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Some applications of the image space analysis to the duality theory for constrained extremum problems

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Abstract By means of the Image Space Analysis, duality properties of a constrained extremum problem are investigated. The analysis of the lower semicontinuity of the perturbation function, related to a right-hand side perturbation of the given problem, leads to a characterization of zero duality gap in the image space.

Keywords Image space · Separation · Perturbation function

1 Introduction

The Image Space Analysis (for short, ISA) [7] has shown to be a unifying scheme for studying constrained extremum problems, variational inequalities, and, more generally, can be applied to any kind of problem, say it (P), that can be expressed under the form of the impossibility of a parametric system. In this approach, the impossibility of such a system is reduced to the disjunction of two suitable subsets $\mathcal{K}$ and $\mathcal{H}$ of the Image Space (for short, IS) associated with (P). $\mathcal{K}$ is defined by the image of the functions involved in (P), while $\mathcal{H}$ is a convex cone that depends only on the class of problems to which (P) belongs.

The disjunction of $\mathcal{K}$ and $\mathcal{H}$ can be proved by showing that they lie in two disjoint level sets of a suitable separating functional.

In the case where (P) is a constrained extremum problem, several theoretical aspects can be developed, as duality, existence of optimal solutions, Lagrangian-type optimality conditions, penalty methods and regularity [6–9,13,16].

The subclass of separating functionals that fulfil the condition that the intersection of their positive level sets coincides with $\mathcal{H}$, is said to be regular.

Duality arises from the existence of a regular separating functional such that $\mathcal{K}$ is included in its non positive level set: this is shown to be equivalent to a saddle point condition for a generalized Lagrangian function associated with (P) [7].

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A weaker condition that guarantees the empty intersection between $\mathcal{K}$ and $\mathcal{H}$ is the existence of a sequence of regular separating functionals such that $\mathcal{K}$ is asimptotically included in their non positive level sets: such an occurrence is closely related to Courant penalty methods [7].

Following similar arguments, necessary optimality conditions can be derived from the linear separation between a convex conical approximation of $\mathcal{K}$ and $\mathcal{H}$ [8].

Let (P) be defined by the following constrained extremum problem:

$$\min f(x) \ \text{ s.t. } \ x \in R := \{x \in X : g(x) \in D\}, \tag{1}$$

where X is a Banach space, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$ $g(x) := (g_1(x), \dots, g_m(x))$, and D is a closed convex cone in $\mathbb{R}^m$. Suppose that $\bar{x} \in R$; define $A_{\bar{x}} : X \longrightarrow \mathbb{R}^{1+m}$, $A_{\bar{x}} := (f(\bar{x}) - f(x), g(x))$ and the sets

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R}^{1+m} : (u, v) = A_{\bar{x}},\ x \in X\}$$
$$\mathcal{H} := \{(u, v) \in \mathbb{R}^{1+m} : u > 0,\ v \in D\}.$$

$\mathcal{K}_{\bar{x}}$ is called the *image* of (P), while the space $\mathbb{R}^{1+m}$ is the *image space* associated with (P). In order to simplify the notation, when there is no fear of confusion, we will understand the dependence of the set $\mathcal{K}$ on $\bar{x}$. We observe that $\bar{x}$ is an optimal solution for (P) iff the generalized system

$$A_{\bar{x}}(x) \in \mathcal{H}, \quad x \in X,$$

is impossible, or, equivalently,

$$\mathcal{K} \cap \mathcal{H} = \emptyset. \tag{2}$$

The dual problem of (P), that will be denoted by (D), is defined by:

$$v_D := \sup_{\lambda \in D^*} \inf_{x \in X} L(x; \lambda), \tag{3}$$

where

$$L(x; \lambda) := f(x) - \langle \lambda, g(x) \rangle$$

is the Lagrangian function associated with (P). By symmetry, (P) is associated with the problem

$$\inf_{x \in X} \sup_{\lambda \in D^*} L(x; \lambda), \tag{4}$$

which, under the assumption $R \neq \emptyset$, is equivalent to (P). (3) and (4) fulfil the inequality:

$$\sup_{\lambda \in D^*} \inf_{x \in X} L(x; \lambda) \le \inf_{x \in X} \sup_{\lambda \in D^*} L(x; \lambda), \tag{5}$$

so that *weak duality* holds, whatever (1) may be. The difference between the right-hand side and left-hand side of (5) is called *duality gap*.

