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Necessary optimality conditions for nonsmooth semi-infinite programming problems

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Abstract In this paper, for a nonsmooth semi-infinite programming problem where the objective and constraint functions are locally Lipschitz, analogues of the Guignard, Kuhn-Tucker, and Cottle constraint qualifications are given. Pshenichnyi-Levin-Valadire property is introduced, and Karush-Kuhn-Tucker type necessary optimality conditions are derived.

Keywords Optimality condition · Semi-infinite programming · Nonsmooth analysis · Constraint qualification

1 Introduction

A *semi-infinite programming problem* (SIP) is an optimization problem on a feasible set described by infinite number of inequality constraints. Recently, semi-infinite optimization became an active field of research in applied mathematics. This is due to the fact that several engineering problems lead to SIP, e.g., robotics, mathematical physics, optimal control, transportation problems, Chebyshev approximation, etc; see [7,10,15,19].

In the present paper we consider a semi-infinite programming problem, defined as follows:

$$\begin{aligned} & \inf f(x) \\ \text{s.t.}\quad & g_i(x) \le 0 \quad i \in I, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{SIP}$$

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where the appearing functions f and g_i, $i \in I$ are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, and the index set I is an arbitrary set, not necessarily finite (but nonempty).

If the set I is finite, necessary conditions of Karush-Kuhn-Tucker (KKT) type for optimality can be established under various constraint qualifications; for instance, Guignard, Abadie, Mangasarian-Fromovitz, Kuhn-Tucker, Cottle, and Zangwill qualifications, are denoted by GCQ, ACQ, MFCQ, KTCQ, CCQ, and ZCQ, respectively. In order to study and compare these CQs in smooth and nonsmooth cases, see the books [3,6,12]. It is well known that in the theory of convex minimization over the solution set of a finite system of convex inequalities the so-called Slater constraint qualification (SCQ) plays an important role (see [11]).

For linear semi-infinite systems, the "Farkas-Minkowski property" has been introduced by Goberna et al. in [8]. In [20], Puente et al. introduced the "locally Farkas-Minkowski (LFM) property" for linear SIPs and its role as constraint qualification was emphasized there. For an excellent study of linear SIPs, see the book [7], and the survey article [19].

In order to establish optimality conditions for convex semi-infinite programming problems, several characterizations of constraint qualifications have been studied. The SCQ has been extended to convex semi-infinite systems by López and Vercher, who have given an optimality condition of KKT type for convex SIP which involves the notion of Lagrangian saddle point [18]. The LFM property was extended to convex SIPs by Fajardo and López, who used the term "convex LFM property" in [5].

The ACQ and related constraint qualifications for semi-infinite systems of convex inequalities and linear inequalities have been studied in [16]. There, characterizations of various constraint qualifications in terms of upper semicontinuity of certain multifunctions are given. In [22], Stein analyzes ACQ and MFCQ in the context of smooth SIPs with compact index set, appealing to Hadamard directional derivatives. Papers [1,4], and [17] deal with convex SIP involving lower semicontinuous proper functions, using the convex subdifferential and the epigraphs on conjugate functions as main tools.

Reference [23] considers nonsmooth SIP where f is a proper lower semicontinuous function while g_is are locally Lipschitzian with respect to x uniformly on i, and continuous with respect to i, using generalized directional derivatives as main tool in the sense of Rockafellar. Article [13] considers locally Lipschitz SIP, and introduce nonsmooth analogue of the ACQ, ZCQ, and LFM property, by use of Clarke subdifferential. There, necessary and sufficient optimality conditions of KKT type for these problems are investigated.

Finally, [14] considered a new form of nonsmooth SIP with a feasible set defined by inequality and equality constraints and a constraint set. There various theorems of the alternative type are developed and Fritz-John and KKT types optimality conditions for them under ACQ and ZCQ are obtained.

In this paper we study some new CQs for nonsmooth SIPs with locally Lipschitz functions, and we investigate necessary optimality conditions of KKT type for them. In our approach we do not need the classical assumptions on the involved functions: convexity and differentiability.

The organization of the paper is as follows. In Sect. 2, basic notations and results of nonsmooth and convex analysis are reviewed. In Sect. 3, we consider nonsmooth semi-infinite systems and introduce the GCQ, CCQ, and KTCQ for them. Moreover, we introduce the notion of "Pshenichnyi-Levin-Valadire (PLV) property", and compare PLV property and CQs using several examples. In Sect. 4, some necessary optimality conditions of KKT type are established, and an example is presented. Furthermore, we compare our corollaries with earlier theorems in Sect. 4.

