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# Piece adding technique for convex maximization problems

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**Abstract** In this article we provide an algorithm, where to escape from a local maximum $y$ of convex function $f$ over $D$, we (locally) solve piecewise convex maximization $\max\{\min\{f(x) - f(y), p_y(x)\} \mid x \in D\}$ with an additional convex function $p_y(\cdot)$. The last problem can be seen as a strictly convex improvement of the standard cutting plane technique for convex maximization. We report some computational results, that show the algorithm efficiency.

**Keywords** Global search algorithm · Local search algorithm · Nonconvex optimization · Convex maximization · Piecewise convex maximization

**Mathematics Subject Classification (2000)** 90C26 · 90C47 · 49M05 · 49M30

## 1 Introduction

In this article we consider the non-convex optimization problem:

$$\begin{cases} \text{maximize} & f(x), \\ \text{subject to} & x \in D \end{cases} \tag{$CM$}$$

where $f : \mathcal{R}^n \to \mathcal{R}$ is a convex continuous function and D is a nonempty, convex compact in $\mathcal{R}^n$ defined by

$$D = \{x \in \mathcal{R}^n \mid Ax \le b\} = \left\{x \in \mathcal{R}^n \mid \left\langle a^i, x\right\rangle \le b_i, \quad i = 1, 2, \ldots, m\right\}.$$

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In the literature, this problem is known also as concave minimization or concave programming. Many applications lead to convex maximization problems; moreover, it turns out that convex maximization techniques also play an important role in other fields of global optimization: fixed charge and economies of scale, integer programming, bilinear programming, complementary problems, multiplicative programs [17]. A good deal of literature exists on methods for solving problem $(CM)$ : vertex, facet enumeration, polyhedral partitions, branch and bound, cutting plane, inner and outer approximation [2,4–6,17,18].

Several interesting necessary and sufficient global optimality conditions characterizing a point $z \in D$ satisfying

$$f(z) \geq f(x) \quad \forall x \in D$$

have been proposed [9,19–21,23].

Here we recall necessary optimality condition for local optimal solution $y$ of $(CM)$:

$$\partial f(y) \cap N(D, y) \neq \emptyset,$$

as well as necessary and sufficient (under assumption $\exists v : f(v) < f(z)$) global optimality condition [23] for $z \in D$ to be a global maximum for $(CM)$ :

$$\partial f(\bar{y}) \cap N(D, \bar{y}) \neq \emptyset \quad \text{for all } \bar{y} \text{ such that } f(\bar{y}) = f(z), \tag{1}$$

where $\partial f(\cdot)$, $N(\cdot, \cdot)$ stand respectively for the subdifferential, the normal cone.

For the state-of-the-art in convex maximization including various algorithms [1,7,8,10] and abundant applications, we refer to the textbooks [17,18] and to survey [3].

It is worth noticing that in spite of being NP-hard, a local search for $(CM)$ is relatively easy due to the method [11,12,22]

$$x^{k+1} = \text{argmax} \left\{ \left\langle \nabla f(x^k), x \right\rangle \mid x \in D \right\}.$$

We call this algorithm shortly as **[CM Local Search** $(x)$**]**, with starting point $x$.

Though the problem $(CM)$ is the most studied one among non-convex problems and have been studied actively over the last four decades, it remains the challenge in global search step: how to escape from a local maximum?

Let us concentrate at the cutting plane method. Let a local maximum $y \in D$ be a vertex of full dimensional polytope $D$. Using $n$ edges at $y$, one can construct hyperplane $\{x \mid \langle c, x\rangle = \gamma\}$, that contains $n$ points $y^1, y^2, \ldots, y^n$, which are intersections of the edges with level set $\{x \mid f(x) = f(y)\}$ [17].

If $y$ is not a global maximum, by the convexity of the objective function $f(\cdot)$ problem $(CM)$ is equivalent to

$$\begin{cases} \text{maximize} & f(x), \\ \text{subject to} & x \in D, \quad \langle c, x\rangle \geq \gamma \end{cases}$$

In other words, one cuts off a part of $D$, where values of function $f(\cdot)$ are less or equal than $f(y)$. The same procedure is then applied to the remaining part of the feasible set whenever this part is not empty.

However, despite such nice theoretical idea, this approach suffers, in practice, from the tailing off effect, i.e. cutting planes become closer or nearly parallel due to rounding errors so that they generate more and more local maxima.