A crucial issue in the duality theory for constrained optimization is whether or not a duality gap exists between (P) and its dual problem (D). It is well-known that, whenever (P) and (D) admit an optimal solution, a zero duality gap is equivalent to the existence of a saddle point of the Lagrangian function. In the IS, this is shown to be equivalent to the existence of a regular linear separation between $\mathcal{K}$ and $\mathcal{H}$.

The existence of a saddle point, however, cannot be ensured in the general case of nonconvex constrained optimization problems. Nonlinear separation arguments in the IS can be a useful tool to overcome this difficulty and to extend such results to nonconvex problems: actually, in [7], it has been shown that nonlinear separation in the image space can be related to the existence of a saddle point of a suitable generalized Lagrangian function.

The aim of this paper is to deepen the analysis, in the IS, of the duality theory for a constrained extremum problem, pointing out some connections with classic topics of mathematical programming such as stability. It will be shown that the properties of the conical extension of the image $\mathcal{K}$ are closely related to those of the epigraph of the perturbation function associated with a right-hand side perturbation of (P). In particular, the lower semicontinuity at zero, of the perturbation function, can be equivalently expressed by the disjunction between the closure of the conical extension of $\mathcal{K}$ and the set $\mathcal{H}$. Such result allows us, in turn, to characterize zero duality gap in the IS, by the disjunction between the closure of the convex hull of the conical extension of $\mathcal{K}$ and $\mathcal{H}$.

The paper is organized as follows. In Sect. 2 we will recall the main concepts concerning linear separation and duality in the IS, while Sect. 3 will be devoted to the analysis of the stability of the problem with respect to a right hand-side perturbation. In Sect. 4 the relations with classic duality and conjugation theory will be investigated, leading to a characterization of zero duality gap, in the IS.

Let us introduce some notations which will be used in the sequel. If $M \subseteq \mathbb{R}^n$; cl M, int M, conv M will denote, the closure, the interior, the convex hull of M, respectively. cone M is the cone generated by M; $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$, while

$$C^* := \{z \in \mathbb{R}^n : \langle y, z \rangle \geq 0, \quad \forall y \in C\}$$

is the positive polar of a cone $C \subseteq \mathbb{R}^n$.

The sets

$$\begin{aligned} \operatorname{epi} f &:= \{(x, u) \in X \times \mathbb{R} : u \geq f(x)\} \\ \operatorname{lev}_{\geq \alpha} f &:= \{x \in X : f(x) \geq \alpha\} \end{aligned}$$

are the epigraph and the level set of the function $f : X \longrightarrow [-\infty, +\infty]$, respectively. *co* f denotes the greatest convex function majorized by f; we recall that the epigraph of *co* f coincides with $conv(epi\ f)$ (see e.g., [15]). We denote by *cl* f the closure of f defined by

$$cl\ f(x) := \begin{cases} \liminf_{x' \to x} f(x'), & \text{for all } x, \text{ if } \liminf_{x' \to x} f(x') > -\infty, \text{ for all } x, \\ -\infty, & \text{for all } x, \text{ if } \liminf_{x' \to x} f(x') = -\infty, \text{ for some } x. \end{cases}$$

A function $g : X \longrightarrow \mathbb{R}^m$ is said to be *D-convex* iff

$$g(\alpha x + (1 - \alpha)y) - \alpha g(x) - (1 - \alpha)g(y) \in D, \quad \forall x, y \in X, \quad \forall \alpha \in (0, 1).$$

g is said to be a *convexlike mapping with respect to* D (for short, D*-convexlike*) iff the set $(g(X) + D)$ is convex. g is said to be a *closely convexlike mapping with respect to* D (for short, *closely* D*-convexlike*) iff the set $cl(g(X) + D)$ is convex.