2 Notations and preliminaries

In this section we briefly overview some notions of variational analysis widely used in formulations and proofs of main results of the paper. For more details, discussion, and applications see [2,11,21].

Given a nonempty set $M \subseteq \mathbb{R}^n$, we denote by $\overline{M}$, $conv(M)$, and $cone(M)$, the closure of M, convex hull and convex cone (containing the origin) generated by M, respectively. The polar cone and strict polar cone of M are defined respectively by:

$$M^0 := \left\{d \in \mathbb{R}^n \mid \langle x, d\rangle \leq 0, \quad \forall x \in M\right\}$$
$$M^s := \left\{d \in \mathbb{R}^n \mid \langle x, d\rangle < 0, \quad \forall x \in M\right\},$$

where $\langle ., .\rangle$ exhibits the standard inner product in $\mathbb{R}^n$. Notice that M^0 is always a closed convex cone. It is easy to show that if $M^s \neq \varnothing$ then $\overline{M^s} = M^0$. The bipolar theorem states that $M^{00} = \overline{cone}(M)$, where $\overline{cone}(M)$ denotes the closed convex cone of M (see [11]).

Definition 2.1 Let $M \subseteq \mathbb{R}^n$ and $\hat{x} \in \overline{M}$.

I: The contingent cone to M at $\hat{x}$ is defined by

$$T(M, \hat{x}) := \Big\{h \in \mathbb{R}^n \mid \text{there are } \{t_k\} \subset \mathbb{R}_+,\ t_k \to 0,\ \{h^k\} \subset \mathbb{R}^n,\ h^k \to h \text{ such that: } \hat{x} + t_k h^k \in M \text{ for all } k \in \mathbb{N}\Big\}.$$

II: The cone of attainable directions to M is defined by

$$A(M, \hat{x}) := \Big\{h \in \mathbb{R}^n \mid \text{for all } \{t_k\} \subset \mathbb{R}_+,\ t_k \to 0,\ \text{there is} \{h^k\} \subset \mathbb{R}^n,\ h^k \to h \text{ such that: } \hat{x} + t_k h^k \in M \text{ for all } k \in \mathbb{N}\Big\}.$$

Notice that $T(M, \hat{x})$ and $A(M, \hat{x})$ are closed cones (generally nonconvex) in $\mathbb{R}^n$, and we always have

$$A(M, \hat{x}) \subseteq T(M, \hat{x}). \tag{1}$$

Definition 2.2 Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function and $\hat{x} \in dom(\varphi)$.

I: The generalized Clarke directional derivative of φ at $\hat{x}$ in the direction $d \in \mathbb{R}^n$ is defined by

$$\varphi^0(\hat{x}; d) := \limsup_{\substack{y \to \hat{x} \\ t \downarrow 0}} \frac{\varphi(y + td) - \varphi(y)}{t}.$$

II: The Clarke subdifferential of φ at $\hat{x}$ is defined by

$$\partial_c \varphi(\hat{x}) := \left\{\xi \in \mathbb{R}^n \mid \varphi^0(\hat{x}; d) \geq \langle \xi, d\rangle, \quad \forall d \in \mathbb{R}^n\right\}.$$

III: We say that φ is regular at $\hat{x}$ if

$$\varphi^0\left(\hat{x}; d\right) = \varphi'\left(\hat{x}, d\right), \qquad \forall d \in \mathbb{R}^n,$$

where $\varphi'(\hat{x}, d)$ denotes the classical directional derivative of φ at $\hat{x}$ in direction d.

Observe that the Clarke subdifferential of a locally Lipschitz function at an interior point of its domain is always nonempty, compact, and convex cone. The Clarke subdifferential reduces to the classical gradient for continuously differentiable functions and to the subdifferential of convex analysis for convex ones.

The finite convex functions and continuously differentiable functions are examples for regular functions. A broader class of regular functions is the class of subsmooth functions; see [21].

Let us recall the following theorems which will be used in the sequel.

