overline{x}, 0), (v, 0)))(k''(C, \overline{x}, v))$ *is convex.*

Then there exist $(y^*, z^*) \in (-Y^+)^o \times Z^*$ with $z^* \neq 0$ such that $\langle y^*, df(\overline{x}, v)\rangle = 0$ and

$$\langle y^*, df(\overline{x}; x) \geq \langle z^*, z\rangle, \ \ for\ all\ x \in k''(E, \overline{x}, v)\ and\ z \in D''_K F((\overline{x}, 0), (v, 0)))(x).$$

Proof The proof follows along the same lines as the proof of Theorem 3.2. □

We may combine Theorem 3.2 and 3.3 into a single result. Using Remark 3.4, the proof of the following Theorem is similar to that of Theorem 3.2 and therefore is omitted.

Theorem 3.4 *Let $\overline{x}$ be a local weak efficient solution of (VP) with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions* (1), (2) *and* (3) *of Theorem 3.1 be satisfied. Moreover, suppose that.*

(i) *The subset C is convex,*
(ii) *The set-valued map $\hat{D}^2_K F((\overline{x}, 0), (v, 0))$ has a convex graph on $X \times Z$,*
(iii) $0 \in core\hat{D}^2_K (Gr(F), (\overline{x}, 0), (v, 0))((\hat{k}^2(C, \overline{x}, v))$
(iv) $(A + IntY^+, \hat{D}^2_K F((\overline{x}, 0), (v, 0)))(\hat{k}^2(C, \overline{x}, v))$ *is convex.*

Then there exists $(y^*, z^*) \in (-Y^+)^o \times Z^*$ with $z^* \neq 0$ such that $\langle y^*, df(\overline{x}, v)\rangle = 0$ and

$$\langle y^*, df(\overline{x}; x) + rd^2 f(\overline{x}; v; 0)\rangle \geq \langle z^*, z\rangle,$$

for all $(x, r) \in \hat{k}^2(C, \overline{x}, v)$ *and* $z \in \hat{D}^2_K F((\overline{x}, 0), (v, 0)))(x, r)$, where $A(x, r) = df(\overline{x}; x) + rd^2 f(\overline{x}; v; 0)$.

4 Second-order optimality conditions in the special case

In this section, we give some applications. Consider the following vector optimization problem

$$(VP1) \quad \begin{cases} \textit{Minimize } f(x) \\ \textit{subject to } x \in C \textit{ and } g(x) \in D, \end{cases}$$

where D is a nonempty subset of Z and g be a mapping from X into Z . Such a formulation encompasses problems in which inequality and equality constraints are present.

Note that the problem $(VP1)$ has been considered by Jiménez and Novo [13] when $C = X$ and when the data of the problem are twice differentiables they established new second-order optimality conditions in the dual form and in the primal form.

From Theorem 3.1, we can derive the primal second-order necessary optimality conditions of the problem $(VP1)$. Theorem 4.1, extends Theorem 4.2 of [13] to the nondifferentiable case.

Theorem 4.1 *Let F be a multifunction defined by $F(x) = -g(x) + D$, $\overline{x}$ be a local weak efficient solution of $(VP1)$ with respect to Y^+ and $v \in K(\overline{x})$. Suppose that*

(1) *f and g are second-order Hadamard directionally differentiables at $\overline{x}$*
(2) *g is locally Lipschitzian at $\overline{x}$*
(3) *F is directionally metrically regular at $((\overline{x}, 0), v)$ with respect to C.*
Then the following system in $x \in X$ is incompatible

$$\text{(a)} \quad \begin{cases} d^2 f(\overline{x}; v; x) \in Int K(-Y^+, df(\overline{x}; v)) \\ x \in k^2(C, \overline{x}, v),\ d^2 g(\overline{x}; v; x) \in K^2(D, g(\overline{x}), dg(\overline{x}; v)). \end{cases}$$

Moreover, suppose that
(4) *f and g are regular at $\overline{x}$*
(5) *f is Lipschitzian at $\overline{x}$.*
Then the following system in $x \in X$ is incompatible

$$\text{(b)} \quad \begin{cases} df(\overline{x}; x) \in Int K(-Y^+, df(\overline{x}; v)) \\ x \in k''(C, \overline{x}, v),\ dg(\overline{x}; x) \in K''(D, g(\overline{x}), dg(\overline{x}; v)). \end{cases}$$