It is known that if g is D-convex, then g is D-convexlike and it is immediate that if g is D-convexlike, then it is closely D-convexlike.

The function $f^* : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ defined by

$$f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) : x \in \mathbb{R}^n\}$$

is called the conjugate of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ while f^{**} will denote the conjugate of the function f^*. We recall that f^{**} coincides with *cl co* f (see [14]).

f is subdifferentiable at $\bar{x} \in \mathbb{R}^n$ iff there exists $\xi \in \mathbb{R}^n$ such that

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \qquad \forall x \in \mathbb{R}^n. \tag{6}$$

The set $\partial f(\bar{x}) := \{\xi \in \mathbb{R}^n : (6) \text{ holds}\}$ is called the "subdifferential" of f at $\bar{x}$.

2 Preliminaries on the image space analysis

In this section, we recall the main features concerning the ISA for the constrained extremum problem (P) defined by (1).

In the ISA the problem of stating whether or not the sets $\mathcal{K}$ and $\mathcal{H}$ have empty intersection plays a crucial role, but proving (2) directly, is in general impracticable. In order to give to the optimality condition (2) a more handy form, we introduce the conical extension of the image $\mathcal{K}$ defined by

$$\mathcal{E} := \mathcal{K} - \mathrm{cl}\, \mathcal{H}. \tag{7}$$

It is known [6] that (2) holds iff

$$\mathcal{E} \cap \mathcal{H} = \emptyset, \tag{8}$$

or, equivalently,

$$\mathcal{E} \cap \mathcal{H}_u = \emptyset. \tag{9}$$

where

$$\mathcal{H}_u := \{(u, v) \in \mathbb{R}^{1+m} : u > 0,\ v = 0\}.$$

The reason that leads to the introduction of the set $\mathcal{E}$ lies in the fact that $\mathcal{E}$ preserves the optimality condition and has, often, better properties than the image $\mathcal{K}$.

In [16] it is proved that $\mathcal{E}$ is convex if and only if the vector function $-A_{\bar{x}}$ is $\mathcal{H}$-convexlike. A consequence of this result is that if (P) is a convex problem (i.e., if f is convex and g is D-convex), then $\mathcal{E}$ is convex, while $\mathcal{K}$ may not.

Such remarkable properties of the conical extension $\mathcal{E}$ allow us to prove (8) by showing that $\mathcal{E}$ and $\mathcal{H}$ lie in two disjoint level sets of a suitable functional: let us consider the case where such a functional can be found linear.

We preliminarly observe that a linear functional separates $\mathcal{K}$ and $\mathcal{H}$, iff it separates $\mathcal{E}$ and $\mathcal{H}$.

Proposition 2.1 [4] *Let* $(\theta^*, \lambda^*) \in \mathcal{H}^* \backslash \{0\}$. *Then the following conditions are equivalent*

(i) $\theta^* u + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{K}$,
(ii) $\theta^* u + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{E}$.

Taking into account the previous result, the analysis of the existence of a linear separation can be equivalently applied to conditions (2) and (8), which leads us to state the following definition [6].

Definition 2.1 The sets $\mathcal{E}$ and $\mathcal{H}$ admit a linear separation, iff there exist $\theta^* \geq 0$ and $\lambda^* \in D^*$ with $(\theta^*, \lambda^*) \neq 0$, such that:

$$\theta^*[f(\bar{x}) - f(x)] + \langle \lambda^*, g(x) \rangle \leq 0, \quad \forall x \in X. \tag{10}$$

If $\theta^* \neq 0$ in (10), then the separation is said to be regular.

When (10) holds, (P) is said to be *linearly separable*.