Theorem 2.3 ([11]) *Let $\{M_\alpha | \alpha \in \Lambda\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$. Then, every non-zero vector of $conv\left(\bigcup_{\alpha\in\Lambda} M_\alpha\right)$ can be expressed as a non-negative linear combination of n or fewer linearly independent vectors, each belonging to a different M_α.*

Theorem 2.4 ([11]) *Let M be a nonempty compact subset of $\mathbb{R}^n$ such that $0 \notin conv(M)$. Then $cone(M)$ is a closed cone.*

Theorem 2.5 ([2]) *Let φ and ψ are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, and $\hat{x} \in dom(\varphi) \cap dom(\psi)$. Then, the following properties hold:*

a: $\varphi^0(\hat{x}; d) = \max\left\{\langle \xi, d\rangle \mid \xi \in \partial_c\varphi(\hat{x})\right\}, \quad \forall d \in \mathbb{R}^n.$

b: *$d \longrightarrow \varphi^0(\hat{x}; d)$ is a convex function, and*

$$\partial_c\varphi(\hat{x}) = \partial\varphi^0(\hat{x}; .)(0),$$

where $\partial\varphi^0(\hat{x}; .)$ denotes the subdifferential of convex function $\varphi^0(\hat{x}; .)$.

c: *$x \longrightarrow \partial_c\varphi(x)$ is an upper semicontinuous set-valued function.*

d: $\partial_c(\varphi + \psi)(\bar{x}) \subseteq \partial_c\varphi(\bar{x}) + \partial_c\psi(\bar{x})$.
Furthermore, if φ and ψ are regular at $\bar{x}$, then $\varphi + \psi$ is also regular at $\bar{x}$, and equality holds above.

e: *If $\hat{x}$ is a minimum point of φ over $\mathbb{R}^n$, then $0 \in \partial_c\varphi(\hat{x})$.*

Theorem 2.6 (Lebourg mean-value [2]) *Let $x, y \in \mathbb{R}^n$, and suppose that φ is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, there exist a point u in the open line segment (x, y), such that*

$$\varphi(y) - \varphi(x) \in \langle \partial_c\varphi(u), y - x\rangle.$$

3 Constraint qualifications

In this section we introduce and compare several constraint qualifications (CQs briefly) for a locally Lipschitz semi-infinite system.

Let I be an arbitrary (but nonempty) index set, and $\{g_i : \mathbb{R}^n \to \mathbb{R} \mid i \in I\}$ be a family of locally Lipschitz functions. Let us consider the following *semi-infinite system* (SIS in brief)

$$\pi := \{g_i(x) \leq 0 \mid i \in I\}.$$

In what follows we shall assume that the feasible set of π is nonempty, i.e.,

$$P := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \forall\, i \in I\} \neq \varnothing.$$

For a given $\bar{x} \in P$, we define $\left(\text{with the convention } \bigcup_{i\in\varnothing} X_i = \varnothing\right)$:

$$I^{\bar{x}} := \{i \in I \mid g_i(\bar{x}) = 0\},$$

and

$$Z(\bar{x}) := \bigcup_{i \in I^{\bar{x}}} \partial_c g_i(\bar{x}).$$

Based on the notions of contingent cone, attainable directions cone, and Clarke subdifferential, we extend the following CQs for π.

Definition 3.1 Let $\bar{x} \in P$. We say that π satisfies

(a) The Guignard CQ (GCQ briefly) at $\bar{x}$, if $Z^0(\bar{x}) \subseteq \overline{conv}(T(P, \bar{x}))$.
(b) The Kuhn-Tucker CQ (KTCQ briefly) at $\bar{x}$, if $Z^0(\bar{x}) \subseteq A(P, \bar{x})$.
(c) The Cottle CQ (CCQ briefly) at $\bar{x}$, if $Z^s(\bar{x}) \neq \varnothing$.

Remark 3.2 The GCQ and its relationships with Abadie, Zangwill, and Basic CQs are studied in [13]. It is also shown that each of these CQs is sufficient for the GCQ to hold.

The following theorem shows that in regular systems GCQ is equivalent to equation $Z^0(\bar{x}) = \overline{conv}(T(P, \bar{x}))$.