In Sect. 2, we give a flavor of how to introduce patches in order to escape from a local solution; in Sect. 3, devoted to piecewise convex maximization problem, we recall some basic definitions and results on the problem studied elsewhere. The remaining sections focus

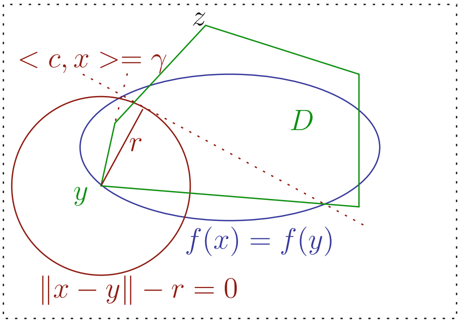


**Fig. 1** From cutting plane to convex cutting

on theoretical necessary conditions to design a global search algorithm and how it behaves on some examples from the literature.

## 2 Escaping from a local solution region

**Definition 1** Let $z$ be the global solution to $(CM)$. A strictly convex function $p_y : \mathcal{R}^n \rightarrow \mathcal{R}$ is a called **a patch around a local solution** $y$ of the problem $(CM)$ if

- $p_y(y) < 0$;
- $p_y(x) \geq 0$ for all $x \in D$ such that $f(x) \geq f(y), x \neq y$;
- $p_y(z) > 0$.

*Remark 1* If $p_y(x)$ is a patch, then the reverse convex constraint

$$p_y(x) \geq 0$$

cuts off the local solution $y$, therefore it can be considered like *a convex cutting* since $p_y(z) > 0$ for the global solution $z$.

**Definition 2** A strictly convex function $p_y : \mathcal{R}^n \rightarrow \mathcal{R}$ is a called **a pseudo patch** at a local solution $y$ of the problem $(CM)$ if

- $p_y(y) < 0$ and
- there is $u \in D$ such that $f(u) > f(y)$ and $p_y(u) \geq 0$.

**Lemma 1** *Let $\langle c, x\rangle \geq \gamma$ be a cutting plane of a local solution $y$ to the convex maximization problem $(CM)$ such that $\langle c, y\rangle < \gamma$. Then the following function*

$$p_y(x) = \|x - y\|^2 - \frac{\gamma - \langle c, y\rangle}{\|c\|^2}$$

*is a patch at $y$* (Fig. 1).

*Proof* As one can easily see the scalar

$$r = \frac{\gamma - \langle c, y\rangle}{\|c\|^2}$$

is the minimal distance from $y$ to the hyperplane $\langle c, x\rangle = \gamma$.

For all $x \in D$ such that $f(x) \geq f(y)$, inequality $\langle c, x\rangle \geq \gamma$ implies

$$\|x - y\|^2 - \frac{\gamma - \langle c, y\rangle}{\|c\|^2} \geq 0.$$

On the other hand, by construction of the cutting plane, since $y$ is not global maximum, the following inequality

$$\langle c, z\rangle > \gamma$$

holds for the global maximum $z$. The latter, together with

$$p_y(y) = -\frac{\gamma - \langle c, y\rangle}{\|c\|^2} < 0$$

prove that the function

$$p_y(x) = \|x - y\|^2 - \frac{\gamma - \langle c, y\rangle}{\|c\|^2}$$

is a patch. □

We will use further notations

$$I(x) = \left\{i \mid \left\langle a^i, x\right\rangle = b_i\right\}, \quad J(x) = \left\{i \mid \left\langle a^i, x\right\rangle < b_i\right\} \tag{2}$$

for respectively, set of active and nonactive constraints at given point $x \in D$.

We calculate the distance from given $x \in D$ to the nearest nonactive constraint like

$$\rho = \min\{\|x^j - x\|^2 \mid j \in J(x)\}, \tag{3}$$

where $x^j$ are projections of $x \in D$ on hyperplanes $\left\langle a^j, \cdot\right\rangle = b_j$, more precisely

$$x^j = x + \frac{b_j - \left\langle a^j, x\right\rangle}{\|a^j\|^2} a^j.$$

**Theorem 1** *Let $f(\cdot)$ be strictly convex, $y$ be a local maximum different from the global maximum $z$ with $J(y) \neq \emptyset$ and $\rho$ be calculated by (3).*

*Then function $p_y(x) = \|x - y\|^2 - \rho$ is a pseudo patch.*

*Proof* $J(y) \neq \emptyset$ implies that $\rho > 0$, which proves $p_y(y) = -\rho < 0$.