Proof Since g is locally Lipschitzian at $\overline{x}$ then F is pseudo-Lipschitzian at $(\overline{x}, 0)$. Because

$$D^2_K F((\overline{x}, 0), (v, 0))(x) = -d^2 g(\overline{x}; v; x) + K^2(D, g(\overline{x}), dg(\overline{x}; v))$$

and

$$D''_K F((\overline{x}, 0), (v, 0))(x) = -dg(\overline{x}; x) + K''(D, g(\overline{x}), dg(\overline{x}; v))$$

then the proof of Theorem 4.1 is complete by Theorem 3.1. □

Now, From Proposition 3.1, we can derive second-order sufficient optimality conditions in the primal form of the problem $(VP1)$. This result can be compared with Theorem 4.8 of [13]

Proposition 4.1 *Let $\overline{x} \in E$, F be a multifunction defined by $F(x) = -g(x) + D$ and X be finite dimensional. Let the assumptions* (1), (2), (4) *and* (5) *of Theorem 4.1 be satisfied. If for any $v \in K(E, \overline{x}) \setminus \{0\}$ satisfying $df(\overline{x}; v) \in -Y^+$ such that the following systems*

(a) $$\begin{cases} df(\overline{x}; x) + d^2 f(\overline{x}; v; 0) \in Int K(-Y^+, df(\overline{x}; v)) \\ x \in K''(C, \overline{x}, v),\ dg(\overline{x}; x) + d^2 g(\overline{x}; v; 0) \in K''(D, g(\overline{x}), dg(\overline{x}; v)) \end{cases}$$

and

(b) $$\begin{cases} df(\overline{x}; x) \in Int K(-Y^+, df(\overline{x}; v)) \\ x \in k''(C, \overline{x}, v),\ dg(\overline{x}; x) \in K''(D, g(\overline{x}), dg(\overline{x}; v)) \end{cases}$$

in $x \in X$ are incompatible then $\overline{x}$ is a strict local efficient solution of order two for $(VP1)$.

Now, we shall derive dual second-order necessary optimality conditions of $(VP1)$ in terms of Lagrange Kuhn-Tucker multipliers. To this end we prove the following lemma.

Lemma 4.1 *For any convex subset H of X containing $\overline{x}$ and any $v \in X$, the following hold.*

(a) $\hat{K}^2(H, \overline{x}, v) + K(K(H, \overline{x}), v) \times \{0\} \subset \hat{K}^2(H, \overline{x}, v) \subset K(K(H, \overline{x}), v) \times \mathbb{R}^+$,
(b) $\hat{k}^2(H, \overline{x}, v) + K(K(H, \overline{x}), v) \times \{0\} \subset \hat{k}^2(H, \overline{x}, v) \subset K(K(H, \overline{x}), v) \times \mathbb{R}^+$,
(c) $k^2(H, \overline{x}, v) + K(K(H, \overline{x}), v) \subset k^2(H, \overline{x}, v) \subset K(K(H, \overline{x}), v)$,
(d) $K^2(H, \overline{x}, v) + K(K(H, \overline{x}), v) \subset K^2(H, \overline{x}, v) \subset K(K(H, \overline{x}), v)$,
(e) $K''(H, \overline{x}, v) + K(K(H, \overline{x}), v) \subset K''(H, \overline{x}, v) \subset K(K(H, \overline{x}), v)$,
(f) $k''(H, \overline{x}, v) + K(K(H, \overline{x}), v) \subset k''(H, \overline{x}, v) \subset K(K(H, \overline{x}), v)$,
(g) $k''(H, \overline{x}, v) \neq \emptyset$ *then* $k''(H, \overline{x}, v) = K(K(H, \overline{x}), v)$,
(h) $K''(H, \overline{x}, v) \neq \emptyset$ *then* $K''(H, \overline{x}, v) = K(K(H, \overline{x}), v)$.

Proof For part (a) see Proposition 2.3 of Penot [19]. Part (b) follows by a similar competition to the one of Proposition 2.3 in Penot [19]. Parts (c) and (f) follow from (b). Part (d) and (e) follow from (a). Part (g) follows from (f). Finally part (h) follows from (d). □

The following result improves and extends for instance the one of [13] and [24] to the nondifferentiable case.