Next result [6] states that linear separation in the IS is equivalent to the existence of a saddle point for the generalized Lagrangian associated with (1), defined by

$$\mathcal{L}(\theta; \lambda, x) := \theta f(x) - \langle \lambda, g(x) \rangle.$$

Proposition 2.2 *$\mathcal{E}$ and $\mathcal{H}$ admit a linear separation and $\bar{x} \in R$, iff there exist $\theta^* \geq 0$ and $\lambda^* \in D^*$ with $(\theta^*, \lambda^*) \neq 0$, such that $(\lambda^*, \bar{x})$ is a saddle point for $\mathcal{L}(\theta^*; \lambda, x)$ on $D^* \times X$. Moreover,*

$$\theta^* u + \langle \lambda^*, v \rangle = 0, \tag{11}$$

is the equation of the separating hyperplane.

Remark 2.1 The previous theorem has the important consequence that the duality gap between (P) and (D) is equal to zero and, moreover, both problems have an optimal solution if and only if $\mathcal{E}$ and $\mathcal{H}$ admit a regular linear separation.

Next theorem [4] characterizes the existence of a regular linear separation.

Theorem 2.1 *The sets $\mathcal{E}$ and $\mathcal{H}$ admit a regular linear separation, iff*

$$cl\ cone(conv\ \mathcal{E}) \cap \mathcal{H}_u = \emptyset. \tag{12}$$

The analysis of linear separation in the IS can be extended to generalized systems (see [1,7,12]).

3 Separation and perturbation function

In this section we will refer to the following parametric problem:

$$\inf f(x) \text{ s.t. } x \in R(y) := \{x \in X : g(x) \in D + y\}, \tag{$P(y)$}$$

where $y \in \mathbb{R}^m$.

The perturbation function associated with $P(y)$ is defined by $p : \mathbb{R}^m \longrightarrow [-\infty, +\infty]$,

$$p(y) := \begin{cases} \inf_{x \in R(y)} f(x), & \text{if } R(y) \neq \emptyset \\ +\infty, & \text{otherwise.} \end{cases}$$

Our aim is to study, by the ISA, the properties of the perturbation function p. In particular, we will analyse the lower semicontinuity of p at $y = 0$ and the relationships between the existence of a linear separation for the image of (P) and the subdifferentiability of p at $y = 0$.

We will suppose that (P) admits an optimal solution $\bar{x}$, so that $P(0)$ coincides with P.

Consider the image $\mathcal{K}$ associated with (P) at the point $\bar{x} \in R$, and its conical extension, defined by (7), which can be explicitly written as

$$\mathcal{E} = \{(u, v) \in \mathbb{R}^{1+m} : u \leq f(\bar{x}) - f(x), v = g(x) - w, w \in D, x \in X\}.$$

We will show that there exists a close relation between the epigraph of the perturbation function p and the set $\mathcal{E}$.

Let us consider the following transformation $B : \mathbb{R}^{1+m} \longrightarrow \mathbb{R}^{m+1}$ from the image space into itself, defined by

$$B(u, v) := (v, -u + f(\bar{x})). \tag{13}$$

It is immediate that B is an isometry, and an homeomorfism in the IS.

Proposition 3.1 *Let $B : \mathbb{R}^{1+m} \longrightarrow \mathbb{R}^{m+1}$ be defined by* (13). *Then*

$$B(\mathcal{E}) \subseteq \ epi \ p \subseteq cl \ B(\mathcal{E});$$

and, in particular,

$$cl \ B(\mathcal{E}) = cl \ epi \ p.$$

Proof We have that

$$B(\mathcal{E}) = \{(v, u) \in \mathbb{R}^{m+1} : v = g(x) - w, u \geq f(x), w \in D, x \in X\}.$$

Consider the problem $P(y)$ and the related perturbation function p. By definition

$$epi \ p = \{(y, t) \in \mathbb{R}^{m+1} : t \geq \inf_{x \in R(y)} f(x), R(y) \neq \emptyset\}.$$

Observe that, since $x \in R(y)$ iff $y = g(x) - w, w \in D, x \in X$, then

$$epi \ p = \{(y, t) \in \mathbb{R}^{m+1} : y = g(x) - w, t \geq \inf_{x \in R(y)} f(x), w \in D, x \in \ X\}.$$

Therefore

$$B(\mathcal{E}) \subseteq \ epi \ p \subseteq cl \ B(\mathcal{E}). \tag{14}$$

Taking the closures in (14) we complete the proof. □

We will now characterize, in the IS, the lower semicontinuity of the perturbation function. We need the following preliminary lemma.

