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# Proximal methods for a class of bilevel monotone equilibrium problems

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**Abstract** We consider a bilevel problem involving two monotone equilibrium bifunctions and we show that this problem can be solved by a simple proximal method. Under mild conditions, the weak convergence of the sequences generated by the algorithm is obtained. Using this result we obtain corollaries which improve several corresponding results in this field.

**Keywords** Bilevel problem · Variational inequality · Monotonicity · Equilibrium problem · Proximal method

**AMS Subject Classification (2000)** 90C25 · 49M45 · 65C25

## 1 Introduction

Throughout, $\mathcal{H}$ is a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $| \, . \, |$. Let $C$ be a nonempty closed convex subset of $\mathcal{H}$, our interest is in finding a solution to the following problem

$$\text{find} \quad \bar{x} \in S_F \quad \text{such that} \quad H(\bar{x}, y) \geq 0, \quad \forall y \in S_F, \tag{1.1}$$

where $F, H : C \times C \to \mathbb{R}$ are two bifunctions and $S_F$ denotes the set of solutions to the following equilibrium problem

$$\text{find} \quad u \in C \quad \text{such that} \quad F(u, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

This class is very interesting because it covers mathematical programs and optimization problems over equilibrium constraints, hierarchical minimization problems, variational inequality, complementarity problems... On the other hand, the framework is general enough and permits

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us to treat in a unified way several iterative schemes, recovering, developing and improving some known related convergence results in this field. We wish also to point out that by setting $H(x, y) = \varphi(y) - \varphi(x)$ and $F(x, y) = \sup_{z \in T(x)} \langle z, y - x \rangle$ with $T$ is for example a maximal monotone operator, problem (1.1) reduces to

$$\min_{0 \in \tilde{T}(x)} \tilde{\varphi}(x),$$

a mathematical program with generalized equation constraint [7], where $\tilde{\varphi} = \varphi + \delta_C$ and $\tilde{T} = T + N_C$, $\delta_C$ being the indicator function of $C$ and $N_C$ standing for the normal cone to $C$. Now, by taking $F(x, y) = \varphi(y) - \varphi(x)$ (resp. $F(x, y) = \langle x - P(x), y - x \rangle$ with $FixP = \{x \in C; x = P(x)\}$) and $H(x, y) = \psi(y) - \psi(x)$, where $\psi, \varphi$ are two lower semi-continuous convex functions, the latter reduces to the following hierarchical minimization problem considered in Cabot [3], see also Solodov [14] (resp. to a minimization problem with respect to a fixed point set studied in [19], see also [9]):

$$\min_{x \in argmin \tilde{\varphi}} \tilde{\psi}(x) \quad (\text{resp. } \min_{x \in FixP} \tilde{\psi}(x)).$$

In recent years, methods for solving equilibrium problems have been studied extensively. In [10], Moudafi extended the proximal method to monotone equilibrium problems and in [6], Konnov used the proximal method to solve equilibrium problems with weakly monotone bifunctions. Recently, Mastroeni in [8] extended the so-called auxiliary problem principle to strong monotone equilibrium problems. Other solution methods such as bundle methods and extragradient methods are extended to equilibrium problems in [17] and [18] and very recently, a numerical viscosity approach was considered in [16].

In this paper, we are interested in approximating a solution of the bilevel problem (1.1), where the bifunction $F$ and $H$ satisfy the following usual conditions:

(A1) $K(x, x) = 0$ for all $x, y \in C$;
(A2) $K$ is monotone, i.e., $K(x, y) + K(y, x) \leq 0$ for all $x, y \in C$;
(A3) $\lim_{t \downarrow 0} K(tz + (1 - t)x, y) \leq K(x, y)$ for any $x, y, z \in C$;
(A4) for each $x \in C$, $y \to K(x, y)$ is convex and lower-semicontinuous.

To this end, we show that the bilevel problem (1.1) can be solved by a very simple proximal method, iteratively applied to the parameterized family of bifunctions

$$G_\varepsilon = F + \varepsilon H, \quad \varepsilon > 0,$$

where $\varepsilon$ varies along the iterations. Namely, if $x_n$, $(n \geq 0)$ is the current iterate and $\varepsilon_n$ the current parameter, then compute the solution $x_{n+1} \in C$ of the regularized problem

$$F(x_{n+1}, y) + \varepsilon_n H(x_{n+1}, y) + \frac{1}{r_n} \langle x_{n+1} - x_n, y - x_{n+1} \rangle \geq 0 \quad \forall y \in C, \tag{1.3}$$

where $x_0 \in C$, $(r_n)$ and $(\varepsilon_n)$ are two positive sequences.