Now, we show that there is no other vertex $v \in D$ such that $p_y(v) < 0$.

Suppose the contrary : there is vertex $v \neq y$ such $p_y(v) < 0$ Since $y \neq v$ we have $I(v) \neq I(y)$, that implies $\exists k \in V$ and $k \in J(y)$.

$$\rho > \|v - y\|^2 \geq \|y^k - y\|^2 \geq \rho,$$

a contradiction.

We know that the global maximum $z$ of $f(\cdot)$ over a convex set is attained at a extremum point of $D$, at a vertex, in the case of polytope. Taking $z$ for $u$ we obtain

$$u \in D, \quad p_y(u) \geq 0, \quad f(u) > f(y),$$

which conclude that $p_y(\cdot)$ is a pseudo patch. □

So for finding a global solution to convex maximization problem, taking account the obtained results, we propose to solve the following so-called piecewise convex maximization problem at each local maximum $y$ found.

$$\begin{cases} \text{maximize} & F(x), \\ \text{subject to} & x \in D \end{cases}$$

where objective function of $(CM)$ is replaced by

$$F(x) = \min\{f(x) - f(y), p_y(x)\}.$$

Here by adding a convex piece into objective function we cut off "virtually" a part of $D$ containing $y$, where values of function $f(\cdot)$ are less or equal than $f(y)$. The same procedure is then applied at the next better local solution.

### 2.1 Example

We consider the one dimensional convex maximization problem:

$$\begin{cases} \text{maximize} & x^2 - 2x, \\ \text{subject to} & 0 \leq x \leq 3. \end{cases}$$

If we apply most local search algorithms with a starting point $x^0 \in [0, 1]$, then we (almost surely) will stop at $y = 0$, that is not the global maximum. To escape from this local solution region, it is reasonable to choose $x$ such that $(x - y)^2 - r \geq 0$ for some radius $r > 0$ issued, say from cutting plane technique. Somehow, "escaping from $y$ region" suggests to maximize convex function $(x - y)^2 - r$.

So, if we solve the following problem with additional convex piece $x^2 - 4$ into objective function,

$$\begin{cases} \text{maximize} & \min\{x^2 - 2x, x^2 - 4\}, \\ \text{subject to} & 0 \leq x \leq 3, \end{cases}$$

it is clear (Fig. 2) that any local search algorithm easily finds the global solution $z = 3$ from any starting point.

## 3 Piecewise convex maximization problem

Piecewise convex maximization problems have been studied in our earlier works [15,24]. Therein necessary and sufficient global optimality conditions have been established and furthermore local, global search algorithms are described. Convergence and further details of an implementation are discussed also. Moreover, in [16] the algorithms have showed to be promising for solving multiknapsack problem, the celebrated example from combinatorial optimization.

In order to be self contained, we recall from [15,24] two definitions and local search algorithm, that we need further of the article.

Let $M$ be a finite index set.

**Definition 3** A function $F : R^n \to R$ is called a piecewise convex function if it can be decomposed into :

$$F(x) = \min\{f_j(x) \mid j \in M\}, \tag{4}$$

where $f_j : R^n \to R$ is convex for all $j \in M = \{1, 2, \ldots, m\}$.

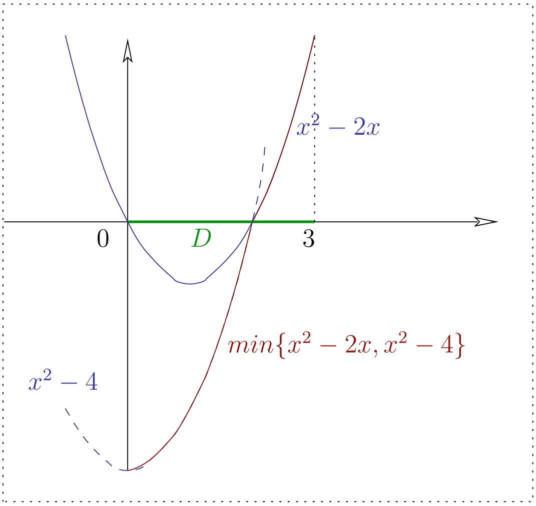


**Fig. 2** Piecewise Adding Technique (PATCH) idea

**Definition 4** A problem

$$\begin{cases} \text{maximize} & F(x), \\ \text{subject to} & x \in D \end{cases} \tag{$PCMP$}$$

is called a piecewise convex maximization problem, if $F(\cdot)$ is a piecewise convex function.