Lemma 3.1 *Let $f : X \longrightarrow [-\infty, +\infty]$ be finite at $x^0 \in X$. Then f is lower semicontinuous (for short, $l.s.c.$) at x^0 if and only if the set $S := \{(x^0, y) \in X \times \mathbb{R} : y < f(x^0)\}$ has the following property:*

$$cl(epi \ f) \cap S = \emptyset. \tag{15}$$

Proof Necessity. By definition, a function f is l.s.c. at x^0 if and only if

$$\liminf_{x \to x^0} f(x) \geq f(x^0).$$

Let us consider any sequence $(x^k, y^k) \in epi \ f$ such that $x^k \to x^0$; since $y^k \geq f(x^k)$, $\forall k \in \mathbb{N}$, it follows that

$$\liminf_{k \to +\infty} y^k \geq \liminf_{k \to +\infty} f(x^k) \geq \liminf_{x \to x^0} f(x) \geq f(x^0).$$

This implies that the sequence (x^k, y^k) cannot have any subsequence convergent to any point of the set S (even not proper).

Sufficiency. Ab absurdo, suppose that f is not l.s.c. at x^0, that is

$$\liminf_{x \to x^0} f(x) < f(x^0).$$

This implies that there exists a sequence $(x^k, y^k) \in X \times \mathbb{R}$, with

$$(x^k, y^k) \to (x^0, y^*), \quad y^* < f(x^0), \quad y^k = f(x^k),$$

(we may have $y^* = -\infty$). Since $(x^k, y^k) \in epi\ f$, if $y^* \neq -\infty$, then we immediately obtain a contradiction.

If $y^* = -\infty$, then we can consider another sequence $(x^k, t^k) \in epi\ f$ such that

$$(x^k, t^k) \to (x^0, t^*), \quad t^* < f(x^0).$$

This is possible due to the properties of $epi\ f$. Also in this case, we achieve a contradiction. □

Theorem 3.1 *Assume that $\bar{x}$ is an optimal solution for (P). Then the perturbation function $p(y)$ is l.s.c. at $y = 0$ if and only*

$$cl\ \mathcal{E} \cap \mathcal{H}_u = \emptyset \tag{16}$$

Proof From Lemma 3.1, we have that p is l.s.c. at $y = 0$ iff

$$cl(epi\ p) \cap S = \emptyset, \tag{17}$$

with $S := \{(0, y) \in X \times \mathbb{R} : y < p(0)\}$.

From Proposition 3.1, we obtain that (17) is equivalent to

$$cl\ B(\mathcal{E}) \cap S = \emptyset. \tag{18}$$

Consider now the transformation $B^{-1} : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{1+m}$ defined by

$$B^{-1}(v, u) = (-u + p(0), v).$$

Since B^{-1} is an isometry, we have that (18) holds if and only if

$$cl\ \mathcal{E} \cap B^{-1}(S) = \emptyset. \tag{19}$$

Since $p(0) = f(\bar{x})$, we obtain $B^{-1}(S) = \mathcal{H}_u$, which completes the proof. □

A further consequence of Proposition 3.1 is that a regular linear separation between $\mathcal{E}$ and $\mathcal{H}$ is equivalent to the subdifferentiability of $p(y)$ at $y = 0$.