Theorem 3.3 *Suppose that $\bar{x} \in P$ and for each $i \in I^{\bar{x}}$, g_i is a regular function at $\bar{x}$. Then*

$$\overline{conv}(T(P, \bar{x})) \subseteq Z^0(\bar{x}).$$

Proof Let $d \in T(P, \bar{x})$ be given arbitrarily. Then, there exist $t_k \downarrow 0$ and $d^k \to d$ such that $\bar{x} + t_k d^k \in P$. Thus, by the definition of P, we obtain that

$$g_i\left(\bar{x} + t_k d^k\right) \leq 0, \qquad \forall i \in I.$$

Suppose that $i_0 \in I^{\bar{x}}$. Then, $g_{i_0}(\bar{x}) = 0$ and hence

$$\frac{g_{i_0}\left(\bar{x} + t_k d^k\right) - g_{i_0}\left(\bar{x} + t_k d\right)}{t_k} + \frac{g_{i_0}\left(\bar{x} + t_k d\right) - g_{i_0}\left(\bar{x}\right)}{t_k} = \frac{g_{i_0}\left(\bar{x} + t_k d^k\right)}{t_k} \leq 0.$$

We denote the Lipschitz constant of g_i near $\bar{x}$ as L_i. As t_k is sufficiently close to 0, we see

$$\left|\frac{g_{i_0}\left(\bar{x} + t_k d^k\right) - g_{i_0}\left(\bar{x} + t_k d\right)}{t_k}\right| \leq L_{i_0} \parallel d^k - d \parallel \to 0 \quad (\text{when } t_k \to 0).$$

Taking into consideration the regularity of g_{i_0} at $\bar{x}$, we conclude

$$\begin{aligned} g_{i_0}^0(\bar{x}; d) &= g'_{i_0}(\bar{x}, d) \\ &= \lim_{t_k \to 0} \frac{g_{i_0}(\bar{x} + t_k d) - g_{i_0}(\bar{x})}{t_k} \\ &= \lim_{t_k \to 0} \frac{g_{i_0}\left(\bar{x} + t_k d^k\right) - g_{i_0}(\bar{x} + t_k d)}{t_k} + \lim_{t_k \to 0} \frac{g_{i_0}(\bar{x} + t_k d) - g_{i_0}(\bar{x})}{t_k} \\ &\leq 0. \end{aligned}$$

Since i_0 is an arbitrary index in $I^{\bar{x}}$, and with regard to Theorem 2.5(a) we obtain that

$$d \in Z^0(\bar{x}).$$

Therefore,

$$T(P, \bar{x}) \subseteq Z^0(\bar{x}).$$

Owing to the convexity and closedness property of $Z^0(\bar{x})$, the proof is complete. □

Set

$$G(x) := \sup_{i\in I} g_i(x), \qquad \forall x \in P.$$

One reason for difficulty of extending the results from a finite inequality system to SIS is that in the finite case $G(.)$ is locally Lipschitz and we have (see [2, Proposition 2.3.12])

$$\partial_c G(x) \subseteq conv\left(\bigcup_{i\in I^x} \partial_c g_i(x)\right) = conv\,(Z(\bar{x}))\,, \qquad \forall x \in P, \tag{2}$$

but in general, (2) does not hold if I is infinite (see [2, Theorem 2.8.2]). We are thus led to the following definition.

Definition 3.4 We say that π has the Pshenichnyi-Levin-Valadire (PLV) property at $x \in P$, if $G(.)$ is Lipschitz around x, and (2) holds.

Note that the above definition is in agreement with [16] for convex SISs.

Remark 3.5 An interesting sufficient condition ensuring the Lipschitzian property of G around x, can be found in [21, Theorem 9.2].

Remark 3.6 It is known that if for all $i \in I$, g_i is convex, I is a compact set in some metric space, and for each fixed $\widetilde{x} \in P$ the function $i \to g_i(\widetilde{x})$ is upper semicontinuous on I, then π has the PLV property at every $x \in P$ (see [pp. 267][11]).

There is no relation of implication between the CQs and the PLV property. Indeed, for any finite I the PLV property is trivially true, but it may not satisfy each of the CQs; while in the following example the system actually satisfies all the CQs at $\bar{x} = 0$, but the PLV property does not hold at this point.

Example Let $I = \mathbb{N} \cup \{0\}$, $\bar{x} = 0$, and

$$\begin{aligned} g_0(x) &= 2x, \\ g_{2k+1}(x) &= x - \frac{1}{k+1}, \qquad k = 0, 1, 2, \ldots \\ g_{2k}(x) &= 3x - \frac{1}{k}, \qquad k = 1, 2, \ldots. \end{aligned}$$

We observe that:

$$P = (-\infty, 0], \qquad I^{\bar{x}} = \{0\}\,, \qquad Z(\bar{x}) = \{2\}\,,$$

and

$$G(x) = \sup_{i\in\mathbb{N}} \{g_0(x), g_i(x)\} = \begin{cases} x & \text{if } x < 0 \\ 3x & \text{if } x \geq 0. \end{cases}$$

Since

$$\begin{aligned} Z^s(\bar{x}) &= (-\infty, 0) \neq \varnothing, \\ Z^0(\bar{x}) &= (-\infty, 0] = \overline{conv}(T(P, \bar{x})) = A(P, \bar{x}), \\ \partial_c G(\bar{x}) &= [1, 3] \nsubseteq conv\,(Z(\bar{x}))\,, \end{aligned}$$

the system does not have the PLV property but CCQ, KTCQ and GCQ are satisfied at $\bar{x}$.