Theorem 4.2 *Let F be a multifunction defined by $F(x) = -g(x) + D$, $\overline{x}$ be a local weak efficient solution of $(VP1)$ with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions* (1), (2) *and* (3) *of Theorem 4.1 be satisfied. Suppose also that the assumptions* (i), (ii) *and* (iii) *of Theorem 3.2 hold. Then there exists $(y^*, z^*) \in \big((-Y^+)^o \setminus \{0\}\big) \times dom\ s(., K^2(D, g(\overline{x}), dg(\overline{x}; v)))$ such that*

(a) $\langle y^*, df\ (\overline{x}; v) \rangle = 0$,
(b) $\langle y^*, d^2 f(\overline{x}; v; w) \rangle + \langle z^*, d^2 g(\overline{x}; v; w) \rangle \geq s(z^*, K^2(D, g(\overline{x}), dg(\overline{x}; v)))$, *for all $w \in k^2(C, \overline{x}, v)$.*
Moreover, if the set D is convex then $z^* \in N(D, g(\overline{x}))$ and $\langle z^*, dg(\overline{x}; v) \rangle = 0$.

Proof By Theorem 3.2, there exists $(y^*, z^*) \in (-Y^+) \times Z^*$ with $y^* \neq 0$ such that $\langle y^*, df(\overline{x}; v) \rangle = 0$ and

$$\langle y^*, d^2 f(\overline{x}; v; w) \rangle \geq \langle z^*, z \rangle \quad for\ all\ w \in k^2(C; \overline{x}, v) \quad and \quad z \in D^2_K F((\overline{x}, 0), (v, 0))(w).$$

As $D^2_K F((\overline{x}, 0), (v, 0))(w) = -d^2 g(\overline{x}; v, w) + K^2(D, g(\overline{x}), dg(\overline{x}; v))$, then

$$\langle y^*, d^2 f(\overline{x}; v, w) \rangle + \langle z^*, d^2 g(\overline{x}; v; w) \rangle \geq s(z^*, K^2(D, g(\overline{x}), dg(\overline{x}; v))), \forall w \in k^2(C, \overline{x}, v).$$

Therefore $z^* \in dom\ s(., K^2(D, g(\overline{x}), dg(\overline{x}; v)))$. Now Suppose that the set D is convex. Then

$$dom\ s(., K^2(D, g(\overline{x}), dg(\overline{x}; v))) = N(K(D, g(\overline{x})), dg(\overline{x}; v))$$

by Lemma 4.1. Since $dg(\overline{x}; v) \in K(D, g(\overline{x}))$, then

$$N(K(D, g(\overline{x})), dg(\overline{x}; v)) = \left\{z^* \in N(D, g(\overline{x})) : \langle z^*, dg(\overline{x}; v)\rangle = 0\right\}.$$

So the proof of Theorem 4.2 is complete. □

We can now derive the following consequence of Theorem 3.3 which extends Theorem 5.3 of [13] to the nondifferentiables case.

Theorem 4.3 *Let F be a multifunction defined by $F(x) = -g(x) + D$, $\overline{x}$ be a local weak efficient solution of $(VP1)$ with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions* (1), (2) (3) (4) *and* (5) *of Theorem 4.1 be satisfied. Suppose also that the assumptions (i), (ii) and (iii) of Theorem 3.3 hold. Then there exists $(y^*, z^*) \in \left((-Y^+)^o \setminus \{0\}\right) \times \left(K''(D, g(\overline{x}), dg(\overline{x}; v))\right)^o$ such that*

(a) $\langle y^*, df\ (\overline{x}; v)\rangle = 0$,
(b) $\langle y^*, df(\overline{x}; w - v)\rangle + \langle z^*, dg(\overline{x}; w - v)\rangle \geq 0$ *for all* $w \in K(C, \overline{x})$.
Moreover, if the set D is convex then $z^* \in N(D, g(\overline{x}))$ and $\langle z^*, dg(\overline{x}; v)\rangle = 0$.

Proof By Theorem 3.3 there exists $(y^*, z^*) \in (-Y^+)^o \times Z^*$ with $y^* \neq 0$ such that $\langle y^*, df(\overline{x}, v)\rangle = 0$ and

$$\langle y^*, df(\overline{x}; x) \geq \langle z^*, z\rangle \quad \forall x \in k''(C, \overline{x}, v), \forall z \in D''_K F((\overline{x}, 0), (v, 0))(x).$$