### 3.1 Local search algorithm for (PCMP) [15]

Let $x^k$ be a feasible point in (PCMP), for $k = 0, 1, 2, \ldots$.

We define Lebesgue's set like

$$\mathcal{L}_f(\alpha) = \{x \mid f(x) \leq \alpha\}$$

and similarly to (2) we keep the following notations respectively for active and nonactive index sets:

$$I(x) = \{i \in M \mid f_i(x) = F(x)\}$$
$$J(x) = \{j \in M \mid f_j(x) > F(x), \quad \mathcal{L}_{f_j}(F(x)) \neq \emptyset\}.$$

By definition of $F(\cdot)$ both $I(x^k) \neq \emptyset$ and $x^k \notin \mathcal{L}_{f_j}(F(x^k))$ for all $j \in J(x^k)$ are true.

We introduce a set of polytopes $P^k$ related to current point $x^k$:

$$P^k = \left\{ x \in \mathcal{R}^n \mid \begin{array}{ll} \left\langle \nabla f_j(x^k), x \right\rangle \geq \left\langle \nabla f_j(x^k), v^j \right\rangle, & j \in J(x^k) \\ \left\langle \nabla f_i(x^k), x \right\rangle \geq \left\langle \nabla f_i(x^k), x^k \right\rangle, & i \in I(x^k) \end{array} \right\}$$

where $v^j = \arg\max \left\{ \left\langle \nabla f_j(x^k), x \right\rangle \mid x \in \mathcal{L}_{f_j}(F(x^k)) \right\}, \quad j \in J.$

It is proven [15] that a sequence $\{x^k\}$ generated by the following algorithm converges to a stationary point of (PCMP), moreover numerical sequence $\{F(x^k)\}$ is nondecreasing and convergent.

**[PCMP Local Search** $(D, M)$**]**

Let $x^k \in D$ be given, $k = 0, 1, 2, \ldots$

**for all** $m \in M$
  $w^m = \arg\max \left\langle \nabla f_m(x^k), x \right\rangle \text{s.t. } x \in D \cap P^k$
**endforall**
$w^r = \arg\max\{\min_{i \in M} f_i(w^m) \mid m \in M\}$
**if** $\| w^r - x^k \| \leq \epsilon$
  **then** stop /* local solution found */
  **else** $k = k + 1; x^k = w^r;$

## 4 Necessary global optimality condition checking

For a differentiable convex function $f(\cdot)$ the necessary global optimality condition (1) can be written like

$$z = \operatorname{argmax}(CM) \Rightarrow \langle \nabla f(\bar{y}), x - \bar{y} \rangle \leq 0, \quad \forall x \in D, \ \forall \bar{y} : f(\bar{y}) = f(z).$$

Let $y$ be a local maximum of $(CM)$. Obviously, $y$ is not the global maximum, if there are $\bar{y} : f(\bar{y}) = f(y), \quad u \in D$ such that $0 < \langle \nabla f(\bar{y}), u - \bar{y} \rangle$.

For $j \in J(y)$ we consider the following convex problem :

$$\begin{cases} \text{maximize} & \left\langle a^j, x \right\rangle, \\ \text{subject to} & f(x) \leq f(y), x \in D_j, \end{cases}$$

where $a^j \in \mathcal{R}^n$ is the normal vector of $j$-th constraint and $D_j$ defined like

$$D_j = \left\{ x \mid \left\langle a^k, x \right\rangle \leq b_k; \quad \text{for all } k \neq j \right\}.$$

**Proposition 1** *Let* $w^j = \operatorname{argmax}\{\left\langle a^j, x \right\rangle \mid f(x) \leq f(y), x \in D_j\}$.
*If* $\left\langle a^j, w^j \right\rangle < b_j$, *then* $y$ *is not the global solution to* $(CM)$.