Theorem 3.2 *$p(y)$ is subdifferentiable at $y = 0$ if and only if $\mathcal{E}$ and $\mathcal{H}$ admit a regular linear separation. Moreover $\lambda \in \partial p(0)$ if and only if $(1, \lambda)$ is the gradient of the separating hyperplane.*

Proof Suppose that $\mathcal{E}$ and $\mathcal{H}$ admit a regular linear separation, that is, there exists $\lambda \in D^*$ such that

$$f(\bar{x}) - f(x) + \langle \lambda, g(x) \rangle \leq 0, \quad \forall x \in X. \tag{20}$$

From (20) it follows that $\bar{x}$ is a global minimum point of (P) and moreover

$$\langle \lambda, y \rangle \leq \langle \lambda, g(x) \rangle \leq f(x) - f(\bar{x}), \quad \forall x \in R(y),$$

where the first inequality is due to the fact that

$$\langle \lambda, g(x) - y \rangle \geq 0, \qquad \forall x \in R(y),$$

since $\lambda \in D^*$. Taking the infimum on the set $R(y)$ and recalling that $f(\bar{x}) = p(0)$, we get

$$\langle \lambda, y \rangle \leq \; p(y) - p(0), \quad \forall y \in \mathbb{R}^m, \tag{21}$$

that is, $\lambda \in \partial p(0)$.

Vice versa, suppose that p is subdifferentiable at $y = 0$, i.e., that there exists $\lambda \in \mathbb{R}^m$ such that (21) holds. Observe that, being D a convex cone, it results $D \subseteq D + y$, $\forall y \in -D$, so that

$$0 \geq p(y) - p(0) \geq \langle \lambda, y \rangle, \qquad \forall y \in -D,$$

which implies that $\lambda \in D^*$.

Let $\gamma(y) := \langle \lambda, y \rangle + p(0)$. Since $(y, p(y)) \geq (y, \gamma(y))$, $\forall y \in \mathbb{R}^m$, then $epi\ p \subseteq epi\ \gamma$. From Proposition 3.1, it follows that

$$\mathcal{E} \subseteq B^{-1}(epi\ p) \subseteq B^{-1}(epi\ \gamma),$$

where, we recall, $B^{-1}(v, u) = (-u + p(0), v)$. Let us compute $B^{-1}(epi\ \gamma)$, we obtain:

$$B^{-1}(epi\ \gamma) = \{(u, y) \in \mathbb{R}^{1+m} : u \leq -\gamma(y) + p(0)\} = \{(u, y) : u \leq -\langle \lambda, y \rangle\}.$$

Since $\mathcal{E} \subseteq B^{-1}(epi\ \gamma)$ we have that

$$u + \langle \lambda, v \rangle \leq 0, \quad \forall (u, v) \in \mathcal{E},$$

and the theorem is proved. □

The previous theorem allows us to generalize classic results concerning the relations between the Lagrangean dual of (P) and the perturbation function $p(y)$. In particular, we can prove, without any assumption on the functions involved in problem (P), the following statement which has been established in [2] in the presence of convexity hypotheses.

Proposition 3.2 *Suppose that $\bar{x}$ is a solution for (P). Then the duality gap between (P) and (D) is zero and (D) possesses a solution if and only if $p(y)$ is subdifferentiable at $y = 0$.*

Proof It is known (see e.g. [14]) that the duality gap is zero and both problems (P) and (D) possess a solution if and only if there exist $\theta^* > 0$ and $\lambda^* \in D^*$ such that $(\lambda^*, \bar{x})$ is a saddle point for $\mathcal{L}(\theta; x, \lambda)$ on $X \times D^*$. By Proposition 2.2, this is equivalent to the existence of a regular linear separation: therefore the proposition follows from Theorem 3.2. □

Remark 3.1 It is interesting to notice that in the presence of a linear separation, from Theorem 3.2, it immediately follows that:

$$\lambda \in D^* \text{ is an optimal solution for } (D) \text{ iff } \lambda \in \partial p(0).$$

4 A characterization of zero duality gap in the Image Space

As noted in [12], a consequence of the linear separation is that, considering a nonlinear optimization problem where $-A_{\bar{x}}$ is an $\mathcal{H}$-convexlike function, a duality theorem with zero duality gap can be proved.

In [10], zero duality gap is studied for an infinite-dimensional constrained extremum problem where $-A_{\bar{x}}$ is a nearly $\mathcal{H}$-convex mapping.

By means of the results obtained in Sect. 3, we propose to analyse conditions equivalent to the zero duality gap in the IS, without assuming that (D) admits an optimal solution.