Since $D''_K F((\overline{x}, 0), (v, 0)))(x) = -dg(\overline{x}; x) + K''(D, g(\overline{x}), dg(\overline{x}; v))$ then $z^* \in \left(K''(D, g(\overline{x}), dg(\overline{x}; v))\right)^o$. Moreover by Lemma 4.1 one has that

$$\langle y^*, df(\overline{x}; w - v)\rangle + \langle z^*, dg(\overline{x}; w - v)\rangle \geq 0 \ \ \forall w \in K(C, \overline{x})$$

since $k''(C, \overline{x}, v) \neq \emptyset$. Now suppose that D is convex. As $dg(\overline{x}; v) \in K(D, g(\overline{x}))$, then

$$N(K(D, g(\overline{x})), dg(\overline{x}; v)) = \left\{z^* \in N(D, g(\overline{x})) : \langle z^*, dg(\overline{x}; v)\rangle = 0\right\}.$$

So the proof is complete by lemma 4.1 because $K''(D, g(\overline{x}), dg(\overline{x}; v)) \neq \emptyset$. □

In order to combine Theorems 4.2 and 4.3 into a single result, we prove the following Lemma.

Lemma 4.2 *Let $g : X \to Z$ be a mapping which is Lipschitzian, regular and second order Hadamard directionally differentiable at $\overline{x}$, F be a multifunction from X into Z defined by $F(x) = -g(x) + D$, $(\overline{x}, 0) \in Gr(F)$ and $v \in X$. Then for each $(w, r) \in X \times \mathbb{R}^+$ one has $z \in \hat{D}^2_K F(((\overline{x}, 0), (v; 0))(w, r)$ if and only if*

$$(z + dg(\overline{x}; w) + rd^2 g(\overline{x}; v; 0), r) \in \hat{K}^2(D, g(\overline{x}), dg(\overline{x}; v)),$$

where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

Proof Let $(w, r) \in X \times \mathbb{R}^+$ and $z \in \hat{D}^2_K F(((\overline{x}, 0), (v; 0))(w, r)$. We have two cases. (a) $r \neq 0$. By Lemma 2.2 one has $z \in r D^2_K F(((\overline{x}, 0), (v; 0))(w/r))$. Since

$$D^2_K F(((\overline{x}, 0), (v; 0))(w/r)) = -dg(\overline{x}; w/r) - d^2 g(\overline{x}; v; 0) + K^2(D, g(\overline{x}), dg(\overline{x}; v))$$

we get

$$z + rdg(\overline{x}; w/r) + rd^2g(\overline{x}; v; 0) \in rK^2(D, g(\overline{x}), dg(\overline{x}; v)).$$

As $x \mapsto dg(\overline{x}; x)$ is positively homogeneous, it follows that

$$z + dg(\overline{x}; w) + rd^2g(\overline{x}; v; 0) \in rK^2(D, g(\overline{x}), dg(\overline{x}; v)).$$

(b) $r = 0$. By Lemma 2.2 one has $z \in D''_K F(((\overline{x}, 0), (v; 0))(w))$. Since

$$D''_K F(((\overline{x}, 0), (v; 0))(w)) = -dg(\overline{x}; w) + K''(D, g(\overline{x}), dg(\overline{x}; v))$$

we get

$$z + dg(\overline{x}; w) \in K''(D, g(\overline{x}), dg(\overline{x}; v))$$

Therefore by the cases (a) and (b), and Remark 2.2 we obtain

$$(z + dg(\overline{x}; w) + rd^2g(\overline{x}; v; 0), r) \in \hat{K}^2(D, g(\overline{x}), dg(\overline{x}; v)).$$

The reverse implication is obvious. □

We can now derive the following consequence of Theorem 3.4, Lemmas 4.1 and 4.2. Theorem 4.4 can be compared for instance with Theorem 5.5 of [13].

Theorem 4.4 *Let F be a multifunction defined by $F(x) = -g(x) + D$, $\overline{x}$ be a local weak efficient solution of $(VP1)$ with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions* (1), (2), (3) (4) *and* (5) *of Theorem 4.1 be satisfied. Suppose also that the assumptions (i), (ii) and (iii) of Theorem 3.4 hold. Then there exists $(y^*, z^*) \in \big((-Y^+)^o \setminus \{0\}\big) \times Z^*$ such that*

(a) $\langle y^*, df(\overline{x}; v)\rangle = 0$,
(b) $\langle y^*, df(\overline{x}, w - v)\rangle + \langle z^*, dg(\overline{x}, w - v)\rangle$ *for all* $w \in K(C, \overline{x})$,
(c) $\langle y^*, df(\overline{x}, w) + rd^2f(\overline{x}; v, 0)\rangle + \langle z^*, dg(\overline{x}, w) + rd^2g(\overline{x}; v, 0)\rangle \geq \langle z^*, z\rangle$ *for all* $(w, r) \in \hat{k}(C, \overline{x}, v)$ *and* $(z, r) \in \hat{K}^2(D, g(\overline{x}), dg(\overline{x}; v))$.
Moreover, if the set D is convex then $z^* \in N(D, g(\overline{x}))$ *and* $\langle z^*, dg(\overline{x}; v)\rangle = 0$.
The following Proposition 4.2 will be used in the next.