*Proof* By first order optimality conditions, there are $\lambda \geq 0, \mu_k \geq 0, k = 1, \ldots, m, k \neq j$ such that

$$\begin{cases} a^j - \lambda f(w^j) - \sum_{k \neq j} \mu_k a^k = 0, \\ \lambda(f(w^j) - f(y)) = 0, \\ \mu_k \left( \left\langle a^k, w^j \right\rangle - b_k \right) = 0, \quad \forall k = 1, \ldots, m, \ k \neq j. \end{cases}$$

Hence, for any $x \in D$ satisfying $\left\langle a^j, x \right\rangle = b_j$, the inequality $\left\langle a^j, w^j \right\rangle < b_j$ implies

$$0 < \left\langle a^j, x - w^j \right\rangle = \lambda \left\langle \nabla f(w^j), x - w^j \right\rangle + \sum_{k \neq j} \mu_k \left\langle a^k, x - w^j \right\rangle.$$

On the other hand, it is not difficult to see that for $x \in D_j$

$$\sum_{k \neq j} \mu_k \left\langle a^k, x - w^j \right\rangle \leq \sum_{k \neq j} \mu_k \left( b_k - \left\langle a^k, w^j \right\rangle \right) = 0,$$

and therefore

$$0 < \left\langle a^j, x - w^j \right\rangle \leq \lambda \left\langle \nabla f(w^j), x - w^j \right\rangle$$

Since $f(\cdot)$ is convex, one has

$$0 < \left\langle \nabla f(w^j), x - w^j \right\rangle \leq f(x) - f(w^j) = f(x) - f(y),$$

which proves the proposition $\exists x \in D : f(x) > f(y)$. □

**Corollary 1** *Let* $\bar{w}^j = \text{argmax}\{\langle a^j, x\rangle \mid f(x) \leq f(y)\}$.
*If* $\left\langle a^j, \bar{w}^j \right\rangle < b_j$, *then y is not the global solution to* $(CM)$.

*Remark 2* We note that in the case of quadratic function

$$f(x) = \frac{1}{2}\langle Qx, x\rangle - \gamma,$$

with positive definite matrix $Q$ and $\gamma > 0$, the above problem is solved analytically:

$$\bar{w}^j = \left( \frac{\langle Qy, y\rangle}{\langle Q^{-1}a^j, a^j\rangle} \right)^{\frac{1}{2}} Q^{-1}a^j.$$

## 5 Piece Adding TeCHnique or PATCH algorithm

The idea of piece adding technique (or PATCH algorithm) is that once a local solution is found, the method constructs so-called "patch" around the local solution to prevent returning back its region and does a local search for piecewise convex maximization problem with additional piece.

More precisely, for escaping from a local solution region we propose to solve locally the following (PCMP)

$$\begin{cases} \text{maximize} & \min\{f_1(x), \ldots, f_m(x)\} \\ \text{subject to} & x \in D, \end{cases}$$

with

$$f_1(x) = f(x) - f(\hat{y})$$
$$f_k(x) = p_{k-1}(x), k = 1, \ldots, m$$

where $p_k(x) = p_{y^k}(x)$, a patch around local solution $y^k$ of $(CM)$ and $\hat{y}$ is the best between $\{y^1, \ldots, y^m\}$.
Now we are in position to describe our global search algorithm.

**[Global Search** $(x)$**]**

1. $y$ = [CM Local Search $(x)$]
2. necessary global optimality conditions checking at $y$
   **if** $(\exists j : \langle a^j, w^j\rangle < b_j)$
   **then** $x = Pr_D(w^j)$ and goto 1.
   **else** construct a ( pseudo ) patch $p_y(\cdot)$

3. u = [PCMP Local Search $(D, M)$]
   **if** $f(u) \geq f(y)$
   **then** $x = u$ and goto 1.
   **else** pop(patches) and goto 3.

### 5.1 The algorithm convergence

Assume that $m = 2$ and we compare two problems

$$(CM)\begin{cases} \text{maximize} & f(x) \\ \text{subject to} & x \in D, \end{cases} \qquad (CMP)\begin{cases} \text{maximize} & \min\{f(x) - f(y), p_y(x)\} \\ \text{subject to} & x \in D, \end{cases}$$

We make the following assumption.

**Assumption 1** There is a function $p_y(x)$ defined like a patch, that satisfies inequality

$$p_y(z) \geq f(z) - f(y) \tag{A}$$

at the global maximum $z$.