Our main purpose is to study the asymptotic convergence of the sequence $(x_n)$ generated by scheme (1.3). Under suitable conditions on the parameters, we establish the weak convergence of $(x_n)$ to a solution of (1.1). In some way, our proposal is related to [3], where a proximal point algorithm for a two-level minimization problem has been considered and a convergence result was derived, in finite dimension setting, relying on conditioning assumptions. More precisions will be given in the Remark 2.1 below. Some other proximal methods and the related weak convergence results can be derived from our main theorem by particularizing the bifunctions $F$ and $H$.

## 2 Main convergence result

In order to prove our main convergence theorem, we start with two key preliminary results, see [2], [5] or [11] and [1] or [9].

**Lemma 2.1** *Let $C$ be a nonempty closed convex subset of $\mathcal{H}$ and let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4).*
*Let $r > 0$ and $x \in \mathcal{H}$. Then there exists an unique element $z \in C$ such that:*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

**Lemma 2.2** *Let $(\xi_n)_{n\geq 0}$, $(\beta_n)_{n\geq 0}$ and $(\varepsilon_n)_{n\geq 0}$ be sequences in $\mathbb{R}_+$ such that $(\varepsilon_n)_{n\geq 0} \in l^1$ and*

$$(\forall n \in \mathbb{N})\ \xi_{n+1} \leq \xi_n - \beta_n + \varepsilon_n.$$

*Then $(\xi_n)_{n\geq 0}$ converges and $(\beta_n)_{n\geq 0} \in l^1$.*

Now, we are in a position to prove the following main theorem:

**Theorem 2.3** *Suppose that $S_F \neq \emptyset$ and that for each $y \in C$ the function $x \to H(x, y)$ is upper semicontinuous and bounded for each $y \in S_F$. Assume that $\liminf_n r_n > 0$ and $\sum_{n=0}^{+\infty} r_n\varepsilon_n < +\infty$. Then*

$$\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$$

*and the sequence $(x_n)$ generated by (1.3) weakly converges to a point in $S_F$. If in addition $\|x_{n+1} - x_n\| = o(\varepsilon_n)$, then $(x_n)$ weakly converges to a solution of (1.1).*

*Proof* Let $\bar{x} \in S_F$. Thanks to the monotonicity of $F$, we have

$$-F(x_{n+1}, \bar{x}) \geq 0.$$

On the other hand, by replacing $y$ by $\bar{x}$ in (1.3), we also have

$$F(x_{n+1}, \bar{x}) + \varepsilon_n H(x_{n+1}, \bar{x}) + \frac{1}{r_n}\langle x_{n+1} - x_n, \bar{x} - x_{n+1}\rangle \geq 0.$$

Combining the two last inequalities, we obtain

$$\langle x_{n+1} - x_n, x_{n+1} - \bar{x}\rangle \leq r_n\varepsilon_n H(x_{n+1}, \bar{x}), \tag{2.1}$$

which can be rewritten as

$$\varphi_{n+1} - \varphi_n + \frac{1}{2}\|x_{n+1} - x_n\|^2 \leq r_n\varepsilon_n H(x_{n+1}, \bar{x}),$$

where $\varphi_n = \frac{1}{2}\|x_n - \bar{x}\|^2$.

Since the function $x \to H(x, y)$ is bounded for each $y \in S_F$ and $\sum_{n=0}^{+\infty} r_n\varepsilon_n < +\infty$, the sequence $(\varphi_n)$ is convergent by virtue of Lemma 2.2 Suppose that

$$\lim_{n\to+\infty} \varphi_n = l(\bar{x}), \tag{2.2}$$

where $l(\bar{x}) \geq 0$. Hence $(x_n)$ is bounded. Furthermore, we have $\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$ which implies that $(x_n)$ is asymptotically regular, that is

$$\lim_{n \to +\infty} \|x_n - x_{n+1}\| = 0.$$

Relation (1.3) together with the property (A2) yields

$$\varepsilon_n H(x_{n+1}, y) + \frac{1}{r_n} \langle x_{n+1} - x_n, y - x_{n+1} \rangle \geq -F(x_{n+1}, y) \geq F(y, x_{n+1}) \quad \forall y \in C.$$

Let $x^*$ be a weak cluster point of $(x_n)$, then there exists a subsequence $(x_{n_k})$ which weakly converges to $x^*$. By replacing $n$ by $n_k - 1$ in the last inequalities, we obtain

$$\varepsilon_{n_k-1} H(x_{n_k}, y) + \frac{1}{r_{n_k-1}} \langle x_{n_k} - x_{n_k-1}, y - x_{n_k} \rangle \geq F(y, x_{n_k}) \quad \forall y \in C.$$