To this aim, we first recall a classical result, proved by Rockafellar in a more general context (see [14], Theorem 7), which relates the optimal value v_D of (D) with the biconjugate of the perturbation function p associated with the parametric problem $P(y)$, introduced in Sect. 3.

Theorem 4.1 $\quad v_D = p^{**}(0) = (cl\ co\ p)(0).$

Proof Define $F : X \times \mathbb{R}^m \longrightarrow [-\infty, +\infty]$:

$$F(x, y) = \begin{cases} f(x), & \text{if } g(x) \in D + y, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the Lagrangian function can be written as

$$L(x, \lambda) = \inf_{y \in \mathbb{R}^m} \{F(x, y) - \langle \lambda, y \rangle\} = \begin{cases} f(x) - \langle \lambda, g(x) \rangle, & \text{if } \lambda \in D^*, \\ -\infty, & \text{otherwise.} \end{cases}$$

and the perturbation function as,

$$p(y) = \inf_{x \in X} F(x, y).$$

Therefore

$$\inf_{x \in X} L(x, \lambda) = \inf_{y \in \mathbb{R}^m} \{\inf_{x \in X} F(x, y) - \langle \lambda, y \rangle\} = - \sup_{y \in \mathbb{R}^m} \{\langle \lambda, y \rangle - p(y)\} = -p^*(\lambda).$$

Finally,

$$\sup_{\lambda \in D^*} \inf_{x \in X} L(x, \lambda) = \sup_{\lambda \in \mathbb{R}^m} \inf_{x \in X} L(x, \lambda) = \sup_{\lambda \in \mathbb{R}^m} [-p^*(\lambda)] = p^{**}(0).$$ □

From now on, we will assume that $\bar{x} \in R$ is an optimal solution for (P).

Next lemma characterizes the epigraph of p^{**} in the IS.

Lemma 4.1 *Let* $B : \mathbb{R}^{1+m} \longrightarrow \mathbb{R}^{m+1}$ *be defined by* (13). *Then*

$$cl\ conv\ B(\mathcal{E}) = epi\ p^{**}.$$

Proof It is known [14] that $epi\ p^{**} = cl\ conv\ (epi\ p)$. By Proposition 3.1,we have

$$B(\mathcal{E}) \subseteq\ epi\ p \subseteq cl\ B(\mathcal{E});$$

taking the convex hulls of the previous sets we obtain

$$conv\ B(\mathcal{E}) \subseteq\ conv(epi\ p) \subseteq conv(cl\ B(\mathcal{E})) \subseteq cl\ conv\ B(\mathcal{E}). \tag{22}$$

The last inclusion follows from the fact that for every set $S \subset \mathbb{R}^n$, it holds [15]:

$$conv(cl\ S) \subseteq cl\ conv(S).$$

Taking the closures in (22) we complete the proof. □

Lemma 4.2 *The following statements are equivalent:*

$$cl\ conv\ \mathcal{E} \cap \mathcal{H}_u = \emptyset, \tag{23}$$

$$p(0) = p^{**}(0). \tag{24}$$

Proof Since B is an isometry, then (23) is equivalent to

$$cl\ conv\ B(\mathcal{E}) \cap B(\mathcal{H}_u) = \emptyset \tag{25}$$

Suppose that (25) holds. As

$$B(\mathcal{H}_u) = \{(0, u) \in \mathbb{R}^{m+1} : u < p(0)\},$$

taking into account Lemma 4.1, we have that (25) implies that $p^{**}(0) \geq p(0)$.

Recalling that

$$p^{**}(y) \leq p(y), \quad \forall y \in \mathbb{R}^m,$$

we obtain (24).

Vice versa, from Lemma 4.1, we immediately obtain that (24) implies (25). □

By Theorem 4.1, the optimal value v_D of the dual problem is equal to $p^{**}(0)$. Therefore Lemma 4.2 leads to the following characterization of zero duality gap in the IS.