Theorem 3.7 *Suppose that π satisfies the PLV property at $\bar{x}$. Then, the CCQ implies the KTCQ at $\bar{x}$.*

Proof Let $d \in Z^s(\bar{x})$. Since

$$Z^s(\bar{x}) = (conv\,(Z(\bar{x})))^s,$$

PLV property leads to

$$d \in (conv\,(Z(\bar{x})))^s \subseteq (\partial_c G(\bar{x}))^s.$$

Hence

$$G^0(\bar{x}; d) < 0,$$

and consequently, there exists a scalar $\delta > 0$ such that

$$G(\bar{x} + \beta d) < G(\bar{x}) \leq 0, \qquad \forall\, \beta \in (0, \delta].$$

Thus, for all $i \in I$ and for all $\beta \in (0, \delta]$, we conclude

$$g_i(\bar{x} + \beta d) < 0.$$

Therefore, for all $\beta \in (0, \delta]$ we have $\bar{x} + \beta d \in P$, which implies

$$d \in A(P, \bar{x}).$$

We have thus proved

$$Z^s(\bar{x}) \subseteq A(P, \bar{x}).$$

Hence, we can obtain that

$$Z^0(\bar{x}) = \overline{Z^s(\bar{x})} \subseteq \overline{A(P, \bar{x})} = A(P, \bar{x}),$$

and the proof is complete. □

Remark 3.8 Since $A(P, \bar{x}) \subseteq \overline{conv}\,(T(P, \bar{x}))$, the KTCQ implies the GCQ at $\bar{x}$. The following example shows that the converse of this result is not true.

Example Let $I = \mathbb{N}$, and

$$\begin{aligned} g_1(x_1, x_2) &= -x_1, \\ g_2(x_1, x_2) &= -x_2 \\ g_k(x_1, x_2) &= x_1 x_2 - \frac{1}{k}, \qquad\qquad k = 3, 4, \ldots. \end{aligned}$$

If we consider the point $\bar{x} = (0, 0)$, we obtain

$$\begin{aligned} P &= (\{0\} \times [0, +\infty)) \cup ([0, +\infty) \times \{0\}), \\ I^{\bar{x}} &= \{1, 2\}, \\ Z(\bar{x}) &= \{(-1, 0), (0, -1)\}, \\ A(P, \bar{x}) &= P = T(P, \bar{x}), \\ \overline{conv}\,(T(P, \bar{x})) &= [0, +\infty) \times [0, +\infty) = Z^0(\bar{x}). \end{aligned}$$

Thus, this system satisfies the GCQ at $\bar{x}$, but does not satisfy the KTCQ at this point.

Recall the following definition from [18, Definition 3.6].

Definition 3.9 We say that π satisfies the Slater constraint qualification (SCQ), if

- for all $i \in I$, g_i is convex function,
- $I \subseteq \mathbb{R}^m$ is compact set,
- $g_i(x)$ is a continuous function of (i, x) in $I \times \mathbb{R}^n$
- there is a point $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$, for all $i \in I$.

Theorem 3.10 *Suppose that π satisfies the SCQ, and $\bar{x} \in P$. Then,*

(i) *: π satisfies the CCQ at $\bar{x}$.*
(ii)*: cone $(Z(\bar{x}))$ is a closed cone.*

(the closedness of *cone* $(Z(\bar{x}))$ is an essential condition in next section.)

Proof (i): By the definition of SCQ, there is an x_0 such that

$$g_i(x_0) < 0, \qquad \text{for all } i \in I.$$

Let $i_0 \in I^{\bar{x}}$ and $\xi \in \partial g_{i_0}(\bar{x})$. Then,

$$\langle \xi, x_0 - \bar{x} \rangle \leq g_{i_0}(x_0) - g_{i_0}(\bar{x}) = g_{i_0}(x_0) < 0.$$

Hence, $x_0 - \bar{x} \in Z^s(\bar{x})$.