Proposition 4.2 *Let g be a mapping from X into Y which is strictly differentiable at $\overline{x}$, F be a multifunction defined by $F(x) = -g(x) + D$, and $v \in X$. Suppose that C and D are convex sets and that*

$$0 \in Core\,[\nabla g(\overline{x})(K(K(C, \overline{x}), v)) - K(K(D, g(\overline{x})), \nabla g(\overline{x})v)]$$

(this qualification condition is a slight extension of the one introduced in [2] and [13]).

(a) *If* $k^2(C, \overline{x}, v) \neq \emptyset$ *and* $K^2(D, g(\overline{x}), \nabla g(\overline{x})v) \neq \emptyset$ *then* $0 \in Int\, D^2_K F((\overline{x}, 0), (v, 0))(k^2(C, \overline{x}, v))$,
(b) *If* $k''(C, \overline{x}, v) \neq \emptyset$ *and* $K''(D, g(\overline{x}), \nabla g(\overline{x})v) \neq \emptyset$ *then* $0 \in Int\, D''_K F((\overline{x}, 0), (v, 0))(k''(C, \overline{x}, v))$,
(c) *If* $\Big\{(x, z, r) \in X \times Z \times \mathbb{R}^+ : (x, r) \in \hat{k}^2(C, \overline{x}, v),\ (z, r) \in \hat{K}^2(D, g(\overline{x}), \nabla g(\overline{x})v\Big\} \neq \emptyset$ *then*

$$Int\, \hat{D}^2_K F((\overline{x}, 0), (v, 0))(\hat{k}^2(C, \overline{x}; v)) \neq \emptyset.$$

Proof (a) The condition $0 \in Core\,[\nabla g(\overline{x})K(K(C,\overline{x}),v)) - K(K(D,g(\overline{x})),\nabla g(\overline{x})v)]$ is equivalent to

$$Z = \nabla g(\overline{x})(K(K(C,\overline{x}),v)) - K(K(D,g(\overline{x})),\nabla g(\overline{x})v).$$

Since $k^2(C,\overline{x},v) \neq \emptyset$ and $K^2(D,g(\overline{x}),\nabla g(\overline{x})v) \neq \emptyset$, then by Lemma 4.1 one has

$$Z = \nabla g(\overline{x})(k^2(C,\overline{x},v)) - K^2(D,g(\overline{x}),\nabla g(\overline{x})v).$$

As $D^2_K F((\overline{x},0),(v,0))(k^2(C,\overline{x},v)) = -\nabla g(\overline{x})(k^2(C,\overline{x},v)) - d^2 g(\overline{x},v,0)$ $+ K^2(D,g(\overline{x}),\nabla g(\overline{x})v)$, we conclude that

$$Z = -\nabla g(\overline{x})(k^2(C,\overline{x},v)) - d^2 g(\overline{x},v,0) + K^2(D,g(\overline{x}),\nabla g(\overline{x})v),$$

which implies that $0 \in Int D^2_K F((\overline{x},0),(v,0))(k^2(C,\overline{x},v))$.

(c) Let $z \in Z$, $(w,r) \in \hat{k}^2(C,\overline{x};v)$ and $(z',r) \in \hat{K}^2(D,g(\overline{x}),\nabla g(\overline{x})v)$. There exists $w' \in K(K(C,\overline{x}),v))$ such that

$$z + \nabla g(\overline{x})(w + w') + r d^2 g(\overline{x};v;0) - z' \in K(K(D,g(\overline{x})),\nabla g(\overline{x})v).$$

Finally (c) follows by Lemma 4.1 and 4.2. The proof of (b) is similar to that of (a). □

In the next of this section, we give some Corollaries whenever the data of the problem $(VP1)$ are strictly differentiables. The following result can be compared with Theorem 5.2 of [13] and the one of [24].