Theorem 4.2 *The duality gap between* (P) *and* (D) *is zero if and only if*

$$cl\ conv\ \mathcal{E} \cap \mathcal{H}_u = \emptyset. \tag{26}$$

As expected, (26) is strictly weaker than (12), since (26) does not guarantee that the dual problem (D) admits an optimal solution, as shown by the following example.

Example 4.1 Let $X = \mathbb{R}$, $f(x) = x$, $g(x) = -x^2$, $D = [0, +\infty]$ and set $\bar{x} = 0$. It is simple to show that

$$\mathcal{E} = \{(u, v) \in \mathbb{R} \times \mathbb{R} : v \leq -u^2, u \geq 0\} \cup \{(u, v) \in \mathbb{R} \times \mathbb{R} : v \leq 0, u < 0\},$$

is a closed convex set, and

$$cl\ cone(conv\ \mathcal{E}) = \{(u, v) \in \mathbb{R} \times \mathbb{R} : v \leq 0\}.$$

Since $\mathcal{H}_u = \{(u, v) \in \mathbb{R} \times \mathbb{R} : u > 0, v = 0\}$, then (26) is fulfilled, while (12) is not.

As an application of the analysis developed in the IS, we now prove a characterization of the optimal value v_D of the dual problem, in the presence of generalized convexity assumptions on the function $A_{\bar{x}}$. A similar result has been obtained by Frenk and Kassay [3], following a different approach.

Theorem 4.3 *Assume that the function* $-A_{\bar{x}}$ *is* $(cl\mathcal{H})$*-closely convexlike. Then*

$$v_D = \liminf_{y \longrightarrow 0} p(y).$$

Proof By the hypothesis the set $cl(-A_{\bar{x}}(X) + cl\mathcal{H}) = cl(-\mathcal{E})$ is convex. Therefore $cl\ \mathcal{E}$ is convex, which implies, by Proposition 3.1, that $cl(epi\ p)$ is convex, taking into account that the transformation B is an isometry. Since $epi\ p \subseteq epi(co\ p)$ then

$$cl(epi\ p) \subseteq cl\ epi(co\ p).$$

Moreover,

$$epi(co\ p) = conv(epi\ p) \subseteq conv(cl\ epi\ p) = cl\ epi\ p,$$

which implies

$$cl(epi\ p) \supseteq cl\ epi(co\ p),$$

so that $cl(epi\ p) = cl\ epi(co\ p)$. Recall that

$$cl(epi\ p) = epi(lsc\ p),$$

where

$$lsc\ p(y) := \liminf_{y' \longrightarrow y} p(y'),$$

(see e.g., [14]). Applying Theorem 4.1, we complete the proof. □

A further consequence of Theorems 3.1 and 4.2, is the following generalization of a classic result on zero duality gap, which can also be derived from Theorem 4.3.

Theorem 4.4 *Assume that $-A_{\bar{x}}$ is $(cl\mathcal{H})$-closely convexlike. Then the duality gap between (P) and (D) is zero if and only if $p(y)$ is lower semicontinuous at $y = 0$.*

Proof Similarly to the proof of Theorem 4.3, we have that $cl\ \mathcal{E}$ is convex, which implies that

$$cl\ \mathcal{E} = cl\ conv\ \mathcal{E}.$$

Applying Theorems 3.1 and 4.2, we complete the proof. □

5 Concluding remarks

Classic Lagrangian duality may receive a natural interpretation by means of the linear separation between two sets, $\mathcal{K}$ and $\mathcal{H}$, in the Image Space. We have considered sensitivity analysis for a right-hand side perturbation of the constraints and have shown that there exists a close relation between the epigraph of the perturbation function and the conical extension of the image $\mathcal{K}$ associated with the original (non perturbed) extremum problem.

In particular, we have characterized the lower semicontinuity at 0 of the perturbation function p, and we have shown that the existence of a regular linear separation between $\mathcal{K}$ and $\mathcal{H}$, is equivalent to the fact that p is subdifferentiable at 0.

The analysis of the properties of the perturbation function has, in turn, led to a characterization of zero duality gap in the Image Space.

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