(ii): Since $Z^s(\bar{x}) \neq \varnothing$, it is easy to see $0 \notin conv\,(Z(\bar{x}))$. On the other hand, according to [11, Theorem 4.4.1], $Z(\bar{x})$ is a compact set. Owing to the Theorem 2.4, the proof is complete.

□

The following diagram summarizes the connections between CQs and PLV property:

$$\text{SCQ} \Longrightarrow \text{CCQ} + \text{PLV} \Longrightarrow \text{KTCQ} \Longrightarrow \text{GCQ}.$$

As an immediate consequence of Theorems 3.7, 3.10, and Remark 3.6, we can obtain the following Corollary which was proved in [18, Theorem 3.8].

Corollary 3.11 *Suppose that π satisfies the SCQ. If $\bar{x} \in P$ is arbitrary, then π satisfies the KTCQ at $\bar{x}$ and cone $(Z(\bar{x}))$ is closed.*

4 Necessary optimality conditions

In this section we apply the CQs which were introduced in the previous section, and give the Karush-Kuhn-Tucker type necessary optimality conditions for SIP.

We say that the SIP satisfies a CQ at $\bar{x}$ when the corresponding constraint system satisfies that CQ at $\bar{x}$.

Lemma 4.1 *Suppose that $\bar{x}$ is an optimal solution of SIP. If $d \longrightarrow f^0(\bar{x}; d)$ is concave function, then*

$$(\partial_c f(\bar{x}))^s \cap \overline{conv}\,(T(P, \bar{x})) = \varnothing.$$

Proof The proof falls naturally into three parts.

- Let $\widehat{d}$ be an arbitrary vector in $T(P, \bar{x})$. By the definition of $T(P, \bar{x})$, there exists a sequence $\left(t_k, d^k\right) \to \left(0^+, \widehat{d}\right)$ such that

$$\left\{\bar{x} + t_k d^k\right\}_{k=1}^{\infty} \subset P.$$

Using Theorem 2.6, one can conclude that for each $k \in \mathbb{N}$ there exists an u^k in the open line segment $\left(\bar{x}, \bar{x} + t_k d^k\right)$, and there is a $\xi^k \in \partial_c f(u^k)$, such that

$$f\left(\bar{x} + t_k d^k\right) - f(\bar{x}) = t_k \left\langle \xi^k, d^k \right\rangle, \qquad \forall\, k \in \mathbb{N}. \tag{3}$$

since $\bar{x} + t_k d^k \in P$ and $\bar{x}$ is an optimal solution of SIP, we have

$$f\left(\bar{x} + t_k d^k\right) - f(\bar{x}) \geq 0. \tag{4}$$

By virtue of (3) and (4) we get

$$\left\langle \xi^k, d^k \right\rangle \geq 0. \tag{5}$$

On the other hand, since $u^k \to \bar{x}$ and the mapping $x \longmapsto \partial_c f(x)$ is upper semicontinuous, there is a subsequence ξ^{k_m} of ξ^k such that $\xi^{k_m} \to \xi$ and $\xi \in \partial_c f(\bar{x})$. Hence, with regard to (5) we see

$$\left\langle \xi, \widehat{d} \right\rangle \geq 0, \qquad \forall\, \widehat{d} \in T(P, \bar{x}).$$

In sum, in view of Theorem 2.5(a) we can conclude that

$$f^0\left(\bar{x}; \widehat{d}\right) \geq 0, \qquad \forall\, \widehat{d} \in T(P, \bar{x}). \tag{6}$$

• Let $d \in conv\,(T(P, \bar{x}))$. Then, there exist scalars $\alpha_1, \ldots, \alpha_l \geq 0$ and vectors $\widehat{d}^1, \ldots, \widehat{d}^l \in T(P, \bar{x})$, such that

$$\sum_{v=1}^{l} \alpha_v = 1, \qquad d = \sum_{v=1}^{l} \alpha_v \widehat{d}^v.$$

Using the concavity of $f^0(\bar{x}; .)$ and inequality (6), we get

$$f^0(\bar{x}; d) = f^0\left(\bar{x}; \sum_{v=1}^{l} \alpha_v \widehat{d}^v\right) = \sum_{v=1}^{l} \alpha_v f^0\left(\bar{x}; \widehat{d}^v\right) \geq 0. \tag{7}$$

Therefore

$$(\partial_c f(\bar{x}))^s \cap conv\,(T(P, \bar{x})) = \varnothing. \tag{8}$$

• Taking into consideration the continuity of $f^0(\bar{x}; .)$ and the validity of (8), the proof is complete.