Corollary 4.1 *Let $\overline{x}$ be a local weak efficient of $(VP1)$ with respect to Y^+ and $v \in K(\overline{x})$. Suppose that*

(1) *C and D are convex,*
(2) *f and g are strictly differentiables and second-order Hadamard directionally differentiables at $\overline{x}$,*
(3) $0 \in Core\,[\nabla g(\overline{x})K(K(C,\overline{x}),v)) - K(K(D,g(\overline{x})),\nabla g(\overline{x})v)]$.
If $k^2(C,\overline{x},v) \neq \emptyset$ then for each nonempty subset $\tau(v)$ of $K^2(D,g(\overline{x}),\nabla g(\overline{x})v)$, there exist $y^* \in (-Y^+)\backslash\{0\}$ and $z^* \in N(D,g(\overline{x}))$ such that

   (a) $\langle y^*, \nabla f(\overline{x})v\rangle = 0$, $\langle z^*, \nabla g(\overline{x})v\rangle = 0$,
   (b) $-y^* o \nabla f(\overline{x}) - z^* o \nabla g(\overline{x}) \in N(C,\overline{x})$,
   (c) $\langle y^*, \nabla f(\overline{x})x + d^2 f(\overline{x};v;0)\rangle + \langle z^*, \nabla g(\overline{x})x + d^2 g(\overline{x};v;0)\rangle \geq s(z^*,\tau(v))$ *for all* $x \in k^2(C,\overline{x},v)$.

Proof The proof is a direct consequence of Proposition 4.2, Theorem 4.2, Corollary 2.1 and Lemmas 4.1. □

The following result can be compared with the one of [13].

Corollary 4.2 *Let $\overline{x}$ be a local weak efficient of $(VP1)$ with respect to Y^+ and $v \in K(\overline{x})$. Let the assumptions* (1), (2) *and* (3) *of Corollary 4.1 be satisfied. If $k''(C,\overline{x},v) \neq \emptyset$ and $K''(D,g(\overline{x}),\nabla g(\overline{x})v) \neq \emptyset$ then there exist $y^* \in (-Y^+)\backslash\{0\}$ and $z^* \in N(D,g(\overline{x}))$ such that*

(a) $\langle y^*, \nabla f(\overline{x})v\rangle = 0$, $\langle z^*, \nabla g(\overline{x})v\rangle = 0$,
(b) $-y^* o \nabla f(\overline{x}) - z^* o \nabla g(\overline{x}) \in N(C,\overline{x})$.

Proof The proof is a direct consequence of Proposition 4.2 and Theorem 4.3. □

Now we may combine Corollaries 4.1 and 4.2 into a single result. This result can be compared with Theorem 5.5 of [13].

Corollary 4.3 *Let* $\overline{x}$ *be a local weak efficient of* $(VP1)$ *with respect to* Y^+ *and* $v \in K(\overline{x})$. *Let the assumptions* (1),(2) *and* (3) *of Corollary 4.1 be satisfied. If* $\hat{k}^2(C, \overline{x}, v) \neq \emptyset$ *then for each nonempty subset* $\hat{\tau}(v)$ *of* $\hat{K}^2(D, g(\overline{x}), \nabla g(\overline{x})v)$ *with*

$$\left\{(x, z, r) \in X \times Z \times \mathbb{R}^+ : (x, r) \in \hat{k}^2(C, \overline{x}, v),\ (z, r) \in \hat{\tau}(v)\right\} \neq \emptyset$$

there exist $y^* \in (-Y^+) \setminus \{0\}$ *and* $z^* \in N(D, g(\overline{x}))$ *such that*

(a) $\langle y^*, \nabla f(\overline{x})v\rangle = 0, \langle z^*, \nabla g(\overline{x})v\rangle = 0,$
(b) $-y^* o \nabla f(\overline{x}) - z^* o \nabla g(\overline{x}) \in N(C, \overline{x}),$
(c) $\langle y^*, \nabla f(\overline{x})x + rd^2 f(\overline{x}; v, 0)\rangle + \langle z^*, \nabla g(\overline{x})x + rd^2 g(\overline{x}; v; 0)\rangle \geq \langle z^*, z\rangle,$
for all $(z, r) \in \hat{\tau}(v)$ *and* $(x, r) \in \hat{k}^2(C, \overline{x}, v)$.

Proof The proof is a direct consequence of Proposition 4.2 and Theorem 4.4. □

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