**Proposition 2** *If* $z \in \operatorname{argmax}(CM)$ *then there exists some patch* $p_y(\cdot)$ *fulfilling assumption (A) such that* $z \in \operatorname{argmax}(CMP)$

*Proof* Suppose that, for all $p_y(\cdot)$ such that $p_y(z) \geq f(z) - f(y)$, $z \notin \operatorname{argmax}(CMP)$ then

$$0 \leq f(z) - f(y) = \min\{f(z) - f(y), p_y(z)\} < \max_{x \in D} \min\{f(x) - f(y), p_y(x)\}$$

From patch definition and strict greater than 0 inequality, we get both $p_y(x) \geq 0$ and $f(x) - f(y) \geq 0$. Therefore, there exists $\bar{x} \in D$ such that

$$0 \leq f(z) - f(y) < \min\{f(\bar{x}) - f(y), p_y(\bar{x})\}$$

Either, $f(\bar{x}) - f(y) \leq p_y(\bar{x})$ then $f(z) < f(\bar{x})$ contradicts global maximum at $z$ for (CM), or $p_y(\bar{x}) < f(\bar{x}) - f(y)$ and $f(z) - f(y) < p_y(\bar{x})$ implies $f(z) < f(y) + p_y(\bar{x}) < f(\bar{x})$, the same contradiction. □

*Remark 3*

– For the converse (sufficient condition), let us consider $z \in \operatorname{argmax}(CMP)$ then

$$\min\{f(x) - f(y), p_y(x)\} \leq \min\{f(z) - f(y), p_y(z)\} \quad \text{for all } x \in D \tag{5}$$

From assumption (A), $\min\{f(z) - f(y), p_y(z)\} = f(z) - f(y)$ therefore

$$\min\{f(x) - f(y), p_y(x)\} \leq f(z) - f(y) \quad \text{for all } x \in D,$$

equivalently

$$\min\{f(x), p_y(x) + f(y)\} \leq f(z) \quad \text{for all } x \in D,$$

However, at this point, without any further information about the actual patch $p_y(\cdot)$ we cannot conclude that $z \in \operatorname{argmax}(CM)$.

– On the other hand, (CM) remains invariant under the patch constraint $p_y(x) \geq 0$; obviously, it is not true for a pseudopatch constraint. According to Lagrangian rules, there exists some multiplier $\lambda \geq 0$ such that a solution of $(CM)$ maximizes also the Lagrangian function :

$$f(x) - f(y) + \lambda p_y(x) = \min\{f(x) - f(y), \lambda p_y(x)\} + \max\{f(x) - f(y), \lambda p_y(x)\}.$$

Intuitively speaking, at global optimum, second term should cancel so that both $(CM)$, $(CMP)$ problems are equivalent upto $\lambda$; moreover, from $p_y(z) \geq f(z) - f(y) > 0$, we could see that $p_y(z)$ measures the quality of the local solution $y$.

## 6 Computational experiments

The patch algorithm has been tested on a bunch of convex maximization problems taken from "A collection of test problems for constrained global optimization algorithms" , "Handbook of test problems in local and global optimization" [13,14].

We present the numerical results in the table below and the meanings for all columns in the table follow:

- number of variables;
- number of constraints ;
- number of $(CM)$ local searches;
- number of $(PCMP)$ local searches;
- the best value found
- the global optimal value.

| Problem | $n$ | $m$ | CM LS | PCMP LS | The best value | Optimal value |
|---|---|---|---|---|---|---|
| fp1 | 5 | 11 | 3 | 73 | 17.0 | 17.0 |
| fp2 | 6 | 13 | 1 | 0 | 361.5 | 361.5 |
| fp3 | 13 | 35 | 1 | 0 | 195.0 | 195.0 |
| fp4 | 6 | 10 | 4 | 2 | 11.0 | 11.0 |
| fp5 | 10 | 31 | 2 | 2 | 268.01463 | 268.01463 |
| fp6 | 10 | 25 | 2 | 2 | 39.0 | 39.0 |
| fp7-1 | 20 | 30 | 10 | 93 | 394.0814 | **394.7506** |
| fp7-5 | 20 | 30 | 22 | 220 | 4150.41013 | 4150.41013 |

The problem 7 is formulated with the sets of parameters and these parameter values $(\lambda_i = 1, \alpha_i = 2)(\lambda_i = i, \alpha_i = 2)$ provide the problems fp7-1, fp7-5 respectively.

The global optimal solutions are found for almost all test problems considered and it confirms the promisses of such a technique whose generalization to (PCMP) itself is straightforward.

## 7 Concluding remarks

In this article, we improve standard linear separation for convex maximization by using a strictly convex separation function instead. The global search algorithm incrementally adds patch function derived from (CM) $\Rightarrow$ (CMP) therefore, there is no proof certificate of global optimality. The patch function actually forbids to return back to a local maximum, while pseudopatch guesses a good candidate for improvement. Any global search algorithm for convex maximization somehow *enumerates* the extremal points in the domain and its efficiency highly depends on the ordering. Here, the pseudopatches early detect good candidates due to their deep cutting capability.

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