□

Lemma 4.2 *Let $\bar{x}$ be an optimal solution of SIP. then*

$$(\partial_c f(\bar{x}))^s \cap A(P, \bar{x}) = \varnothing$$

Proof In the previous lemma, the concavity of $f^0(\bar{x}; .)$ is used only for (7). Therefore, with regard to (1) and (6), the proof is complete. □

We can now formulate our main results.

Theorem 4.3 (KKT condition) *Suppose that $\bar{x}$ is an optimal solution of SIP, and assume that the KTCQ is fulfilled at $\bar{x}$.*

(a) :

$$0 \in \partial_c f(\bar{x}) + \overline{cone}\,(Z(\bar{x}))\,. \tag{9}$$

(b): *If, in addition, cone* $(Z(\bar{x}))$ *is closed, then there exist nonnegative scalars* $\lambda_i, \ i \in I^{\bar{x}}$*, finite number of them not vanishing, such that*

$$0 \in \partial_c f(\bar{x}) + \sum_{i \in I^{\bar{x}}} \lambda_i \partial_c g_i(\bar{x}). \tag{10}$$

Proof (a): Let $d \in Z^0(\bar{x})$. We conclude $f^0(\bar{x}; d) \geq 0$ in view of Lemma 4.2 and KTCQ. Since

$$Z^0(\bar{x}) = (\overline{cone}\,(Z(\bar{x})))^0\,,$$

we can obtain

$$f^0(\bar{x}; d) \geq 0, \qquad \forall d \in (\overline{cone}\,(Z(\bar{x})))^0 := V(\bar{x}).$$

Thus, the following convex function attains its minimum at $\bar{d} = 0$,

$$\Psi(.) := \Phi_{V(\bar{x})}(.) + f^0(\bar{x}; .),$$

where $\Phi_X(.)$ denotes the indicator function of $X \subseteq \mathbb{R}^n$, it is defined as follows

$$\Phi_X(y) := \begin{cases} 0 & \text{if } y \in X \\ +\infty & \text{if } y \notin X. \end{cases}$$

Hence, by parts (b), (d), and (f) of Theorem 2.5, we have

$$0 \in \partial\Psi(0) = \partial\Phi_{V(\bar{x})}(0) + \partial f^0(\bar{x}; .)(0) = \overline{cone}(Z(\bar{x})) + \partial_c f(\bar{x}).$$

(b): It follows from (9) and Theorem 2.3.

□

Theorem 4.4 (KKT condition) *Let* $\bar{x}$ *be an optimal solution of SIP, and assume that the GCQ satisfies at* $\bar{x}$*. Furthermore, suppose that* $f^0(\bar{x}; .)$ *is a concave function. Then,* (a) *and* (b) *in Theorem* 4.3 *are satisfied.*

Proof It follows from Lemma 4.1 and proof of Theorem 4.3. □

As consequences of Theorems 3.7 and 4.3, we obtain the following result.

Corollary 4.5 *Suppose that* $\bar{x}$ *is a optimal solution of SIP, the PLV property satisfies at* $\bar{x}$*, and CCQ holds at* $\bar{x}$*. Then,* (a) *and* (b) *in Theorem* 4.3 *are satisfied.*

The following result is immediate from Theorem 4.3 and Corollary 3.11, which was proved in [18].

Corollary 4.6 *Suppose that* $\bar{x}$ *is a optimal solution of a convex SIP, and SCQ holds at* $\bar{x}$*. Then, there exist nonnegative scalers* $\lambda_i, \ i \in I^{\bar{x}}$*, finite number of them not vanishing, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in I^{\bar{x}}} \lambda_i \partial g_i(\bar{x}).$$

Note that *cone* $(Z(\bar{x}))$ is assumed to be closed in Theorems 4.3(b) and 4.4(b). The following example shows that this assumption can not be dropped.