As $(x_{n_k})$ is a bounded sequence, by passing to the limit in this inequality and by taking into account that $\liminf_n r_n > 0$ and the fact that $\frac{x_{n_k} - x_{n_k-1}}{r_{n_k-1}} \to 0$, $x_{n_k} \rightharpoonup x^*$ weakly and that the function $F(y, \cdot)$ is weak lower-semicontinuous, we deduce that

$$0 \geq F(y, x^*).$$

Now, by setting $y_t = ty + (1-t)x^*$ (for $t \in (0, 1]$) and thanks to the fact that $y \in C$ and $x^* \in C$, we have $y_t \in C$ so that $F(y_t, x^*) \leq 0$.
Hence by virtue of (A1) and (A4), we get

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, x^*) \leq tF(y_t, y).$$

Consequently, we deduce

$$F(y_t, y) \geq 0,$$

which in the light of (A3) entails

$$F(x^*, y) \geq 0, \quad \forall y \in C,$$

thus $x^* \in S_F$.

The fact that there is no more than one weak cluster point is a consequence of (2.2) and follows by applying Opial's Lemma [13]. The proof is presented here for completeness. Indeed, let $\tilde{x}$ be another weak-cluster point of $(x_n)$, we will show that $\tilde{x} = x^*$. From

$$\|x_n - \tilde{x}\|^2 = \|x_n - x^*\|^2 + \|x^* - \tilde{x}\|^2 + 2\langle x_n - x^*, \tilde{x} - x^* \rangle, \tag{2.3}$$

we see that the limit of $(\langle x_n - x^*, x^* - \tilde{x} \rangle)$ must exists and has to be zero because $x^*$ is a weak cluster point of $(x_n)$. Hence, at the limit

$$l(\tilde{x}) = l(x^*) + \|x^* - \tilde{x}\|^2.$$

Reversing the role of $x^*$ and $\tilde{x}$ we also have

$$l(x^*) = l(\tilde{x}) + \|\tilde{x} - x^*\|^2.$$

This implies that $x^* = \tilde{x}$.
It remains to show that $H(x^*, y) \geq 0 \quad \forall y \in S_F$. Taking (1.3) again with $y \in S_F$, we obtain

$$H(x_{n+1}, y) + \frac{1}{r_n \varepsilon_n} \langle x_{n+1} - x_n, y - x_{n+1} \rangle \geq \frac{1}{\varepsilon_n} F(y, x_{n+1}) \geq 0,$$

which in the light of the upper semicontinuity of the function $x \to H(x, y)$ and the fact that $\|x_{n+1} - x_n\| = o(\varepsilon_n)$ implies

$$H(x^*, y) \geq 0, \quad \forall y \in S_F.$$

This completes the proof. □

*Remark 2.1*

- By taking $H \equiv 0$, we recover the main result in [8] and in the case where $F(x, y) = \varphi(y) - \varphi(x)$ and $H = \psi(y) - \psi(x)$, we obtain a result in [3].
- Remember that by taking $F(x, y) = \varphi(y) - \varphi(x)$ and $H(x, y) = \psi(y) - \psi(x)$, where $\psi, \varphi$ are two lower semicontinuous convex functions, the problem under consideration reduces to $\min_{x \in argmin \tilde{\varphi}} \tilde{\psi}(x)$. In this case conditions on the bifunction $H$ are satisfied if $\psi$ is bounded from below and the proposed algorithm can be rewritten as
$$x_{n+1} = arg \min_{y \in C} \left\{ \varphi(y) + \varepsilon_n \psi(y) + \frac{1}{2r_n} \|y - x_n\|^2 \right\} := \text{prox}^C_{r_n(\varphi + \varepsilon_n \psi)} x_n.$$
In this case, Cabot [3] proved that $\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$ provided that $(r_n)$ is bounded, $(\varepsilon_n)$ is a non-increasing sequence which converges to 0 and $\psi$ is a convex finite valued function bounded from below. If in addition $dim\mathcal{H} < +\infty$ and $(x_n)$ is bounded, he proved that the distance between $(x_n)$ and the solution set tends to zero and that $\lim_n(\varphi(x_n), \psi(x_n)) = (\min \varphi, \min_{argmin\varphi} \psi)$.
- The proposed method is implicit but can be accompanied computationally by numerically realistic approximation rules and specific implementable schemes for satisfying these rules, see for example [12,15].
- Some other methods and the related weak convergence results can be derived from our main theorem by particularizing the bifunctions $F$ and $H$.

To conclude, our hope is that this analysis can serve as a guide to the analysis of similar algorithms. An open question is to remove condition $\|x_{n+1} - x_n\| = o(\varepsilon_n)$, since one does not know the convergence-rate of the sequence $\|x_{n+1} - x_n\|$. Consequently, it is not easy to choose such a sequence $(\varepsilon_n)$. We conjecture that this assumption can be probably removed via the introduction of a conditioning notion for equilibrium bifunctions.

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