Example For any $i \in I = \{2, 3, \ldots\}$, let

$$D_i = conv\left\{\left(-h, -h^i\right) \mid 0 \le h \le 1\right\}.$$

Suppose that $f(x_1, x_2) = x_1$, $\bar{x} = (0, 0)$, and g_i is the support function of D_i, i.e.,

$$g_i(x) = \sup_{b \in D_i} \langle b, x \rangle .$$

We obtain:

$$\begin{aligned}
P &= \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0,\ x_1 + x_2 \ge 0\right\},\\
I^{\bar{x}} &= I,\\
\partial_c g_i(\bar{x}) &= D_i,\\
Z(\bar{x}) &= \left\{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_1 < 0,\ x_1 \le x_2 < -x_1^2\right\} \cup \{(0, 0), (1, 1)\},\\
Z^0(\bar{x}) &= P = A(P, \bar{x}) = \overline{conv}\,(T(P, \bar{x})),\\
\nabla f(\bar{x}) &= (1, 0),
\end{aligned}$$

and

$$cone\,(Z(\bar{x})) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \ge x_1,\ x_1 < 0,\ x_2 < 0\right\} \cup \{(0, 0)\}.$$

Note that $cone\,(Z(\bar{x}))$ is not closed, GCQ and KTCQ are satisfied at $\bar{x}$, $f^0(\bar{x}; .)$ is linear, and $\bar{x}$ is an optimal solution of the problem. It is easy to see that there is no sequence of scalars as in Theorem 4.3(b) satisfying (10). Moreover, a short calculation shows that

$$(0, 0) \in \nabla f(\bar{x}) + \overline{cone}\,(Z(\bar{x})).$$

Remark 4.7 On CCQ in min-max smooth SIP
We consider the following feasible set

$$\mathrm{M}^{\max} := \left\{x \in \mathbb{R}^n \mid \max_{0 \le k \le s} g_k(x, y) \ge 0,\ \text{for all } y \in \mathbb{R}^m\right\},$$

with continuously differentiable functions $g_k(.,.)$.
$\mathrm{M}^{\max}$ is a particular feasible set for nonsmooth SIP. The CCQ reads at $\bar{x} \in \mathrm{M}^{\max}$:

$$\text{there exists } \xi \in \mathbb{R}^n \text{ such that } \langle \xi, v \rangle > 0 \text{ for all } v \in Z(\bar{x}), \tag{11}$$

where

$$\begin{aligned}
Z(\bar{x}) &= \bigcup_{y \in M(\bar{x})} \partial_c^x \left(\max_{0 \le k \le s} g_k\right)(\bar{x}, y) = \bigcup_{y \in M(\bar{x})} conv\,(D_x g_k(\bar{x}, y), k \in K_0(\bar{x}, y)),\\
M(\bar{x}) &= \left\{y \in \mathbb{R}^m \mid \max_{0 \le k \le s} g_k(\bar{x}, y) = 0\right\},\\
K_0(\bar{x}, y) &= \{k \in \{0, 1, \ldots, s\} \mid g_k(\bar{x}, y) = 0\},\ \ y \in M(\bar{x}).
\end{aligned}$$

However, in the recent paper [9] it was suggested that in contrast to (11) only some special convex combinations from $Z(\bar{x})$ are needed. Namely, the sym-MFCQ is said to be fulfilled at $\bar{x} \in \mathrm{M}^{\max}$ if

$$\text{there exists } \xi \in \mathbb{R}^n \text{ such that } \langle \xi, v \rangle > 0 \text{ for all } v \in V(\bar{x}), \tag{12}$$

where

$$V(\bar{x}) = \bigcup_{y \in M(\bar{x})} \left\{ \sum_{k \in K_0(\bar{x},y)} \lambda_k D_x g_k(\bar{x}, y) \middle| \sum_{k \in K_0(\bar{x},y)} \lambda_k D_y g_k(\bar{x}, y) = 0, \right.$$

$$\left. \sum_{k \in K_0(\bar{x},y)} \lambda_k = 1, \ \lambda_k \geq 0 \right\}.$$

It was proven there that sym-MFCQ is stable and generic with respect to C^1-perturbations of defining functions g_k. Moreover, under sym-MFCQ a KKT type condition is obtained. We point out that the explicit dependence on y in $V(\bar{x})$ is crucial (see also discussion in [9]).

We conclude that in some particular situations (e.g. min-max smooth SIP) the set $Z(\bar{x})$ from above can be replaced by a smaller set ($V(\bar{x})$, here). This gives rise to formulate corresponding weaker constraint qualifications (cf. (12)).

Certainly, similar consideration can't be done in general case of nonsmooth SIP, since the dependence of $g_i(.)$ on a parameter i is not given explicitly.

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